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# On Hadwiger Numbers of Direct Products of Convex Bodies

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ABSTRACT. The Hadwiger number $H(K)$ of a $d$-dimensional convex body $K$ is the maximum number of mutually nonoverlapping translates of $K$ that can touch $K$. We define $H^*(K)$ analogously, with the restriction that all touching translates of $K$ are pairwise disjoint. In this paper, we verify a conjecture of Zong [1997] by showing that for any $d_1, d_2 \geq 3$ there exist convex bodies $K_1$ and $K_2$ such that $K_i$ is $d_i$-dimensional, $i = 1, 2$, and $H(K_1 \times K_2) > (H(K_1)+1)(H(K_2)+1)-1$ holds, where $K_1 \times K_2$ denotes the direct product of $K_1$ and $K_2$. To obtain the inequality, we prove that if $K$ is the direct product of $n$ convex discs in the plane and there are exactly $k$ parallelograms among its factors, then $H^*(K) = 4^k(4{\cdot}6^{n-k}+1)/5$. Based on this formula, we also establish that for every $d \geq 3$ there exists a strictly convex $d$-dimensional body $K$ fulfilling $H(K) \geq \frac{16}{35}(\sqrt{7})^{d-1}$.

## 1. Introduction and Main Results

The *Hadwiger number* $H(K)$ of a $d$-dimensional convex body $K$ is the maximum number of mutually nonoverlapping translates of $K$ that can be arranged so that all touch $K$. Often $H(K)$ is called the *translative kissing number of* $K$ as well. $H^*(K)$ is defined analogously with the restriction that all touching translates of $K$ are pairwise disjoint. Trivially, $H^*(K) \leq H(K)$. It is known that $H(K) \leq 3^d - 1$ [Hadwiger 1957], with equality attained only for parallelotopes [Groemer 1961].

Let $A_i \subseteq \mathbb{R}^{d_i}$, $i = 1, 2 \ldots, n$, for some positive integer $n$. We denote by $A_1 \times A_2 \times \ldots \times A_n$ the direct product of the $A_i$'s in their given order, which is the collection of the ordered $n$-tuples $\{(x_1, x_2, \ldots x_n) \mid x_i \in A_i, 1 \leq i \leq n\}$, and

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it is identified with a subset of $\mathbb{R}^d$, where $d = \sum_{i=1}^{n} d_i$, by writing the coordinates of the $x_i$'s consecutively in one $d$-tuple. It is also called the Cartesian product of the $A_i$'s, and sometimes it is denoted by $\prod_{i=1}^{n} A_i$ as well. Clearly, the direct product of convex bodies is also a convex body. If $A \subseteq \mathbb{R}^d$, then $A^n$ stands for the direct product of $n$ copies of $A$.

Let $K_i$ be a $d_i$-dimensional convex body, for $i = 1, 2$. Observe that if $\mathcal{C}_i$ is a packing with translates of $K_i$, $i = 1, 2$, then $\mathcal{C}_1(\times)\mathcal{C}_2 = \{A \times B \mid A \in \mathcal{C}_1, B \in \mathcal{C}_2\}$ is a packing with translates of $K_1 \times K_2$. By looking at which translates of $K_1 \times K_2$ touch each other in $\mathcal{C}_1(\times)\mathcal{C}_2$, we get the general inequality

$$H(K_1 \times K_2) \geq (H(K_1) + 1)(H(K_2) + 1) - 1. \tag{1–1}$$

Zong [1997] proved that there is equality in (1–1) when $\min(d_1, d_2) \leq 2$. He also conjectured that there are some large integers $d_1, d_2$ for which inequality (1–1) is strict for suitable $d_i$-dimensional convex bodies $K_i$, $i = 1, 2$. In the following theorem, we verify Zong's conjecture, and we even show that more is true: We provide examples for a strict inequality in (1–1) for every $d_1, d_2 \geq 3$.

Theorem 1.1. *For every $d_1, d_2 \geq 3$, there is a $d_1$-dimensional convex body $K_1$ and a $d_2$-dimensional convex body $K_2$ such that*

$$H(K_1 \times K_2) \geq (H(K_1) + 1)(H(K_2) + 1) + 16 \cdot 3^{d_1 + d_2 - 6} - 1 \tag{1–2}$$

*holds.*

To prove Theorem 1.1, we rely on the value of $H^*(K)$ when $K$ is the direct product of two circles. In the following proposition, we determine that quantity in a more general setting when the convex body is the direct product of finitely many arbitrary convex discs. (By a convex disc we always mean a two-dimensional convex body.)

Proposition 1.2. *Let $D_1, D_2, \ldots D_n$ be convex discs, $n \geq 1$. If there are exactly $k$ parallelograms among the discs, then*

$$H^*(D_1 \times D_2 \times \ldots \times D_n) = 4^k \left( \frac{4(6^{n-k}) + 1}{5} \right) \tag{1–3}$$

*holds.*

Note that one can prove

$$H^*(K_1 \times K_2) \geq H^*(K_1) H^*(K_2) \tag{1–4}$$

the same way as (1–1). Proposition 1.2 shows that there can be strict inequality in (1–4), e.g., that is the case when $K_1$ and $K_2$ are convex discs that are different from parallelograms.

Since $H(K) < 3^d - 1$ for any convex body $K$ different from a parallelotope [Groemer 1961], one may ask how large $H(K)$ can be, when $K$ belongs to some specific class of convex bodies, in which the shapes of the bodies are very different from a parallelotope, for example, when the bodies are strictly convex, i.e., their boundaries contain no segment. Based on Proposition 1.2, for every $d \geq 3$ we are able to show the existence of a strictly convex $d$-dimensional body, having relatively large Hadwiger number, as $d \to \infty$.

THEOREM 1.3. *For every $d \geq 3$ there exists a strictly convex $d$-dimensional body $S$ such that*

$$H(S) \geq \frac{16}{35}(\sqrt{7})^d \approx 2.6457^{d-o(d)} \tag{1–5}$$

*holds.*

The *lattice kissing number* $H_L(K)$ is the maximum number of those translates that touch $K$ in a lattice packing of $K$. Trivially, $H_L(K) \leq H(K)$. Although in several cases $H(K) = H_L(K)$ holds (for example, it holds for every convex disc by [Grünbaum 1961]), it can happen that $H(K) - H_L(K) > 0$ [Zong 1994]. In fact, $H(K) - H_L(K) \geq 2^{d-1}$ holds for some $d$-dimensional convex body $K$, for every $d \geq 3$ [Talata 1998a], showing that there can be exponentially large gap between $H(K)$ and $H_L(K)$. Minkowski [1896/1910] (see also [Cassels 1959]) showed that $H_L(K) \leq 2^{d+1} - 2$ holds for strictly convex $d$-dimensional bodies; thus Theorem 1.3 implies new asymptotic bounds showing that both the gap and the ratio between $H(K)$ and $H_L(K)$ can be relatively large.

COROLLARY 1.4. *For every integer $d \geq 1$, denote by $\mathcal{K}_d$ the collection of all $d$-dimensional convex bodies. Then*

$$\max_{K \in \mathcal{K}_d} (H(K) - H_L(K)) \geq (\sqrt{7})^{d-1} - 2^{d+1} + 2 \approx 2.6457^{d-o(d)} \tag{1–6}$$

*and*

$$\max_{K \in \mathcal{K}_d} (H(K)/H_L(K)) \geq \tfrac{8}{35}\left(\sqrt{7}/2\right)^d \approx 1.3228^{d-o(d)} \tag{1–7}$$

*hold.*

We would like to note that the proofs of Theorems 1.1 and 1.3 (and the proof of the lower bound part for $H^*\left(\prod_{i=1}^n D_i\right)$ in Proposition 1.2) are constructive: thus, general methods are given to construct bodies of different shapes, based on some initially given convex discs and some parameters, implying for example, that with respect to the Hausdorff metric, in every neighbourhood of any $d$-dimensional parallelotope there are convex bodies which possess the properties described in the Theorems 1.1 and 1.3. Furthermore, when the initial discs are unit circles, we can calculate the actual values for those parameters to make the definitions of those bodies.

We conclude this section with two conjectures. First, we suggest that there may not be any kind of analogue of Zong's formula [1997] for $H(K_1 \times K_2)$ when $K_1$ and $K_2$ are sufficiently high dimensional convex bodies. That is, we conjecture that $H(K_1 \times K_2)$ can not be expressed as a function of $H(K_1)$ and $H(K_2)$ in general.

Conjecture 1.1. *For some $d_1, d_2 \geq 3$, there exist a pair $(K_1, K_1')$ of $d_1$-dimensional convex bodies and a pair $(K_2, K_2')$ of $d_2$-dimensional convex bodies such that $H(K_1) = H(K_1')$ and $H(K_2) = H(K_2')$, but $H(K_1 \times K_2) \neq H(K_1' \times K_2')$.*

Second, we consider a quantity similar to $H(K)$: The touching number $t(K)$ of a $d$-dimensional convex body $K$ is defined as the maximum number of mutually touching translates of $K$. We have $t(K) \leq 2^d$, with equality exactly for parallelotopes [Danzer and Grünbaum 1962]. It is conjectured in [Füredi et al. 1991] that for strictly convex bodies $t(K) \leq (2-\varepsilon)^d$ holds for some $\varepsilon > 0$. We conjecture the analogous inequality for $H(K)$ in case of strictly convex bodies.

Conjecture 1.2. *There exists an absolute constant $\varepsilon > 0$ such that*

$$H(K) \leq (3-\varepsilon)^d \tag{1–8}$$

*holds whenever $K$ is a strictly convex $d$-dimensional body.*

We organize the remaining part of the paper in the following way: In Section 2 we introduce notation and recall some facts. Then we prove Proposition 1.2, Theorem 1.1 and Theorem 1.3 in Sections 3, 4 and 5, respectively. In those sections we usually prove various statements organized in lemmas and propositions so that we can combine them to get the desired theorem or proposition.

## 2. Preliminaries

For arbitrary $A, B \subseteq \mathbb{R}^d$ and $\alpha, \beta \in \mathbb{R}$, let $\alpha A + \beta B = \{\alpha a + \beta b \mid a \in A, b \in B\}$. We write $A + v$ instead of $A + \{v\}$, and further, we write $A - B$ instead of $A + (-1)B$. The notation $\operatorname{conv}(\cdot)$ stands for the convex hull and $[a, b]$ stands for the segment whose endpoints are $a, b \in \mathbb{R}^d$. If $c \in \mathbb{R}$, then $\{c\}$ denotes the fractional part of $c$, that is, $\{c\} = c - [c]$, where $[c]$ is the largest integer which does not exceed $c$. In the text, we always avoid confusion with the similar notation for a one-element set by using fractional parts only in inequalities. We denote by $|S|$ the cardinality of a set $S$. We use the notation $o_d$ for the origin of $\mathbb{R}^d$.

We denote by $\partial K$ the boundary of a convex body $K$. For an $o_d$-symmetric convex body $K$, let $\operatorname{dist}_K$ be the distance function of the Minkowski metric whose unit ball is $K$. Note that we denote the usual Euclidean distance simply by dist. Recall that $\operatorname{dist}_{K_1 \times K_2 \times \ldots \times K_n} = \max_{1 \leq i \leq n}(\operatorname{dist}_{K_i})$. In a metric space, a set $S$ is called *$r$-discrete* for some $r > 0$ if the distance between any two points of $S$ is at least $r$ in the given metric. If the distance is larger than $r$ we say $S$ is *$r^+$-discrete.*

A *Hadwiger configuration* of a convex body $K$ is a collection of mutually nonoverlapping translates of $K$ which all touch $K$. It is easy to see that any collection $\{K + v_i\}_{i=1}^n$ of translates of a convex body $K$ is a Hadwiger configuration of $K$ if and only if $v_i \in \partial(K - K)$ and $\operatorname{dist}_{K-K}(v_i, v_j) \geq 1$, for every $i \neq j$ [Talata 1998b]. Clearly, $K + v_i$ and $K + v_j$ are touching if and only if

$\text{dist}_{K-K}(v_i, v_j) = 1$. Thus $H(K)$ is the maximum cardinality of a 1-discrete subset of $\partial(K-K)$ in the metric $\text{dist}_{K-K}$. Furthermore, it is not difficult to see that $H(K)+1$ is the maximum cardinality of a 1-discrete subset $S \subseteq K-K$ in the metric $\text{dist}_{K-K}$, and $|S| = H(K)+1$ holds only if $o_d \in S$. (To see this, observe that if $S \subseteq K-K$ is 1-discrete with respect to $\text{dist}_{K-K}$, then replacing each $p \in S \setminus \{o_d\}$ with that $q \in \partial(K-K)$ for which $p \in [o_d, q]$, we get a 1-discrete set with respect to $\text{dist}_{K-K}$.) Similarly, one can obtain that $H^*(K)$ is the maximum cardinality of a $1^+$-discrete subset of $K-K$ in the metric $\text{dist}_{K-K}$. Note if $K$ is $o_d$-symmetric, then $K-K$ can be replaced by $K$ in the preceding characterizations of $H(K)$ and $H^*(K)$, since then $K-K = 2K$.

## 3. Determining $H^*(K)$ for Direct Products of Convex Discs

In this section, we prove Proposition 1.2. First we prove several lemmas, then we combine those to get the actual proof of the proposition. Note that in some cases we even allow 0-dimensional convex bodies to appear as factors in direct products for sake of completeness. Observe that $K_1 \times K_2 \cong K_2$ when $K_1$ is a 0-dimensional convex body (i.e., $K_1$ is a point).

Lemma 3.1. *Let $K$ be a $d$-dimensional convex body, $d \geq 0$, and let $I$ be a segment. Then $H^*(K \times I) = 2H^*(K)$.*

Proof. We may assume that $K$ is $o_d$-symmetric and $I = [-1, 1]$. If $S \subseteq K$ is $1^+$-discrete in the metric $\text{dist}_K$, then $S \times \{-1, 1\}$ is $1^+$-discrete in the metric $\text{dist}_{K \times I}$, implying $H^*(K \times I) \geq 2H^*(K)$. On the other hand, if $S \subseteq K \times I$ is $1^+$-discrete in the metric $\text{dist}_{K \times I}$, then let $S_1 = S \cap (K \times [-1, 0])$, and $S_2 = S \cap (K \times [0, 1])$. Now, let $\pi : K \times I \to K$ be the projection of the direct product body to the first factor. Then both $\pi(S_1)$ and $\pi(S_2)$ are $1^+$-discrete subsets of $K$ in the metric $\text{dist}_K$, implying $H^*(K \times I) \leq 2H^*(K)$. $\square$

Observe that an immediate consequence of Lemma 3.1 is that $H^*(K \times P) = 2^n H(K)$ holds if $P$ is an $n$-dimensional parallelotope, $n \geq 1$.

Lemma 3.2. *Let $K$ be a $d$-dimensional convex body, $d \geq 0$, and let $D$ be a convex disc. Then $H^*(K \times D) \leq 6H^*(K) - 1$.*

Proof. We may assume that both $K$ and $D$ are symmetric about the origin. Let $S \subseteq K \times D$ be $1^+$-discrete in the metric $\text{dist}_{K \times D}$, and let $s_0 \in S$. Define $\pi_1 : K \times D \to K$ and $\pi_2 : K \times D \to D$ as projections of the direct product body to its first and second factor, respectively. Consider an affine regular hexagon $H$ inscribed to $D$, having vertices $v_1, v_2, \ldots, v_6$. We may even assume that $H$ is chosen in a way that $\pi_2(s_0) \in [o_2, v_1]$; see [Fejes Tóth 1972]. Then the segments $[o_2 v_i]$, $1 \leq i \leq 6$, divide $H$ into six regions $U_i$, $1 \leq i \leq 6$, such that their diameters are equal to 1 in the metric $\text{dist}_D$, and $\bigcup_{i=1}^{6} U_i = D$. Let $S_i = S \cap (K \times U_i)$. Then $\pi_1(S_i) \subseteq K$ is $1^+$-discrete in the metric $\text{dist}_K$, implying $|S_i| \leq H^*(K)$ for each $i$. But $s_0$ is contained in two $S_i$'s, implying $H^*(K \times I) \leq 6H^*(K) - 1$. $\square$

LEMMA 3.3. *Let $C$ be a centrally symmetric convex disc, different from a parallelogram, and let $k$ be a positive integer. Let $m = 6k-1$. Then there exists a sequence $S = \langle s_i \rangle_{i=0}^{m-1}$ of points of $\partial C$ such that for every $i \geq j$, $\mathrm{dist}_C(s_i, s_j) > 1$ holds if and only if $\frac{1}{6} < \{\frac{i-j}{m}\} < \frac{5}{6}$ is satisfied.*

PROOF. If $C$ is a circle, then it is easy to check that $\langle s_i \rangle_{i=0}^{m-1}$ can be chosen as consecutive vertices of a regular $m$-gon. For general $C$, we describe a little bit more sophisticated construction for $S$. Pick an affine regular hexagon $H$ that is inscribed to $C$. Since $C$ is not a parallelogram, it can be seen that we can choose $H$ in a way that no side of $H$ is longer than any segment in $\partial C$ which is parallel to that side. Fix a positive constant $\varepsilon < 1$. If $s_i'$ is already defined, let $v_{i+1}$ be the first point chosen on $\partial C$ in counterclockwise direction for which $\mathrm{dist}_C(v_i, v_{i+1}) = 1 + \varepsilon$. Let $V = \langle v_i \rangle_{i=0}^{m-1}$. It is easy to check that if $\varepsilon$ is small enough, then every point of $V$ lies in a small neighbourhood of some vertex of $H$, and $v_0, v_6, v_{12}, \ldots v_1, v_7 \ldots$ etc. are consecutive points on $\partial C$. Now, to get $S$, order the points of $V$ consecutively in counterclockwise direction, starting with $v_0$, so $s_0 = v_0, s_1 = v_6, s_2 = v_{12} \ldots$ etc. It is easy to check that for any $i \geq j$, $\mathrm{dist}_C(s_i, s_j) > 1$ holds if and only if $\min(|i-j|, |m+j-i|) \geq k$, from which one can get that $S$ has the property required in the lemma. □

LEMMA 3.4. *Let $n$ and $q$ be integers, $n \geq 1$, $q \geq 3$. Let*

$$m = \frac{(q-2) \cdot q^n + 1}{q-1}.$$

*Then for every positive integer $j \leq m-1$, there is an integer $i$, $1 \leq i \leq n$, such that*

$$\frac{1}{q} < \left\{ \frac{q^{n-i} j}{m} \right\} < \frac{q-1}{q}.$$

PROOF. Define a sequence by $a_0 = 1$ and $a_i = q a_{i-1} - 1$, for every $i \geq 1$. It is easy to check that $a_n = q^n - \sum_{i=0}^{n-1} q^i = ((q-2)(q^n)+1)/(q-1)$, for every $n \geq 0$. That is, $m = a_n$.

Let $1 \leq j \leq a_n - 1$. We claim that there are integers $t, z, j_1 \geq 0$ such that $j = (q-2)q^t z + j_1$, $1 \leq t \leq n-1$, $a_{t-1} \leq j_1 \leq a_t - a_{t-1}$ and $z \leq \sum_{i=0}^{n-t-1} q^i$ hold. To see this, we express $j$ in the number system of base $q$ as $(b_{n-1}, b_{n-2}, \ldots, b_1, b_0)_q$. That is, $j = \sum_{i=0}^{n-1} b_i q^i$, where $b_i \in \{0, 1, \ldots, q-1\}$ for every $i$. Note $a_n - 1 = \sum_{i=0}^{n-1} (q-2) q^i$. If $k \leq n-1$, then $a_k - a_{k-1} = (q-2)q^{k-1}$ and $a_{k-1} = (q-1) + \sum_{i=1}^{k-2} (q-2) q^i$ also hold. We distinguish three cases. First, if $b_i \in \{0, q-2\}$ for every $i$, then let $t = 1 + \min\{k \mid b_k = q-2\}$. Second, if $k_0 \neq q-1$, where $k_0 = \max\{k \mid b_k \notin \{0, q-2\}\}$, then let $t = k_0 + 1$. Third, if $k_0 = q-1$, then let $t = \min\{k \mid k > k_0, b_k = 0\}$. In all cases, let $j_1 = \sum_{i=0}^{t-1} b_i q^i$ and $z = \sum_{i=0}^{n-t-1} c_i q^i$, where $c_i = b_{i+t}/(q-2) \in \{0, 1\}$ for $1 \leq i \leq n-t-1$. It is easy to check that the defined $t, z$ and $j_1$ all have the claimed properties.

We show that the lemma holds for $i = t$. Clearly, $q^{n-t} j = (q-2) q^n z + q^{n-t} j_1$ holds, thus $(q-2)q^n z = ((q-1)a_n - 1)z$ implies the equality $\{q^{n-t} j / a_n\} =$

$\{(q^{n-t}j_1 - z)/a_n\}$. Now, on one hand, $q^{n-t}j_1/a_n < (q-1)/q$ by $j_1 \le (q-2)q^{t-1}$. On the other hand, $(q^{n-t}j_1 - z)/a_n \ge (q^{n-t}a_{t-1} - \sum_{i=0}^{n-t-1} q^i)/a_n$. Observe that $a_k = q^{k-i}a_i - \sum_{i=0}^{k-i-1} q^i$, for every $k > i$. This can be proved by induction on $k-i$. Consequently, $(q^{n-t}j_1 - z)/a_n \ge a_{n-1}/a_n > 1/q$. This completes the proof of the lemma. □

Proof of Proposition 1.2. From Lemma 3.1 follows that $H^*(K \times P) = 4H^*(K)$ for any parallelogram $P$, implying that in the following it is enough to consider (1–3) for $k = 0$. Assume that $K = \prod_{i=1}^n D_i$ is a direct product of convex discs all different from a parallelogram. We may also assume that all the discs $D_i$ are symmetric about $o_2$. On the one hand, for $d = 0$, Lemma 3.2 implies $H^*(D_1) \le 5$, thus repeated applications of Lemma 3.2 yield $H^*(K) \le c_n$, where $c_n$ is defined as $c_0 = 1$, $c_i = 6c_{i-1} - 1$ for every $i \ge 1$. Since $c_n = (4(6^n) + 1)/5$, consequently $H^*(K) \le (4(6^n) + 1)/5$. On the other hand, applying Lemma 3.3 for $k = c_{n-1}$, $m = c_n$ and $C = D_i$, for any $1 \le i \le n$, we obtain a sequence $\langle s_i(j)\rangle_{j=0}^{m-1}$ of points of $\partial D_i$ such that for every $j$ and $j_0$, $\mathrm{dist}_{D_i}(s_i(j), s_i(j_0)) > 1$ is equivalent with $1/6 < \{(j - j_0)/m\} < 5/6$. Now, define a point $p_j = \prod_{i=1}^n s_i(b(i,j))$, for every $0 \le j \le m-1$, where $0 \le b(i,j) \le m-1$, $b(i,j) \equiv 6^{n-i}j \pmod{m}$. Then $S = \{p_j\}_{j=1}^{m-1} \subseteq \partial K$, and $S$ is $1^+$-discrete in the metric $\mathrm{dist}_K = \max_{1\le i\le n}(\mathrm{dist}_{D_i})$, since for every $j_1 \ne j_2$, by applying Lemma 3.4 for $q = 6$ and $j = j_1 - j_2$, there is an index $i$ such that $1/6 < \{6^{n-i}(j_1 - j_2)/m\} < 5/6$, that is equivalent with $\mathrm{dist}_{D_i}\big(s_i(b(i,j_1)), s_i(b(i,j_2))\big) > 1$. Consequently, $H^*(K) \ge c_n$. □

## 4. Verifying a Conjecture of Zong

In this section, we prove Theorem 1.1. For a set $S \subseteq \mathbb{R}^d$ and any $x \in \mathbb{R}$ we denote by $(S, x)$ the set $S \times \{x\} \subseteq \mathbb{R}^{d+1}$. Further, if $C \subseteq \mathbb{R}^2$, then let $C(r) = r \cdot C$. From now on, $I$ stands for the interval $[-1, 1]$ in the paper. Let $C$ be a centrally symmetric convex disc, $0 < \varepsilon < 1$, $0 < \delta \le 1$. We define a three-dimensional convex body as the convex hull of four suitable homothetic copies of $C$ placed in $\mathbb{R}^3$: Let $B(C, \varepsilon, \delta) = \mathrm{conv}(C_1, C_2, -C_2, -C_1)$, where $C_1 = ((1-\delta)C) \times \{1\}$ and $C_2 = C \times \{1 - \varepsilon\}$. Next we prove two lemmas. Combining these, first we get Theorem 1.1 for $d_1 = d_2 = 3$ in Proposition 4.3, then we prove it in general.

Lemma 4.1. *Let $C$ be an arbitrary centrally symmetric convex disc that is different from a parallelogram, $0 < \varepsilon \le 1/3$, and $0 < \delta < \delta_0$, where $\delta_0 < 1$ is a positive constant that depends on $C$ only (when $C$ is a circle, one can choose $\delta_0 = 1 - (2\sin(\frac{\pi}{5}))^{-1} \approx 0.1493$). If $B = B(C, \varepsilon, \delta)$, then $H(B) = 16$ holds.*

Proof. We may assume that $C$ is symmetric about the origin. Let $\alpha_n$ be the largest possible value for the minimum distance occuring in a set of $n$ points of $\partial C$ with respect to the metric $\mathrm{dist}_C$, for any $n \ge 1$. If $C$ is a circle, then $\alpha_n$ is the side length of a regular $n$-gon insribed into $C$. Observe that $H(C) = 6$

implies $\alpha_6 \geq 1$ and $\alpha_8 < 1$. By Proposition 1.2 we have $H^*(C) = 5$, from which $\alpha_5 > 1$ follows. Let $\delta_0 = \min(\alpha_8, 1 - (1/\alpha_5))$.

First we show $H(B) \geq 16$. Let $V_i \subseteq \partial C$ be a set of $i$ points that is $\alpha_i$-discrete in the metric $\mathrm{dist}_C$, for $i = 5, 6$. Let $V = ((1-\delta)V_5, 1) \cup (V_6, 0) \cup (-(1-\delta)V_5, -1)$. It is easy to check that $V$ is a 1-discrete subset of $\partial B$ in the metric $\mathrm{dist}_B$. Thus $H(B) \geq 16$.

Next we prove $H(B) \leq 16$. First we introduce further notation. We define the projection functions $h : \mathbb{R}^3 \to \mathbb{R}$ and $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ by $h(x) = x_3$ and $\pi(x) = (x_1, x_2)$ if $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. For $c \in \mathbb{R}$, let $P(c)$ be the plane $\{x \in \mathbb{R}^3 \mid h(x) = c\}$. Denote by $P^+(c)$ the open halfspace $\{x \in \mathbb{R}^3 \mid h(x) > c\}$.

Let $\mathcal{C}$ be a Hadwiger configuration of $B$. We can partition $\mathcal{C}$ into $\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3\}$ in a way that for $B + v \in \mathcal{C}$ we have $B + v \in \mathcal{C}_1$ if $h(v) \geq 2\varepsilon$, $B + v \in \mathcal{C}_2$ if $h(v) \leq -2\varepsilon$, and $B + v \in \mathcal{C}_3$ otherwise. Let $n_i = |\mathcal{C}_i|$, $i = 1, 2, 3$. We may assume $n_1 \geq n_2$. It is clear that for every $B' \in \mathcal{C}_1$, $B' \cap P(1+\varepsilon)$ is a translate of $(C, 1+\varepsilon)$, and $B' \cap P^+(1+\varepsilon) \neq \varnothing$, so $\mathcal{U} = \{\pi(B') \mid B' \in \mathcal{C}_1\}$ is a packing of $n_1$ translates of $C$, each having common point with the $C$. This immediately gives $n_1 \leq 7$.

Now we consider when $n_1 \geq 6$. Let $\mathcal{C}' = \{B' \in \mathcal{C} \mid B' \cap P \neq \varnothing, B' \cap P^+ \neq \varnothing\}$, where $P = P(1-2\varepsilon)$ and $P^+ = P(1-2\varepsilon)^+$. Observe that $\{\pi(B' \cap P) \mid B' \in \mathcal{C}'\}$ is a packing of sets all containing a translate of $C(1-\delta)$, where $\delta < \alpha_8$, and having centers of symmetry on $\partial(2C)$. This implies $|\mathcal{C}'| \leq 7$. If $C \in \mathcal{U}$, then at least five other members of $\mathcal{U}$ touch $C$. But if $B' = B + v \in \mathcal{C}_1$, and $\pi(B')$ is such a touching disc, then $B'$ touches $B$ at a point $p = (1/2)v$ for which $\varepsilon \leq h(p) \leq 1-\varepsilon$, and thus $h(v) \leq 2-2\varepsilon$. Therefore $|\mathcal{C}' \cap \mathcal{C}_1| \geq 5$. By $\varepsilon \leq 1/3$ we have $\mathcal{C}_3 \subseteq \mathcal{C}'$. Thus $n_3 + 5 \leq |\mathcal{C}'|$, implying $n_3 \leq 2$. Therefore $|\mathcal{C}| \leq 7+2+7 = 16$. If $C \notin \mathcal{U}$, then $n_1 = 6$, and there are at least three members of $\mathcal{U}$ that touch $C$. (To see this, one can replace $\mathcal{U}$ by a Hadwiger configuration of $C$ similarly as we did in Section 2 by "pushing out" the translates, and then use the description of all possible Hadwiger configurations of six translates by [Swanepoel 2000] to observe that at most three translates can be "pushed back". Note that for circles the claim be easily shown directly, using angles determined by the translation vectors). Similarly to the case $C \in \mathcal{U}$, we get $|\mathcal{C}' \cap \mathcal{C}_1| \geq 3$, implying $n_3 + 3 \leq |\mathcal{C}'|$ and thus $n_3 \leq 4$. Then $|\mathcal{C}| \leq 6 + 4 + 6 = 16$.

Finally, we consider when $n_1 \leq 5$. Since for $\{\pi(B' \cap P(0)) \mid B' \in \mathcal{C}_3\}$ one can easily show by $\varepsilon \leq 1/3$ that it is a packing of translates of $C$, all touching $C$, we have $n_3 \leq 6$. Combining the upper bounds, we get $|\mathcal{C}| = 5 + 6 + 5 \leq 16$. □

Lemma 4.2. *Let $\{C_i\}_{i=1}^n$ be a collection of $n$ centrally symmetric convex discs that are different from parallelograms, $n \geq 1$. If $0 < \varepsilon < 1$, $0 < \delta \leq \gamma$, where $\gamma < 1$ is a positive constant that depends on $\{C_i\}_{i=1}^n$ only, and $B_i = B(C_i, \varepsilon, \delta)$, $1 \leq i \leq n$, then*

$$H\left(\prod_{i=1}^n B_i\right) \geq \frac{4(19)^n + 9^n}{5} - 1. \tag{4–1}$$

*If every* $C_i$ *is a circle, one can choose* $\gamma = \delta_n = 1 - \left(2\sin\left(\frac{\pi}{6} + \frac{5\pi}{4(6^{n+1})+6}\right)\right)^{-1}$. *In particular, if* $\delta \le \gamma$, *then* $H(B_1 \times B_2) \ge 304$. (*Note if* $C_1, C_2$ *are circles, then one can choose* $\gamma = \delta_2 = 1 - \left(2\sin\left(\frac{5\pi}{29}\right)\right)^{-1} \approx 0.030169$.)

Proof. Let $K = \prod_{i=1}^n B_i$. Let $A_i(j) = (C_i(1-\delta), j)$, for $j = -1, 1$, and let $A_i(0) = (C_i, j)$, $1 \le i \le n$. Clearly, $A_i(j) \subseteq B_i$. Moreover, if $p \in A_i(j)$ and $q \in A_i(k)$, $j \ne k$, then $\mathrm{dist}_{B_i}(p,q) \ge 1$. Let $D = \prod_{i=1}^n M_i$, where $M_i \in \{A_i(j)\}_{j=-1,0,1}$, and $M_i$ is chosen in an arbitrary way. Then, there is a permutation $\pi$ of the $3n$ coordinates so that $\pi(D) = U \times W \times Z$, where $U = \prod_{M_i \ne A_i(0)} C_i(1-\delta)$, $W = \prod_{M_i = A_i(0)} C_i$ and $Z$ is a single vector having coordinates from the set $\{-1,0,1\}$. Denote by $2m$ the dimension of $U$. By Proposition 1.2, for some $\gamma_0 > 0$ there is a $(1/(1-\gamma_0))$-discrete set $S_1 \subseteq U$ in the metric $\mathrm{dist}_U$ having cardinality $c_n = (4(6^m)+1)/5$, and by $H(C_i) = 6$, there is a 1-discrete set $S_2 \subseteq W$ in the metric $\mathrm{dist}_W$ having cardinality $7^{n-m}$. Let $X = \pi^{-1}(S_1 \times S_2 \times Z)$. Let $Y$ be the union of such sets $X$ when $M_i$'s are choosen all possible ways, and let $\gamma$ be the minimum of all occuring $\gamma_0$'s. Clearly, $Y \subseteq K$ and $Y$ is 1-discrete in the metric $\mathrm{dist}_K$ if $(1-\delta)/(1-\gamma) \ge 1$, that is, $\delta \le \gamma$. Thus $H(K) + 1 \ge |Y|$. If every $C_i$ is a unit circle, then $S_1$ is a subset of the direct products of inscribed regular $c_n$-gons $G_i$, and $1/(1-\gamma)$ can be chosen as the minimum distance that is larger than 1 and occurs among the vertices of $G_i$. Corresponding to the choices of the sets $M_i$, we can count the cardinality of $Y$:

$$|Y| = \sum_{m=0}^{n} \binom{n}{m} \frac{4(6^{n-m})+1}{5} (7^m)(2^{n-m}) = \frac{4}{5}(19^n) + \frac{1}{5}(9^n), \qquad (4\text{–}2)$$

Finally, based on $H(K) \ge |Y| - 1$ and (4–2), we get (4–1). □

Combining Lemma 4.1 and Lemma 4.2 for $n = 2$, it readily implies the following.

Proposition 4.3. *Let* $C_1, C_2$ *be arbitrary convex discs that are different from parallelograms*, $0 < \varepsilon \le 1/3$, *and* $0 < \delta < \mu$, *where* $\mu < 1$ *is a positive constant that depends on* $C_1, C_2$ *only. Then* $B_i = B(C_i, \varepsilon, \delta)$, $i = 1, 2$, *satisfies*

$$H(B_1 \times B_2) \ge (H(B_1)+1)(H(B_2)+1) + 15. \qquad (4\text{–}3)$$

*If* $C_1, C_2$ *are circles, then one can choose* $\mu = 0.03$.

Proof of Theorem 1.1. For any $d_1, d_2 \ge 3$, let $K_i = B \times I^{d_i - 3}$, $i = 1, 2$, where $I^{d-3}$ is a $(d-3)$-dimensional cube. Since $H(K \times I^n) + 1 = 3^n(H(K)+1)$ holds for every convex body $K$ and positive integer $n$ by Zong [1997], from Proposition 4.3 one can immediately deduce (1–2). □

## 5. Hadwiger Numbers of Strictly Convex Bodies

In this section, we prove Theorem 1.3. First we show that for every odd integer $d \ge 3$ there exists a $d$-dimensional convex body for which $H^*(K)$ is relatively large. After that we prove a similar statement for arbitrary $d \geq 3$, which will imply Theorem 1.3.

PROPOSITION 5.1. *For every odd integer $d \geq 3$ there exists a $d$-dimensional convex body $K$ such that*

$$H^*(K) \geq \frac{8(\sqrt{7})^{d-1} + 2(\sqrt{2})^{d-1}}{5} \tag{5–1}$$

*holds.*

PROOF. Let $n = (d-1)/2$. Consider $K_0 = \prod_{i=1}^{n} D_i$ where every $D_i$ is a strictly convex disc that is symmetric about the origin. Let $\pi_i : K_0 \to D_i$ be the projection to the $i$th factor of the direct product. Denote by $J$ an arbitrary subset of $N = \{1, 2, \ldots, n\}$. Let $m = |J|$, $P_J = \prod_{i \in J} D_i$, and let $Q_J = \prod_{i \in N \setminus J} D_i$. Then $g_J(K_0) = P_J \times Q_J$ for some permutation $g_J$ of the coordinates. By Proposition 1.2, there is a set $S_J \subseteq \partial P_J$ of cardinality $(4(6^m) + 1)/5$ which is a $1^+$-discrete set in the metric $\mathrm{dist}_{P_J}$. Let $T_i(J) = \pi_i(S)$, for $i \in J$, and let $V_i = \bigcup_{\{J : i \in J\}} T_i(J)$, for every $1 \leq i \leq n$. We may assume that $\pi_i(S_J) \subseteq \partial D_i$ holds for every $i$ and $J$, by moving out the points of $\pi_i(S_J)$ towards $\partial D_i$ on a ray emanating from the center $o_2$ if necessary. We can even perturb the elements of every occuring set $S_J$ if necessary so that $o_2 \notin (p+q)/2$ holds for every $p, q \in V_i$, $p \notin q$, and $S_J$ still remains $1^+$-discrete in the metric $\mathrm{dist}_{K_1}$ and $\pi_i(S_J) \subseteq \partial D_i$ is still holds for every $i$. Let $W_i = \mathrm{conv}(V_i)$, $1 \leq i \leq n$, $W = \prod_{i=1}^{n} W_i$, and let $K = \mathrm{conv}((W, 1), (-W, -1))$. Denote by $X_J$ the set $g_J^{-1}(S_J \times \{o_{n-m}\})$, and let $X = \bigcup_{J \subseteq N} X_J$. Observe that if $p, q \in X$, $p \neq q$, then either $p, q \in X_J$ for some $J$, or $p \in X_J$, $q \in X_M$ for some $J, M \subseteq N$, $J \neq M$. In the first case, there is an index $i \in J$ for which $\mathrm{dist}_{D_i}(\pi_i(p), \pi_i(q)) > 1$. In the second case, there is an index $i \in (J \setminus M) \cup (M \setminus J)$, for which either $\pi_i(p) = o_2$ and $\pi_i(q) \in \partial D_i$, or $\pi_i(q) = o_2$ and $\pi_i(p) \in \partial D_i$ holds, implying $\mathrm{dist}_{(W_i - W_i)/2}(\pi_i(p), \pi_i(q)) > 1$. Let $Y = (X, 1) \cup (-X, -1)$. It is easy to see that $Y \subseteq K$ and $Y$ is $1^+$-discrete in the metric $\mathrm{dist}_K$. Counting the cardinality of $Y$ by the corresponding choices of $J$, we get

$$|Y| = 2 \sum_{m=0}^{n} \binom{n}{m} \frac{4(6^m) + 1)}{5} = \frac{8(7^n) + 2^{n+1}}{5}. \tag{5–2}$$

By $H^*(K) \geq |Y|$, we obtain (5–1). □

If $d \geq 4$ is even, then one can apply Proposition 5.1 in dimension $d-1$ and combine that with the cylindrical construction of Lemma 3.1 to get a $d$-dimensional convex body $K$ with $H^*(K) \geq (16(\sqrt{7})^{d-2} + 4(\sqrt{2})^{d-2})/5$. Comparing this formula with (5–1), we get the following.

COROLLARY 5.2. *For every integer $d \geq 3$ there exists a $d$-dimensional convex body $K$ such that*

$$H^*(K) \geq \frac{16(\sqrt{7})^{d-2} + 4(\sqrt{2})^{d-2}}{5} \geq \frac{16}{35}(\sqrt{7})^d \approx 2.6457^{d-o(d)}. \tag{5–3}$$

Proof of Theorem 1.3. Consider the collection $\mathcal{K}_d$ of all $d$-dimensional convex bodies equipped with the Hausdorff metric [Schneider 1993]. Note that $H^*(K)$ is not decreasing in a sufficiently small neighbourhood of $K$ (this latter is obvious by the description of $H^*(K)$ in terms of $1^+$-discrete subsets, see Section 2), and the strictly convex bodies form a dense set in $\mathcal{K}_d$. Therefore we can apply Corollary 5.2 to get a convex body $K \in \mathcal{K}_d$ for which (5–3) holds, and we can pick a strictly convex body $S$ sufficiently close to it in the Hausdorff metric so that $H(S) \geq H^*(S) \geq H^*(K)$. $\square$

Remark 1. Instead of proving only existence, one can also construct strictly convex bodies of various shapes having the properties of Theorem 1.3: By the proof of Proposition 5.1 and the paragraph following that we have a description of an $o_d$-symmetric convex polytope $K$ that fulfils (5–3), for every $d \geq 3$. We also have a description of a $1^+$-discrete set $Y \subseteq \partial K$ in the metric $\mathrm{dist}_K$ whose cardinality is at least the lower bound appearing in (5–3). Denote by $\tau$ the minimum distance occuring in $Y$ with respect to the metric $\mathrm{dist}_K$. Then $K \subseteq \mathrm{int}(\tau K)$, therefore to each facet $F$ of $\tau K$ we can find a Euclidean ball $B(F)$ which touches $F$ at a point $p \in \mathrm{relint} F$ and contains $K$, just the radius of the ball needs to be sufficiently large. Let $S = \bigcap\{B(F) \mid \text{F is a facet of } \tau K\}$. Then $S$ is strictly convex, and (1–5) holds.

Remark 2. In particular, when $K$ is constructed in the proof of Proposition 5.1 by applying Proposition 1.2 for direct products of unit circles, then one can explicitely define a strictly convex body $S$ fulfilling (1–5): If $d$ is odd, then $K$ is chosen as $\mathrm{conv}((W, 1), (-W, -1))$ where $W$ is the direct product of $n$ copies of a regular $c_n$-gon, inscribed into a unit circle. One can check $\tau = 2/(1+\cos(\pi/c_n))$, and for every facet $F$ of $\tau K$, the body $K$ is contained in a ball $B(F)$ that touches $F$ at its baricenter (that is, at $(1/|\mathrm{vert}(F)|) \sum_{v \in \mathrm{vert}(F)} v$) and has radius $(n+4)\tau^2/(\tau^2-1)$, so $S$ can be chosen as the intersection of such balls. To make the definiton more expicit, one may even calculate the centers of the balls $B(F)$ in terms of the vertices of $G$. The case when $d$ is even can be treated similarly.

Remark 3. Finally, we note that similarly to the proof of Theorem 1.3, one can show that every dense subcollection of the space of all $d$-dimensional convex bodies contains a member $S$ for which (1–5) holds.

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