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Causes of stretching of Birkhoff sums and mixing in flows on surfaces

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On the Anniversary of Anatole Katok, my Friend and Teacher.

ABSTRACT. We study causes of stretching of Birkhoff sums and study their action in the mixing of various surface flows. In so doing, we succeed in amplifying the result of Khanin and Sinai about mixing in the Arnold's example of flow with nonsingular fixed points on a two-dimensional torus.

1. Introduction

There are three known kinds of mixing flows on two-dimensional surfaces: continuous flows without fixed points on a torus, smooth flows with singular fixed points, and smooth flows with nonsingular fixed points (Arnold's example). However, in the last case mixing arises not on the whole torus but only on an ergodic component.

We suggest a special flow S^t, constructed over a circle rotation or an interval exchange transformation (which we denote by T) and under some positive "roof" function, as an ergodic relative of a Borel measure-preserving flow on a two-dimensional surface. In such special flows the only possible cause of mixing is the difference in the times that various points take to get from the "floor" to the "roof". This can cause, as time passes, a small rectangle to be strongly stretched and almost uniformly distributed along trajectories and hence over the phase space.

The divergence of adjacent points is described via Birkhoff sums of the "roof" function

$$f^r(x) = \sum_{k=0}^{r-1} f(T^k x).$$

This is obvious from the relation $S^t(x, y) = S^{y+t-f^r(x)}(T^r x, 0)$, where (x, y) denotes a point in phase space. Strong and almost uniform distribution of a little rectangle over the phase space is ensured by strong almost uniform stretching of Birkhoff sums for $r \approx t$.

It is obvious → This is obvious (is this what you mean?)

Formally this is described by the next theorem. In order to state it, we introduce, for $x \in \mathbb{T}^1$ and $t > 0$, the notation $\mathcal{R}(t, x)$ for the number of jumps that the point $(x; 0)$ undergoes under the action of S^t over a time t. For any measurable $X \subset \mathbb{T}^1$ we set

$$\mathcal{R}(t, X) = \bigcup_{x \in X} \mathcal{R}(t, x).$$

THEOREM 1 (SUFFICIENT CONDITION FOR MIXING). *Let T be an ergodic circle rotation and suppose $t_0 > 0$. Assume that the following objects are fixed for each $t > t_0$:*

– *a finite partial partition ξ_t of the circle $\mathbb{T}^1$ into closed intervals: $\xi_t = \{C\}$ with*

$$\lim_{t \to +\infty} \max_{C \in \xi_t} |C| = 0, \qquad \lim_{t \to +\infty} \mu([\xi_t]) = 1,$$

where $[\xi_t]$ denotes the union of elements of ξ_t; and

– *positive functions ε and H such that $\varepsilon(t) \to 0$ and $H(t) \to +\infty$ for $t \to +\infty$.*

If for each $t > t_0$ for any $C \in \xi_t$ and any $r \in \mathcal{R}(t, [\xi_t])$, the Birkhoff sum $f^r|C$ is $(\varepsilon(t), H(t))$-uniformly distributed, then the special flow constructed over T and under the function f is mixing.

That $f^r|C$ is (ε, H)-uniformly distributed means that the function $f^r|C$ is, in some sense, ε-uniformly distributed in an interval of length no less then H. The exact definition of an ε-uniform distribution varies slightly with the circumstances.

We identify three causes of stretching of Birkhoff sums which we may tentatively call ergodic, resonant and individual. They exert various influences on point divergence in various kinds of mixing flows.

2. Flows without fixed points

It is shown in my paper [5] that if f is of bounded variation, the special flow over the circle rotation with roof function f cannot be mixing. A. Katok [3] generalized this result to special flows over interval exchange transformations.

In this situation one can say that the absence of mixing is a corollary of an ergodic effect, the effect of averaging, which results in the Birkhoff sums f^r, for certain values of r, having relatively small variation on sufficiently large sets.

It is possible to gain mixing for a special flow over a circle rotation and under continuous functions at the expense of a resonance condition on T and f.

do you mean "at the cost of"? (That is, provided with we assume...)?

The main idea is the following. Let p_n/q_n be the sequence of convergents of the rotation angle ρ of the circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$. Choose ρ and positive sequences a_n and t_n such that for $n \to +\infty$

$$a_n t_n \nearrow +\infty, \quad a_{n+1} t_n \searrow 0, \quad \frac{a_n q_n}{a_{n+1} q_{n+1}} \to 0, \quad t_n/q_n \to 0.$$

Set

$$u_0(x) = \min(\{x\}, \{1-x\}) - \tfrac{1}{4}, \quad u_n(x) = a_n u_0(q_n x), \tag{2–1}$$

$$f = F + \sum_{n=1}^{\infty} u_n, \tag{2–2}$$

where F is a positive Lipschitz function on $\mathbb{T}^1$ with unit integral.

THEOREM 2. *The special flow over the rotation of the circle by ρ and under the function f constructed above is mixing.*

For instance, u_0 could be the functions $\sin 2\pi x$ or $\cos 2\pi x$.

The stretching and almost uniform distribution of Birkhoff sums f^r for $r \in (t_n/2, 2t_{n+1})$ is ensured by the term u_n, because it "almost resonates" with the rotation through the angle ρ: the period of u_n is $1/q_n$, and thus $u_n(x+\rho) \approx u_n(x)$, and

$$u_n^r(x) \approx r u_n(x) \quad \text{for } r \leqslant 2t_{n+1} \ll q_{n+1}),$$

which yields vertical stretching of u_n^r in each segment of length $1/q_n$, the requirement $a_n t_n \nearrow +\infty$ guaranteeing strong stretching for $r \in (t_n/2, 2t_{n+1})$ and n sufficiently large, since the amplitude of u_n^r is approximately equal to $r a_n/4$. One can call this effect ergodic.

However, the growth of f^r cannot be ensured indefinitely by a single term. In further iterations, due to the accumulation of errors in the shifts the growth of u_n^r breaks down. For example, if $r \approx q_{n+1}/2$, the shifted function $u_n(T^r x)$ is almost exactly half a period out of phase with $u_n(x)$. Hence the next term u_{n+1} must be charged with the stretching in the next interval $r \in (t_{n+1}/2, 2t_{n+2})$; moreover, for $r \approx t_{n+1}$ the term u_{n+1}^r has to grow enough, and therefore it must be also be taken into account in the estimates.

This fact and the requirement that the function $(u_n^r + u_{n+1}^r)|C$ (where C is an element of the partition) be stretched and almost uniformly distributed for $r \in (t_n/2, 2t_{n+1})$ impose the additional constraints on a_n and q_n given above.

Due to the rapid change of a_n and q_n all other terms can, in effect, be discarded: the preceding ones because their derivatives are relatively small and the following ones because their amplitudes are not yet sufficiently large.

The construction of a mixing special flow over an arbitrary ergodic automorphism and under a continuous function [6] is also based on this two-term model, and the terms in this case are related to the Rokhlin towers.

Using the model above we can construct mixing flows over circle rotations and under the roof functions with additional regularity [9].

THEOREM 3. *For any sufficiently regular modulus of continuity weaker than the Lipschitz condition, there exists a mixing special flow over some circle rotation and under a roof function with this modulus of continuity.*

THEOREM 4. *For any $\gamma \in (0,1)$ and $\theta > 0$, and any circle rotation through the angle ρ satisfying $q_{n+1} \asymp q_n^{1+\theta}$, there exists a positive function $f \in C^\gamma(\mathbb{T}^1)$ such that the special flow S^t over this circle rotation and under f is mixing with power-rate behavior, that is, there exist an exponent $\beta > 0$, a constant M and a time moment t^* such that*

$$|\mu_2(S^t Q_1 \cap Q_2) - \mu_2(Q_1)\cdot\mu_2(Q_2)| < M t^{-\beta}.$$

for any rectangles Q_1, Q_2 and any $t > t^*$.

For example we'll construct the flow satisfying this theorem.

Let $\theta > 0$ and $0 < \gamma < 1$ be given. Choose ρ such that the sequence q_n of denominators of convergents to ρ satisfies

$$q_{n+1} \asymp q_n^{1+\theta}.$$

Choose a sequence a_n satisfying

$$a_n \asymp q_n^{-\gamma},$$

and then construct f by (2–1) and (2–2). The times for switching from one term to another are given by $t_n = q_n^{\varkappa}$, where $\varkappa$ and the exponent β of the mixing rate are found from some system of inequalities; moreover β depends on θ and γ. For example, if $\theta = 1$ and $\gamma = 1/2$, we may set $\beta = 1/9$, and for $\theta = 0.754$, $\gamma = 0.57$ one may set $\beta \approx 0.118$.

Bassam Fayad cleverly implemented this two-term model in the construction of an analytical mixing special flow over a translation on $\mathbb{T}^2$ [2]. He represents each term in (2–2) as $u_n(x,y) = X_n(x) + Y_n(y)$ and thus arranges the shifts in each direction on $\mathbb{T}^2$ so that two successive terms do not interfere: the terms

$X_n(x)$, $Y_n(y)$, $X_{n+1}(x)$, $Y_{n+1}(y)$ consequently charge with stretching, and functions of different variables change one another; as a result u_{n+1} can be substantially smaller than u_n, and f can be made analytical.

To summarize this section we can say that the mixing in the model above is obtained with a resonant effect which is in a certain sense stronger than the ergodic one. In this case rapid growth is needed for q_n.

The natural question is for which moduli of continuity is it possible to obtain a mixing special flow over a circle rotation by a typical angle. Perhaps it is necessary to construct another model realizing the resonant effect, if possible.

3. Singular fixed points

Another variant of mixing flow on a surface is a smooth flow with singular fixed points. Such a flow is isomorphic to a special flow over an interval exchange with a roof function which is smooth everywhere except the break point of T, which are power singularities of the function. point→points (is this what you mean?)

To describe precisely the effects arising from singular points, we introduce after [7] (with some simplifications) a class of functions, denoted by $\mathcal{F}(a,b)$.

Let $M : (0,1] \to \mathbb{R}_+$ be a nondecreasing function with $M(1) \geqslant 1$, let and $\omega : (0,1) \to \mathbb{R}_+$ be a nondecreasing function such that $\lim_{x\to+0}\omega(x) = 0$. We say that $\varphi \in \mathcal{F}_{M,\omega}(a,b)$ if

(1) $\varphi \in C^2(a,b)$,

(2) for any $x, y \in (a,b)$ and any $\theta \in (0,1)$, if $\theta(x-a) \leqslant y-a \leqslant \dfrac{x-a}{\theta}$ then

$$\frac{\varphi''(x)}{M(\theta)} \leqslant \varphi''(y) \leqslant M(\theta)\varphi''(x),$$

and

(3) for any $x \in (a,b)$,

$$\varphi''(x) \geqslant \frac{1}{(x-a)^2\omega(x-a)}.$$

Then we set

$$\mathcal{F}(a,b) = \bigcup_{M,\omega} \mathcal{F}_{M,\omega}(a,b).$$

THEOREM 5. *Let T be an ergodic interval exchange of the circle $\mathbb{T}^1$, and let $\bar{x}_1,\dots,\bar{x}_K$ be a finite set of points containing all the break points of T. Assume that for $x \in \mathbb{T}^1 \setminus \cup_i \bar{x}_i$ we have $f(x) \geqslant c > 0$ and*

$$f(x) = f_0(x) + \sum_{i=1}^{K} \left(f_i(\{x-\bar{x}_i\}) + g_i(\{\bar{x}_i - x\})\right),$$

where $f_i,\ g_i \in \mathcal{F}(0,1)$, and $f_0 \in C^2(\mathbb{T}^1)$. Then the special flow over T with roof function f is mixing.

(Functions of type $x^\alpha(A + o(1))$ for $\alpha \in (0,1)$ and $A > 0$ belong to $\mathcal{F}(0,1)$, so flows with singular fixed points are mixing.)

I'm put this in parentheses since I think "In this case" below doesn't not refer only to the specific example of this paragraph

In this case, for large r, the strong and almost uniform stretching of Birkhoff sums f^r in the interval of continuity (a,b) is provided by two terms having singularities at the points a and b; the other terms do not oppose it. One can say that the mixing in the flow is provided by the *individual* effect of fixed points.

This statement is supported by two following facts (for simplicity we suppose that $a = 0$).

LEMMA 1. *Suppose $\varphi \in \mathcal{F}_{M,\omega}(0,1)$, $0 < b < 1$, and $0 \leqslant h_j < 1 - b$ for $j = 0, \ldots, N$. Then*

$$\sum_{j=0}^{N} \varphi(x + h_j) \in \mathcal{F}_{M,\omega}(0,b).$$

LEMMA 2. *Suppose $\varphi, \psi \in \mathcal{F}_{M,\omega}(0,b)$. If b is small enough then $\varphi(x) + \psi(b - x)|(0,b)$ is almost uniformly distributed in a long enough interval.*

(For exact statements see [7].)

This fact is interesting since it implies that the presence of nonsingular fixed points in the flow on surfaces (or logarithmic singularities of roof function) isn't sufficient for mixing. is this what you mean?

4. Functions with logarithmic singularities

We say that a roof function has *logarithmic singularities* if it suffices the next conditions:

(1) f has K singular points $\bar{x}_1, \ldots, \bar{x}_K$.

(2) $f \in C^1\big(\mathbb{T}^1 \setminus \bigcup_{i=1}^{K} \bar{x}_i\big)$ and $f(x) \geqslant c > 0$.

(3) For any $i = 1, \ldots, K$

$$f'(x) = \frac{1}{\{x - \bar{x}_i\}}(-A_i + o(1)) \quad \text{for} \quad x \to \bar{x}_i + 0,$$
$$f'(x) = \frac{1}{\{\bar{x}_i - x\}}(B_i + o(1)) \quad \text{for} \quad x \to \bar{x}_i - 0,$$

where $A_i,\ B_i \geqslant 0$.

We set

$$A = \sum_{i=1}^{K} A_i, \quad B = \sum_{i=1}^{K} B_i.$$

The function f is called *symmetric* if $A = B \neq 0$, *asymmetric* if $A \neq B$, and *strongly asymmetric* if

$$\text{sign}(A_i - B_i) = \text{sign}(A - B) \neq 0 \quad \text{for any } i.$$

For symmetric functions there is the following theorem [8].

THEOREM 6. *If*

$$f(x) = f_0(x) + \sum_{i=1}^{K} \left(A_i \log \frac{1}{\{x - \bar{x}_i\}} + B_i \log \frac{1}{\{\bar{x}_i - x\}} \right),$$

where f_0 has a bounded variation, $A = B$, and ρ allows approximation by rationals with rate $const/(q^2 \log q)$*, then the special flow over the circle rotation by ρ with roof function f is not mixing.*

So, the deceleration of points and the stretching of a little rectangle in the neighborhood of a regular fixed point are not sufficient for mixing: the stretching produced by moving on one side of a fixed point is compensated while moving on the other side.

Examples of smooth flows on the two-dimensional torus with nonsingular fixed points appear naturally in Arnold's paper [1]. The phase space of such a flow decomposes into cells bounded by closed separatrices of regular fixed points and filled with periodic orbits, and an ergodic component in which orbits move on one side of a fixed point frequently then on the other. The ergodic component of such a flow is isomorphic to a special flow over a circle rotation and under a roof function with an asymmetric logarithmic singular point. A conjecture about the possibility of mixing in such flows was proposed in [8]. Khanin and Sinai proved it with a certain restriction on the rotation angle. “and an... the other” is garbled

Due to estimates of the ergodic and resonant effects on the stretching of Birkhoff sums, it is possible to weaken this restriction in the case of an asymmetric function and to prove mixing for any irrational angle in the case of a *strongly asymmetric* function.

Let $\rho = [k_1, \ldots, k_s, \ldots]$ be the expansion ρ in a continued fraction. Let p_s/q_s be the s-th convergent to ρ.

THEOREM 7. [10] *Let f be an asymmetric function with logarithmic singularities and ρ an irrational satisfying*

$$\log k_{n+1} = o(\log q_n). \qquad (*)$$

Then the special flow over the circle rotation through the angle ρ under the roof function f is mixing.

In [4], the restriction on ρ is stronger: $k_{n+1} \leqslant \text{const}\, n^{1+\gamma}$, where $0 < \gamma < 1$. It is easy to show that, if for some $\gamma > 0$, perhaps great than 1, we have $k_{n+1} \leqslant \text{const}\, n^{1+\gamma}$ for all n, then ρ satisfies $(*)$.

THEOREM 8. [10] *If f is a strongly asymmetric function with logarithmic singularities and ρ is an arbitrary irrational angle then the special flow over a circle rotation through ρ with roof function f is mixing.*

In a special flow with an asymmetric roof function, a new relationship between the ergodic and resonant effects is detected. To illustrate this we'll describe the main ideas of the proof of the previous two theorems. The full presentation takes about fifty pages.

To estimate the stretching of Birkhoff sums f^r, we estimate their derivatives $(f^r)'$. The idea is that, since

$$\text{v.p.} \int_{\mathbb{T}^1} f'(x)\, d\, x = +\infty \text{ or } -\infty,$$

it would be very nice if "according to the ergodic theorem" almost everywhere $(f^r)' \to +\infty$ or $(f^r)' \to -\infty$, and this would ensure the stretching of Birkhoff sums. Moreover, one would want to give this "proof" for the interval exchange. As we'll see, the ergodic component in the expansion of $(f^r)'$ is really present. But the additional term arising from the frequent return of the orbit to a neighborhood of the singularity can essentially violate the "ergodic theorem for non-summable functions". This additional term is large when ρ is well approximable by rationals, and we call this term "resonant".

For such a flow over a circle rotation one can estimate ergodic, resonant and other terms and prove mixing in the cases stated in the theorems.

Let $u, v : \mathbb{R} \to \mathbb{R}$ be the functions with period 1 defined thus:

$$u(x) = 1/x \qquad \text{if } x \in (0, 1],$$
$$v(x) = 1/(1-x) \qquad \text{if } x \in [0, 1).$$

The point $x_0 = 0$ in $\mathbb{T}^1$ is a singular point of both. One can show, that for every x, except singular points of f^r, Is this what you mean?

$$(f^r)'(x) = \sum_{i=1}^{K} \left(u^r(x - \bar{x}_i)(-A_i + \alpha_i^-(r, x)) + v^r(x - \bar{x}_i)(B_i + \alpha_i^+(r, x)) \right), \tag{4–1}$$

where $|\alpha_i^{\pm}(r, x)| \leqslant \alpha(r)$, $\alpha(r) \to 0$ for $r \to +\infty$.

Let

$$q_s \leqslant r < q_{s+1}, \qquad r = l_s q_s + \ldots + l_0 q_0$$

be the expansion of r by denominators q_n with integer nonnegative coefficients, such that

$$1 \leqslant l_s \leqslant k_{s+1}, \quad 0 \leqslant l_n \leqslant k_{n+1} \text{ for } n = 0, 1, \ldots, s-1,$$
$$l_{n-1} q_{n-1} + \ldots + l_0 q_0 < q_n \text{ for } n = 1, \ldots, s.$$

Then

$$u^r(x) = u^{l_s q_s}(x) + \ldots + u^{l_n q_n}(T^{r_{n+1}} x) + \ldots + u^{l_0 q_0}(T^{r_1} x),$$

where $r_n = l_s q_s + \ldots + l_n q_n$.

We set

$$\Delta_n = \min_{0 \leqslant i < j < q_n} |T^i x_0 - T^j x_0|, \quad \delta_n = q_n \rho - p_n.$$

Let us expand u in two terms:

$$\hat{u}_n(x) = u(x), \quad \check{u}_n(x) = 0 \text{ for } x \in (0, \Delta_n),$$

$$\hat{u}_n(x) = 0, \quad \check{u}_n(x) = u(x) \text{ for } x \notin (0, \Delta_n).$$

Then $u^{l_n q_n}(T^{r_{n+1}} x) = \hat{u}_n^{l_n q_n}(T^{r_{n+1}} x) + \check{u}_n^{l_n q_n}(T^{r_{n+1}} x)$.

One can show that for any x

$$\check{u}_n^{l_n q_n}(x) = l_n q_n \log q_n + P^-_{l_n q_n}(T^{r_{n+1}} x), \quad |P^-_{l_n q_n}(T^{r_{n+1}} x)| < 4 l_n q_n.$$

The term $l_n q_n \log q_n$ is called the *ergodic* component of $u^{l_n q_n}$. This component does not depend on x and for a given q_n is proportional to l_n.

We'll present the term $\hat{u}_n^{l_n q_n}$ as a sum

$$\hat{u}_n^{l_n q_n}(T^{r_{n+1}} x) = I_n^-(x) + Z_n^-(x)$$

by the following way. We denote by $x_n^-(x)$ the singular point of the function $\hat{u}^{l_n q_n}(T^{r_{n+1}} x)$ nearest to x on its left hand side, if such exists, and set

$$I_n^-(x) = u(x - x_n^-(x)), \quad Z_n^-(x) = \hat{u}^{l_n q_n}(T^{r_{n+1}} x) - I_n^-(x).$$

If $x_n^-(x)$ does not exist, then we set $I_n^-(x) = 0$, $Z_n^-(x) = 0$. Thus, we obtain the expansion

$$u^{l_n q_n}(T^{r_{n+1}} x) = l_n q_n \log q_n + I_n^-(x) + Z_n^-(x) + P_n^-(T^{r_{n+1}} x).$$

We conditionally call the term $Z_n^-(x)$ *resonant*. Its value essentially depends on arrangement of the point x and the singular points of $\hat{u}_n^{l_n q_n}(T^{r_{n+1}} x)$. Its maximal value is approximately $q_{n+1} \log k_{n+1}$, which depends on the precision of the approximation of ρ by p_n/q_n. We take the sum over n of the expansion above and denote

$$e(r) = \sum_{n=1}^{s} l_n q_n \log q_n, \quad Z^-(x) = \sum_{n=0}^{s} Z_n^-(x) \quad P^-(x) = \sum_{n=0}^{s} P_n^-(T^{r_{n+1}}x).$$

It is not difficult to show that

$$\sum_{n=0}^{s} I_n^-(x) = I^-(x) + I_{rem}(x),$$

where $0 \leqslant I^-(x) \leqslant 2/\{x^-(x) - x\}$, $0 \leqslant I_{rem}(x) \leqslant 2q_s(\log s + 1)$.

Similar estimates are valid for v^r. Thus we get

$$u^r(x) = e(r) + Z^-(x) + I^-(x) + o(e(r)),$$
$$v^r(x) = e(r) + Z^+(x) + I^+(x) + o(e(r)),$$

where

$$Z^+(x) = \sum_{n=0}^{s} Z_n^+(x), \quad Z^-(x) = \sum_{n=0}^{s} Z_n^-(x).$$

For $|P^\pm(x)|$ and $I^\pm(x)$ we have the estimates

$$|P^\pm(x)| \leqslant 4r, \quad 0 < I^\pm(x) < \frac{2}{|x^\pm(x) - x|},$$

where $x^-(x)$ and $x^+(x)$ are the nearest singular points of u^r and v^r respectively to x on its left and right hand sides respectively.

In the expansion given, all three components are present: ergodic $e(r)$, resonant $Z^\pm$ and individual $I^\pm$. The last becomes inessential after a slight restriction of the set on which it is considered. The ergodic component would be enough for stretching of Birkhoff sums if the resonant terms "wouldn't oppose" or "would help" it. It turns out that in the case $\log k_{n+1} = o(\log q_n)$ the resonant terms are small in comparison with the ergodic term, and in the case of a strongly asymmetric function they go with the ergodic term on a large set.

We consider this assertion more explicitly. Choose a sequence σ_n, $n \in \mathbb{Z}_+$, depending on ρ and satisfying the conditions

$$\sigma_n \searrow 0; \quad \sigma_n > (\log q_n)^{-1/4} \quad (\text{ for } n > 1), \quad \sigma_n^2 \log q_n \nearrow +\infty.$$

If $\log k_{n+1} = o(\log q_n)$, it is easy to show that σ_n can satisfy an additional condition $\log k_{n+1} \leqslant \sigma_n^2 \log q_n$.

Fix t large enough and choose m such that $\sqrt{2}q_m \leqslant t < \sqrt{2}q_{m+1}$. Define

$$V_m = \{x : |x - T^{-j}x_0| \geqslant 3\sigma_m/q_m \qquad \text{for } j = 0, 1, \dots, 2q_m - 1\},$$
$$V'_m = \{x : |x - T^{-j}x_0| \geqslant 3\sigma_m/q_{m+1} \qquad \text{for } j = 0, 1, \dots, 2q_{m+1} - 1\}.$$

Also set

$$m' = m,\ V(t) = V_m,\ \text{for } \sqrt{2}q_m \leqslant t < \sqrt{2}\sigma_m q_{m+1},$$
$$m' = m + 1,\ V(t) = V'_m,\ \text{for } \sqrt{2}\sigma_m q_{m+1} \leqslant t < \sqrt{2}q_{m+1}.$$

The set $V(t)$ consists of disjoint closed intervals (or isolated points); the number of these intervals is no more than $2q_{m'}$; $\mu(V(t)) > 1 - 12\sigma_m$; the length of each interval is no more $2/q_{m'}$.

It is not difficult to show that for any $x \in V(t)$ and any $r \in (t/\sqrt{2}, \sqrt{2}t)$ the set $X(r)$ of singular points of u^r and v^r together with their $\sigma_m/q_{m'}$-neighborhoods does not intersect $V(t)$, and thus

$$I^{\pm}(x) < \frac{2}{\sigma_m^2 \log q_m} e(r).$$

(Note that $\sigma_m^2 \log q_m \to +\infty$ for $m \to +\infty$.)

One more object is necessary to describe the properties of the resonant terms. We decompose the set $X^{(n)} = X^{(n)}(r)$ of singular points of $u^{l_n q_n}(T^{r_{n+1}}x)$ into subsets

$$X_i^{(n)} = \left\{T^{-r_{n+1}-i-jq_n},\ j = 0, \dots, l_n - 1\right\}, \quad i = 0, \dots, q_n - 1,$$

which we call *clusters* of rank n. Each cluster consists of l_n points, producing an arithmetic progression with the step δ_n. By $[X_i^{(n)}]$ we denote the minimal segment containing $X_i^{(n)}$, $[X^{(n)}] = \bigcup_i [X_i^{(n)}]$, $\partial[X^{(n)}]$ is the bound of $[X^{(n)}]$. The length of each segment is

$$|X_i^{(n)}| = (l_n - 1)|\delta_n| \approx \frac{l_n/k_{n+1}}{q_n}, \quad \mu([X^{(n)}]) \approx l_n/k_{n+1}.$$

The Segments $[X_i^{(n)}]$ are disjoint, so $\partial[X^{(n)}]$ is the union of the ends of $[X_i^{(n)}]$.

For the set W, we define $U(\varepsilon, W) = \bigcup_{x \in W} U(\varepsilon, x)$, where $U(\varepsilon, x)$ is ε-neighborhood of x.

THEOREM 9 (ABOUT THE MAIN RESONANT TERM). *For m sufficiently large, there are the following possible situations.*

(1) *The main resonant term is absent: for any $s < m$*

$$q_{s+1} \log k_{s+1} \leqslant \sigma_m t \log q_m$$

and additionally $\log k_{m+1} \leqslant \sigma_m^2 \log q_m$ *or* $\sqrt{2}q_m \leqslant t < \sqrt{2}\sigma_m q_{m+1}$. *Then for any* $r \in (t/\sqrt{2}, \sqrt{2}t)$ *and* $x \in V(t)$

$$0 \leqslant \sum_n Z_n^{\pm}(x) < 20\sigma_m e(r).$$

(2) *The main resonant term is of rank* m*: for any* $s < m$ $\quad q_{s+1} \log k_{s+1} \leqslant \sigma_m t \log q_m$, *and*

$$\log k_{m+1} > \sigma_m^2 \log q_m, \quad \sqrt{2}\sigma_m q_{m+1} \leqslant t < \sqrt{2} q_{m+1}.$$

Then for any x

$$\sum_{n \neq m} Z_n^{\pm}(x) < 16\sigma_m e(r),$$

and for $Z_m^{\pm}(x) = Z_m^{\pm}(r, x)$, *when* $r \in (t/\sqrt{2}, \sqrt{2}t)$ *and*

$$x \in V(t) \setminus U(\sigma_m/q_m, \partial[X^{(m)}(r)]),$$

there is an alternative:
— *if* $x \notin [X^{(m)}(r)]$, *then* $Z_m^{\pm}(x) < \sigma_m e(r)$;
— *if* $x \in [X^{(m)}(r)]$, *then* $q_{m+1} \ln k_{m+1} - \sigma_m e(r) < Z_m^{\pm}(x) < q_{m+1} \ln k_{m+1} + \sigma_m e(r)$.

(3) *The main resonant term is of rank* $s < m$*: there exists* $s < m$ *such that* $q_{s+1} \log k_{s+1} > \sigma_m t \log q_m$. *Then for any* $r \in (t/\sqrt{2}, \sqrt{2}t)$ *and* $x \in V(t)$

$$\sum_{n \neq s} Z_n^{\pm}(x) < \sigma_m e(r);$$

for $Z_s^{\pm}(x) = Z_s^{\pm}(r, x)$, *when* $r \in (t/\sqrt{2}, \sqrt{2}t)$ *and*

$$x \in V(t) \setminus U(\sigma_s/q_s, \partial[X^{(s)}(r)]),$$

there is an alternative:
— *if* $x \notin [X^{(s)}(r)]$, *then* $Z_s^{\pm}(x) < \sigma_m e(r)$;
— *if* $x \in [X^{(s)}(r)]$, *then* $q_{s+1} \log k_{s+1} - \sigma_m e(r) < Z_s^{\pm}(x) < q_{s+1} \log k_{s+1} + \sigma_m e(r)$.

Now we may define the functions $\varepsilon(t) \to 0$, $H(t) \to +\infty$ for $t \to +\infty$, for the sufficient condition of mixing (Theorem 1). Let

$$\alpha_t = \max_{r \in (t/\sqrt{2}, \sqrt{2}t)} \alpha(r),$$

$$\varepsilon_e(t) = \frac{21(A+B)}{|B-A|}\sigma_m + \frac{4K}{|B-A|}\alpha_t, \quad \varepsilon_L(t) = \max_{1 \leqslant i \leqslant K} \frac{2}{|A_i - B_i|}\alpha_t,$$

where K is the number of singular points of f, $\alpha(r)$ is the infinitesimal sequence defined in (4–1).

Next set

$$\varepsilon(t) = 2\max(\varepsilon_e(t), \varepsilon_L(t)), \quad H(t) = \frac{|B-A|\sigma_m^2 \log q_m}{4}.$$

For each of the situations (1)–(3) we'll define a partial partition ξ_t such that each element is a segment and for $t \to +\infty$ they satisfies the following conditions:

(1) $\mu([\xi_t]) \to 1$.
(2) $\max_{C \in \xi_t} |C| \to 0$.
(3) For any element $C \in \xi_t$ there exists a constant $L(C) \geqslant 0$, such that for any $r \in \mathcal{R}(t, [\xi_t])$ and $x \in C$,

There was a comment: "the next equation is corrected"

$$(f^r)'(x) = (B-A)(e(r) + L(C))(1+\gamma(r,x))), \quad |\gamma(r,x)| < \varepsilon(t)/2. \quad (4–2)$$

(4) For any element $C \in \xi_t$ and $r \in (t/\sqrt{2}, \sqrt{2}t)$,

$$|C|e(r) > \frac{\sigma_m^2 \log q_m}{2}.$$

Thus we'll verify the sufficient condition for mixing.

In situation 1 we set

$$\overline{V}(t) = \bigcap_{i=1}^{K}(V(t) + \bar{x}_i), \quad L(C) = 0.$$

Then for any $x \in \overline{V}(t)$ and any i $x - \bar{x}_i \in V(t)$, and substituting the expansions of $u^r(x - \bar{x}_i)$ and $v^r(x - \bar{x}_i)$ to (4–1) it is easy to make sure that for any $r \in (t/\sqrt{2}, \sqrt{2}t)$ the relation (4–2) is satisfied.

As elements of the partition ξ_t we take those connected components of $\overline{V}(t)$ whose length is at least $\sigma_m/q_{m'}$.

If $\log k_{n+1} = o(\log q_n)$, then for each t situation 1 is realized.

In situations 2 and 3, the principle of construction of ξ_t is the same, but it is necessary to slightly narrow the set $\overline{V}(t)$. We show how to do it in the situation 3, for situation 2 it is necessary to replace the index s by the index m everywhere.

Let $\bar{r} = \min \mathcal{R}(t, \overline{V}(t))$. Set

$$\widetilde{V}(t) = \big(V(t) \setminus U(\sigma_m/q_s, \partial[X^{(s)}(\bar{r})])\big) \cap \big(V_s - \bar{r}\rho\big),$$
$$\overline{\overline{V}}(t) = \bigcap_{i=1}^{K} (\widetilde{V}(t) + \bar{x}_i).$$

As elements of the partition ξ_t we take those connected components of $\overline{\overline{V}}(t)$ whose length is at least $\sigma_m/q_{m'}$.

For $C \in \xi_t$ set

$$L(C) = \left(\sum_{i=1}^{K} \frac{B_i - A_i}{B - A} \chi_i(x) \right) q_{s+1} \log k_{s+1},$$

where χ_i is the indicator function of the set $[X^{(s)}] + \bar{x}_i$, $x \in C$. This definition does not depend on the choice of representative $x \in C$, as it follows the construction of ξ_t. Also the inequality $L(C) \geqslant 0$ follows from the construction and the condition that $\operatorname{sign}(B_i - A_i) = \operatorname{sign}(B - A) \neq 0$.

Note, that the partition ξ_t depends on some fixed $\bar{r}$ in the situations 2 and 3, and thus (4–2) is valid only for r closed to $\bar{r}$, whereas the sufficient condition of mixing requires it for any $r \in \mathcal{R}(t, [\xi_t])$. By estimating the oscillation of f^r in the set $\overline{V}(t)$ it is possible to prove that the range $\mathcal{R}(t, [\xi_t])$ is not too large, and (4–2) is valid for the whole range.

5. Some problems

(1) It is not known whether the restriction $(*)$ on the angle in the theorem 7 is appreciable. For angles which don't satisfy $(*)$, it is possible to construct achievable? an asymmetric function f with logarithmic singularities, such that for an unbounded set of moments t and corresponding r, the oscillation of f^r on each element $C \in \xi_t : C \subset ([X^{(m)}]) + \bar{x}_i$ is small since $e(r) + L(C) = o(q_{m+1})$ in the expansion (4–2), but the oscillation of f^r on each set

$$[X_i^{(m)}] \setminus U(\sigma_m, \partial[X_i^{(m)}]) \cap [\xi_t]$$

is large enough; also the distribution of $f^r | \big([X_i^{(m)}] \setminus U(\sigma_m, \partial[X_i^{(m)}])\big)$ is not almost uniform, it is almost discrete. Thus we cannot use the sufficient condition for mixing given above, and yet cannot prove the absence of mixing.

(2) It is not known whether the theorem 6 for symmetric function with logarithmic singularities is right for angles satisfying $k_{n+1} = o(\log q_n)$ (or the same $\log k_{n+1} = o(\log \log q_n)$).

Using techniques from Fourier analysis, M. Lemańczyk has slightly extended the class of functions considered but not the class of angles.

(3) It would be interesting to clarify how for mixing special flows the modulus of continuity of the roof function relates to the speed of rational approximation of the angle of the rotation in the base. Maybe there exist models other than the model of "two terms" described above, in which more terms are simultaneously involved in the stretching of Birkhoff sums.

(4) It is also not known what the maximum rate of mixing is for special flows over ergodic rotations and under continuous roof functions. This question

is interesting in connection with the existence of such flows with Lebesgue spectrum.

(5) Is the presence of only one singular fixed point, even in the presence of other nonsingular fixed points, sufficient for mixing of an ergodic flow on a surface ? It seems the answer should be positive but it has not yet been proved.

(6) Is the mixing in the above flows mixing of all orders ?

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