Quantitative symplectic geometry

KAI CIELIEBAK, HELMUT HOFER,
JANKO LATSCHEV, AND FELIX SCHLENK

Dedicated to Anatole Katok on the occasion of his sixtieth birthday

A symplectic manifold (M, ω) is a smooth manifold M endowed with a nondegenerate and closed 2-form ω. By Darboux's Theorem such a manifold looks locally like an open set in some $\mathbb{R}^{2n} \cong \mathbb{C}^n$ with the standard symplectic form

$$\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j, \qquad (0\text{--}1)$$

and so symplectic manifolds have no local invariants. This is in sharp contrast to Riemannian manifolds, for which the Riemannian metric admits various curvature invariants. Symplectic manifolds do however admit many global numerical invariants, and prominent among them are the so-called symplectic capacities.

Symplectic capacities were introduced in 1990 by I. Ekeland and H. Hofer [18; 19] (although the first capacity was in fact constructed by M. Gromov [39]). Since then, lots of new capacities have been defined [16; 29; 31; 43; 48; 58; 59; 88; 97] and they were further studied in [1; 2; 8; 9; 25; 20; 27; 30; 34; 36; 37; 40; 41; 42; 45; 47; 49; 51; 55; 56; 57; 60; 61; 62; 63; 65; 71; 72; 73; 86; 87; 89; 90; 92; 95; 96]. Surveys on symplectic capacities are [44; 49; 54; 66; 95]. Different capacities are defined in different ways, and so relations between capacities often lead to surprising relations between different aspects of symplectic geometry and Hamiltonian dynamics. This is illustrated in Section 2, where we discuss some examples of symplectic capacities and describe a few consequences of their existence. In Section 3 we present an attempt to

better understand the space of all symplectic capacities, and discuss some further general properties of symplectic capacities. In Section 4, we describe several new relations between certain symplectic capacities on ellipsoids and polydiscs. Throughout the discussion we mention many open problems.

As illustrated below, many of the quantitative aspects of symplectic geometry can be formulated in terms of symplectic capacities. Of course there are other numerical invariants of symplectic manifolds which could be included in a discussion of quantitative symplectic geometry, such as the invariants derived from Hofer's bi-invariant metric on the group of Hamiltonian diffeomorphisms, [43; 79; 82], or Gromov–Witten invariants. Their relation to symplectic capacities is not well understood, and we will not discuss them here.

We start out with a brief description of some relations of symplectic geometry to neighboring fields.

1. Symplectic geometry and its neighbors

Symplectic geometry is a rather new and vigorously developing mathematical discipline. The "symplectic explosion" is described in [21]. Examples of symplectic manifolds are open subsets of $\left(\mathbb{R}^{2n}, \omega_0\right)$, the torus $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ endowed with the induced symplectic form, surfaces equipped with an area form, Kähler manifolds like complex projective space $\mathbb{CP}^n$ endowed with their Kähler form, and cotangent bundles with their canonical symplectic form. Many more examples are obtained by taking products and through more elaborate constructions, such as the symplectic blow-up operation. A diffeomorphism φ on a symplectic manifold (M, ω) is called *symplectic* or a *symplectomorphism* if $\varphi^*\omega = \omega$.

A fascinating feature of symplectic geometry is that it lies at the crossroad of many other mathematical disciplines. In this section we mention a few examples of such interactions.

Hamiltonian dynamics. Symplectic geometry originated in Hamiltonian dynamics, which originated in celestial mechanics. A time-dependent Hamiltonian function on a symplectic manifold (M, ω) is a smooth function $H: \mathbb{R} \times M \to \mathbb{R}$. Since ω is nondegenerate, the equation

$$\omega(X_H, \cdot) = dH(\cdot)$$

defines a time-dependent smooth vector field X_H on M. Under suitable assumption on H, this vector field generates a family of diffeomorphisms φ_H^t called the *Hamiltonian flow* of H. As is easy to see, each map φ_H^t is symplectic. A *Hamiltonian diffeomorphism* φ on M is a diffeomorphism of the form φ_H^1.

Symplectic geometry is the geometry underlying Hamiltonian systems. It turns out that this geometric approach to Hamiltonian systems is very fruitful. Explicit examples are discussed in Section 2 below.

Volume geometry. A volume form Ω on a manifold M is a top-dimensional nowhere vanishing differential form, and a diffeomorphism φ of M is *volume preserving* if $\varphi^*\Omega = \Omega$. Ergodic theory studies the properties of volume preserving mappings. Its findings apply to symplectic mappings. Indeed, since a symplectic form ω is nondegenerate, ω^n is a volume form, which is preserved under symplectomorphisms. In dimension 2 a symplectic form is just a volume form, so that a symplectic mapping is just a volume preserving mapping. In dimensions $2n \geq 4$, however, symplectic mappings are much more special. A geometric example for this is Gromov's Nonsqueezing Theorem stated in Section 2.2 and a dynamical example is the (partly solved) Arnol'd conjecture stating that Hamiltonian diffeomorphisms of closed symplectic manifolds have at least as many fixed points as smooth functions have critical points. For another link between ergodic theory and symplectic geometry see [81].

Contact geometry. Contact geometry originated in geometrical optics. A contact manifold (P, α) is a $(2n - 1)$-dimensional manifold P endowed with a 1-form α such that $\alpha \wedge (d\alpha)^{n-1}$ is a volume form on P. The vector field X on P defined by $d\alpha(X, \cdot) = 0$ and $\alpha(X) = 1$ generates the so-called Reeb flow. The restriction of a time-independent Hamiltonian system to an energy surface can sometimes be realized as the Reeb flow on a contact manifold. Contact manifolds also arise naturally as boundaries of symplectic manifolds. One can study a contact manifold (P, α) by symplectic means by looking at its symplectization $\left(P \times \mathbb{R}, d(e^t \alpha)\right)$, see e.g. [46; 22].

Algebraic geometry. A special class of symplectic manifolds are Kähler manifolds. Such manifolds (and, more generally, complex manifolds) can be studied by looking at holomorphic curves in them. M. Gromov [39] observed that some of the tools used in the Kähler context can be adapted for the study of symplectic manifolds. One part of his pioneering work has grown into what is now called Gromov–Witten theory, see e.g. [70] for an introduction.

Many other techniques and constructions from complex geometry are useful in symplectic geometry. For example, there is a symplectic version of blowing-up, which is intimately related to the symplectic packing problem, see [64; 68] and 4.1.2 below. Another example is Donaldson's construction of symplectic submanifolds [17]. Conversely, symplectic techniques proved useful for studying problems in algebraic geometry such as Nagata's conjecture [5; 6; 68] and degenerations of algebraic varieties [7].

Riemannian and spectral geometry. Recall that the differentiable structure of a smooth manifold M gives rise to a canonical symplectic form on its cotangent bundle T^*M. Giving a Riemannian metric g on M is equivalent to prescribing its unit cosphere bundle $S_g^*M \subset T^*M$, and the restriction of the canonical 1-form from T^*M gives S^*M the structure of a contact manifold. The Reeb flow on S_g^*M is the geodesic flow (free particle motion).

In a somewhat different direction, each symplectic form ω on some manifold M distinguishes the class of Riemannian metrics which are of the form $\omega(J\cdot,\cdot)$ for some almost complex structure J.

These (and other) connections between symplectic and Riemannian geometry are by no means completely explored, and we believe there is still plenty to be discovered here. Here are some examples of known results relating Riemannian and symplectic aspects of geometry.

Lagrangian submanifolds. A middle-dimensional submanifold L of (M, ω) is called *Lagrangian* if ω vanishes on TL.

(i) Volume. Endow complex projective space $\mathbb{CP}^n$ with the usual Kähler metric and the usual Kähler form. The volume of submanifolds is taken with respect to this Riemannian metric. According to a result of Givental–Kleiner–Oh, the standard $\mathbb{RP}^n$ in $\mathbb{CP}^n$ has minimal volume among all its Hamiltonian deformations [74]. A partial result for the Clifford torus in $\mathbb{CP}^n$ can be found in [38]. The torus $S^1 \times S^1 \subset S^2 \times S^2$ formed by the equators is also volume minimizing among its Hamiltonian deformations, [50]. If L is a closed Lagrangian submanifold of $(\mathbb{R}^{2n}, \omega_0)$, there exists according to [98] a constant C depending on L such that

$$\mathrm{vol}\left(\varphi_H(L)\right) \geq C \quad \text{for all Hamiltonian deformations of } L. \qquad (1\text{--}1)$$

(ii) Mean curvature. The mean curvature form of a Lagrangian submanifold L in a Kähler–Einstein manifold can be expressed through symplectic invariants of L, see [15].

The first eigenvalue of the Laplacian. Symplectic methods can be used to estimate the first eigenvalue of the Laplace operator on functions for certain Riemannian manifolds [80].

Short billiard trajectories. Consider a bounded domain $U \subset \mathbb{R}^n$ with smooth boundary. There exists a periodic billiard trajectory on $\overline{U}$ of length l with

$$l^n \leq C_n \mathrm{vol}(U) \qquad (1\text{--}2)$$

where C_n is an explicit constant depending only on n, see [98; 30].

2. Examples of symplectic capacities

In this section we give the formal definition of symplectic capacities, and discuss a number of examples along with sample applications.

2.1. Definition. Denote by $Symp^{2n}$ the category of all symplectic manifolds of dimension $2n$, with symplectic embeddings as morphisms. A *symplectic category* is a subcategory $\mathscr{C}$ of $Symp^{2n}$ such that $(M, \omega) \in \mathscr{C}$ implies $(M, \alpha\omega) \in \mathscr{C}$ for all $\alpha > 0$.

CONVENTION. *We will use the symbol $\hookrightarrow$ to denote symplectic embeddings and $\to$ to denote morphisms in the category $\mathscr{C}$ (which may be more restrictive).*

Let $B^{2n}(r^2)$ be the open ball of radius r in $\mathbb{R}^{2n}$ and $Z^{2n}(r^2) = B^2(r^2) \times \mathbb{R}^{2n-2}$ the open cylinder (the reason for this notation will become apparent below). Unless stated otherwise, open subsets of $\mathbb{R}^{2n}$ are always equipped with the canonical symplectic form $\omega_0 = \sum_{j=1}^{n} dy_j \wedge dx_j$. We will suppress the dimension $2n$ when it is clear from the context and abbreviate

$$B := B^{2n}(1), \qquad Z := Z^{2n}(1).$$

Now let $\mathscr{C} \subset Symp^{2n}$ be a symplectic category containing the ball B and the cylinder Z. A *symplectic capacity* on $\mathscr{C}$ is a covariant functor c from $\mathscr{C}$ to the category $([0, \infty], \leq)$ (with $a \leq b$ as morphisms) satisfying

(MONOTONICITY): $c(M, \omega) \leq c(M', \omega')$ if there exists a morphism $(M, \omega) \to (M', \omega')$;

(CONFORMALITY): $c(M, \alpha\omega) = \alpha\, c(M, \omega)$ for $\alpha > 0$;

(NONTRIVIALITY): $0 < c(B)$ and $c(Z) < \infty$.

Note that the (Monotonicity) axiom just states the functoriality of c. A symplectic capacity is said to be *normalized* if

(NORMALIZATION): $c(B) = 1$.

As a frequent example we will use the set Op^{2n} of open subsets in $\mathbb{R}^{2n}$. We make it into a symplectic category by identifying $(U, \alpha^2\omega_0)$ with the symplectomorphic manifold $(\alpha U, \omega_0)$ for $U \subset \mathbb{R}^{2n}$ and $\alpha > 0$. We agree that the morphisms in this category shall be symplectic embeddings induced by *global* symplectomorphisms of $\mathbb{R}^{2n}$. With this identification, the (Conformality) axiom above takes the form

(CONFORMALITY)$'$: $c(\alpha U) = \alpha^2 c(U)$ for $U \in Op^{2n}$, $\alpha > 0$.

2.2. Gromov radius. In view of Darboux's Theorem one can associate with each symplectic manifold (M, ω) the numerical invariant

$$c_B(M, \omega) := \sup \{\alpha > 0 \mid B^{2n}(\alpha) \hookrightarrow (M, \omega)\}$$

called the *Gromov radius* of (M, ω), [39]. It measures the symplectic size of (M, ω) in a geometric way, and is reminiscent of the injectivity radius of a Riemannian manifold. Note that it clearly satisfies the (Monotonicity) and (Conformality) axioms for a symplectic capacity. It is equally obvious that $c_B(B) = 1$.

If M is 2-dimensional and connected, then $\pi c_B(M, \omega) = \int_M \omega$, i.e. c_B is proportional to the volume of M, see [89]. The following theorem from Gromov's seminal paper [39] implies that in higher dimensions the Gromov radius is an invariant very different from the volume.

NONSQUEEZING THEOREM (GROMOV, 1985). *The cylinder $Z \in Symp^{2n}$ satisfies $c_B(Z) = 1$.*

Therefore the Gromov radius is a normalized symplectic capacity on $Symp^{2n}$. Gromov originally obtained this result by studying properties of moduli spaces of pseudo-holomorphic curves in symplectic manifolds.

It is important to realize that the existence of at least one capacity c with $c(B) = c(Z)$ also *implies* the Nonsqueezing Theorem. We will see below that each of the other important techniques in symplectic geometry (such as variational methods and the global theory of generating functions) gave rise to the construction of such a capacity, and hence an independent proof of this fundamental result.

It was noted in [18] that the following result, originally established by Eliashberg and by Gromov using different methods, is also an easy consequence of the existence of a symplectic capacity.

THEOREM (ELIASHBERG, GROMOV). *The group of symplectomorphisms of a symplectic manifold (M, ω) is closed for the compact-open C^0-topology in the group of all diffeomorphisms of M.*

2.3. Symplectic capacities via Hamiltonian systems. The next four examples of symplectic capacities are constructed via Hamiltonian systems. A crucial role in the definition or the construction of these capacities is played by the action functional of classical mechanics. For simplicity, we assume that $(M, \omega) = (\mathbb{R}^{2n}, \omega_0)$. Given a Hamiltonian function $H: S^1 \times \mathbb{R}^{2n} \to \mathbb{R}$ which is periodic in the time-variable $t \in S^1 = \mathbb{R}/\mathbb{Z}$ and which generates a global flow φ_H^t, the

action functional on the loop space $C^\infty(S^1, \mathbb{R}^{2n})$ is defined as

$$\mathscr{A}_H(\gamma) = \int_\gamma y \, dx - \int_0^1 H(t, \gamma(t)) dt. \qquad (2\text{–}1)$$

Its critical points are exactly the 1-periodic orbits of φ_H^t. Since the action functional is neither bounded from above nor from below, critical points are saddle points. In his pioneering work [83; 84], P. Rabinowitz designed special minimax principles adapted to the hyperbolic structure of the action functional to find such critical points. We give a heuristic argument why this works. Consider the space of loops

$$E = H^{1/2}(S^1, \mathbb{R}^{2n}) = \left\{ z \in L^2\left(S^1; \mathbb{R}^{2n}\right) \ \bigg| \ \sum_{k \in \mathbb{Z}} |k| \, |z_k|^2 < \infty \right\}$$

where $z = \sum_{k \in \mathbb{Z}} e^{2\pi k t J} z_k$, $z_k \in \mathbb{R}^{2n}$, is the Fourier series of z and J is the standard complex structure of $\mathbb{R}^{2n} \cong \mathbb{C}^n$. The space E is a Hilbert space with inner product

$$\langle z, w \rangle = \langle z_0, w_0 \rangle + 2\pi \sum_{k \in \mathbb{Z}} |k| \langle z_k, w_k \rangle,$$

and there is an orthogonal splitting $E = E^- \oplus E^0 \oplus E^+$, $z = z^- + z^0 + z^+$, into the spaces of $z \in E$ having nonzero Fourier coefficients $z_k \in \mathbb{R}^{2n}$ only for $k < 0$, $k = 0$, $k > 0$. The action functional $\mathscr{A}_H : C^\infty(S^1, \mathbb{R}^{2n}) \to \mathbb{R}$ extends to E as

$$\mathscr{A}_H(z) = \left(\tfrac{1}{2} \left\| z^+ \right\|^2 - \tfrac{1}{2} \left\| z^- \right\|^2 \right) - \int_0^1 H(t, z(t)) \, dt. \qquad (2\text{–}2)$$

Notice now the hyperbolic structure of the first term $\mathscr{A}_0(x)$, and that the second term is of lower order. Some of the critical points $z(t) \equiv const$ of $\mathscr{A}_0$ should thus persist for $H \neq 0$.

2.3.1. Ekeland–Hofer capacities. The first construction of symplectic capacities via Hamiltonian systems was carried out by Ekeland and Hofer [18; 19]. To give the heuristics, we consider a bounded domain $U \subset \mathbb{R}^{2n}$ with smooth boundary ∂U. A *closed characteristic* γ on ∂U is an embedded circle in ∂U tangent to the characteristic line bundle

$$\mathscr{L}_U = \{(x, \xi) \in T \partial U \mid \omega_0(\xi, \eta) = 0 \ \text{for all} \ \eta \in T_x \partial U\}.$$

If ∂U is represented as a regular energy surface $\{x \in \mathbb{R}^{2n} \mid H(x) = \text{const}\}$ of a smooth function H on $\mathbb{R}^{2n}$, then the Hamiltonian vector field X_H restricted to ∂U is a section of $\mathscr{L}_U$, and so the traces of the periodic orbits of X_H on ∂U are

the closed characteristics on ∂U. The *action* of a closed characteristic γ on ∂U is defined as $\mathscr{A}(\gamma) = \left| \int_\gamma y\,dx \right|$. The set

$$\Sigma\,(U) \;=\; \{k\,\mathscr{A}(\gamma) \mid k = 1, 2, \ldots; \; \gamma \text{ is a closed characteristic on } \partial U\}$$

is called the *action spectrum* of U. Now one would like to associate with U suitable elements of $\Sigma\,(U)$. Without further assumptions on U, however, the set $\Sigma\,(U)$ may be empty (see [32; 33; 35]), and there is no obvious way to achieve (Monotonicity). To salvage this naive idea, Ekeland and Hofer considered for each bounded open subset U of $\mathbb{R}^{2n}$ the space $\mathscr{F}(U)$ of time-independent Hamiltonian functions $H: \mathbb{R}^{2n} \to [0, \infty)$ satisfying

- $H \equiv 0$ on some open neighbourhood of $\overline{U}$, and
- $H(z) = a|z|^2$ for $|z|$ large, where $a > \pi$, $a \notin \mathbb{N}\pi$.

Notice that the circle S^1 acts on the Hilbert space E by time-shift $x(t) \mapsto x(t + \theta)$ for $\theta \in S^1 = \mathbb{R}/\mathbb{Z}$. The special form of $H \in \mathscr{F}(U)$ guarantees that for each $k \in \mathbb{N}$ the equivariant minimax value

$$c_{H,k} := \inf \left\{ \sup_{\gamma \in \xi} \mathscr{A}_H(\gamma) \mid \xi \subset E \text{ is } S^1\text{-equivariant and } \mathrm{ind}(\xi) \geq k \right\}$$

is a critical value of the action functional (2–2). Here, $\mathrm{ind}(\xi)$ denotes a suitable Fadell–Rabinowitz index [26; 19] of the intersection $\xi \cap S^+$ of ξ with the unit sphere $S^+ \subset E^+$. The k-th *Ekeland–Hofer capacity* c_k^{EH} on the symplectic category Op^{2n} is now defined as

$$c_k^{\mathrm{EH}}(U) := \inf \left\{ c_{H,k} \mid H \in \mathscr{F}(U) \right\}$$

if $U \subset \mathbb{R}^{2n}$ is bounded and as

$$c_k^{\mathrm{EH}}(U) := \sup \left\{ c_k^{\mathrm{EH}}(V) \mid V \subset U \text{ bounded} \right\}$$

in general. It turns out that these numbers are indeed symplectic capacities. Moreover, they realize the naive idea of picking out suitable elements of $\Sigma(U)$ for many U: A bounded open subset U of $\mathbb{R}^{2n}$ is said to be of *restricted contact type* if its boundary ∂U is smooth and if there exists a vector field v on $\mathbb{R}^{2n}$ which is transverse to ∂U and whose Lie derivative satisfies $L_v \omega_0 = \omega_0$. Examples are bounded star-shaped domains with smooth boundary.

PROPOSITION (EKELAND AND HOFER, 1990). *If U is of restricted contact type, then $c_k^{\mathrm{EH}}(U) \in \Sigma(U)$ for each $k \in \mathbb{N}$.*

Since the index appearing in the definition of $c_{H,k}$ is monotone, it is immediate from the definition that $c_1^{\mathrm{EH}} \leq c_2^{\mathrm{EH}} \leq c_3^{\mathrm{EH}} \leq \ldots$ form an increasing sequence. Their values on the ball and cylinder are

$$c_k^{\mathrm{EH}}(B) = \left[\frac{k+n-1}{n}\right]\pi \qquad \text{and} \qquad c_k^{\mathrm{EH}}(Z) = k\pi,$$

where $[x]$ denotes the largest integer $\leq x$. Hence the existence of c_1^{EH} gives an independent proof of Gromov's Nonsqueezing Theorem. Using the capacity c_n^{EH}, Ekeland and Hofer [19] also proved the following nonsqueezing result.

THEOREM (EKELAND AND HOFER, 1990). *The cube*

$$P = B^2(1) \times \ldots \times B^2(1) \subset \mathbb{C}^n$$

can be symplectically embedded into the ball $B^{2n}(r^2)$ if and only if $r^2 \geq n$.

Other illustrations of the use of Ekeland–Hofer capacities in studying embedding problems for ellipsoids and polydiscs appear in Section 4.

2.3.2. Hofer–Zehnder capacity. (See [48; 49].) Given a symplectic manifold (M, ω) we consider the class $\mathcal{S}(M)$ of simple Hamiltonian functions $H\colon M \to [0, \infty)$ characterized by the following properties:

- $H = 0$ near the (possibly empty) boundary of M;
- The critical values of H are 0 and $\max H$.

Such a function is called *admissible* if the flow φ_H^t of H has no nonconstant periodic orbits with period $T \leq 1$.

The *Hofer–Zehnder capacity* c_{HZ} on $Symp^{2n}$ is defined as

$$c_{\mathrm{HZ}}(M) := \sup\{\max H \mid H \in \mathcal{S}(M) \text{ is admissible}\}$$

It measures the symplectic size of M in a dynamical way. Easily constructed examples yield the inequality $c_{\mathrm{HZ}}(B) \geq \pi$. In [48; 49], Hofer and Zehnder applied a minimax technique to the action functional (2–2) to show that $c_{\mathrm{HZ}}(Z) \leq \pi$, so

$$c_{\mathrm{HZ}}(B) = c_{\mathrm{HZ}}(Z) = \pi,$$

providing another independent proof of the Nonsqueezing Theorem. Moreover, for every symplectic manifold (M, ω) the inequality $\pi c_B(M) \leq c_{\mathrm{HZ}}(M)$ holds.

The importance of understanding the Hofer–Zehnder capacity comes from the following result proved in [48; 49].

THEOREM (HOFER AND ZEHNDER, 1990). *Let $H\colon (M, \omega) \to \mathbb{R}$ be a proper autonomous Hamiltonian. If $c_{\mathrm{HZ}}(M) < \infty$, then for almost every $c \in H(M)$ the energy level $H^{-1}(c)$ carries a periodic orbit.*

Variants of the Hofer–Zehnder capacity which can be used to detect periodic orbits in a prescribed homotopy class where considered in [59; 88].

2.3.3. Displacement energy (See [43; 55].) Next, let us measure the symplectic size of a subset by looking at how much energy is needed to displace it from itself. Fix a symplectic manifold (M, ω). Given a compactly supported Hamiltonian $H : [0, 1] \times M \to \mathbb{R}$, set

$$\|H\| := \int_0^1 \left(\sup_{x \in M} H(t, x) - \inf_{x \in M} H(t, x) \right) dt.$$

The *energy* of a compactly supported Hamiltonian diffeomorphism φ is

$$E(\varphi) := \inf \left\{ \|H\| \mid \varphi = \varphi_H^1 \right\}.$$

The *displacement energy* of a subset A of M is now defined as

$$e(A, M) := \inf \{ E(\varphi) \mid \varphi(A) \cap A = \varnothing \}$$

if A is compact and as

$$e(A, M) := \sup \{ e(K, M) \mid K \subset A \text{ is compact} \}$$

for a general subset A of M.

Now consider the special case $(M, \omega) = (\mathbb{R}^{2n}, \omega_0)$. Simple explicit examples show $e(Z, \mathbb{R}^{2n}) \leq \pi$. In [43], H. Hofer designed a minimax principle for the action functional (2–2) to show that $e(B, \mathbb{R}^{2n}) \geq \pi$, so that

$$e(B, \mathbb{R}^{2n}) = e(Z, \mathbb{R}^{2n}) = \pi.$$

It follows that $e(\cdot, \mathbb{R}^{2n})$ is a symplectic capacity on the symplectic category Op^{2n} of open subsets of $\mathbb{R}^{2n}$.

One important feature of the displacement energy is the inequality

$$c_{\text{HZ}}(U) \leq e(U, M) \tag{2–3}$$

holding for open subsets of many (and possibly all) symplectic manifolds, including $(\mathbb{R}^{2n}, \omega_0)$. Indeed, this inequality and the Hofer–Zehnder Theorem imply existence of periodic orbits on almost every energy surface of any Hamiltonian with support in U provided only that U is displaceable in M. The proof of this inequality uses the spectral capacities introduced in Section 2.3.4 below.

As an application, consider a closed Lagrangian submanifold L of $(\mathbb{R}^{2n}, \omega_0)$. Viterbo [98] used an elementary geometric construction to show that

$$e\left(L, \mathbb{R}^{2n}\right) \leq C_n \left(\text{vol}(L)\right)^{2/n}$$

for an explicit constant C_n. By a result of Chekanov [12], $e\left(L, \mathbb{R}^{2n}\right) > 0$. Since $e\left(\varphi_H(L), \mathbb{R}^{2n}\right) = e\left(L, \mathbb{R}^{2n}\right)$ for every Hamiltonian diffeomorphism of L, we obtain Viterbo's inequality (1–1).

2.3.4. Spectral capacities. (See [31; 45; 49; 75; 76; 77; 86; 97].) For simplicity, we assume again $(M, \omega) = (\mathbb{R}^{2n}, \omega_0)$. Denote by $\mathcal{H}$ the space of compactly supported Hamiltonian functions $H \colon S^1 \times \mathbb{R}^{2n} \to \mathbb{R}$. An *action selector* σ selects for each $H \in \mathcal{H}$ the action $\sigma(H) = \mathcal{A}_H(\gamma)$ of a "topologically visible" 1-periodic orbit γ of φ_H^t in a suitable way. Such *action selectors* were constructed by Viterbo [97], who applied minimax to generating functions, and by Hofer and Zehnder [45; 49], who applied minimax directly to the action functional (2–2). An outline of their constructions can be found in [30].

Given an action selector σ for $(\mathbb{R}^{2n}, \omega_0)$, one defines the *spectral capacity* c_σ on the symplectic category Op^{2n} by

$$c_\sigma(U) := \sup\left\{\sigma(H) \mid H \text{ is supported in } S^1 \times U\right\}.$$

It follows from the defining properties of an action selector (not given here) that $c_{\mathrm{HZ}}(U) \le c_\sigma(U)$ for any spectral capacity c_σ. Elementary considerations also imply $c_\sigma(U) \le e(U, \mathbb{R}^{2n})$, see [30; 45; 49; 97]. In this way one in particular obtains the important inequality (2–3) for $M = \mathbb{R}^{2n}$.

Here is another application of action selectors:

THEOREM (VITERBO, 1992). *Every nonidentical compactly supported Hamiltonian diffeomorphism of $\left(\mathbb{R}^{2n}, \omega_0\right)$ has infinitely many nontrivial periodic points.*

Moreover, the existence of an action selector is an important ingredient in Viterbo's proof of the estimate (1–2) for billiard trajectories.

Using the Floer homology of (M, ω) filtered by the action functional, an action selector can be constructed for many (and conceivably for all) symplectic manifolds (M, ω), [31; 75; 76; 77; 86]. This existence result implies the energy-capacity inequality (2–3) for arbitrary open subsets U of such (M, ω), which has many applications [87].

2.4. Lagrangian capacity. In [16] a capacity is defined on the category of $2n$-dimensional symplectic manifolds (M, ω) with $\pi_1(M) = \pi_2(M) = 0$ (with symplectic embeddings as morphisms) as follows. The *minimal symplectic area* of a Lagrangian submanifold $L \subset M$ is

$$A_{\min}(L) := \inf\left\{\int_\sigma \omega \,\Big|\, \sigma \in \pi_2(M, L), \int_\sigma \omega > 0\right\} \in [0, \infty].$$

The *Lagrangian capacity* of (M, ω) is defined as

$$c_L(M, \omega) := \sup\{A_{\min}(L) \mid L \subset M \text{ is an embedded Lagrangian torus}\}.$$

Its values on the ball and cylinder are

$$c_L(B) = \pi/n, \qquad c_L(Z) = \pi.$$

As the cube $P = B^2(1) \times \ldots \times B^2(1)$ contains the standard Clifford torus $T^n \subset \mathbb{C}^n$, and is contained in the cylinder Z, it follows that $c_L(P) = \pi$. Together with $c_L(B) = \pi/n$ this gives an alternative proof of the nonsqueezing result of Ekeland and Hofer mentioned in Section 2.3.1. There are also applications of the Lagrangian capacity to Arnold's chord conjecture and to Lagrangian (non)embedding results into uniruled symplectic manifolds [16].

3. General properties and relations between symplectic capacities

In this section we study general properties of and relations between symplectic capacities. We begin by introducing some more notation. Define the *ellipsoids* and *polydiscs*

$$E(a) := E(a_1, \ldots, a_n) := \left\{ z \in \mathbb{C}^n \,\middle|\, \frac{|z_1|^2}{a_1} + \ldots + \frac{|z_n|^2}{a_n} < 1 \right\}$$

$$P(a) := P(a_1, \ldots, a_n) := B^2(a_1) \times \ldots \times B^2(a_n)$$

for $0 < a_1 \leq \ldots \leq a_n \leq \infty$. Note that in this notation the ball, cube and cylinder are $B = E(1, \ldots, 1)$, $P = P(1, \ldots, 1)$ and $Z = E(1, \infty, \ldots, \infty) = P(1, \infty, \ldots, \infty)$.

Besides $Symp^{2n}$ and Op^{2n}, two symplectic categories that will frequently play a role below are

Ell^{2n}: the category of ellipsoids in $\mathbb{R}^{2n}$, with symplectic embeddings induced by global symplectomorphisms of $\mathbb{R}^{2n}$ as morphisms,

Pol^{2n}: the category of polydiscs in $\mathbb{R}^{2n}$, with symplectic embeddings induced by global symplectomorphisms of $\mathbb{R}^{2n}$ as morphisms.

3.1. Generalized symplectic capacities. From the point of view of this work, it is convenient to have a more flexible notion of symplectic capacities, whose axioms were originally designed to explicitly exclude such invariants as the volume. We thus define a *generalized symplectic capacity* on a symplectic category $\mathscr{C}$ as a covariant functor c from $\mathscr{C}$ to the category $([0, \infty], \leq)$ satisfying only the (Monotonicity) and (Conformality) axioms of Section 2.1.

Now examples such as the *volume capacity* on $Symp^{2n}$ are included into the discussion. It is defined as

$$c_{\text{vol}}(M, \omega) := \left(\frac{\text{vol}(M, \omega)}{\text{vol}(B)} \right)^{1/n},$$

where $\text{vol}(M, \omega) := \int_M \omega^n / n!$ is the symplectic volume. For $n \geq 2$ we have $c_{\text{vol}}(B) = 1$ and $c_{\text{vol}}(Z) = \infty$, so c_{vol} is a normalized generalized capacity but not a capacity. Many more examples appear below.

3.2. Embedding capacities. Let $\mathscr{C}$ be a symplectic category. Every object (X, Ω) of $\mathscr{C}$ induces two generalized symplectic capacities on $\mathscr{C}$,

$$c_{(X,\Omega)}(M, \omega) := \sup \{\alpha > 0 \mid (X, \alpha\Omega) \to (M, \omega)\},$$
$$c^{(X,\Omega)}(M, \omega) := \inf \{\alpha > 0 \mid (M, \omega) \to (X, \alpha\Omega)\},$$

Here the supremum and infimum over the empty set are set to 0 and ∞, respectively. Note that

$$c_{(X,\Omega)}(M, \omega) = \left(c^{(M,\omega)}(X, \Omega) \right)^{-1}. \tag{3-1}$$

EXAMPLE 1. Suppose that $(X, \alpha\Omega) \to (X, \Omega)$ for some $\alpha > 1$. Then

$$c_{(X,\Omega)}(X, \Omega) = \infty \quad \text{and} \quad c^{(X,\Omega)}(X, \Omega) = 0,$$

so

$$c_{(X,\Omega)}(M, \omega) = \begin{cases} \infty & \text{if} \quad (X, \beta\Omega) \to (M, \omega) \text{ for some } \beta > 0, \\ 0 & \text{if} \quad (X, \beta\Omega) \to (M, \omega) \text{ for no } \beta > 0, \end{cases}$$

$$c^{(X,\Omega)}(M, \omega) = \begin{cases} 0 & \text{if} \quad (M, \omega) \to (X, \beta\Omega) \text{ for some } \beta > 0, \\ \infty & \text{if} \quad (M, \omega) \to (X, \beta\Omega) \text{ for no } \beta > 0. \end{cases}$$

The following fact follows directly from the definitions.

FACT 1. *Suppose that there exists no morphism $(X, \alpha\Omega) \to (X, \Omega)$ for any $\alpha > 1$. Then $c_{(X,\Omega)}(X, \Omega) = c^{(X,\Omega)}(X, \Omega) = 1$, and for every generalized capacity c with $0 < c(X, \Omega) < \infty$,*

$$c_{(X,\Omega)}(M, \omega) \leq \frac{c(M, \omega)}{c(X, \Omega)} \leq c^{(X,\Omega)}(M, \omega) \qquad \text{for all } (M, \omega) \in \mathscr{C}.$$

In other words, $c_{(X,\Omega)}$ (resp. $c^{(X,\Omega)}$) is the minimal (resp. maximal) generalized capacity c with $c(X, \Omega) = 1$.

Important examples on $Symp^{2n}$ arise from the ball $B = B^{2n}(1)$ and cylinder $Z = Z^{2n}(1)$. By Gromov's Nonsqueezing Theorem and volume reasons we have for $n \geq 2$:

$$c_B(Z) = 1, \qquad c^Z(B) = 1, \qquad c^B(Z) = \infty, \qquad c_Z(B) = 0.$$

In particular, for every normalized symplectic capacity c,

$$c_B(M, \omega) \leq c(M, \omega) \leq c(Z)\, c^Z(M, \omega) \qquad \text{for all } (M, \omega) \in Symp^{2n}. \quad (3\text{–}2)$$

Recall that the capacity c_B is the Gromov radius defined in Section 2.2. The capacities c_B and c^Z are not comparable on Op^{2n}: Example 3 below shows that for every $k \in \mathbb{N}$ there is a bounded star-shaped domain U_k of $\mathbb{R}^{2n}$ such that

$$c_B(U_k) \leq 2^{-k} \qquad and \qquad c^Z(U_k) \geq \pi k^2,$$

see also [42].

We now turn to the question which capacities can be represented as *embedding capacities* $c_{(X,\Omega)}$ or $c^{(X,\Omega)}$.

EXAMPLE 2. Consider the subcategory $\mathscr{C} \subset Op^{2n}$ of connected open sets. Then every generalized capacity c on $\mathscr{C}$ can be represented as the capacity $c^{(X,\Omega)}$ of embeddings into a (possibly uncountable) union (X, Ω) of objects in $\mathscr{C}$.

For this, just define (X, Ω) as the disjoint union of all (X_ι, Ω_ι) in the category $\mathscr{C}$ with $c(X_\iota, \Omega_\iota) = 0$ or $c(X_\iota, \Omega_\iota) = 1$.

PROBLEM 1. *Which (generalized) capacities can be represented as $c^{(X,\Omega)}$ for a* connected *symplectic manifold (X, Ω)?*

PROBLEM 2. *Which (generalized) capacities can be represented as the capacity $c_{(X,\Omega)}$ of embeddings* from *a symplectic manifold (X, Ω)?*

EXAMPLE 3. Embedding capacities give rise to some curious generalized capacities. For example, consider the capacity c^Y of embeddings into the symplectic manifold $Y := \amalg_{k \in \mathbb{N}} B^{2n}(k^2)$. It only takes values 0 and ∞, with $c^Y(M, \omega) = 0$ if and only if (M, ω) embeds symplectically into Y; see Example 1. If M is connected, $\mathrm{vol}(M, \omega) = \infty$ implies $c^Y(M, \omega) = \infty$. On the other hand, for every $\varepsilon > 0$ there exists an open subset $U \subset \mathbb{R}^{2n}$, diffeomorphic to a ball, with $\mathrm{vol}(U) < \varepsilon$ and $c^Y(U) = \infty$. To see this, consider for $k \in \mathbb{N}$ an open neighbourhood U_k of volume $< 2^{-k}\varepsilon$ of the linear cone over the Lagrangian torus $\partial B^2(k^2) \times \ldots \times \partial B^2(k^2)$. The Lagrangian capacity of U_k clearly satisfies $c_L(U_k) \geq \pi k^2$. The open set $U := \cup_{k \in \mathbb{N}} U_k$ satisfies $\mathrm{vol}(U) < \varepsilon$ and $c_L(U) = \infty$, hence U does not embed symplectically into any ball. By appropriate choice of the U_k we can arrange that U is diffeomorphic to a ball; see [86, Proposition A.3]. $\diamondsuit$

Special embedding spaces. Given an arbitrary pair of symplectic manifolds (X, Ω) and (M, ω), it is a difficult problem to determine or even estimate $c_{(X,\Omega)}(M, \omega)$ and $c^{(X,\Omega)}(M, \omega)$. We thus consider two special cases.

1. *Embeddings of skinny ellipsoids.* Assume (M, ω) is an ellipsoid $E(a, \ldots, a, 1)$ with $0 < a \le 1$, and (X, Ω) is connected and has finite volume. Upper bounds for the function

$$e^{(X,\Omega)}(a) = c^{(X,\Omega)}\left(E(a, \ldots, a, 1)\right), \quad a \in (0, 1],$$

are obtained from symplectic embedding results of ellipsoids into (X, Ω), and lower bounds are obtained from computing other (generalized) capacities and using Fact 1. In particular, the volume capacity yields

$$\frac{\left(e^{(X,\Omega)}(a)\right)^n}{a^{n-1}} \ge \frac{\mathrm{vol}(B)}{\mathrm{vol}(X, \Omega)}.$$

The only known general symplectic embedding results for ellipsoids are obtained via multiple symplectic folding. The following result is part of Theorem 3 in [86], which in our setting reads

FACT 2. *Assume that (X, Ω) is a connected $2n$-dimensional symplectic manifold of finite volume. Then*

$$\lim_{a \to 0} \frac{\left(e^{(X,\Omega)}(a)\right)^n}{a^{n-1}} = \frac{\mathrm{vol}(B)}{\mathrm{vol}(X, \Omega)}.$$

For a restricted class of symplectic manifolds, Fact 2 can be somewhat improved. The following result is part of Theorem 6.25 of [86].

FACT 3. *Assume that X is a bounded domain in $\left(\mathbb{R}^{2n}, \omega_0\right)$ with piecewise smooth boundary or that (X, Ω) is a compact connected $2n$-dimensional symplectic manifold. If $n \le 3$, there exists a constant $C > 0$ depending only on (X, Ω) such that*

$$\frac{\left(e^{(X,\Omega)}(a)\right)^n}{a^{n-1}} \le \frac{\mathrm{vol}(B)}{\mathrm{vol}(X, \Omega)\left(1 - Ca^{1/n}\right)} \quad \text{for all } a < \frac{1}{C^n}.$$

These results have their analogues for polydiscs $P(a, \ldots, a, 1)$. The analogue of Fact 3 is known in all dimensions.

2. *Packing capacities.* Given an object (X, Ω) of $\mathscr{C}$ and $k \in \mathbb{N}$, we denote by $\coprod_k (X, \Omega)$ the disjoint union of k copies of (X, Ω) and define

$$c_{(X,\Omega;k)}(M, \omega) := \sup\left\{\alpha > 0 \,\middle|\, \coprod_k (X, \alpha\Omega) \hookrightarrow (M, \omega)\right\}.$$

If $\mathrm{vol}(X, \Omega)$ is finite, we see as in Fact 1 that

$$c_{(X,\Omega;k)}(M,\omega) \leq \frac{1}{c_{\mathrm{vol}}\left(\coprod_k (X,\Omega)\right)} c_{\mathrm{vol}}(M,\omega). \qquad (3\text{–}3)$$

We say that (M,ω) admits a *full k-packing* by (X,Ω) if equality holds in (3–3). For $k_1, \ldots, k_n \in \mathbb{N}$ a full $k_1 \cdots k_n$-packing of $B^{2n}(1)$ by $E\left(\frac{1}{k_1}, \ldots, \frac{1}{k_n}\right)$ is given in [94]. E. Ophstein recently showed in [78] that for every closed symplectic manifold (M,ω) with $[\omega] \in H^2(M; \mathbb{Q})$ there exists a full 1-packing by some ellipsoid. Full k-packings by balls and obstructions to full k-packings by balls are studied in [3; 4; 39; 53; 63; 68; 86; 94].

Assume that also $\mathrm{vol}(M,\omega)$ is finite. Studying the capacity $c_{(X,\Omega;k)}(M,\omega)$ is equivalent to studying the *packing number*

$$p_{(X,\Omega;k)}(M,\omega) = \sup_\alpha \frac{\mathrm{vol}\left(\left(\coprod_k (X, \alpha\Omega)\right)\right)}{\mathrm{vol}(M,\omega)}$$

where the supremum is taken over all α for which $\coprod_k (X, \alpha\Omega)$ symplectically embeds into (M,ω). Clearly, $p_{(X,\Omega;k)}(M,\omega) \leq 1$, and equality holds if and only if equality holds in (3–3). Results in [68] together with the above-mentioned full packings of a ball by ellipsoids from [94] imply

FACT 4. *If X is an ellipsoid or a polydisc, then*

$$p_{(X,k)}(M,\omega) \to 1 \ \ as \ k \to \infty$$

for every symplectic manifold (M,ω) of finite volume.

Note that if the conclusion of Fact 4 holds for X and Y, then it also holds for $X \times Y$.

PROBLEM 3. *For which bounded convex subsets X of $\mathbb{R}^{2n}$ is the conclusion of Fact 4 true?*

In [68] and [3; 4], the packing numbers $p_{(X,k)}(M)$ are computed for $X = B^4$ and $M = B^4$ or $\mathbb{C}P^2$. Moreover, the following fact is shown in [3; 4]:

FACT 5. *If $X = B^4$, then for every closed connected symplectic 4-manifold (M,ω) with $[\omega] \in H^2(M; \mathbb{Q})$ there exists $k_0(M,\omega)$ such that*

$$p_{(X,k)}(M,\omega) = 1 \ \ for \ all \ k \geq k_0(M,\omega).$$

PROBLEM 4. *For which bounded convex subsets X of $\mathbb{R}^{2n}$ and which connected symplectic manifolds (M,ω) of finite volume is the conclusion of Fact 5 true?*

3.3. Operations on capacities. We say that a function $f\colon[0,\infty]^n \to [0,\infty]$ is *homogeneous* and *monotone* if

$$f(\alpha x_1,\ldots,\alpha x_n) = \alpha f(x_1,\ldots,x_n) \qquad \text{for all } \alpha > 0,$$
$$f(x_1,\ldots,x_i,\ldots,x_n) \le f(x_1,\ldots,y_i,\ldots,x_n) \qquad \text{for } x_i \le y_i.$$

If f is homogeneous and monotone and $c_1,\ldots,c_n$ are generalized capacities, then $f(c_1,\ldots,c_n)$ is again a generalized capacity. If in addition

$$0 < f(1,\ldots,1) < \infty$$

and $c_1,\ldots,c_n$ are capacities, then $f(c_1,\ldots,c_n)$ is a capacity. Compositions and pointwise limits of homogeneous monotone functions are again homogeneous and monotone. Examples include $\max(x_1,\ldots,x_n)$, $\min(x_1,\ldots,x_n)$, and the weighted (arithmetic, geometric, harmonic) means

$$\lambda_1 x_1 + \ldots + \lambda_n x_n, \qquad x_1^{\lambda_1}\cdots x_n^{\lambda_n}, \qquad \frac{1}{\frac{\lambda_1}{x_1} + \ldots + \frac{\lambda_n}{x_n}},$$

with $\lambda_1,\ldots,\lambda_n \ge 0$, $\lambda_1 + \ldots + \lambda_n = 1$.

There is also a natural notion of convergence of capacities. We say that a sequence c_n of generalized capacities on $\mathcal{C}$ *converges pointwise* to a generalized capacity c if $c_n(M,\omega) \to c(M,\omega)$ for every $(M,\omega) \in \mathcal{C}$.

These operations yield lots of dependencies between capacities, and it is natural to look for generating systems. In a very general form, this can be formulated as follows.

PROBLEM 5. *For a given symplectic category $\mathcal{C}$, find a minimal generating system $\mathcal{G}$ for the (generalized) symplectic capacities on $\mathcal{C}$. This means that every (generalized) symplectic capacity on $\mathcal{C}$ is the pointwise limit of homogeneous monotone functions of elements in $\mathcal{G}$, and no proper subcollection of $\mathcal{G}$ has this property.*

This problem is already open for Ell^{2n} and Pol^{2n}. One may also ask for generating systems allowing fewer operations, e.g. only max and min, or only positive linear combinations. We will formulate more specific versions of this problem below. The following simple fact illustrates the use of operations on capacities.

FACT 6. *Let $\mathcal{C}$ be a symplectic category containing B (resp. P). Then every generalized capacity c on $\mathcal{C}$ with $c(B) \ne 0$ (resp. $c(P) \ne 0$) is the pointwise limit of capacities.*

Indeed, if $c(B) \ne 0$, then c is the pointwise limit as $k \to \infty$ of the capacities

$$c_k = \min\left(c, k\, c_B\right),$$

and likewise with $c(P)$, c_P instead of $c(B)$, c_B.

EXAMPLE 4. (i) The generalized capacity $c \equiv 0$ on Op^{2n} is not a pointwise limit of capacities, and so the assumption $c(B) \neq 0$ in Fact 6 cannot be omitted.

(ii) The assumption $c(B) \neq 0$ is not always necessary:

(a) Define a generalized capacity c on Op^{2n} by

$$c(U) = \begin{cases} 0 & \text{if vol}(U) < \infty, \\ c_B(U) & \text{if vol}(U) = \infty. \end{cases}$$

Then $c(B) = 0$ and $c(Z) = 1$, and c is the pointwise limit of the capacities

$$c_k = \max\left(c, \tfrac{1}{k}c_B\right).$$

(b) Define a generalized capacity c on Op^{2n} by

$$c(U) = \begin{cases} 0 & \text{if } c_B(U) < \infty, \\ \infty & \text{if } c_B(U) = \infty. \end{cases}$$

Then $c(B) = 0 = c(Z)$ and $c(\mathbb{R}^{2n}) = \infty$, and $c = \lim_{k\to\infty}\tfrac{1}{k}c_B$.

(iii) We do not know whether the generalized capacity $c_{\mathbb{R}^{2n}}$ on Op^{2n} is the pointwise limit of capacities.

PROBLEM 6. *Given a symplectic category $\mathscr{C}$ containing B or P and Z, characterize the generalized capacities which are pointwise limits of capacities.*

3.4. Continuity. There are several notions of continuity for capacities on open subsets of $\mathbb{R}^{2n}$, see [1; 18]. For example, consider a *smooth family of hypersurfaces* $(S_t)_{-\varepsilon<t<\varepsilon}$ in $\mathbb{R}^{2n}$, each bounding a compact subset with interior U_t. Recall that S_0 is said to be of *restricted contact type* if there exists a vector field v on $\mathbb{R}^{2n}$ which is transverse to S_0 and whose Lie derivative satisfies $L_v\omega_0 = \omega_0$. Let c be a capacity on Op^{2n}. As the flow of v is conformally symplectic, the (Conformality) axiom implies the following (see [49, p. 116]):

FACT 7. *If S_0 is of restricted contact type, the function $t \mapsto c(U_t)$ is Lipschitz continuous at 0.*

Fact 7 fails without the hypothesis of restricted contact type. For example, if S_0 possesses no closed characteristic (such S_0 exist by [32; 33; 35]), then by Theorem 3 in Section 4.2 of [49] the function $t \mapsto c_{\mathrm{HZ}}(U_t)$ is not Lipschitz continuous at 0. V. Ginzburg [34] presents an example of a smooth family of hypersurfaces (S_t) (albeit not in $\mathbb{R}^{2n}$) for which the function $t \mapsto c_{\mathrm{HZ}}(U_t)$ is not smoother than $1/2$-Hölder continuous. These considerations lead to

PROBLEM 7. *Are capacities continuous on all smooth families of domains bounded by smooth hypersurfaces?*

3.5. Convex sets. Here we restrict to the subcategory $Conv^{2n} \subset Op^{2n}$ of convex open subsets of $\mathbb{R}^{2n}$, with embeddings induced by global symplectomorphisms of $\mathbb{R}^{2n}$ as morphisms. Recall that a subset $U \subset \mathbb{R}^{2n}$ is *star-shaped* if U contains a point p such that for every $q \in U$ the straight line between p and q belongs to U. In particular, convex domains are star-shaped.

FACT 8. (Extension after Restriction Principle [18]) *Assume that $\varphi \colon U \hookrightarrow \mathbb{R}^{2n}$ is a symplectic embedding of a bounded star-shaped domain $U \subset \mathbb{R}^{2n}$. Then for any compact subset K of U there exists a symplectomorphism Φ of $\mathbb{R}^{2n}$ such that $\Phi|_K = \varphi|_K$.*

This principle continues to hold for some, but not all, symplectic embeddings of unbounded star-shaped domains, see [86]. We say that a capacity c defined on a symplectic subcategory of Op^{2n} has the *exhaustion property* if

$$c(U) = \sup\{ c(V) \mid V \subset U \text{ is bounded } \}. \qquad (3\text{--}4)$$

The capacities introduced in Section 2 all have this property, but the capacity in Example 3 does not. By Fact 8, all statements about capacities defined on a subcategory of $Conv^{2n}$ and having the exhaustion property remain true if we allow all symplectic embeddings (not just those coming from global symplectomorphisms of $\mathbb{R}^{2n}$) as morphisms.

FACT 9. *Let U and V be objects in $Conv^{2n}$. Then there exists a morphism $\alpha U \to V$ for every $\alpha \in (0, 1)$ if and only if $c(U) \leq c(V)$ for all generalized capacities c on $Conv^{2n}$.*

Indeed, the necessity of the condition is obvious, and the sufficiency follows by observing that $\alpha U \to U$ for all $\alpha \in (0, 1)$ and $1 \leq c_U(U) \leq c_U(V)$. What happens for $\alpha = 1$ is not well understood, see Section 3.6 for related discussions. The next example illustrates that the conclusion of Fact 9 is wrong without the convexity assumption.

EXAMPLE 5. Consider the open annulus $A = B(4) \setminus B(1)$ in $\mathbb{R}^2$. If $\frac{3}{4} < \alpha^2 < 1$, then αA cannot be embedded into A by a global symplectomorphism. Indeed, volume considerations show that any potential such global symplectomorphism would have to map A homotopically nontrivially into itself. This would force the image of the ball $\alpha B(1)$ to cover all of $B(1)$, which is impossible for volume reasons. $\diamondsuit$

Assume now that c is a normalized symplectic capacity on $Conv^{2n}$. Using John's ellipsoid, Viterbo [98] noticed that there is a constant C_n depending only on n such that

$$c^Z(U) \leq C_n \, c_B(U) \quad \text{ for all } U \in Conv^{2n}$$

and so, in view of (3–2),

$$c_B(U) \leq c(U) \leq C_n \, c(Z) \, c_B(U) \quad \text{for all } U \in Conv^{2n}. \qquad (3\text{–}5)$$

In fact, $C_n \leq (2n)^2$ and $C_n \leq 2n$ on centrally symmetric convex sets.

PROBLEM 8. *What is the optimal value of the constant C_n appearing in (3–5)? In particular, is $C_n = 1$?*

Note that $C_n = 1$ would imply uniqueness of capacities satisfying $c(B) = c(Z) = 1$ on $Conv^{2n}$. In view of Gromov's Nonsqueezing Theorem, $C_n = 1$ on Ell^{2n} and Pol^{2n}. More generally, this equality holds for all convex Reinhardt domains [42]. In particular, for these special classes of convex sets

$$\pi c_B = c_1^{EH} = c_{HZ} = e(\cdot, \mathbb{R}^{2n}) = \pi c^Z.$$

3.6. Recognition. One may ask how complete the information provided by all symplectic capacities is. Consider two objects (M, ω) and (X, Ω) of a symplectic category $\mathscr{C}$.

QUESTION 1. *Assume $c(M, \omega) \leq c(X, \Omega)$ for all generalized symplectic capacities c on $\mathscr{C}$. Does it follow that $(M, \omega) \hookrightarrow (X, \Omega)$ or even that $(M, \omega) \to (X, \Omega)$?*

QUESTION 2. *Assume $c(M, \omega) = c(X, \Omega)$ for all generalized symplectic capacities c on $\mathscr{C}$. Does it follow that (M, ω) is symplectomorphic to (X, Ω) or even that $(M, \omega) \cong (X, \Omega)$ in the category $\mathscr{C}$?*

Note that if $(M, \alpha\omega) \to (M, \omega)$ for all $\alpha \in (0, 1)$ then, under the assumptions of Question 1, the argument leading to Fact 9 yields $(M, \alpha\omega) \to (X, \Omega)$ for all $\alpha \in (0, 1)$.

EXAMPLE 6. (i) Set $U = B^2(1)$ and $V = B^2(1) \backslash \{0\}$. For each $\alpha < 1$ there exists a symplectomorphism of $\mathbb{R}^2$ with $\varphi(\alpha U) \subset V$, so that monotonicity and conformality imply $c(U) = c(V)$ for all generalized capacities c on Op^2. Clearly, $U \hookrightarrow V$, but $U \not\to V$, and U and V are not symplectomorphic.

(ii) Set $U = B^2(1)$ and let $V = B^2(1) \backslash \{(x, y) \mid x \geq 0, \ y = 0\}$ be the slit disc. As is well-known, U and V are symplectomorphic. Fact 8 implies $c(U) = c(V)$ for all generalized capacities c on Op^2, but clearly $U \not\to V$. In dimensions $2n \geq 4$ there are bounded convex sets U and V with smooth boundary which are symplectomorphic while $U \not\to V$, see [24].

(iii) Let U and V be ellipsoids in Ell^{2n}. The answer to Question 1 is unknown even for Ell^4. For $U = E(1, 4)$ and $V = B^4(2)$ we have $c(U) \leq c(V)$ for all generalized capacities that can presently be computed, but it is unknown whether

$U \hookrightarrow V$, (see 4.1.2 below). By Fact 10 below the answer to Question 2 is "yes" on Ell^{2n}.

(iv) Let U and V be polydiscs in Pol^{2n}. Again, the answer to Question 1 is unknown even for Pol^4. However, in this dimension the Gromov radius together with the volume capacity determine a polydisc, so that the answer to Question 2 is "yes" on Pol^4. $\diamond$

PROBLEM 9. *Are two polydiscs in dimension $2n \geq 6$ with equal generalized symplectic capacities symplectomorphic?*

To conclude this section, we mention a specific example in which $c(U) = c(V)$ for all known (but possibly not for all) generalized symplectic capacities.

EXAMPLE 7. Consider the subsets

$$U = E(2, 6) \times E(3, 3, 6) \quad \text{and} \quad V = E(2, 6, 6) \times E(3, 3)$$

of $\mathbb{R}^{10}$. Then $c(U) = c(V)$ whenever $c(B) = c(Z)$ by the Nonsqueezing Theorem, the volumes agree, and $c_k^{\mathrm{EH}}(U) = c_k^{EH}(V)$ for all k by the product formula (3–8). It is unknown whether $U \hookrightarrow V$ or $V \hookrightarrow U$ or $U \rightarrow V$. Symplectic homology as constructed in [28; 93] does not help in these problems because a computation based on [29] shows that all symplectic homologies of U and V agree.

3.7. Hamiltonian representability. Consider a bounded domain $U \subset \mathbb{R}^{2n}$ with smooth boundary of restricted contact type (see Section 2.3.1 for the definition). As in 2.3.1 we consider the action spectrum

$$\Sigma(U) = \left\{ k \left| \int_\gamma y \, dx \right| \mid k = 1, 2, \ldots; \gamma \text{ is a closed characteristic on } \partial U \right\}$$

of U. This set is nowhere dense in $\mathbb{R}$ (compare [49, Section 5.2]), and it is easy to see that $\Sigma(U)$ is closed and $0 \notin \Sigma(U)$. For many capacities c constructed via Hamiltonian systems, such as Ekeland–Hofer capacities c_k^{EH} and spectral capacities c_σ, one has $c(U) \in \Sigma(U)$, see [19; 41]. Moreover,

$$c_{\mathrm{HZ}}(U) = c_1^{\mathrm{EH}}(U) = \min(\Sigma(U)) \quad \text{if } U \text{ is convex.} \tag{3–6}$$

One might therefore be tempted to ask

QUESTION 3. *Is it true that $\pi c(U) \in \Sigma(U)$ for every normalized symplectic capacity c on Op^{2n} and every domain U with boundary of restricted contact type?*

The following example due to D. Hermann [42] shows that the answer to Question 3 is "no".

EXAMPLE 8. Choose any U with boundary of restricted contact type such that

$$c_B(U) < c^Z(U). \tag{3–7}$$

Examples are bounded star-shaped domains U with smooth boundary which contain the Lagrangian torus $S^1 \times \ldots \times S^1$ but have small volume: According to [91], $c^Z(U) \geq 1$, while $c_B(U)$ is as small as we like. Now notice that for each $t \in [0, 1]$,

$$c_t = (1-t)c_B + tc^Z$$

is a normalized symplectic capacity on Op^{2n}. By (3–7), the interval

$$\{c_t(U) \mid t \in [0, 1]\} = [c_B(U), c^Z(U)]$$

has positive measure and hence cannot lie in the nowhere dense set $\Sigma(U)$. $\Diamond$

D. Hermann also pointed out that the argument in Example 8 together with (3–6) implies that the question "$C_n = 1$?" posed in Problem 8 is equivalent to Question 3 for convex sets.

3.8. Products. Consider a family of symplectic categories $\mathcal{C}^{2n}$ in all dimensions $2n$ such that

$$(M, \omega) \in \mathcal{C}^{2m}, \quad (N, \sigma) \in \mathcal{C}^{2n} \implies (M \times N, \omega \oplus \sigma) \in \mathcal{C}^{2(m+n)}.$$

We say that a collection $c \colon \coprod_{n=1}^{\infty} \mathcal{C}^{2n} \to [0, \infty]$ of generalized capacities has the *product property* if

$$c(M \times N, \omega \oplus \sigma) = \min\{c(M, \omega), c(N, \sigma)\}$$

for all $(M, \omega) \in \mathcal{C}^{2m}, (N, \sigma) \in \mathcal{C}^{2n}$. If $\mathbb{R}^2 \in \mathcal{C}^2$ and $c(\mathbb{R}^2) = \infty$, the product property implies the *stability property*

$$c(M \times \mathbb{R}^2, \omega \oplus \omega_0) = c(M, \omega)$$

for all $(M, \omega) \in \mathcal{C}^{2m}$.

EXAMPLE 9. (i) Let Σ_g be a closed surface of genus g endowed with an area form ω. Then

$$c_B\left(\Sigma_g \times \mathbb{R}^2, \omega \oplus \omega_0\right) = \begin{cases} c_B\left(\Sigma_g, \omega\right) = \frac{1}{\pi}\omega\left(\Sigma_g\right) & \text{if } g = 0, \\ \infty & \text{if } g \geq 1. \end{cases}$$

While the result for $g = 0$ follows from Gromov's Nonsqueezing Theorem, the result for $g \geq 1$ belongs to Polterovich [69, Exercise 12.4] and Jiang [52]. Since c_B is the smallest normalized symplectic capacity on $Symp^{2n}$, we find that no collection c of symplectic capacities defined on the family $\coprod_{n=1}^{\infty} Symp^{2n}$ with $c\left(\Sigma_g, \omega\right) < \infty$ for some $g \geq 1$ has the product or stability property.

(ii) On the family of polydiscs $\coprod_{n=1}^{\infty} Pol^{2n}$, the Gromov radius, the Lagrangian capacity and the unnormalized Ekeland–Hofer capacities c_k^{EH} all have the product property (see Section 4.2). The volume capacity is not stable.

(iii) Let $U \in Op^{2m}$ and $V \in Op^{2n}$ have smooth boundary of restricted contact type (see Section 3.4 for the definition). The formula

$$c_k^{\mathrm{EH}}(U \times V) = \min_{i+j=k}\left(c_i^{\mathrm{EH}}(U) + c_j^{\mathrm{EH}}(V)\right), \qquad (3\text{--}8)$$

in which we set $c_0^{\mathrm{EH}} \equiv 0$, was conjectured by Floer and Hofer [95] and has been proved by Chekanov [13] as an application of his equivariant Floer homology. Consider the collection of sets $U_1 \times \ldots \times U_l$, where each $U_i \in Op^{2n_i}$ has smooth boundary of restricted contact type, and $\sum_{i=1}^{l} n_i = n$. We denote by RCT^{2n} the corresponding category with symplectic embeddings induced by global symplectomorphisms of $\mathbb{R}^{2n}$ as morphisms. If v_i are vector fields on $\mathbb{R}^{2n_i}$ with $L_{v_i}\omega_0 = \omega_0$, then $L_{v_1+\ldots+v_l}\omega_0 = \omega_0$ on $\mathbb{R}^{2n}$. Elements of RCT^{2n} can therefore be exhausted by elements of RCT^{2n} with smooth boundary of restricted contact type. This and the exhaustion property (3–4) of the c_k^{EH} shows that (3–8) holds for all $U \in RCT^{2m}$ and $V \in RCT^{2n}$, implying in particular that Ekeland–Hofer capacities are stable on $RCT := \coprod_{n=1}^{\infty} RCT^{2n}$. Moreover, (3–8) yields that

$$c_k^{\mathrm{EH}}(U \times V) \leq \min\left(c_k^{\mathrm{EH}}(U), c_k^{\mathrm{EH}}(V)\right),$$

and it shows that c_1^{EH} on RCT has the product property. Using (3–8) together with an induction over the number of factors and $c_2^{\mathrm{EH}}(E(a_1, \ldots, a_n)) \leq 2a_1$ we also see that c_2^{EH} has the product property on products of ellipsoids. For $k \geq 3$, however, the Ekeland–Hofer capacities c_k^{EH} on RCT do not have the product property. As an example, for $U = B^4(4)$ and $V = E(3, 8)$ we have

$$c_3^{\mathrm{EH}}(U \times V) = 7 < 8 = \min\left(c_3^{\mathrm{EH}}(U), c_3^{\mathrm{EH}}(V)\right).$$

PROBLEM 10. *Characterize the collections of (generalized) capacities on polydiscs that have the product (resp. stability) property.*

Next consider a collection c of generalized capacities on open subsets Op^{2n}. In general, it will not be stable. However, we can stabilize c to obtain stable generalized capacities $c^{\pm} \colon \coprod_{n=1}^{\infty} Op^{2n} \to [0, \infty]$,

$$c^+(U) := \limsup_{k \to \infty} c(U \times \mathbb{R}^{2k}), \qquad c^-(U) := \liminf_{k \to \infty} c(U \times \mathbb{R}^{2k}).$$

Notice that $c(U) = c^+(U) = c^-(U)$ for all $U \in \coprod_{n=1}^{\infty} Op^{2n}$ if and only if c is stable. If c consists of capacities and there exist constants $a, A > 0$ such that

$$a \leq c\left(B^{2n}(1)\right) \leq c\left(Z^{2n}(1)\right) \leq A \qquad \text{for all } n \in \mathbb{N},$$

then $c^{\pm}$ are collections of capacities. Thus there exist plenty of stable capacities on Op^{2n}. However, we have

PROBLEM 11. *Decide stability of specific collections of capacities on $Conv^{2n}$ or Op^{2n}, e.g.: Gromov radius, Ekeland–Hofer capacity, Lagrangian capacity, and the embedding capacity c_P of the unit cube.*

PROBLEM 12. *Does there exist a collection of capacities on $\coprod_{n=1}^{\infty} Conv^{2n}$ or $\coprod_{n=1}^{\infty} Op^{2n}$ with the product property?*

3.9. Higher order capacities? Following [44], we briefly discuss the concept of higher order capacities. Consider a symplectic category $\mathscr{C} \subset Symp^{2n}$ containing Ell^{2n} and fix $d \in \{1, \ldots, n\}$. A *symplectic d-capacity* on $\mathscr{C}$ is a generalized capacity satisfying

(d-NONTRIVIALITY): $0 < c(B)$ and

$$\begin{cases} c\big(B^{2d}(1) \times \mathbb{R}^{2(n-d)}\big) < \infty, \\ c\big(B^{2(d-1)}(1) \times \mathbb{R}^{2(n-d+1)}\big) = \infty. \end{cases}$$

For $d = 1$ we recover the definition of a symplectic capacity, and for $d = n$ the volume capacity c_{vol} is a symplectic n-capacity.

PROBLEM 13. *Does there exist a symplectic d-capacity on a symplectic category $\mathscr{C}$ containing Ell^{2n} for some $d \in \{2, \ldots, n-1\}$?*

Problem 13 on $Symp^{2n}$ is equivalent to the following symplectic embedding problem.

PROBLEM 14. *Does there exist a symplectic embedding*

$$B^{2(d-1)}(1) \times \mathbb{R}^{2(n-d+1)} \hookrightarrow B^{2d}(R) \times \mathbb{R}^{2(n-d)} \tag{3–9}$$

for some $R < \infty$ and $d \in \{2, \ldots, n-1\}$?

Indeed, the existence of such an embedding would imply that no symplectic d-capacity can exist on $Symp^{2n}$. Conversely, if no such embedding exists, then the embedding capacity $c^{Z_{2d}}$ into $Z_{2d} = B^{2d}(1) \times \mathbb{R}^{2(n-d)}$ would be an example of a d-capacity on $Symp^{2n}$. The Ekeland–Hofer capacity c_d^{EH} shows that R is at least 2 if a symplectic embedding (3–9) exists. The known symplectic embedding techniques are not designed to effectively use the unbounded factor of the target space in (3–9). E.g., multiple symplectic folding only shows that there exists a function $f : [1, \infty) \to \mathbb{R}$ with $f(a) < \sqrt{2a} + 2$ such that for each $a \geq 1$ there exists a symplectic embedding

$$B^2(1) \times B^2(a) \times \mathbb{R}^2 \hookrightarrow B^4\big(f(a)\big) \times \mathbb{R}^2$$

of the form $\varphi \times id_2$, see [86, Section 4.3.2].

4. Ellipsoids and polydiscs

In this section we investigate generalized capacities on the categories of ellipsoids Ell^{2n} and polydiscs Pol^{2n} in more detail. All (generalized) capacities c in this section are defined on some symplectic subcategory of Op^{2n} containing at least one of the above categories and are assumed to have the exhaustion property (3–4).

4.1. Ellipsoids.

4.1.1. Arbitrary dimension. We first describe the values of the capacities introduced in Section 2 on ellipsoids.

The values of the Gromov radius c_B on ellipsoids are

$$c_B\big(E(a_1,\ldots,a_n)\big) = \min\{a_1,\ldots,a_n\}.$$

More generally, monotonicity implies that this formula holds for all symplectic capacities c on Op^{2n} with $c(B) = c(Z) = 1$ and hence also for $\frac{1}{\pi}c_1^{\mathrm{EH}}$, $\frac{1}{\pi}c_{\mathrm{HZ}}$, $\frac{1}{\pi}e(\cdot,\mathbb{R}^{2n})$ and c^Z.

The values of the Ekeland–Hofer capacities on the ellipsoid $E(a_1,\ldots,a_n)$ can be described as follows [19]. Write the numbers $m\,a_i\pi$, $m \in \mathbb{N}$, $1 \leq i \leq n$, in increasing order as $d_1 \leq d_2 \leq \ldots$, with repetitions if a number occurs several times. Then

$$c_k^{\mathrm{EH}}\big(E(a_1,\ldots,a_n)\big) = d_k.$$

The values of the Lagrangian capacity on ellipsoids are presently not known. In [16], Cieliebak and Mohnke make the following conjecture:

CONJECTURE 1.

$$c_L\big(E(a_1,\ldots,a_n)\big) = \frac{\pi}{1/a_1 + \ldots + 1/a_n}.$$

Since $\mathrm{vol}\big(E(a_1,\ldots,a_n)\big) = a_1\cdots a_n\mathrm{vol}(B)$, the values of the volume capacity on ellipsoids are

$$c_{\mathrm{vol}}\big(E(a_1,\ldots,a_n)\big) = (a_1\cdots a_n)^{1/n}.$$

In view of conformality and the exhaustion property, a (generalized) capacity on Ell^{2n} is determined by its values on the ellipsoids $E(a_1,\ldots,a_n)$ with $0 < a_1 \leq \ldots \leq a_n = 1$. So we can view each (generalized) capacity c on ellipsoids as a function

$$c(a_1,\ldots,a_{n-1}) := c\,(E(a_1,\ldots,a_{n-1},1))$$

on the set $\{0 < a_1 \leq \ldots \leq a_{n-1} \leq 1\}$. By Fact 7, this function is continuous. This identification with functions yields a notion of *uniform convergence* for capacities on Ell^{2n}.

For what follows, it is useful to have normalized versions of the Ekeland–Hofer capacities, so in dimension $2n$ we define

$$\bar{c}_k := \frac{c_k^{\mathrm{EH}}}{[\frac{k+n-1}{n}]\pi}.$$

PROPOSITION 1. *As $k \to \infty$, for every $n \geq 2$ the normalized Ekeland–Hofer capacities $\bar{c}_k$ converge uniformly on Ell^{2n} to the normalized symplectic capacity c_∞ given by*

$$c_\infty \left(E(a_1, \ldots, a_n) \right) = \frac{n}{1/a_1 + \ldots + 1/a_n}.$$

REMARK. Note that Conjecture 1 asserts that c_∞ agrees with the normalized Lagrangian capacity $\bar{c}_L = n c_L / \pi$ on Ell^{2n}.

PROOF OF PROPOSITION 1. Fix $\varepsilon > 0$. We need to show that $|\bar{c}_k(a) - c_\infty(a)| \leq \varepsilon$ for every vector $a = (a_1, \ldots, a_n)$ with $0 < a_1 \leq a_2 \leq \ldots \leq a_n = 1$ and all sufficiently large k. Abbreviate $\delta = \varepsilon/n$.

Case 1. $a_1 \leq \delta$. Then

$$c_k^{\mathrm{EH}}(a) \leq k\delta\pi, \qquad \bar{c}_k(a) \leq n\delta, \qquad c_\infty(a) \leq n\delta$$

from which we conclude $|\bar{c}_k(a) - c_\infty(a)| \leq n\delta = \varepsilon$ for all $k \geq 1$.

Case 2. $a_1 > \delta$. Let $k \geq 2\frac{n-1}{\delta} + 2$. For the unique integer l with

$$\pi l\, a_n \leq c_k^{\mathrm{EH}}(a) < \pi(l+1)a_n$$

we then have $l \geq 2$. In the increasing sequence of the numbers $m\, a_i$ ($m \in \mathbb{N}$, $1 \leq i \leq n$), the first $[l\, a_n/a_i]$ multiples of a_i occur no later than $l\, a_n$. By the description of the Ekeland–Hofer capacities on ellipsoids given above, this yields the estimates

$$\frac{(l-1)\, a_n}{a_1} + \ldots + \frac{(l-1)\, a_n}{a_n} \leq k \leq \frac{(l+1)\, a_n}{a_1} + \ldots + \frac{(l+1)\, a_n}{a_n}.$$

With $\gamma := a_n/a_1 + \ldots + a_n/a_n$ this becomes

$$(l-1)\gamma \leq k \leq (l+1)\gamma.$$

Using $\gamma \geq n$, we derive the inequalities

$$\left[\frac{k+n-1}{n}\right] \leq \frac{k}{n} + 1 \leq \frac{(l+1)\gamma + n}{n} \leq \frac{(l+2)\gamma}{n},$$

$$\left[\frac{k+n-1}{n}\right] \geq \frac{k}{n} \geq \frac{(l-1)\gamma}{n}.$$

With the definition of $\bar{c}_k$ and the estimate above for c_k^{EH}, we find

$$\frac{n\,l\,a_n}{(l+2)\gamma} \le \bar{c}_k(a) = \frac{c_k^{EH}(a)}{\left[\frac{k+n-1}{n}\right]\pi} \le \frac{n(l+1)a_n}{(l-1)\gamma}.$$

Since $c_\infty(a) = n\,a_n/\gamma$, this becomes

$$\frac{l}{l+2}c_\infty(a) \le \bar{c}_k(a) \le \frac{l+1}{l-1}c_\infty(a),$$

which in turn implies

$$|\bar{c}_k(a) - c_\infty(a)| \le \frac{2c_\infty(a)}{l-1}.$$

Since $a_1 > \delta$ we have

$$\gamma \le \frac{n}{\delta}, \qquad l+1 \ge \frac{k}{\gamma} \ge \frac{k\delta}{n},$$

from which we conclude

$$|\bar{c}_k(a) - c_\infty(a)| \le \frac{2}{l-1} \le \frac{2n}{k\delta - 2n} \le \varepsilon$$

for k sufficiently large. $\qquad\square$

We turn to the question whether Ekeland–Hofer capacities generate the space of all capacities on ellipsoids by suitable operations. First note some easy facts.

FACT 10. *An ellipsoid $E \subset \mathbb{R}^{2n}$ is uniquely determined by its Ekeland–Hofer capacities $c_1^{EH}(E), c_2^{EH}(E), \ldots$.*

Indeed, if $E(a)$ and $E(b)$ are two ellipsoids with $a_i = b_i$ for $i < k$ and $a_k < b_k$, then the multiplicity of a_k in the sequence of Ekeland–Hofer capacities is one higher for $E(a)$ than for $E(b)$, so not all Ekeland–Hofer capacities agree.

FACT 11. *For every $k \in \mathbb{N}$ there exist ellipsoids E and E' with $c_i^{EH}(E) = c_i^{EH}(E')$ for $i < k$ and $c_k^{EH}(E) \ne c_k^{EH}(E')$.*

For example, we can take $E = E(a)$ and $E' = E(b)$ with $a_1 = b_1 = 1$, $a_2 = k - 1/2$, $b_2 = k + 1/2$, and $a_i = b_i = 2k$ for $i \ge 3$. So formally, every generalized capacity on ellipsoids is a function of the Ekeland–Hofer capacities, and the Ekeland–Hofer capacities are functionally independent. However, Ekeland–Hofer capacities do not form a generating system for symplectic capacities on Ell^{2n} (see Example 10 below), and on bounded ellipsoids each finite set of Ekeland–Hofer capacities is determined by the (infinitely many) other Ekeland–Hofer capacities:

LEMMA 1. *Let $d_1 \leq d_2 \leq \ldots$ be an increasing sequence of real numbers obtained from the sequence $c_1^{\mathrm{EH}}(E) \leq c_2^{\mathrm{EH}}(E) \leq \ldots$ of Ekeland–Hofer capacities of a bounded ellipsoid $E \in Ell^{2n}$ by removing at most N_0 numbers. Then E can be recovered uniquely.*

PROOF. We first consider the special case in which $E = E(a_1, \ldots, a_n)$ is such that $a_i/a_j \in \mathbb{Q}$ for all i, j. In this case, the sequence $d_1 \leq d_2 \leq \ldots$ contains infinitely many blocks of n consecutive equal numbers. We traverse the sequence until we have found $N_0 + 1$ such blocks, for each block $d_k = d_{k+1} = \ldots = d_{k+n-1}$ recording the number $g_k := d_{k+n} - d_k$. The minimum of the g_k for the $N_0 + 1$ first blocks equals a_1. After deleting each occurring positive integer multiple of a_1 once from the sequence $d_1 \leq d_2 \leq \ldots$, we can repeat the same procedure to determine a_2, and so on.

In general, we do not know whether or not $a_i/a_j \in \mathbb{Q}$ for all i, j. To reduce to the previous case, we split the sequence $d_1 \leq d_2 \leq \ldots$ into (at most n) subsequences of numbers with rational quotients. More precisely we traverse the sequence, grouping the d_i into increasing subsequences $s_1, s_2, \ldots$, where each new number is added to the first subsequence s_j whose members are rational multiples of it. Furthermore, in this process we record for each sequence s_j the maximal length l_j of a block of consecutive equal numbers seen so far. We stop when

(i) the sum of the l_j equals n, and
(ii) each subsequence s_j contains at least $N_0 + 1$ blocks of l_j consecutive equal numbers.

Now the previously described procedure in the case that $a_i/a_j \in \mathbb{Q}$ for all i, j can be applied for each subsequence s_j separately, where l_j replaces n in the above argument. $\square$

REMARK. If the volume of E is known, one does not need to know N_0 in Fact 1. The proof of this is left to the interested reader. $\diamondsuit$

The set of Ekeland–Hofer capacities does *not* form a generating system for symplectic capacities on Ell^{2n}. Indeed, the volume capacity c_{vol} is not the pointwise limit of homogeneous monotone functions of Ekeland–Hofer capacities:

EXAMPLE 10. Consider the ellipsoids $E = E(1, \ldots, 1, 3^n + 1)$ and $F = E(3, \ldots, 3)$ in Ell^{2n}. As is easy to see,

$$c_k^{\mathrm{EH}}(E) < c_k^{\mathrm{EH}}(F) \qquad \text{for all } k. \tag{4–1}$$

Assume that f_i is a sequence of homogeneous monotone functions of Ekeland–Hofer capacities which converge pointwise to c_{vol}. By (4–1) and the monotonicity of the f_i we would find that $c_{\text{vol}}(E) \leq c_{\text{vol}}(F)$. This is not true.

PROBLEM 15. *Do the Ekeland–Hofer capacities together with the volume capacity form a generating system for symplectic capacities on Ell^{2n}?*

If the answer to this problem is "yes", this is a very difficult problem as Lemma 2 below illustrates.

4.1.2. Ellipsoids in dimension 4. A generalized capacity on ellipsoids in dimension 4 is represented by a function $c(a) := c\big(E(a, 1)\big)$ of a single real variable $0 < a \leq 1$. This function has the following two properties.

(i) The function $c(a)$ is nondecreasing.
(ii) The function $c(a)/a$ is nonincreasing.

The first property follows directly from the (Monotonicity) axiom. The second property follows from (Monotonicity) and (Conformality): For $a \leq b$, $E(b, 1) \subset E\left(\frac{b}{a}a, \frac{b}{a}\right)$, hence $c(b) \leq \frac{b}{a}c(a)$. Note that property (ii) is equivalent to the estimate

$$\frac{c(b) - c(a)}{b - a} \leq \frac{c(a)}{a} \tag{4–2}$$

for $0 < a < b$, so the function $c(a)$ is Lipschitz continuous at all $a > 0$. We will restrict our attention to *normalized* (generalized) capacities, so the function c also satisfies

(iii) $c(1) = 1$.

An ellipsoid $E(a_1, \ldots, a_n)$ embeds into $E(b_1, \ldots, b_n)$ by a *linear* symplectic embedding only if $a_i \leq b_i$ for all i, see [49]. Hence for normalized capacities on the category $LinEll^4$ of ellipsoids with *linear* embeddings as morphisms, properties (i), (ii) and (iii) are the only restrictions on the function $c(a)$. On Ell^4, nonlinear symplectic embeddings ("folding") yield additional constraints which are still not completely known; see [86] for the presently known results.

By Fact 1, the embedding capacities c_B and c^B are the smallest, resp. largest, normalized capacities on ellipsoids. By Gromov's Nonsqueezing Theorem, $c_B(a) = \bar{c}_1(a) = a$. The function $c^B(a)$ is not completely known. Fact 1 applied to $\bar{c}_2$ yields

$$c^B(a) = 1 \ \text{ if } a \in \left[\tfrac{1}{2}, 1\right] \quad \text{and} \quad c^B(a) \geq 2a \ \text{ if } a \in \left(0, \tfrac{1}{2}\right],$$

and Fact 1 applied to c_{vol} yields $c^B(a) \geq \sqrt{a}$. Folding constructions provide upper bounds for $c^B(a)$. Lagrangian folding [94] yields $c^B(a) \leq l(a)$ where

$$
l(a) = \begin{cases} (k+1)a & \text{for } \dfrac{1}{k(k+1)} \leq a \leq \dfrac{1}{(k-1)(k+1)} \\[2ex] \dfrac{1}{k} & \text{for } \dfrac{1}{k(k+2)} \leq a \leq \dfrac{1}{k(k+1)} \end{cases}
$$

and multiple symplectic folding [86] yields $c^B(a) \leq s(a)$ where the function $s(a)$ is as shown in Figure 1. While symplectically folding once yields $c^B(a) \leq a + 1/2$ for $a \in (0, 1/2]$, the function $s(a)$ is obtained by symplectically folding "infinitely many times", and it is known that

$$
\liminf_{\varepsilon \to 0^+} \frac{c^B\left(\frac{1}{2}\right) - c^B\left(\frac{1}{2} - \varepsilon\right)}{\varepsilon} \geq \frac{8}{7}.
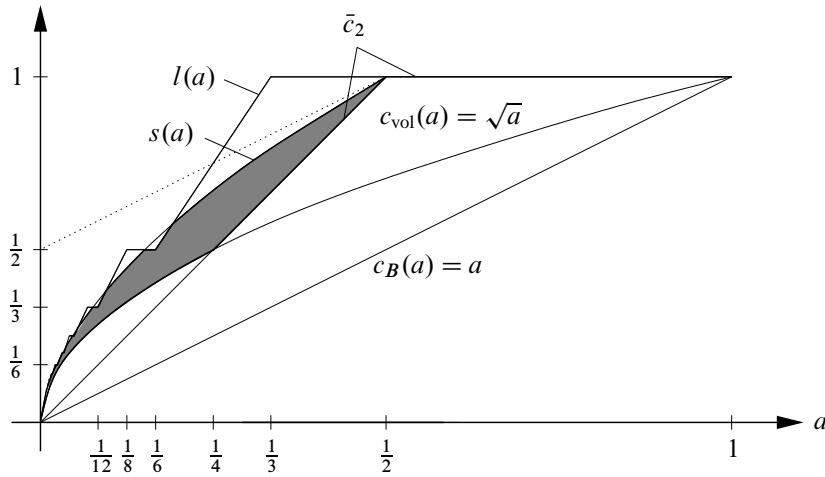$$

Figure 1. Lower and upper bounds for $c^B(a)$.

Let us come back to Problem 15.

LEMMA 2. *If the Ekeland–Hofer capacities and the volume capacity form a generating system for symplectic capacities on Ell^{2n}, then $c^B\left(\frac{1}{4}\right) = \frac{1}{2}$.*

We recall that $c^B\left(\frac{1}{4}\right) = \frac{1}{2}$ means that the ellipsoid $E(1, 4)$ symplectically embeds into $B^4(2 + \varepsilon)$ for every $\varepsilon > 0$.

PROOF OF LEMMA 2. We can assume that all capacities are normalized. By assumption, there exists a sequence f_i of homogeneous and monotone functions in the $\bar{c}_k$ and in c_{vol} forming normalized capacities which pointwise converge to

c^B. As is easy to see, $\bar{c}_k\left(E\left(\frac{1}{4}, 1\right)\right) \leq \bar{c}_k\left(B^4\left(\frac{1}{2}\right)\right)$ for all k, and $c_{\mathrm{vol}}\left(E\left(\frac{1}{4}, 1\right)\right) = c_{\mathrm{vol}}\left(B^4\left(\frac{1}{2}\right)\right)$. Since the f_i are monotone and converge in particular at $E\left(\frac{1}{4}, 1\right)$ and $B^4\left(\frac{1}{2}\right)$ to c^B, we conclude that $c^B\left(\frac{1}{4}\right) = c^B\left(E\left(\frac{1}{4}, 1\right)\right) \leq c^B\left(B^4\left(\frac{1}{2}\right)\right) = \frac{1}{2}$, which proves Lemma 2. $\qquad\square$

In view of Lemma 2, the following problem is a special case of Problem 15.

PROBLEM 16. *Is it true that $c^B\left(\frac{1}{4}\right) = \frac{1}{2}$?*

The best upper bound for $c^B\left(\frac{1}{4}\right)$ presently known is $s\left(\frac{1}{4}\right) \approx 0.6729$. Answering Problem 16 in the affirmative means to construct for each $\varepsilon > 0$ a symplectic embedding $E\left(\frac{1}{4}, 1\right) \to B^4\left(\frac{1}{2} + \varepsilon\right)$. We do not believe that such embeddings can be constructed "by hand". A strategy for studying symplectic embeddings of 4-dimensional ellipsoids by algebrogeometric tools is proposed in [6].

Our next goal is to represent the (normalized) Ekeland–Hofer capacities as embedding capacities. First we need some preparations.

From the above discussion of c^B it is clear that capacities and folding also yield bounds for the functions $c^{E(1,b)}$ and $c_{E(1,b)}$. We content ourselves with noting

LEMMA 3. *Let $N \in \mathbb{N}$ be given. Then for $N \leq b \leq N + 1$ we have*

$$c^{E(1,b)}(a) = \begin{cases} 1/b & \text{for} \quad 1/(N+1) \leq a \leq 1/b, \\ a & \text{for} \quad 1/b \leq a \leq 1 \end{cases} \tag{4–3}$$

and

$$c_{E(1,b)}(a) = \begin{cases} a & \text{for} \quad 0 < a \leq 1/b, \\ 1/b & \text{for} \quad 1/b \leq a \leq 1/N, \end{cases} \tag{4–4}$$

see Figure 2.

REMARK. Note that (4–4) completely describes $c_{E(1,b)}$ on the whole interval $(0, 1]$ for $1 \leq b \leq 2$.

PROOF. As both formulas are proved similarly, we only prove (4–3). The first Ekeland–Hofer capacity gives the lower bound $c^{E(1,b)}(a) \geq a$ for all $a \in (0, 1]$. Note that for $a \geq 1/b$ this bound is achieved by the standard embedding, so that the second claim follows.

For $1/(N+1) \leq a \leq 1/N$ we have $\bar{c}_{N+1}(E(a, 1)) = 1$ and $\bar{c}_{N+1}(E(1, b)) = b$. By Fact 1 we see that $c^{E(1,b)} \geq 1/b$ on this interval, and this bound is again achieved by the standard embedding. This completes the proof of (4–3). $\qquad\square$
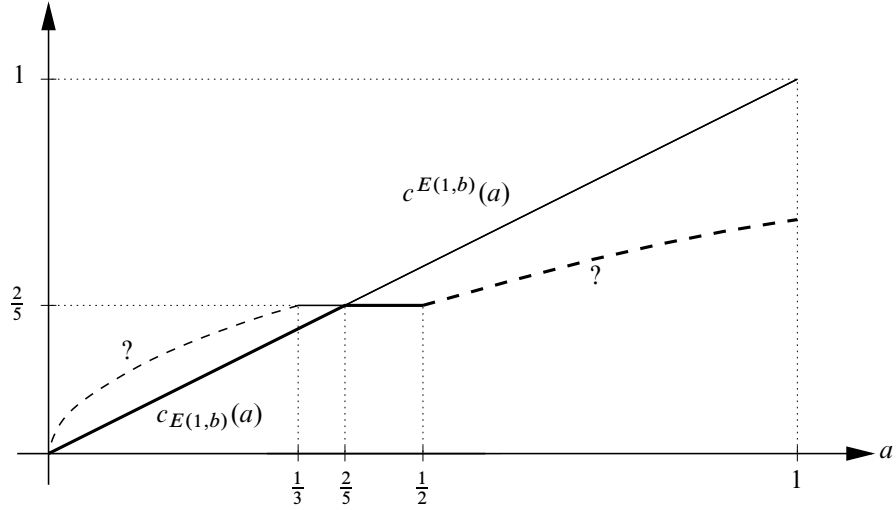
Figure 2. The functions $c^{E(1,b)}(a)$ and $c_{E(1,b)}(a)$ for $b = \frac{5}{2}$.

REMARK. Consider the functions

$$e^b(a) := c^{E(1,b)}(a), \quad a \in (0, 1], \; b \geq 1.$$

Notice that $e^1 = c^B$. By Gromov's Nonsqueezing Theorem and monotonicity,

$$a = c_B(a) = c^Z(a) \leq e^b(a) \leq c^B(a), \quad a \in (0, 1], \; b \geq 1.$$

Since $e^b(a) = \big(c_{E(a,1)}\big(E(1, b)\big)\big)^{-1}$ by equation (3–1), we see that for each $a \in (0, 1]$ the function $b \mapsto e^b(a)$ is monotone decreasing and continuous. By (4–3), it satisfies $e^b(a) = a$ for $a \geq 1/b$. In particular, we see that the family of graphs $\big\{\text{graph}(e^b) \mid 1 \leq b < \infty\big\}$ fills the whole region between the graphs of c_B and c^B; see Figure 1. $\diamond$

The normalized Ekeland–Hofer capacities are represented by piecewise linear functions $\bar{c}_k(a)$. Indeed, $\bar{c}_1(a) = a$ for all $a \in (0, 1]$, and for $k \geq 2$ the following formula follows straight from the definition.

LEMMA 4. *Setting* $m := \left[\dfrac{k+1}{2}\right]$, *the function* $\bar{c}_k \colon (0, 1] \to (0, 1]$ *is given by*

$$\bar{c}_k(a) = \begin{cases} \dfrac{k+1-i}{m}a & \text{for } \dfrac{i-1}{k+1-i} \leq a \leq \dfrac{i}{k+1-i}, \\[2mm] \dfrac{i}{m} & \text{for } \dfrac{i}{k+1-i} \leq a \leq \dfrac{i}{k-i}. \end{cases} \tag{4–5}$$

Here i *takes integer values between* 1 *and* m.

Figure 3 shows the first six of the $\bar{c}_k$ and their limit function c_∞ according to Proposition 1.
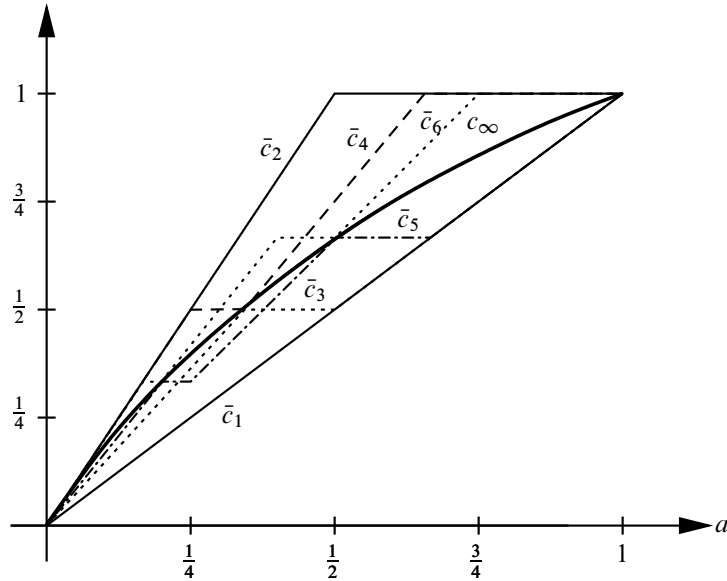
Figure 3. The first six $\bar{c}_k$ and c_∞.

In dimension 4, the uniform convergence $\bar{c}_k \to c_\infty$ is very transparent, as can be seen in Figure 3. One readily checks that $\bar{c}_k - c_\infty \geq 0$ if k is even, in which case $\|\bar{c}_k - c_\infty\| = \frac{1}{k+1}$, and that $\bar{c}_k - c_\infty \leq 0$ if $k = 2m - 1$ is odd, in which case $\|\bar{c}_k - c_\infty\| = \frac{m-1}{mk}$ if $k \geq 3$. Note that the sequences of the even (resp. odd) $\bar{c}_k$ are almost, but not quite, decreasing (resp. increasing). We still have

COROLLARY 1. *For all $r, s \in \mathbb{N}$, we have*

$$\bar{c}_{2rs} \leq \bar{c}_{2r}.$$

This will be a consequence of the following characterization of Ekeland–Hofer capacities.

LEMMA 5. *Fix $k \in \mathbb{N}$ and denote by $[a_l, b_l]$ the interval on which $\bar{c}_k$ has the value $l / \left[\frac{k+1}{2}\right]$. Then*

(a) $\bar{c}_k \leq c$ *for every capacity c such that $\bar{c}_k(a_l) \leq c(a_l)$ for all $l = 1, 2, \ldots, \left[\frac{k+1}{2}\right]$.*
(b) $\bar{c}_k \geq c$ *for every capacity c such that $\bar{c}_k(b_l) \geq c(b_l)$ for all $l = 1, 2, \ldots, \left[\frac{k}{2}\right]$ and*

$$\lim_{a \to 0} \frac{c(a)}{a} \leq \frac{k}{\left[\frac{k+1}{2}\right]}.$$

PROOF. Formula (4–2) and Lemma 4 show that where a normalized Ekeland–Hofer capacity grows, it grows with maximal slope. In particular, going left from the left end point a_l of a plateau a normalized Ekeland–Hofer capacity drops

with the fastest possible rate until it reaches the level of the next lower plateau and then stays there, showing the minimality. Similarly, going right from the right end point b_l of some plateau a normalized Ekeland–Hofer capacity grows with the fastest possible rate until it reaches the next higher level, showing the maximality. $\qquad\square$

PROOF OF COROLLARY 1. The right end points of plateaus for $\bar{c}_{2r}$ are given by $b_i = \frac{i}{2r-i}$. Thus we compute

$$\bar{c}_{2r}\left(\frac{i}{2r-i}\right) = \frac{i}{r} = \frac{i\,s}{r\,s} = \bar{c}_{2rs}\left(\frac{i\,s}{2rs-i\,s}\right) = \bar{c}_{2rs}\left(\frac{i}{2r-i}\right)$$

and the claim follows from the characterization of $\bar{c}_{2r}$ by maximality. $\qquad\square$

Lemma 3 and the piecewise linearity of the $\bar{c}_k$ suggest that they may be representable as embedding capacities into a disjoint union of finitely many ellipsoids. This is indeed the case.

PROPOSITION 2. *The normalized Ekeland–Hofer capacity $\bar{c}_k$ on Ell^4 is the capacity c^{X_k} of embeddings into the disjoint union of ellipsoids*

$$X_k = Z\left(\frac{m}{k}\right) \sqcup \coprod_{j=1}^{\left[\frac{k}{2}\right]} E\left(\frac{m}{k-j}, \frac{m}{j}\right),$$

where $m = \left[\frac{k+1}{2}\right]$.

PROOF. The proposition clearly holds for $k = 1$. We thus fix $k \geq 2$. Recall from Lemma 4 that $\bar{c}_k$ has $\left[\frac{k}{2}\right]$ plateaus, the j-th of which has height $\frac{j}{m}$ and starts at $a_j := j/(k+1-j)$ and ends at $b_j := j/(k-j)$. The j-th ellipsoid in Proposition 2 is found as follows: In view of (4–3) we first select an ellipsoid $E(1, b)$ so that the point $\frac{1}{b}$ corresponds to b_j. This ellipsoid is then rescaled to achieve the correct height $\frac{j}{m}$ of the plateau (note that by conformality, $\alpha c^{E(\alpha, \alpha b)} = c^{E(1,b)}$ for $\alpha > 0$). We obtain the candidate ellipsoid

$$E_j = E\left(\frac{m}{k-j}, \frac{m}{j}\right).$$

The slope of $\bar{c}_k$ following its j-th plateau and the slope of c^{E_j} after its plateau both equal $\frac{k-j}{m}$. The cylinder is added to achieve the correct behavior near $a = 0$. We are thus left with showing that for each $1 \leq j \leq \left[\frac{k}{2}\right]$,

$$\bar{c}_k(a) \leq c^{E_j}(a) \quad \text{for all } a \in (0, 1].$$

According to Lemma 5 (a) it suffices to show that for each $1 \le j \le \left[\frac{k}{2}\right]$ and each $1 \le l \le \left[\frac{k}{2}\right]$ we have

$$\bar{c}_k(a_l) = \frac{l}{m} \le c^{E_j}(a_l). \tag{4-6}$$

For $l > j$, the estimate (4–6) follows from the fact that $\bar{c}_k = c^{E_j}$ near b_j and from the argument given in the proof of Lemma 5 (a), and for $l = j$ the estimate (4–6) follows from (4–3) of Lemma 3 by a direct computation. We will deal with the other cases

$$1 \le l < j \le \left[\frac{k}{2}\right]$$

by estimating $c^{E_j}(a_l)$ from below, using Fact 1 with $c = c_{\mathrm{vol}}$ and $c = \bar{c}_2$.

Fix j and recall that $c_{\mathrm{vol}}(E(x, y)) = \sqrt{xy}$, so

$$c^{E_j}(a_l) \ge \frac{c_{\mathrm{vol}}(E(a_l, 1))}{c_{\mathrm{vol}}\left(E\left(\frac{m}{k-j}, \frac{m}{j}\right)\right)} = \sqrt{\frac{lj(k-j)}{(k+1-l)m^2}} = \frac{l}{m} \cdot \sqrt{\frac{j(k-j)}{(k+1-l)l}}$$

gives the desired estimate (4–6) if $j(k-j) \ge -l^2 + (k+1)l$. Computing the roots $l_\pm$ of this quadratic inequality in l, we find that this is the case if

$$l \le l_- = \tfrac{1}{2}\left(k+1 - \sqrt{1+2k+(k-2j)^2}\right).$$

Computing the normalized second Ekeland–Hofer capacity under the assumption that $a_l \le \frac{1}{2}$, we find that $\bar{c}_2(E(a_l, 1)) = 2a_l = \frac{2l}{k+1-l}$ and $\bar{c}_2(E_j) \le \frac{m}{j}$, so

$$c^{E_j}(a_l) \ge \frac{\bar{c}_2(E(a_l, 1))}{\bar{c}_2\left(E\left(\frac{m}{k-j}, \frac{m}{j}\right)\right)} \ge \frac{2l}{k+1-l} \cdot \frac{j}{m} = \frac{l}{m} \cdot \frac{2j}{k+1-l},$$

which gives the required estimate (4–6) if

$$l \ge k+1-2j.$$

Note that for $\frac{1}{2} \le a_l \le 1$ we have $\bar{c}_2(E(a_l, 1)) = 1$ and hence

$$\frac{\bar{c}_2(E(a_l, 1))}{\bar{c}_2\left(E\left(\frac{m}{k-j}, \frac{m}{j}\right)\right)} \ge \frac{j}{m} > \frac{l}{m}$$

trivially, because we only consider $l < j$.

So combining the results from the two capacities, we find that the desired estimate (4–6) holds provided either $l \leq l_- = \frac{1}{2}\left(k+1 - \sqrt{1+2k+(k-2j)^2}\right)$ or $l \geq k+1-2j$. As we only consider $l < j$, it suffices to verify that

$$\min(j-1, k+1-2j) \leq \frac{1}{2}\left(k+1 - \sqrt{1+2k+(k-2j)^2}\right)$$

for all positive integers j and k satisfying $1 \leq j \leq \left[\frac{k}{2}\right]$. This indeed follows from another straightforward computation, completing the proof of Proposition 2. $\square$

Using the results above, we find a presentation of the normalized capacity $c_\infty = \lim_{k\to\infty} \bar{c}_k$ on Ell^4 as embedding capacity into a countable disjoint union of ellipsoids. Indeed, the space X_{4r} appearing in the statement of Proposition 2 is obtained from X_{2r} by adding r more ellipsoids. Combined with Proposition 1 this yields the presentation

$$c_\infty = c^X \quad \text{on } Ell^4,$$

where $X = \coprod_{r=1}^{\infty} X_{2r}$ is a disjoint union of countably many ellipsoids. Together with Conjecture 1, the following conjecture suggests a much more efficient presentation of c_∞ as an embedding capacity.

CONJECTURE 2. *The restriction of the normalized Lagrangian capacity $\bar{c}_L$ to Ell^4 equals the embedding capacity c^X, where X is the connected subset $B(1) \cup Z(\frac{1}{2})$ of $\mathbb{R}^4$.*

For the embedding capacities *from* ellipsoids, we have the following analogue of Proposition 2.

PROPOSITION 3. *The normalized Ekeland–Hofer capacity $\bar{c}_k$ on Ell^4 is the maximum of finitely many capacities $c_{E_{k,j}}$ of embeddings of ellipsoids $E_{k,j}$,*

$$\bar{c}_k(a) = \max\{c_{E_{k,j}}(a) \mid 1 \leq j \leq m\}, \quad a \in (0,1],$$

where

$$E_{k,j} = E\left(\frac{m}{k+1-j}, \frac{m}{j}\right)$$

with $m = \left[\frac{k+1}{2}\right]$.

PROOF. The ellipsoids $E_{k,j}$ are determined using (4–4) in Lemma 3. According to Lemma 5 (b), this time it suffices to check that for all $1 \leq j \leq l \leq \left[\frac{k}{2}\right]$ the values of the corresponding capacities at the right end points $b_l = \frac{l}{k-l}$ of plateaus of $\bar{c}_k$ satisfy

$$c_{E_{k,j}}(b_l) \leq \frac{l}{m} = \bar{c}_k(b_l). \tag{4–7}$$

The case $l = j$ follows from (4–4) in Lemma 3 by a direct computation. For the remaining cases

$$1 \leq j < l \leq \left[\frac{k}{2} \right]$$

we use three different methods, depending on the value of j. If $j \leq \frac{k-1}{3}$, then Fact 1 with $c = c_{\mathrm{vol}}$ gives (4–7) by a computation similar to the one in the proof of Proposition 2. If $j \geq \frac{k+1}{3}$, then $a_j = \frac{j}{k+1-j} \geq \frac{1}{2}$, so that (4–4) in Lemma 3 shows that $c_{E_{k,j}}$ is constant on $[a_j, 1]$, proving (4–7) in this case. Finally, if $j = \frac{k}{3}$ and $l \geq j + 1$, then $\bar{c}_2(E_{k,j}) = \frac{2m}{k+1-j}$ and $\bar{c}_2(b_l) = 1$, so with Fact 1 we get

$$c_{E_{k,j}}(b_l) \leq \frac{k+1-j}{2m},$$

which is smaller than $\frac{l}{m}$ for the values of j and l we consider here. This completes the proof of Proposition 3. $\square$

Here is the corresponding conjecture for the normalized Lagrangian capacity.

CONJECTURE 3. *The restriction of the normalized Lagrangian capacity $\bar{c}_L$ to Ell^{2n} equals the embedding capacity $c_{P(1/n,\ldots,1/n)}$ of the cube of radius $1/\sqrt{n}$.*

4.2. Polydiscs.

4.2.1. Arbitrary dimension. Again we first describe the values of the capacities in Section 2 on polydiscs.

The values of the Gromov radius c_B on polydiscs are

$$c_B\big(P(a_1, \ldots, a_n)\big) = \min\{a_1, \ldots, a_n\}.$$

As for ellipsoids, this also determines the values of c_1^{EH}, c_{HZ}, $e(\cdot, \mathbb{R}^{2n})$ and c^Z. According to [19], the values of Ekeland–Hofer capacities on polydiscs are

$$c_k^{\mathrm{EH}}\big(P(a_1, \ldots, a_n)\big) = k\pi \min\{a_1, \ldots, a_n\}.$$

Using Chekanov's result [11] that $A_{\min}(L) \leq e(L, \mathbb{R}^{2n})$ for every closed Lagrangian submanifold $L \subset \mathbb{R}^{2n}$, one finds the values of the Lagrangian capacity on polydiscs to be

$$c_L\big(P(a_1, \ldots, a_n)\big) = \pi \min\{a_1, \ldots, a_n\}.$$

Since $\mathrm{vol}\big(P(a_1, \ldots, a_n)\big) = a_1 \cdots a_n \cdot \pi^n$ and $\mathrm{vol}(B^{2n}) = \frac{\pi^n}{n!}$, the values of the volume capacity on polydiscs are

$$c_{\mathrm{vol}}\big(P(a_1, \ldots, a_n)\big) = (a_1 \cdots a_n \cdot n!)^{1/n}.$$

As in the case of ellipsoids, a (generalized) capacity c on Pol^{2n} can be viewed as a function

$$c(a_1, \ldots, a_{n-1}) := c\left(P(a_1, \ldots, a_{n-1}, 1)\right)$$

on the set $\{0 < a_1 \leq \ldots \leq a_{n-1} \leq 1\}$. Directly from the definitions and the computations above we obtain the following easy analogue of Proposition 1.

PROPOSITION 4. *As $k \to \infty$, the normalized Ekeland–Hofer capacities $\bar{c}_k$ converge on Pol^{2n} uniformly to the normalized Lagrangian capacity $\bar{c}_L = nc_L/\pi$.*

Propositions 4 and 1 (together with Conjecture 1) give rise to

PROBLEM 17. *What is the largest subcategory of Op^{2n} on which the normalized Lagrangian capacity is the limit of the normalized Ekeland–Hofer capacities?*

4.2.2. Polydiscs in dimension 4. Again, a normalized (generalized) capacity on polydiscs in dimension 4 is represented by a function $c(a) := c\left(P(a, 1)\right)$ of a single real variable $0 < a \leq 1$, which has the properties (i), (ii), (iii). Contrary to ellipsoids, these properties are not the only restrictions on a normalized capacity on 4-dimensional polydiscs even if one restricts to *linear* symplectic embeddings as morphisms. Indeed, the linear symplectomorphism

$$(z_1, z_2) \mapsto \frac{1}{\sqrt{2}}\,(z_1 + z_2, z_1 - z_2)$$

of $\mathbb{R}^4$ yields a symplectic embedding

$$P(a, b) \hookrightarrow P\left(\frac{a+b}{2} + \sqrt{ab}, \frac{a+b}{2} + \sqrt{ab}\right)$$

for any $a, b > 0$, which implies

FACT 12. *For any normalized capacity c on $LinPol^4$,*

$$c(a) \leq \frac{1}{2} + \frac{a}{2} + \sqrt{a}.$$

Still, we have the following easy analogues of Propositions 2 and 3.

PROPOSITION 5. *The normalized Ekeland–Hofer capacity $\bar{c}_k$ on Pol^4 is the capacity c^{Y_k}, where*

$$Y_k = Z\left(\frac{[\frac{k+1}{2}]}{k}\right),$$

as well as the capacity $c_{Y_k'}$, where

$$Y_k' = B\left(\frac{[\frac{k+1}{2}]}{k}\right).$$

COROLLARY 2. *The identity $\bar{c}_k = c^{X_k}$ of Proposition 2 extends to $Ell^4 \cup Pol^4$.*

PROOF. Note that Y_k is the first component of the space X_k of Proposition 2. It thus remains to show that for each of the ellipsoid components E_j of X_k,

$$\bar{c}_k\left(P(a,1)\right) \leq c^{E_j}\left(P(a,1)\right), \quad a \in (0,1].$$

This follows at once from the observation that for each j we have $c_k^{\mathrm{EH}}\left(E_j\right) = [\frac{k+1}{2}]\pi$, whereas $c_k^{\mathrm{EH}}\left(P(a,1)\right) = ka\pi$. $\qquad\square$

PROBLEM 18. *Does the equality $\bar{c}_k = c^{X_k}$ hold on a larger class of open subsets of $\mathbb{R}^4$?*

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KAI CIELIEBAK
MATHEMATISCHES INSTITUT
LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
THERESIENSTRASSE 39
D-80333 MÜNCHEN
GERMANY
    kai@mathematik.uni-muenchen.de

HELMUT HOFER
COURANT INSTITUTE
NEW YORK UNIVERSITY
251 MERCER STREET
NEW YORK, NY 10012
UNITED STATES
    hofer@cims.nyu.edu

JANKO LATSCHEV
MATHEMATISCHES INSTITUT
LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
THERESIENSTRASSE 39
D-80333 MÜNCHEN
GERMANY
    latschev@mathematik.uni-muenchen.de

FELIX SCHLENK
DÉPARTEMENT DE MATHÉMATIQUES
UNIVERSITÉ LIBRE DE BRUXELLES
BOULEVARD DU TRIOMPHE
1050 BRUXELLES
BELGIUM
    fschlenk@ulb.ac.be