Assessing Mathematical Proficiency
MSRI Publications
Volume **53**, 2007

Chapter 16
Reflections on an Assessment Interview: What a Close Look at Student Understanding Can Reveal

ALAN H. SCHOENFELD

This chapter offers a brief introduction to and commentary on some of the issues raised by Deborah Ball's interview with Brandon Peoples, which was transcribed as Chapter 15 and can be seen in its entirety at http://www.msri.org/publications/ln/msri/2004/assessment/session5/1/index.html. The video and its transcript are fascinating, and they reward close watching and reading. What follows in the next few pages barely scratches the surface of the issues they raise.

Before getting down to substance, one must express admiration for both of the participants in the conversation transcribed in Chapter 15. Brandon, a sixth grader, is remarkably poised and articulate. He responds openly and thoughtfully to questioning from Ball on a wide range of issues, in a conversation that is casual but intense — in front of a very large audience of adults! He also shows a great deal of stamina — the interview lasted an hour and a half! Ball demonstrates extraordinary skill in relating to Brandon, and in establishing a climate in which he feels comfortable enough to discuss mathematics in public, and to reveal what he knows. She covers a huge amount of territory with subtlety and skill, examining different aspects of Brandon's knowledge of fractions, revealing connections and confusions, and spontaneously pursuing issues that open up as Brandon reveals what he knows. The interview is a tour de force, demonstrating the potential of such conversations to reveal the kinds of things that students understand.

To begin, the interview reveals the complexity of both what it means to understand fractions and what it means for a student to come to grips with fractions. Consider the range of topics that was covered, roughly in this order: conversions from fractions to decimals and percents; the algorithm for multiplying or dividing one fraction by another; the meaning of fractions as parts of a whole (and the need for the parts to be equal); equivalent fractions and what it means for fractions to be equivalent; the role of the numerator and denominator in determining the magnitude of a fraction; comparing magnitudes of fractions; improper fractions; rectangular, circular, and more complex area models for fractions; number-line representations of fractions; which fractions can be "reduced" and why; finding the area of subsets of complex geometric figures; algorithms for adding and subtracting proper and improper fractions; models for the multiplication of fractions; approximating the result of computations with fractions.

For the person who understands the mathematics deeply, all of these topics are connected; all the pathways between topics tie them together neatly. For someone learning about the material, however, things are very different. Some connections exist, at various degrees of robustness and some are being formed; some are missing, and some mis-connections exist as well. This is a fact of life, and an interesting one. At its best, assessment serves to reveal this set of invisible mental connections in the same way that first x-rays and now MRI techniques serve to reveal that which is physiologically beneath the surface. Let us take a brief tour of Brandon's interview, to see what is revealed in his case. The idea is not to do an exhaustive commentary, but to highlight the kinds of things that a sensitive assessment interview can reveal.

Broadly speaking, Brandon is quite comfortable with, and competent in, the procedural aspects of working with fractions. At the very beginning of the interview he produces the algorithm for dividing fractions (a topic he has just studied) and demonstrates its use. He demonstrates how to multiply fractions and reduce the result to lowest terms. Although he does make an initial canceling error in multiplying $\frac{2}{3} \times \frac{4}{6}$ (remember, he is a sixth grader performing in front of an auditorium full of adults!), he notes his error, corrects it, and confidently confirms that the answer is correct. He confidently converts simple fractions such as $\frac{1}{2}$ and $\frac{1}{3}$ to percents and decimals. Throughout the interview he easily generates fractions equivalent to a given fraction; and he knows how to add and subtract mixed fractions.

At the beginning of the interview, Brandon demonstrates his understanding of certain area models. He explains after folding a paper in half that the two parts have to be equal in size, and goes on to show that the two halves could each be represented by $\frac{19}{38}$ or $\frac{37}{74}$. In folding a paper into thirds, he works to make the three parts equal. He shows that $\frac{1}{4}$ can also be written as $\frac{2}{8}$, and as 25%. He knows that $\frac{1}{6}$ is between 16% and 17%, and justifies his claim by showing that $6 \times 16 < 100$, while $6 \times 17 > 100$.

Asked how much paper the fraction $\frac{3}{2}$ would represent, Brandon does a long division to show that $\frac{3}{2}$ is equal to "a whole and a half." Yet he pronounces $\frac{3}{2}$ as "three twos, I think," suggesting that he has not spent very much if any time talking about improper fractions. He is able to put the fractions $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$ in increasing order, and justify the ordering, and, with a slight bit of prompting, to show that the figure

can be written as 60%, $\frac{60}{100}$, $\frac{3}{5}$, and $\frac{6}{10}$ — also dividing the figure in half horizontally to reveal six shaded pieces out of ten.

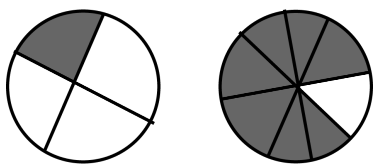
But then, life gets cognitively interesting. Ball asks (turn 267), "of these three fractions you've written [$\frac{60}{100}$, $\frac{3}{5}$, and $\frac{6}{10}$], which one's the largest?" and the following dialogue ensues:

> Brandon: Three-fifths?
>
> Ball: Why is it the largest?
>
> Brandon: 'Cause three-fifths is like like we said earlier, it – 'cause one-hundredths are really small . . .
>
> Ball: Mm-hmm.
>
> Brandon: I mean it – In – I – my opinion it's not – it's not about the numerator, I think it's about the denominator.

Brandon goes on to draw a circle partitioned into five equal pieces and another into ten equal pieces, noting that the tenths are much smaller. He says,

> Brandon: So – so it's six of that, even though the numer – the numerator's bigger than this numerator, it – my opinion is that the denominator – how big the denominator determines how – how big the fraction – the whole fraction is.

Moving to different numbers chosen by Ball, he indicates that $\frac{7}{8}$ is less than $\frac{1}{4}$, by placing a $\frac{7}{8}$ card to the left of the $\frac{1}{4}$ card on the board. He then goes on (turns 292–293) to draw the following two figures:



claiming that is larger than "because it's – you have fourths – I meant – I mean eighths is – eighths are a lot smaller, so seven of them would have to – you have to shade in ’cause you couldn’t put seven into four.” The discussion continues, with Brandon explaining that when two numbers have different denominators, the one with the smaller denominator is larger; but when two numbers have the same denominator, the one with the larger numerator is larger. Ball tries to reconcile these statements by having Brandon compare $\frac{1}{4}$, $\frac{2}{8}$, and $\frac{7}{8}$. Ball reminds Brandon that he had earlier said that $\frac{1}{4}$ and $\frac{2}{8}$ are equal. He draws pictures of those two fractions, and the following dialogue ensues:

> Ball: Mm-hmm. When you look at these two pictures, which one do you think is greater: two-eighths or one fourth?
>
> Brandon: Umm… One-fourth?
>
> Ball: Why do you think one-fourth?
>
> Brandon: Umm. ’Cause it has – it has bigger chunks into it to make fourths, so – but these are all, like li – sorta small, so just one out of four is bigger than two out of eight.

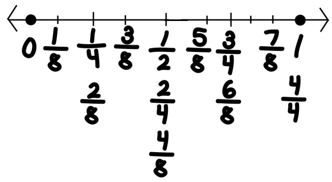
What this indicates is that the various pieces of the fractions puzzle have not yet fallen into place for Brandon. Although he has mastered some of the relevant algorithms and some aspects of the area model (especially with rectangles, where relative sizes can be perceived more readily), he is not confident of his (sometimes incorrect) judgments about relative sizes when it comes to circle models, despite the formal calculation that says that $\frac{1}{4}$ and $\frac{2}{8}$ are equal. And his incorrect algorithm for comparing fractions leads him to claim that $\frac{7}{8} < \frac{1}{4}$, despite the pictures he has produced that suggest the contrary. I stress that these confusions are normal and natural — they are part of coming to grips with a complex subject matter domain — and that they are shared by many students. Part of what this interview reveals is how complex it is to put all the pieces of the puzzle together, and how easy it is to overlook such difficulties, if one focuses on just the procedural aspects of understanding that comprised the first part of the interview.

On turns 332–576 of the interview, Ball conducts what is commonly called a “teaching experiment,” in which she introduces Brandon to some new ideas and monitors what he learns and how he connects it with what he already knows. Brandon says that he is not familiar with the number-line representation of integers and fractions, and Ball takes him into new territory when she introduces him to it. Brandon seems confident and comfortable with labeling fractions between 0 and 1. It is interesting to see him grapple with numbers greater than 1: at first he labels the number half-way between 1 and 2 as $\frac{1}{2}$, and he needs to work to see that it should be labeled $1\frac{1}{2}$. Once he does, however, the rest of the labeling scheme seems to fall into place: he labels 1, $1\frac{1}{4}$, $1\frac{1}{2}$, $1\frac{3}{4}$, and 2. But then there is the following exchange:

> Ball: Okay. All right. So, umm, can you think of any other fractions that you could put up here? Like, is there any fraction that goes between one-fourth and two-fourths, for example?
>
> Brandon: Umm, mm-mm. No.
>
> Ball: No? Er, have we put all the fractions up here that we can?
>
> Brandon: Yeah.

It is worth noting that, earlier, Brandon had put the fractions $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$ in linear order — he had observed that $\frac{1}{2} < \frac{2}{3} < \frac{3}{4}$, — so it should follow that $1\frac{1}{2} < 1\frac{2}{3} < 1\frac{3}{4}$. But what should follow (at least to the cognoscenti) is not necessarily what does follow, when one is new to a domain. This is a critically important fact about learning. At the same time, much of what Brandon understands about proper fractions does transfer well onto his developing understanding of the interval between 0 and 1; he fills in that part of the number line with no difficulty. In short, learning is complex!

Ball then (turns 506–575) revisits the issue of the relationship between $\frac{2}{8}$ and $\frac{1}{4}$ with Brandon, this time using the number line. As in Alice in Wonderland, things get curiouser and curiouser. With Brandon, Ball co-constructs a representation of some of the fractions and their equivalents between 0 and 1, inclusive:



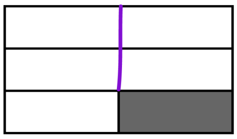
Brandon insists (turn 565) that $\frac{1}{2}$ and $\frac{4}{8}$ are the same size ("I mean, the numbers [in $\frac{4}{8}$] are bigger, but they're both the same [i.e., the fractions] cause they're both the same 'cause they're both half"). But then he goes on to assert that $\frac{1}{4}$ is larger than $\frac{2}{8}$ (turn 575):

> Brandon: 'Cause – from here [pointing to zero and referring to the distance between 0 and $\frac{1}{4}$] it's like – this is like one-fourth and two-fourths and three-fourths and one whole [counting up the number line by fourths], so the space – the space betwe – these are eighth [pointing to the distance between 0 and $\frac{1}{8}$], so – but this is one-fourth so they're in – since it's in eighths, the spaces in between it – it is smaller, so that's why one-fourth would be bigger [i.e. because the space between 0 and $\frac{1}{4}$ is bigger than the spaces between the eighths, one-fourth is bigger than two-eighths].

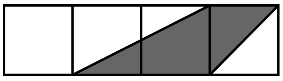
Again, this should not come as a tremendous surprise. To whose who have the big picture, Brandon's reasoning is inconsistent: he is not applying the same logic to the relationship between $\frac{1}{2}$ and $\frac{4}{8}$ and the relationship between $\frac{1}{4}$ and $\frac{2}{8}$. But Brandon does not yet have the big picture; he is in process of constructing it. As he does, he does not have the bird's eye view that enables him to see contradictions, or the tight network of relationships that would constrain him to rethink his judgment. As he sees it, the facts about the two relationships are independent. Hence for him there is no contradiction.

After this discussion, Ball turns once again to area models (turn 576). Given a triangle divided in half, Brandon has no trouble calling each piece a half. Given the figure

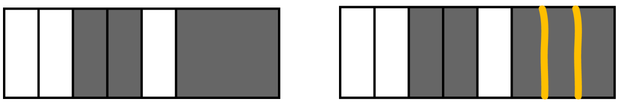
he tentatively identifies the shaded area as one fourth, but given the opportunity to "do something to the picture," he draws a line through the rectangle as below, and calls the shaded area one sixth.



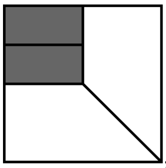
This appears to indicate solid mastery of the "equal parts" part of the definition of fractions. Another more complex figure,



causes him some difficulty, but I suspect that one would cause a lot of people difficulty. On the other hand, he is able to modify the figure on the left below to the figure on the right, and say correctly that the shaded area is $\frac{5}{8}$ of the whole.



So far, so good: Brandon seems to have part-whole down pretty well. But the next figure,

,

throws him for a loop. Possibly because the shaded and unshaded pieces are so different in shape, he loses track of their relative size and claims "two of the segments are shaded, so it would be two fourths, or half." This claim resists a fairly strong examination by Ball.

In the discussion that follows, Brandon correctly produces a rectangular area representation for the fraction $\frac{3}{6}$, "reduces" the fraction to $\frac{1}{2}$, and provides cogent descriptions of what it means to reduce a fraction and when a fraction cannot be reduced. He has thus shown that he is on solid ground with regard to the straightforward representation of some simple areas, but that the knowledge is not yet robust.

At this point (turn 764) the conversation turns to adding and subtracting proper fractions, improper fractions, and mixed numbers. Brandon demonstrates clear mastery of the addition algorithm. Subtraction proves more complex, and Ball once again provides a tutorial — once again revealing the complexities of the learning process, as Brandon works to connect what he already knows (e.g., subtraction of integers and the conversion of integers into fractions) with the new context that calls for their use (e.g., computing $4\frac{2}{6} - 2\frac{3}{6}$). This conversation continues through turn 893.

The next part of their conversation reveals how, when some things are familiar, a student can produce correct answers and seem to have deep understanding; but that going beyond the familiar can reveal the fragility of the underlying knowledge. Ball asks Brandon to draw a picture of half of half a cake, and to say how much of the cake that is. Brandon produces the following picture and says that the area in question (which Ball colors in) is one-fourth of the original cake.

So far, so good. But when Ball asks Brandon, "Okay, so how could she divide that part equally and give you each an equal piece?," he draws the correct figure:

but says that the remaining part is one-third of the whole cake. It is difficult to know why he did this, although one can speculate. First, he must be tired by now; second, the whole cake is not there for him to see; third, the combination of those two factors might cause him to focus just on the picture and forget the "equal size" criterion for fraction definition. The part he is interested in is one part of three in the diagram he sees, and is thus labeled as one-third.

In the final mathematical segment of their conversation (turns 949–978), Ball asks Brandon to estimate the sum

$$\frac{19}{22} + \frac{52}{55}.$$

Brandon — who, earlier, had demonstrated his competence in using the standard algorithm to add fractions with more manageable denominators — now employs the frequently used (and incorrect) procedure

$$\frac{a}{c} + \frac{c}{d} \;\rightarrow\; \frac{a+b}{c+d}$$

to arrive at the approximate sum

$$\frac{20+50}{20+60} = \frac{70}{80} \approx 1.$$

At this point Ball leads Brandon to the observation that each of the two fractions in the original sum is nearly 1 in value, so that their sum is close to 2.

Discussion

Beyond admiring Brandon for his intelligence, bravery, and stamina, and Ball for her skill as interviewer, there are at least two major points to take away from their exchange. But first one must stress that the point of their exchange, and of this chapter, is not to evaluate Brandon. Rather, it is to see what one can learn from their conversation.

One point that comes through with great force is the complexity of what it means to learn and understand a topic such as fractions. Knowing is not a zero-one valued variable. The interview reveals that Brandon knows some things in some mathematical contexts, but not in others; that in some places his knowledge is robust and that in others it is shaky; that some connections are strong and others not; and that he (as can everyone) can have in his mind pieces of information that, when put side by side, can be seen as contradicting each other. When he is on solid ground, for example when he is working with rectangular figures as models of area, he makes certain that all the pieces of a figure are the same size before counting them as n-ths; see turns 588–604 and 635–641. However, when the figures get complex or he gets tired, he loses sight of this constraint: see turns 643–647 and 916–942. Similarly, Brandon correctly uses the standard algorithm to calculate the sum of two fractions when the denominators are relatively small and the task is familiar; but when confronted with an estimation task using unfamiliar denominators he makes the common error of adding the fractions by adding numerators and denominators. In some contexts and with some representations he asserts confidently that $\frac{2}{8}$ is equal to

$\frac{1}{4}$; but in other contexts or using other representations, he will assert that $\frac{2}{8}$ is less than $\frac{1}{4}$. And, he will assert that $\frac{7}{8}$ is less than $\frac{1}{4}$, justifying his statement by saying that whenever fraction A has a larger denominator than fraction B, that A is smaller in value than B.

The point is not that Brandon is confused, or that he does not understand. The point is that he is building a complex network of understandings — we see him doing so in interaction with Ball — and that some partial understandings and misunderstandings are natural and come with the territory. Anyone who thinks that understanding is simple does not understand understanding. That is why assessment is such a subtle art.

The second point to be observed is there are significant differences in the potential, cost, and utility of different kinds of assessments. It should be clear that it would be impossible to reveal the complexity of Brandon's understanding of fractions using a typical paper-and-pencil test. Simple multiple-choice tests can be useful for accountability purposes, providing a rough accounting of what students know at various points during their academic careers; but they are not really useful for diagnostic purposes, or fine-grained enough to support teachers' decision-making in the classroom. More complex "essay questions" of the type discussed in Chapter 14 of this volume provide a much richer picture of the various aspects of student understanding of fractions and can be used for accountability purposes and to support instruction. But these too are incomplete, as Ball's interview with Brandon demonstrates. The more that teachers can "get inside their students' heads" in an ongoing way, in the way that Ball interacted with Brandon, the more they will be able to tailor their instruction to students' needs. To the degree that we can foster such inclinations and skills in all teachers, and add the diagnostic interview to their toolkits (in addition to more formally structured assessments) the richer the possibilities for classroom instruction. This is not to suggest that teachers should do separate 90-minute interviews with each of their students. The idea is that ordinary classroom interactions provide significant opportunities for noticing what students understand — and for asking probing questions, if teachers are prepared to take advantage of these opportunities. The more teachers know what students know, the more they will be able to build on their strengths and to address their needs.