

# A Long Noncrossing Path Among Disjoint Segments in the Plane

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ABSTRACT. Let  $\mathcal{L}$  be a collection of  $n$  pairwise disjoint segments in general position in the plane. We show that one can find a subcollection of  $\Omega(n^{1/3})$  segments that can be completed to a noncrossing simple path by adding rectilinear edges between endpoints of pairs of segments. On the other hand, there is a set  $\mathcal{L}$  of  $n$  segments for which no subset of size  $(2n)^{1/2}$  or more can be completed to such a path.

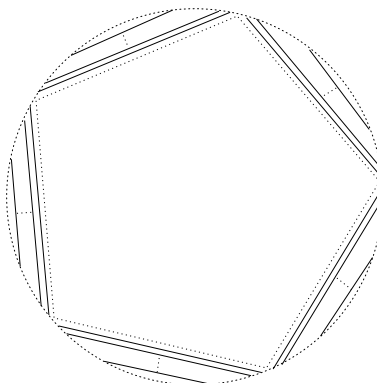
## 1. Introduction

Since the publication of the seminal paper of Erdős and Szekeres [1935], many similar results have been discovered, establishing the existence of various regular subconfigurations in large geometric arrangements. The classical tool for proving such theorems is Ramsey theory [Graham et al. 1990]. However, the size of the regular substructures guaranteed by Ramsey's theorem are usually very small (at most logarithmic) in terms of the size  $n$  of the underlying arrangement. In most cases, the results are far from optimal. One can obtain better bounds ( $n^\varepsilon$  for some  $\varepsilon > 0$ ) by introducing some linear orders on the elements of the arrangement and applying some Dilworth-type theorems [1950] for partially ordered sets [Pach and Törőcsik 1994; Larman et al. 1994; Pach and Tardos 2000]. A simple one-dimensional prototype of such a statement is the Erdős-Szekeres lemma: any sequence of  $n$  real numbers has a monotone increasing or monotone decreasing subsequence of length  $\lceil \sqrt{n} \rceil$ . In this note, we give a new application of this idea.

A collection  $\mathcal{L}$  of segments in the plane is in *general position* if no two elements of  $\mathcal{L}$  are parallel, all of their endpoints are distinct, and no three endpoints are collinear. A polygonal path  $P = p_1p_2 \dots p_n$  is called *simple* if no pair of its vertices coincide, i.e.,  $p_i \neq p_j$  whenever  $i \neq j$ . It is called *noncrossing* if no

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**Figure 1.** An arrangement of segments showing that  $f(n) \leq 2\sqrt{2n}$ .

two edges share an interior point. A polygonal path  $P$  is called *alternating* with respect to  $\mathcal{L}$  if every other edge of  $P$  belongs to  $\mathcal{L}$ .

We consider the following old problem of unknown origin: what is the maximum length  $f(n)$  of an alternating path that can be found in any collection of  $n$  pairwise disjoint segments in the plane in general position? This question appears in a list of open problems in computational geometry collected and annotated by Urrutia [2002]. The easy construction described there can be slightly improved to show that  $f(n) \leq 2\sqrt{2}\sqrt{n}$  for  $n = 2k^2$ . Consider a  $2k$ -gon inscribed in a circle  $C$  and replace each of its edges  $e$  with  $k$  pairwise disjoint chords of  $C$ , almost parallel to  $e$ , that are farther away from the center of  $C$  than  $e$  is. (See Figure 1.) It seems likely that the order of magnitude of this bound is not far from optimal. For some similar problems, see [Hoffmann and Tóth 2003; Mirzaian 1992; Pach and Rivera-Campo 1998; Rappaport et al. 1990].

First we consider the special case when all segments cross the same line.

**THEOREM 1.** *Let  $\mathcal{L}$  be a collection of  $n$  pairwise disjoint segments in general position in the plane, all of whose members cross a given line. Then one can select  $\Omega(n^{1/2})$  segments from  $\mathcal{L}$  that can be completed to a noncrossing simple alternating path.*

**THEOREM 2.** *The maximum length  $f(n)$  of an alternating path that can be found in any collection of  $n$  pairwise disjoint segments in the plane satisfies  $f(n) = \Omega(n^{1/3})$ .*

**PROOF OF THEOREM 2 ASSUMING THEOREM 1.** By the Dilworth theorem (for example), any collection  $\mathcal{L}$  of  $n$  pairwise disjoint segments has a subcollection  $\mathcal{L}_1$  consisting of least  $n^{1/3}$  segments whose projections to the  $x$ -axis are pairwise disjoint, or a subcollection  $\mathcal{L}_2$  consisting of at least  $n^{2/3}$  segments, all of which can be crossed by a line parallel to the  $y$ -axis. In the first case, the elements of  $\mathcal{L}_1$  can be connected to form an alternating path. In the second case, we can apply Theorem 1.  $\square$

### 2. Proof of Theorem 1

Assume without loss of generality that all segments cross the  $y$ -axis, no two of them are parallel, and all  $2n$  coordinates of their endpoints are distinct. The above-below relation between the crossings of the segments with the  $y$ -axis induces a natural linear order on the elements of  $\mathcal{L}$ . We apply the Erdős-Szekeres lemma to find a subsequence of  $\mathcal{L}$  consisting of  $\lceil \sqrt{n} \rceil$  segments with increasing or decreasing slopes with respect to this order. Since we can always flip the plane about the  $y$ -axis, we may assume that the slopes of the elements of this subsequence are monotone increasing. In what follows, for convenience we assume that  $\sqrt{n}$  and all other numbers that appear in the argument (except the coordinates of the endpoints) are integers satisfying the necessary divisibility conditions so that we do not have to use “floor” and “ceiling” operations. This will not effect the asymptotic results obtained in this paper.

To be more precise, we find a sequence of at least  $\sqrt{n}$  segments  $s_1, \dots, s_m$  ( $m = \sqrt{n}$ ) of  $\mathcal{L}$  such that if  $i < j$ , then  $s_i$  is above  $s_j$  and the slope of  $s_i$  is smaller than that of  $s_j$  (see Figure 2).

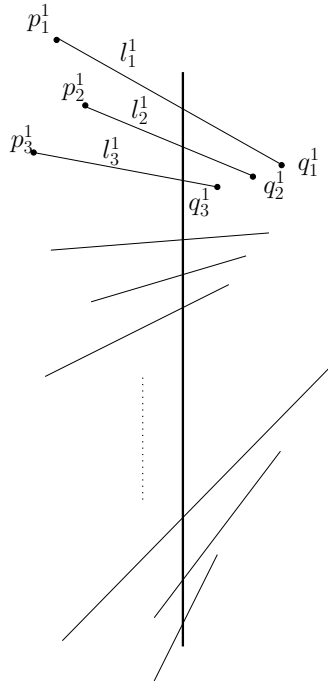
Partition  $s_1, \dots, s_m$  into  $k = m/5$  groups, each consisting of 5 consecutive segments. That is, let  $G_i = \{s_{5(i-1)+1}, \dots, s_{5(i-1)+5}\}$  for every  $1 \leq i \leq k$ . For each  $G_i$ , apply again the Erdős-Szekeres lemma and find a subsequence of 3 segments such that the  $x$ -coordinates of their right endpoints form an increasing or a decreasing sequence. By flipping the plane about the  $x$ -axis, if necessary, we can also assume that for at least half of the  $G_i$ s, these sequences are decreasing. From now on, we disregard all other segments. Summarizing: we now have  $k/2$  groups  $L_1, \dots, L_{k/2}$ , each consisting of 3 elements of  $\mathcal{L}$ . For each  $1 \leq i \leq k/2$ , let  $L_i = \{\ell_1^i, \ell_2^i, \ell_3^i\}$ , where  $\ell_b^a$  is above  $\ell_b^{a'}$  and its slope is smaller, whenever  $a < a'$ , or if  $a = a'$  and  $b < b'$ . Moreover, for a fixed  $a$  and any  $b < b'$ , the  $x$ -coordinate of the right endpoint of  $\ell_b^a$  is larger than that of  $\ell_{b'}^a$ . Let  $\mathcal{S} := L_1 \cup \dots \cup L_{k/2}$ .

Denote by  $p_b^a$  and  $q_b^a$  the left endpoint and the right endpoint of  $\ell_b^a$ , respectively. For any two points  $r, s$ , let  $[r, s]$  stand for the segment connecting  $r$  and  $s$ .

Define a set of auxiliary segments as follows. For  $1 \leq a \leq k/2$  and  $b = 1, 2$ , let  $z_b^a = [q_b^a, q_{b+1}^a]$ . We say that  $z_b^a$  is *bad*, if there is a segment in  $\mathcal{S}$  that meets the interior of  $z_b^a$ . For any segment  $\ell_j^t \in \mathcal{S}$  meeting the interior of  $z_b^a$ , we have  $t > a$ , because all elements of  $\cup_{t < a} L_t$  lie strictly above  $z_b^a$ , otherwise they would cross  $\ell_b^a$ . Define the *witness index* of a bad segment  $z_b^a$  as the smallest index  $t > a$  with the property that there exists an  $\ell_j^t$  meeting the interior of  $z_b^a$ .

LEMMA 2.1. *If the witness index of a bad segment  $z_b^a$  is  $t$ , then  $\ell_1^t$  meets  $z_b^a$ . Moreover,  $q_1^t$  must belong to the interior of the region enclosed by the  $y$ -axis,  $\ell_b^a$ ,  $\ell_{b+1}^a$ , and  $z_b^a$ .*

PROOF. We know that  $t > a$  and that for some  $j$  the segment  $\ell_j^t$  crosses  $z_b^a$ . Assume that  $j > 1$ . Let  $W$  denote the region bounded by the  $y$ -axis,  $\ell_{b+1}^a$ ,  $z_b^a$ , and  $\ell_j^t$ . The segment  $\ell_1^t$  lies above  $\ell_j^t$ , and the  $x$ -coordinate of its right endpoint



**Figure 2.** The segments  $l_1^i, l_2^i, l_3^i$ .

$q_1^t$  is larger than the  $x$ -coordinate of  $q_j^t$ . Clearly, the intersection point  $r$  of  $\ell_j^t$  and  $z_b^a$  is the rightmost corner of the boundary of  $W$ . There is a point on  $\ell_1^t$  whose  $x$ -coordinate is the same as that of  $r$ . This point must lie above  $r$  and outside the region  $W$ . Since  $\ell_1^t$  crosses the  $y$ -axis above  $\ell_j^t$  and below  $\ell_{b+1}^a$ , at a boundary point of  $R$ , and it has a point outside  $W$ , it must have another crossing with the boundary of  $W$ . Using the fact that the elements of  $\mathcal{S}$  are pairwise disjoint, this second crossing must belong to  $z_b^a$ .

As for the second part of the lemma, let  $R$  denote the region bounded by the  $y$ -axis,  $\ell_b^a$ ,  $\ell_{b+1}^a$ , and  $z_b^a$ . We have seen that  $\ell_1^t$  meets the boundary of  $R$  (at a point of  $z_b^a$ ). Since  $\ell_1^t$  is disjoint from both  $\ell_b^a$  and  $\ell_{b+1}^a$ , and it intersects the  $y$ -axis below  $\ell_{b+1}^a$ , it follows that  $\ell_1^t$  cannot cross the boundary of  $R$  a second time. Therefore,  $q_1^t$  must belong to the interior of  $R$ .  $\square$

LEMMA 2.2. *No two different bad segments can have the same witness index.*

PROOF. Assume to the contrary that  $t$  is the witness index of two bad segments,  $z_b^a$  and  $z_{b'}^{a'}$ . Suppose without loss of generality that  $\ell_b^a$  lies above  $\ell_{b'}^{a'}$ . We know that both of them lie above  $\ell_1^t$ . As in the proof of Lemma 2.1, let  $R$  denote the region bounded by the  $y$ -axis,  $\ell_b^a$ ,  $\ell_{b+1}^a$ , and  $z_b^a$ . Similarly, let  $R'$  denote the region bounded by the  $y$ -axis,  $\ell_{b'}^{a'}$ ,  $\ell_{b'+1}^{a'}$ , and  $z_{b'}^{a'}$ .  $R$  and  $R'$  do not overlap. Indeed, since the elements of  $\mathcal{S}$  are pairwise disjoint,  $R$  and  $R'$  could overlap only if  $\ell_{b'}^{a'}$  crossed  $z_b^a$ . However, this would contradict the minimality of  $t$ .

On the other hand, by Lemma 2.1,  $\ell_1^t$  must intersect both  $z_b^a$  and  $z_{b'}^{a'}$ , and its right endpoint  $q_1^t$  must belong to the interiors of both  $R$  and  $R'$ . We thus obtained the desired contradiction.  $\square$

Now we are in a position to prove Theorem 1.

By Lemma 2.2, the number of bad segments is at most  $k/2$ . We say that an index  $i$  ( $1 \leq i \leq k/2$ ) is *good* if at least one of the segments  $z_1^i, z_2^i$  is not bad. Obviously, at least  $k/2 - \frac{1}{2}(k/2) = k/4$  indices between 1 and  $k/2$  are good. Assume without loss of generality that the first  $k/4$  indices are good. To complete the proof it is sufficient to show how to draw a noncrossing simple alternating path  $P$  that uses the segments  $\ell_2^i, \ell_3^i$  (and perhaps even  $\ell_1^i$ ) for  $1 \leq i \leq k/4 = \Omega(\sqrt{n})$ .

Let the first points of  $P$  be  $q_1^1, p_1^1, q_2^1, p_2^1, q_3^1, p_3^1$ , in this order. That is, so far we have built a “zigzag” path that uses the segments  $\ell_1^1, \ell_2^1, \ell_3^1$ . Since 2 is a good index, there exists a segment  $z_j^2$  ( $j = 1$  or  $2$ ) which is not bad. Let us extend  $P$  by adding the vertices  $p_j^2, q_j^2, q_{j+1}^2$ , and hence adding the edges  $\ell_j^2$  (from left to right) and  $z_j^2$ . Next we can add the point  $p_{j+1}^2$  and, if  $j = 1$ , also the points  $q_3^2, p_3^2$ , zigzagging just like before. Continuing in the same manner, we build a path  $P$  using at least two edges from each group  $L_i$  ( $i \leq k/4$ ). It is easy to check that  $P$  is a *noncrossing* path, because (1) its edges belonging to  $\mathcal{L} \subset \mathcal{S}$  are pairwise disjoint; (2) its edges to the left of the  $y$ -axis do not cross any other edge, by the assumption that the slopes of the elements of  $\mathcal{S}$  form an increasing sequence; (3) its edges to the right of the  $y$ -axis are not bad, therefore they do not cross any other edge of  $P$ . This completes the proof of Theorem 1.

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