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Solenoid functions for hyperbolic sets on surfaces

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ABSTRACT. We describe a construction of a moduli space of *solenoid functions* for the C^{1+}-conjugacy classes of hyperbolic dynamical systems f on surfaces with hyperbolic basic sets Λ_f. We explain that if the holonomies are sufficiently smooth then the diffeomorphism f is *rigid* in the sense that it is C^{1+} conjugate to a hyperbolic affine model. We present a moduli space of *measure solenoid functions* for all Lipschitz conjugacy classes of C^{1+}-hyperbolic dynamical systems f which have a invariant measure that is absolutely continuous with respect to Hausdorff measure. We extend *Livšic and Sinai's eigenvalue formula* for Anosov diffeomorphisms which preserve an absolutely continuous measure to hyperbolic basic sets on surfaces which possess an invariant measure absolutely continuous with respect to Hausdorff measure.

CONTENTS

1. Introduction

We say that (f, Λ) is a C^{1+} *hyperbolic diffeomorphism* if it has the following properties:

(i) $f : M \to M$ is a $C^{1+\alpha}$ diffeomorphism of a compact surface M with respect to a $C^{1+\alpha}$ structure $\mathscr{C}_f$ on M, for some $\alpha > 0$.

(ii) Λ is a hyperbolic invariant subset of M such that $f|\Lambda$ is topologically transitive and Λ has a local product structure.

We allow both the case where $\Lambda = M$ and the case where Λ is a proper subset of M. If $\Lambda = M$ then f is Anosov and M is a torus [16; 33]. Examples where Λ is a proper subset of M include the Smale horseshoes and the codimension one attractors such as the Plykin and derived-Anosov attractors.

THEOREM 1.1 (EXPLOSION OF SMOOTHNESS). *Let f and g be any two C^{1+} hyperbolic diffeomorphisms with basic sets Λ_f and Λ_g, respectively. If f and g are topologically conjugate and the conjugacy has a derivative at a point with nonzero determinant, then f and g are C^{1+} conjugate.*

See definitions of topological and C^{1+} conjugacies in Section 2.3. A weaker version of this theorem was first proved by D. Sullivan [47] and E. de Faria [8] for expanding circle maps. Theorem 1.1 follows from [13] using the results presented in [1] and in [13] which apply to Markov maps on train tracks and to nonuniformly hyperbolic diffeomorphisms.

For every C^{1+} hyperbolic diffeomorphism f we denote by $\delta_{f,s}$ the Hausdorff dimension of the stable-local leaves of f intersected with Λ, and we denote by $\delta_{f,u}$ the Hausdorff dimension of the unstable-local leaves of f intersected with Λ. Let $\mathcal{P}$ be the set of all periodic points of Λ under f. For every $x \in \mathcal{P}$, let us denote by $\lambda_{f,s}(x)$ and $\lambda_{f,u}(x)$ the stable and unstable eigenvalues of the periodic orbit containing x. A. Livšic and Ya. Sinai [25] proved that an Anosov diffeomorphism f admits an f-invariant measure that is absolutely continuous with respect to the Lebesgue measure on M if, and only if, $\lambda_{f,s}(x)\lambda_{f,u}(x) = 1$ for every periodic point $x \in \mathcal{P}$. In Theorem 1.1 of [42], it is proved the following extension of Livšic and Sinai's Theorem to C^{1+} hyperbolic diffeomorphisms with hyperbolic sets on surfaces such as Smale horseshoes and codimension one attractors.

THEOREM 1.2 (LIVŠIC AND SINAI'S EXTENDED FORMULA). *A C^{1+} hyperbolic diffeomorphism f admits an f-invariant measure that is absolutely continuous with respect to the Hausdorff measure on Λ if, and only if, for every periodic point $x \in \mathcal{P}$,*

$$\lambda_{f,s}(x)^{\delta_{f,s}} \lambda_{f,u}(x)^{\delta_{f,u}} = 1 .$$

By the *flexibility* of a given topological model of hyperbolic dynamics we mean the extent of different smooth realizations of this model. Thus a typical result provides a moduli space to parameterise these realizations. To be effective it is important that these moduli spaces should be easily characterised. For example, for C^2 Anosov diffeomorphisms of the torus that preserve a smooth invariant measure, the eigenvalue spectrum is a complete invariant of smooth conjugacies as shown by De la Llave, Marco and Moriyon [26; 27; 30; 31]. However, for hyperbolic systems on surface other than Anosov systems the eigenvalue spectra are only a complete invariant of Lipschitz conjugacy (see [42]). In [39], the notions of a *HR-structure* and of a *solenoid function* are used to construct the moduli space.

Consider affine structures on the stable and unstable lamination in Λ. These are defined in terms of a pair of *ratio functions* r^s and r^u (see Section 3.1). If r^s and r^u are Hölder continuous and invariant under f then we call the associated structure a HR-structure (HR for Hölder-ratios). Theorem 5.1 in [39] gives a one-to-one correspondence between HR-structures and C^{1+} conjugacy classes of $f|\Lambda$ (see Theorem 3.1). The main step in the proof of this and related results is to show that, given a HR-structure, there is a canonical construction of a representative in the corresponding conjugacy class. By Theorem 5.3 in [39], this representative has the following maximum smoothness property: the holonomies for the representative are as smooth as those of any diffeomorphism that is C^{1+} conjugate to it. In particular, if there is an affine diffeomorphism with this HR-structure, then this representative is the affine diffeomorphism. In Section 3.9, we present the definition of *stable and unstable solenoid functions* and we introduce the set $\mathcal{PS}(f)$ of all pairs of solenoid functions. To each HR-structure one can associate a pair (σ^s, σ^u) of solenoid functions corresponding to the stable and unstable laminations of Λ, where the solenoid functions σ^s and σ^u are the restrictions of the ratio functions r^s and r^u, respectively, to a set determined by a Markov partition of f. Theorem 6.1 in [39] says that there is a one-to-one correspondence between Hölder solenoid function pairs and HR-structures (see Theorem 3.4). Since these solenoid function pairs form a nice space with a simply characterised completion they provide a good moduli space. For example, in the classical case of Smale horseshoes the moduli space is the set of all pairs of positive Hölder continuous functions with domain $\{0, 1\}^{\mathbb{N}}$.

Let $\mathcal{T}(f, \Lambda)$ be the set of all C^{1+} hyperbolic diffeomorphisms (g, Λ_g) such that (g, Λ_g) and (f, Λ) are topologically conjugate (See definitions of topological and C^{1+} conjugacies in Section 2.3).

THEOREM 1.3 (FLEXIBILITY). *The natural map $c : \mathcal{T}(f, \Lambda) \to \mathcal{PS}(f)$ which associates a pair of solenoid functions to each C^{1+} conjugacy class is a bijection.*

The solenoid functions were first introduced in [36; 39] inspired by the scaling functions introduced by M. Feigenbaum [10; 11] and D. Sullivan [48]. The completion of the image of c is the set of pairs of continuous solenoid functions which is a closed subset of a Banach space. They correspond to f-invariant affine structures on the stable and unstable laminations for which the holonomies are uniformly asymptotically affine (uaa) as defined in [47].

In [41], the moduli space of solenoid functions is used to study the existence of *rigidity* for diffeomorphisms on surfaces. In dynamics, rigidity occurs when simple topological and analytical conditions on the model system imply that there is no flexibility and so a unique smooth realization. One can paraphrase this by saying that the moduli space for such systems is a singleton. For example, a famous result of this type due to Arnol'd, Herman and Yoccoz [3; 20; 51] is that a sufficiently smooth diffeomorphism of the circle with an irrational rotation number satisfying the usual Diophantine condition is C^{1+} conjugate to a rigid rotation. The rigidity depends upon both the analytical hypothesis concerning the smoothness and the topological condition given by the rotation number and if either are relaxed then it fails. The analytical part of the rigidity hypotheses for hyperbolic surface dynamics will be a condition on the smoothness of the holonomies along stable and unstable manifolds.

THEOREM 1.4 (RIGIDITY). *If f is a C^r diffeomorphism with a hyperbolic basic set Λ and the holonomies of f are C^r with uniformly bounded C^r norm and with $r-1$ greater than the Hausdorff dimension along the stable and unstable leaves intersected with Λ then f is C^{1+} conjugated to a hyperbolic affine model.*

See the definition of a *hyperbolic affine model* in Section 4.2. Theorem 4.1 contains a slightly stronger version of Theorem 1.4 using the notion of a HD^ι *complete set of holonomies*. Both theorems are proved in [41]. In these theorems we allow both the case where $\Lambda = M$ (so that f is Anosov and $M \cong \mathbb{T}^2$ [16; 33]) and the case where Λ is a proper subset. In the case of the Smale horseshoe f, as presented in Figure 8, the hyperbolic affine maps $\hat{f}$ topologically conjugate to f, up to affine conjugacy, form a two-dimension set homeomorphic to $\mathbb{R}^+ \times \mathbb{R}^+$. In the case of hyperbolic attractors with $HD^s < 1$, there are no affine maps as proved in [14]. Hence, Theorem 1.4 implies that the stable holonomies can never be smoother than $C^{1+\alpha}$ with α greater than the Hausdorff dimension along the stable leaves intersected with Λ (see [14]). This result is linked with J. Harrison's conjecture of the nonexistence of $C^{1+\alpha}$ diffeomorphisms of the circle with $\alpha > HD$, where HD is the Hausdorff dimension of its nonwandering domain. A weaker version of this conjecture was proved by A. Norton [35] using box dimension instead of Hausdorff dimension. In the case of Anosov diffeomorphisms of the torus, the hyperbolic affine model is a hyperbolic toral automorphism and is unique up to affine conjugacy [15; 16; 29; 33]. In general, the topological conjugacy between such a diffeomorphism and the corresponding hyperbolic affine model is only Hölder continuous and need not be any smoother. This is the case if there is a periodic orbit of f whose eigenvalues differ from those of the hyperbolic affine model. For Anosov diffeomorphisms f of the torus there are the following results, all of the form that if a C^k f has

C^r foliations then f is C^s-*rigid*, i.e. f is C^s-conjugate to the corresponding hyperbolic affine model:

(i) Area-preserving Anosov maps f with $r = \infty$ are C^∞-rigid (Avez [4]).
(ii) C^k area-preserving Anosov maps f with $r = 1 + o(t|\log t|)$ are C^{k-3}-rigid (Hurder and Katok [22]).
(iii) C^1 area-preserving Anosov maps f with $r \geq 2$ are C^r-rigid (Flaminio and Katok [17]).
(iv) C^k Anosov maps f ($k \geq 2$) with $r \geq 1+$Lipshitz are C^k-rigid (Ghys [18]).

The moduli space of solenoid functions is used in [42] to construct classes of smooth hyperbolic diffeomorphisms with an invariant measure μ absolutely continuous with respect to the Hausdorff measure. It is interesting to note that when we consider the C^{1+} hyperbolic diffeomorphisms realising a particular topological model then the stable and unstable ratio functions are indendent in the following sense. If r^s is a stable ratio function for some hyperbolic diffeomorphism and r^u is the unstable ratio function for some other hyperbolic diffeomorphism then there is a hyperbolic diffeomorphism that has the pair (r^s, r^u) as its HR structure. The same is no longer true if we ask the C^{1+} hyperbolic realizations to have an invariant measure μ absolutely continuous with respect to the Hausdorff measure. For $\iota \in \{s, u\}$, let us denote by ι' the element of $\{s, u\}$ which is not ι.

THEOREM 1.5 (MEASURE RIGIDITY FOR ANOSOV DIFFEOMORPHISMS). *For $\iota \in \{s, u\}$, given an ι-solenoid function σ_ι there is a unique ι'-solenoid function such that the C^{1+} Anosov diffeomorphisms determined by the pair (σ_s, σ_u) satisfy the property of having an invariant measure μ absolutely continuous with respect to Lebesgue measure of their hyperbolic sets.*

In the case of Smale horseshoes, the ι'-solenoid function is not anymore unique but belongs to a well-characterized set which is the δ-solenoid equivalence class of the Gibbs measure determined by the ι-solenoid function (see Section 5.6).

THEOREM 1.6 (MEASURE FLEXIBILITY FOR SMALE HORSESHOES). *For $\iota \in \{s, u\}$, given an ι-solenoid function σ_ι there is an infinite dimensional space of solenoid functions $\sigma_{\iota'}$ (but not all) such that the C^{1+} hyperbolic Smale horseshoes determined by the pairs (σ_s, σ_u) have the property of having an invariant measure μ absolutely continuous with respect to the Hausdorff measure of their hyperbolic sets.*

Codimension one attractors partly inherit the properties of Anosov diffeomorphisms and partly those of Smale horseshoes because locally they are a product of lines with Cantor sets embedded in lines.

THEOREM 1.7. (i) (MEASURE FLEXIBILITY FOR CODIMENSION ONE ATTRACTORS) *Given an u-solenoid function σ_u there is an infinite dimensional space of s-solenoid functions σ_s (but not all) such that the C^{1+} hyperbolic codimension one attractors determined by the pairs (σ_s, σ_u) have the property of having an invariant measure μ absolutely continuous with respect to the Hausdorff measure of their hyperbolic sets.*

(ii) (MEASURE RIGIDITY FOR CODIMENSION ONE ATTRACTORS) *Given an s-solenoid function σ_s there is a unique u-solenoid function σ_u such that the C^{1+} hyperbolic codimension one attractors determined by the pair (σ_s, σ_u) have the property of having an invariant measure μ absolutely continuous with respect to the Hausdorff measure of their hyperbolic sets.*

Theorem 5.9 contains a stronger version of Theorems 1.5, 1.6 and 1.7, and it is proved in Lemmas 8.17 and 8.18 in [42].

Since (f, Λ) is a C^{1+} hyperbolic diffeomorphism it admits a Markov partition $\mathcal{R} = \{R_1, \ldots, R_k\}$. This implies the existence of a two-sided subshift of finite type Θ in the symbol space $\{1, \ldots, k\}^{\mathbb{Z}}$, and an inclusion $i : \Theta \to \Lambda$ such that (a) $f \circ i = i \circ \tau$ and (b) $i(\Theta_j) = R_j$ for every $j = 1, \ldots, k$. For every $g \in \mathcal{T}(f, \Lambda)$, the inclusion $i_g = h_{f,g} \circ i : \Theta \to \Lambda_g$ is such that $g \circ i_g = i_g \circ \tau$. We call such a map $i_g : \Theta \to \Lambda_g$ a *marking* of (g, Λ_g).

DEFINITION 1.1. If $(g, \Lambda_g) \in \mathcal{T}(f, \Lambda)$ is a C^{1+} hyperbolic diffeomorphism as above and ν is a Gibbs measure on Θ then we say that (g, Λ_g, ν) is a Hausdorff realisation if $(i_g)_*\nu$ is absolutely continuous with respect to the Hausdorff measure on Λ_g. If this is the case then we will often just say that ν is a Hausdorff realisation for (g, Λ_g).

We note that the Hausdorff measure on Λ_g exists and is unique, and if a Hausdorff realisation exists then it is unique. However, a Hausdorff realisation need not exist.

DEFINITION 1.2. Let $\mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$ be the set of all C^{1+} hyperbolic diffeomorphisms (g, Λ_g) in $\mathcal{T}(f, \Lambda)$ such that (i) $\delta_{g,s} = \delta_s$ and $\delta_{g,u} = \delta_u$; (ii) there is a g-invariant measure μ_g on Λ_g which is absolutely continuous with respect to the Hausdorff measure on Λ_g. We denote by $[\nu] \subset \mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$ the subset of all C^{1+}-realisations of a Gibbs measure ν in $\mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$.

De la Llave, Marco and Moriyon [26; 27; 30; 31] have shown that the set of stable and unstable eigenvalues of all periodic points is a complete invariant of the C^{1+} conjugacy classes of Anosov diffeomorphisms. We extend their result to the sets $[\nu] \subset \mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$.

THEOREM 1.8 (EIGENVALUE SPECTRA). (i) *Any two elements of the set $[\nu] \subset \mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$ have the same set of stable and unstable eigenvalues and these* *sets are a complete invariant of* $[\nu]$ *in the sense that if* $g_1, g_2 \in \mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$ *have the same eigenvalues if, and only if, they are in the same subset* $[\nu]$.

(ii) *The map* $\nu \to [\nu] \subset \mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$ *gives a* $1-1$ *correspondence between* C^{1+}*-Hausdorff realisable Gibbs measures* ν *and Lipschitz conjugacy classes in* $\mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$.

Theorem 1.8 is proved in [42], where it is also proved that the set of stable and unstable eigenvalues of all periodic orbits of a C^{1+} hyperbolic diffeomorphism $g \in \mathcal{T}(f, \Lambda)$ is a complete invariant of each Lipschitz conjugacy class. Furthermore, for Anosov diffeomorphisms every Lipschitz conjugacy class is a C^{1+} conjugacy class.

REMARK 1.9. We have restricted our discussion to Gibbs measures because it follows from Theorem 1.8 that, if $g \in \mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$ has a g-invariant measure μ_g which is absolutely continuous with respect to the Hausdorff measure then μ_g is a C^{1+}-Hausdorff realisation of a Gibbs measure ν so that $\mu_g = (i_g)_*\nu$.

If f is a Smale horseshoe then every Gibbs measure ν is C^{1+}-Hausdorff realisable by a hyperbolic diffeomorphism contained in $\mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$ (see [42]). However, this is not the case for Anosov diffeomorphisms and codimension one attractors. E. Cawley [6] characterised all C^{1+}-Hausdorff realisable Gibbs measures as Anosov diffeomorphisms using cohomology classes on the torus. In [42], it is used measure solenoid functions to classify all C^{1+}-Hausdorff realisable Gibbs measures, in an integrated way, of all C^{1+} hyperbolic diffeomorphisms on surfaces. In Section 5.3, the stable and unstable measure solenoid functions are easily built from the Gibbs measures, and, in Section 5.6, we define the infinite dimensional metric space $\mathcal{SOL}^\iota$.

THEOREM 1.10 (MEASURE SOLENOID FUNCTIONS). *Let* f *be an Anosov diffeomorphism or a codimension one attractor. The following statements are equivalent:*

(i) *The Gibbs measure* ν *is* C^{1+}*-Hausdorff realisable by a hyperbolic diffeomorphism contained in* $\mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$.

(ii) *The* ι*-measure solenoid function* $\sigma_{\nu,\iota} : \mathrm{msol}^s \to \mathbb{R}^+$ *has a nonvanishing Hölder continuous extension to the closure of* msol^s *belonging to* $\mathcal{SOL}^\iota$.

We present a more detailed version of this theorem in Theorems 5.5 and 5.8. These theorems are proved in [42].

By Theorems 1.8 and 1.10, for ι equal to s and u, we obtain that the map $\nu \to \sigma_{\nu,\iota}$ gives a one-to-one correspondence between the sets $[\nu]$ contained in $\mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$ and the space of measure solenoid functions $\sigma_{g,\iota}$ whose continuous extension is contained in $\mathcal{SOL}^\iota$.

COROLLARY 1.11 (MODULI SPACE). *The set $\mathcal{SOL}^\iota$ is a moduli space parameterizing all Lipschitz conjugacy classes $[\nu]$ of C^{1+} hyperbolic diffeomorphisms contained in $\mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$.*

2. Hyperbolic diffeomorphisms

In this section, we present some basic facts on hyperbolic dynamics, that we include for clarity of the exposition.

2.1. Stable and unstable superscripts. Throughout the paper we will use the following notation: we use ι to denote an element of the set $\{s, u\}$ of the stable and unstable superscripts and ι' to denote the element of $\{s, u\}$ that is not ι. In the main discussion we will often refer to objects which are qualified by ι such as, for example, an ι-leaf: This means a leaf which is a leaf of the stable lamination if $\iota = s$, or a leaf of the unstable lamination if $\iota = u$. In general the meaning should be quite clear.

We define the map $f_\iota = f$ if $\iota = u$ or $f_\iota = f^{-1}$ if $\iota = s$.

2.2. Leaf segments. Let d be a metric on M. For $\iota \in \{s, u\}$, if $x \in \Lambda$ we denote the local ι-manifolds through x by

$$W^\iota(x, \varepsilon) = \left\{ y \in M : d(f_\iota^{-n}(x), f_\iota^{-n}(y)) \leq \varepsilon, \text{ for all } n \geq 0 \right\}.$$

By the Stable Manifold Theorem (see [21]), these sets are respectively contained in the stable and unstable immersed manifolds

$$W^\iota(x) = \bigcup_{n \geq 0} f_\iota^n \left(W^\iota \left(f_\iota^{-n}(x), \varepsilon_0 \right) \right)$$

which are the image of a $C^{1+\gamma}$ immersion $\kappa_{\iota,x} : \mathbb{R} \to M$. An *open* (resp. *closed*) *full ι-leaf segment* I is defined as a subset of $W^\iota(x)$ of the form $\kappa_{\iota,x}(I_1)$ where I_1 is an open (resp. closed) subinterval (nonempty) in $\mathbb{R}$. An *open (resp. closed) ι-leaf segment* is the intersection with Λ of an open (resp. closed) full ι-leaf segment such that the intersection contains at least two distinct points. If the intersection is exactly two points we call this closed ι-leaf segment an *ι-leaf gap*. A *full ι-leaf segment* is either an open or closed full ι-leaf segment. An *ι-leaf segment* is either an open or closed ι-leaf segment. The *endpoints* of a full ι-leaf segment are the points $\kappa_{\iota,x}(u)$ and $\kappa_{\iota,x}(v)$ where u and v are the endpoints of I_1. The *endpoints* of an ι-leaf segment I are the points of the minimal closed full ι-leaf segment containing I. The *interior* of an ι-leaf segment I is the complement of its boundary. In particular, an ι-leaf segment I has empty interior if, and only if, it is an ι-leaf gap. A map $c : I \to \mathbb{R}$ is an *ι-leaf chart* of an ι-leaf segment I if has an extension $c_E : I_E \to \mathbb{R}$ to a full ι-leaf segment I_E with the following properties: $I \subset I_E$ and c_E is a homeomorphism onto its image.

2.3. Topological and smooth conjugacies. Let (f, Λ) be a C^{1+} hyperbolic diffeomorphism. Somewhat unusually we also desire to highlight the C^{1+} structure on M in which f is a diffeomorphism. By a C^{1+} *structure on* M we mean a maximal set of charts with open domains in M such that the union of their domains cover M and whenever U is an open subset contained in the domains of any two of these charts i and j then the overlap map $j \circ i^{-1} : i(U) \to j(U)$ is $C^{1+\alpha}$, where $\alpha > 0$ depends on i, j and U. We note that by compactness of M, given such a C^{1+} structure on M, there is an atlas consisting of a finite set of these charts which cover M and for which the overlap maps are $C^{1+\alpha}$ compatible and uniformly bounded in the $C^{1+\alpha}$ norm, where $\alpha > 0$ just depends upon the atlas. We denote by $\mathscr{C}_f$ the C^{1+} structure on M in which f is a diffeomorphism. Usually one is not concerned with this as, given two such structures, there is a homeomorphism of M sending one onto the other and thus, from this point of view, all such structures can be identified. For our discussion it will be important to maintain the identity of the different smooth structures on M.

We say that a map $h : \Lambda_f \to \Lambda_g$ is a *topological conjugacy* between two C^{1+} hyperbolic diffeomorphisms (f, Λ_f) and (g, Λ_g) if there is a homeomorphism $h : \Lambda_f \to \Lambda_g$ with the following properties:

(i) $g \circ h(x) = h \circ f(x)$ for every $x \in \Lambda_f$.
(ii) The pull-back of the ι-leaf segments of g by h are ι-leaf segments of f.

We say that a topological conjugacy $h : \Lambda_f \to \Lambda_g$ is a *Lipschitz conjugacy* if h has a bi-Lipschitz homeomorphic extension to an open neighborhood of Λ_f in the surface M (with respect to the C^{1+} structures $\mathscr{C}_f$ and $\mathscr{C}_g$, respectively).

Similarly, we say that a topological conjugacy $h : \Lambda_f \to \Lambda_g$ is a C^{1+} *conjugacy* if h has a $C^{1+\alpha}$ diffeomorphic extension to an open neighborhood of Λ_f in the surface M, for some $\alpha > 0$.

Our approach is to fix a C^{1+} hyperbolic diffeomorphism (f, Λ) and consider C^{1+} hyperbolic diffeomorphism (g_1, Λ_{g_1}) topologically conjugate to (f, Λ). The topological conjugacy $h : \Lambda \to \Lambda_{g_1}$ between f and g_1 extends to a homeomorphism H defined on a neighborhood of Λ. Then, we obtain the new C^{1+}-realization (g_2, Λ_{g_2}) of f defined as follows: (i) the map $g_2 = H^{-1} \circ g_1 \circ H$; (ii) the basic set is $\Lambda_{g_2} = H^{-1}|\Lambda_{g_1}$; (iii) the C^{1+} structure $\mathscr{C}_{g_2}$ is given by the pull-back $(H)_* \mathscr{C}_{g_1}$ of the C^{1+} structure $\mathscr{C}_{g_1}$. From (i) and (ii), we get that $\Lambda_{g_2} = \Lambda$ and $g_2|\Lambda = f$. From (iii), we get that g_2 is C^{1+} conjugated to g_1. Hence, to study the conjugacy classes of C^{1+} hyperbolic diffeomorphisms (f, Λ) of f, we can just consider the C^{1+} hyperbolic diffeomorphisms (g, Λ_g)

with $\Lambda_g = \Lambda$ and $g|\Lambda_g = f|\Lambda$, which we will do from now on for simplicity of our exposition.

2.4. Rectangles. Since Λ is a hyperbolic invariant set of a diffeomorphism $f : M \to M$, for $0 < \varepsilon < \varepsilon_0$ there is $\delta = \delta(\varepsilon) > 0$ such that, for all points $w, z \in \Lambda$ with $d(w, z) < \delta$, $W^u(w, \varepsilon)$ and $W^s(z, \varepsilon)$ intersect in a unique point that we denote by $[w, z]$. Since we assume that the hyperbolic set has a *local product structure*, we have that $[w, z] \in \Lambda$. Furthermore, the following properties are satisfied: (i) $[w, z]$ varies continuously with $w, z \in \Lambda$; (ii) the bracket map is continuous on a δ-uniform neighborhood of the diagonal in $\Lambda \times \Lambda$; and (iii) whenever both sides are defined $f([w, z]) = [f(w), f(z)]$. Note that the bracket map does not really depend on δ provided it is sufficiently small.

Let us underline that it is a standing hypothesis that all the hyperbolic sets considered here have such a local product structure.

A *rectangle* R is a subset of Λ which is (i) closed under the bracket i.e. $x, y \in R \Longrightarrow [x, y] \in R$, and (ii) proper i.e. is the closure of its interior in Λ. This definition imposes that a rectangle has always to be proper which is more restrictive than the usual one which only insists on the closure condition.

If ℓ^s and ℓ^u are respectively stable and unstable leaf segments intersecting in a single point then we denote by $[\ell^s, \ell^u]$ the set consisting of all points of the form $[w, z]$ with $w \in \ell^s$ and $z \in \ell^u$. We note that if the stable and unstable leaf segments ℓ and ℓ' are closed then the set $[\ell, \ell']$ is a rectangle. Conversely in this 2-dimensional situations, any rectangle R has a product structure in the following sense: for each $x \in R$ there are closed stable and unstable leaf segments of Λ, $\ell^s(x, R) \subset W^s(x)$ and $\ell^u(x, R) \subset W^u(x)$ such that $R = [\ell^s(x, R), \ell^u(x, R)]$. The leaf segments $\ell^s(x, R)$ and $\ell^u(x, R)$ are called *stable and unstable spanning leaf segments* for R (see Figure 1). For $\iota \in \{s, u\}$, we denote by $\partial\ell^\iota(x, R)$ the set consisting of the endpoints of $\ell^\iota(x, R)$, and we denote by $\operatorname{int}\ell^\iota(x, R)$ the set $\ell^\iota(x, R) \setminus \partial\ell^\iota(x, R)$. The *interior of* R is given by $\operatorname{int} R = [\operatorname{int}\ell^s(x, R), \operatorname{int}\ell^u(x, R)]$, and the *boundary of* R is given by $\partial R = [\partial\ell^s(x, R), \ell^u(x, R)] \cup [\ell^s(x, R), \partial\ell^u(x, R)]$.

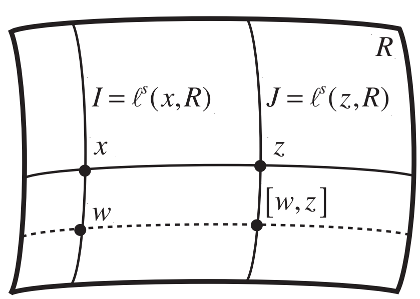

Figure 1. A rectangle.

2.5. Markov partitions. By a *Markov partition of* f we understand a collection $\mathcal{R} = \{R_1, \ldots, R_k\}$ of rectangles such that (i) $\Lambda \subset \bigcup_{i=1}^{k} R_i$; (ii) $R_i \cap R_j = \partial R_i \cap \partial R_j$ for all i and j; (iii) if $x \in \iint R_i$ and $fx \in \iint R_j$ then

(a) $f(\ell^s(x, R_i)) \subset \ell^s(fx, R_j)$ and $f^{-1}(\ell^u(fx, R_j)) \subset \ell^u(x, R_i)$
(b) $f(\ell^u(x, R_i)) \cap R_j = \ell^u(fx, R_j)$ and $f^{-1}(\ell^s(fx, R_j)) \cap R_i = \ell^s(x, R_i)$.

The last condition means that $f(R_i)$ goes across R_j just once. In fact, it follows from condition (a) providing the rectangles R_j are chosen sufficiently small (see [28]). The rectangles making up the Markov partition are called *Markov rectangles*.

We note that there is a Markov partition $\mathcal{R}$ of f with the following *disjointness property* (see [5; 34; 46]):

(i) if $0 < \delta_{f,s} < 1$ and $0 < \delta_{f,u} < 1$ then the stable and unstable leaf boundaries of any two Markov rectangles do not intersect.
(ii) if $0 < \delta_{f,\iota} < 1$ and $\delta_{f,\iota'} = 1$ then the ι'-leaf boundaries of any two Markov rectangles do not intersect except, possibly, at their endpoints.

If $\delta_{f,s} = \delta_{f,u} = 1$, the disjointness property does not apply and so we consider that it is trivially satisfied for every Markov partition. For simplicity of our exposition, we will just consider Markov partitions satisfying the disjointness property.

2.6. Leaf n-cylinders and leaf n-gaps. For $\iota = s$ or u, an *ι-leaf primary cylinder of a Markov rectangle* R is a spanning ι-leaf segment of R. For $n \geq 1$, an *ι-leaf n-cylinder of* R is an ι-leaf segment I such that

(i) $f_\iota^n I$ is an ι-leaf primary cylinder of a Markov rectangle M;
(ii) $f_\iota^n \left(\ell^{\iota'}(x, R)\right) \subset M$ for every $x \in I$.

For $n \geq 2$, an *ι-leaf n-gap G of R* is an ι-leaf gap $\{x, y\}$ in a Markov rectangle R such that n is the smallest integer such that both leaves $f_\iota^{n-1} \ell^{\iota'}(x, R)$ and $f_\iota^{n-1} \ell^{\iota'}(y, R)$ are contained in ι'-boundaries of Markov rectangles; An *ι-leaf primary gap* G is the image $f_\iota G'$ by f_ι of an ι-leaf 2-gap G'.

We note that an ι-leaf segment I of a Markov rectangle R can be simultaneously an n_1-cylinder, $(n_1 + 1)$-cylinder, ..., n_2-cylinder of R if $f^{n_1}(I)$, $f^{n_1+1}(I), \ldots, f^{n_2}(I)$ are all spanning ι-leaf segments. Furthermore, if I is an ι-leaf segment contained in the common boundary of two Markov rectangles R_i and R_j then I can be an n_1-cylinder of R_i and an n_2-cylinder of R_j with n_1 distinct of n_2. If $G = \{x, y\}$ is an ι-gap of R contained in the interior of R then there is a unique n such that G is an n-gap. However, if $G = \{x, y\}$ is contained in the common boundary of two Markov rectangles R_i and R_j then G can be an n_1-gap of R_i and an n_2-gap of R_j with n_1 distinct of n_2. Since the number

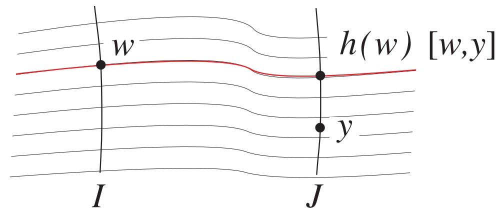

Figure 2. A basic stable holonomy from I to J.

of Markov rectangles $R_1, \ldots, R_k$ is finite, there is $C \geq 1$ such that, in all the above cases for cylinders and gaps we have $|n_2 - n_1| \leq C$.

We say that a leaf segment K is the i-th *mother* of an n-cylinder or an n-gap J of R if $J \subset K$ and K is a leaf $(n-i)$-cylinder of R. We denote K by $m^i J$.

By the properties of a Markov partition, the smallest full ι-leaf $\hat{K}$ containing a leaf n-cylinder K of a Markov rectangle R is equal to the union of all smallest full ι-leaves containing either a leaf $(n+j)$-cylinder or a leaf $(n+i)$-gap of R, with $i \in \{1, \ldots, j\}$, contained in K.

2.7. Metric on Λ. We say that a rectangle R is an (n_s, n_u)-*rectangle* if there is $x \in R$ such that, for $\iota = s$ and u, the spanning leaf segments $\ell^\iota(x, R)$ are either an ι-leaf n_ι-cylinder or the union of two such cylinders with a common endpoint.

The reason for allowing the possibility of the spanning leaf segments being inside two touching cylinders is to allow us to regard geometrically very small rectangles intersecting a common boundary of two Markov rectangles to be small in the sense of having n_s and n_u large.

If $x, y \in \Lambda$ and $x \neq y$ then $d_\Lambda(x, y) = 2^{-n}$ where n is the biggest integer such that both x and y are contained in an (n_s, n_u)-rectangle with $n_s \geq n$ and $n_u \geq n$. Similarly if I and J are ι-leaf segments then $d_\Lambda(I, J) = 2^{-n_{\iota'}}$ where $n_\iota = 1$ and $n_{\iota'}$ is the biggest integer such that both I and J are contained in an (n_s, n_u)-rectangle.

2.8. Basic holonomies. Suppose that x and y are two points inside any rectangle R of Λ. Let $\ell(x, R)$ and $\ell(y, R)$ be two stable leaf segments respectively containing x and y and inside R. Then we define $\theta : \ell(x, R) \to \ell(y, R)$ by $\theta(w) = [w, y]$. Such maps are called the *basic stable holonomies* (see Figure 2). They generate the pseudo-group of all stable holonomies. Similarly we define the basic unstable holonomies.

By Theorem 2.1 in [40], the holonomy $\theta : \ell^\iota(x, R) \to \ell^\iota(y, R)$ has a $C^{1+\alpha}$ extension to the leaves containing $\ell^\iota(x, R)$ and $\ell^\iota(y, R)$, for some $\alpha > 0$.

2.9. Foliated lamination atlas. In this section when we refer to a C^r object r is allowed to take the values $k+\alpha$ where k is a positive integer and $0<\alpha\leq 1$. Two ι-leaf charts i and j are C^r compatible if whenever U is an open subset of an ι-leaf segment contained in the domains of i and j then $j\circ i^{-1}: i(U)\to j(U)$ extends to a C^r diffeomorphism of the real line. Such maps are called *chart overlap maps*. A *bounded C^r ι-lamination atlas $\mathcal{A}^\iota$* is a set of such charts which (a) cover Λ, (b) are pairwise C^r compatible, and (c) the chart overlap maps are uniformly bounded in the C^r norm.

Let $\mathcal{A}^\iota$ be a bounded $C^{1+\alpha}$ ι-lamination atlas, with $0<\alpha\leq 1$. If $i: I\to\mathbb{R}$ is a chart in $\mathcal{A}^\iota$ defined on the leaf segment I and K is a leaf segment in I then we define $|K|_i$ to be the length of the minimal closed interval containing $i(K)$. Since the atlas is bounded, if $j: J\to\mathbb{R}$ is another chart in $\mathcal{A}^\iota$ defined on the leaf segment J which contains K then the ratio between the lengths $|K|_i$ and $|K|_j$ is universally bounded away from 0 and ∞. If $K'\subset I\cap J$ is another such segment then we can define the ratio $r_i(K:K')=|K|_i/|K'|_i$. Although this ratio depends upon i, the ratio is exponentially determined in the sense that if T is the smallest segment containing both K and K' then

$$r_j\left(K:K'\right)\in\left(1\pm\mathcal{O}\left(|T|_i^\alpha\right)\right)r_i\left(K:K'\right) \ .$$

This follows from the $C^{1+\alpha}$ smoothness of the overlap maps and Taylor's Theorem.

A C^r lamination atlas $\mathcal{A}^\iota$ has *bounded geometry* (i) if f is a C^r diffeomorphism with C^r norm uniformly bounded in this atlas; (ii) if for all pairs I_1, I_2 of ι-leaf n-cylinders or ι-leaf n-gaps with a common point, we have that $r_i(I_1:I_2)$ is uniformly bounded away from 0 and ∞ with the bounds being independent of i, I_1, I_2 and n; and (iii) for all endpoints x and y of an ι-leaf n-cylinder or ι-leaf n-gap I, we have that $|I|_i\leq\mathcal{O}\big((d_\Lambda(x,y))^\beta\big)$ and $d_\Lambda(x,y)\leq\mathcal{O}\big(|I|_i^\beta\big)$, for some $0<\beta<1$, independent of i, I and n.

A C^r bounded lamination atlas $\mathcal{A}^\iota$ is *C^r foliated* (i) if $\mathcal{A}^\iota$ has bounded geometry; and (ii) if the basic holonomies are C^r and have a C^r norm uniformly bounded in this atlas, except possibly for the dependence upon the rectangles defining the basic holonomy. A bounded lamination atlas $\mathcal{A}^\iota$ is *C^{1+} foliated* if $\mathcal{A}^\iota$ is C^r foliated for some $r>1$.

2.10. Foliated atlas $\mathcal{A}^\iota(g,\rho)$. Let $g\in\mathcal{T}(f,\Lambda)$ and $\rho=\rho_g$ be a C^{1+} Riemannian metric in the manifold containing Λ. The *ι-lamination atlas $\mathcal{A}^\iota(g,\rho)$ determined by ρ* is the set of all maps $e: I\to\mathbb{R}$ where $I=\Lambda\cap\hat{I}$ with $\hat{I}$ a full ι-leaf segment, such that e extends to an isometry between the induced Riemannian metric on $\hat{I}$ and the Euclidean metric on the reals. We call the maps $e\in\mathcal{A}^\iota(\rho)$ the *ι-lamination charts*. If I is an ι-leaf segment (or a full ι-leaf segment) then by $|I|_\rho$ we mean the length in the Riemannian metric ρ of the minimal full ι-leaf containing I. By Theorem 2.2 in [40], the lamination atlas $\mathcal{A}^\iota(g,\rho)$ is C^{1+}foliated for $\iota = \{s,u\}$.

3. Flexibility

In this section, we construct the stable and unstable solenoid functions, and we show an equivalence between C^{1+} hyperbolic diffeomorphisms and pairs of stable and unstable solenoid functions.

3.1. HR-Hölder ratios. A *HR-structure* associates an affine structure to each stable and unstable leaf segment in such a way that these vary Hölder continuously with the leaf and are invariant under f.

An affine structure on a stable or unstable leaf is equivalent to a *ratio function* $r(I:J)$ which can be thought of as prescribing the ratio of the size of two leaf segments I and J in the same stable or unstable leaf. A *ratio function* $r(I:J)$ is positive (we recall that each leaf segment has at least two distinct points) and continuous in the endpoints of I and J. Moreover,

$$r(I:J) = r(J:I)^{-1} \text{ and } r(I_1 \cup I_2 : K) = r(I_1 : K) + r(I_2 : K) \qquad (3\text{–}1)$$

provided I_1 and I_2 intersect at most in one of their endpoints.

We say that r is an *ι-ratio function* if (i) for all ι-leaf segments K, $r(I:J)$ defines a ratio function on K, where I and J are ι-leaf segments contained in K; (ii) r is invariant under f, i.e. $r(I:J) = r(fI:fJ)$ for all ι-leaf segments; and (iii) for every basic ι-holonomy $\theta : I \to J$ between the leaf segment I and the leaf segment J defined with respect to a rectangle R and for every ι-leaf segment $I_0 \subset I$ and every ι-leaf segment or gap $I_1 \subset I$,

$$\left| \log \frac{r(\theta I_0 : \theta I_1)}{r(I_0 : I_1)} \right| \le \mathcal{O}\left((d_\Lambda(I,J))^\varepsilon \right) \qquad (3\text{–}2)$$

where $\varepsilon \in (0,1)$ depends upon r and the constant of proportionality also depends upon R, but not on the segments considered.

A *HR-structure* on Λ invariant by f is a pair (r_s, r_u) consisting of a stable and an unstable ratio function.

3.2. Realised ratio functions. Let $(g,\Lambda) \in \mathcal{T}(f,\Lambda)$ and let $\mathcal{A}(g,\rho)$ be an ι-lamination atlas which is C^{1+} foliated. Let $|I| = |I|_\rho$ for every ι-leaf segment I. By hyperbolicity of g on Λ, there are $0 < \nu < 1$ and $C > 0$ such that for all ι-leaf segments I and all $m \ge 0$ we get $|g_{\iota'}^m I| \le C\nu^m |I|$. Thus, using the mean value theorem and the fact that g_ι is C^r, for all short leaf segments K and all leaf segments I and J contained in it, the ι-realised ratio function $r_{\iota,g}$ given by

$$r_{\iota,g}(I:J) = \lim_{n\to\infty} \frac{|g_{\iota'}^n I|}{|g_{\iota'}^n J|}$$

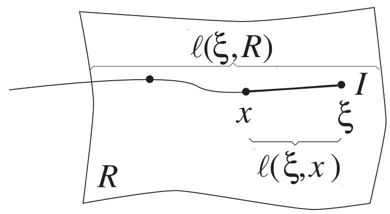

Figure 3. The embedding $e : I \to \mathbb{R}$.

is well-defined, where $\alpha = \min\{1, r-1\}$. This construction gives the HR-structure on Λ determined by g, and so also invariant by f. By [39], we get the following equivalence:

THEOREM 3.1. *The map $g \to (r_{s,g}, r_{u,g})$ determines a one-to-one correspondence between C^{1+} conjugacy classes in $\mathcal{T}(f, \Lambda)$ and HR-structures on Λ invariant by f.*

3.3. Foliated atlas $\mathscr{A}(r)$. Given an ι-ratio function r, we define the embeddings $e : I \to \mathbb{R}$ by

$$e(x) = r(\ell(\xi, x), \ell(\xi, R)) \tag{3–3}$$

where ξ is an endpoint of the ι-leaf segment I and R is a Markov rectangle containing ξ (see Figure 3). For this definition it is not necessary that R contains I. We denote the set of all these embeddings e by $\mathscr{A}(r)$.

The embeddings e of $\mathscr{A}(r)$ have overlap maps with affine extensions. Therefore, the atlas $\mathscr{A}(r)$ extends to a $C^{1+\alpha}$ lamination structure $\mathscr{L}(r)$. By Proposition 4.2 in [40], we obtain that $\mathscr{A}(r)$ is a C^{1+}foliated atlas.

Let $g \in \mathcal{T}(f, \Lambda)$ and $\mathscr{A}(g, \rho)$ a C^{1+} foliated ι-lamination atlas determined by a Riemmanian metric ρ. Combining Proposition 2.5 and Proposition 3.5 of [39], we get that the overlap map $e_1 \circ e_2^{-1}$ between a chart $e_1 \in \mathscr{A}(g, \rho)$ and a chart $e_2 \in \mathscr{A}(r_{\iota,g})$ has a C^{1+} diffeomorphic extension to the reals. Therefore, the atlasses $\mathscr{A}(g, \rho)$ and $\mathscr{A}(r_{\iota,g})$ determine the same C^{1+} foliated ι-lamination. In particular, for all short leaf segments K and all leaf segments I and J contained in it, we obtain that

$$r_{\iota,g}(I : J) = \lim_{n\to\infty} \frac{|g^n_{\iota'} I|_\rho}{|g^n_{\iota'} J|_\rho} = \lim_{n\to\infty} \frac{|g^n_{\iota'} I|_{i_n}}{|g^n_{\iota'} J|_{i_n}}$$

where i_n is any chart in $\mathscr{A}(r_{\iota,g})$ containing the segment $g^n_{\iota'} K$ in its domain.

3.4. Realised solenoid functions. For $\iota = s$ and u, let sol^ι denote the set of all ordered pairs (I, J) of ι-leaf segments with the following properties:

(i) The intersection of I and J consists of a single endpoint.
(ii) If $\delta_{\iota,f} = 1$ then I and J are primary ι-leaf cylinders.

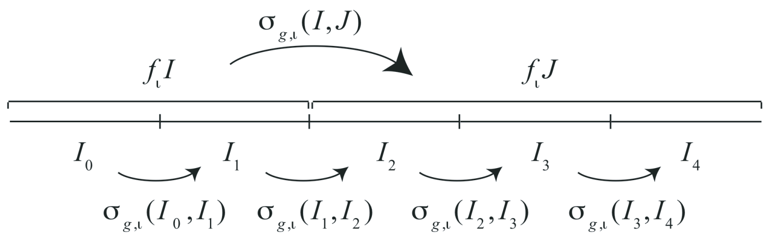

Figure 4. The f-matching condition for ι-leaf segments.

(iii) If $0 < \delta_{\iota,f} < 1$ then $f_{\iota'} I$ is an ι-leaf 2-cylinder of a Markov rectangle R and $f_{\iota'} J$ is an ι-leaf 2-gap also of the same Markov rectangle R.

(See section 2.4 for the definitions of leaf cylinders and gaps). Pairs (I, J) where both are primary cylinders are called *leaf-leaf pairs*. Pairs (I, J) where J is a gap are called *leaf-gap pairs* and in this case we refer to J as a *primary gap*. The set sol^ι has a very nice topological structure. If $\delta_{\iota',f} = 1$ then the set sol^ι is isomorphic to a finite union of intervals, and if $\delta_{\iota',f} < 1$ then the set sol^ι is isomorphic to an embedded Cantor set on the real line.

We define a pseudo-metric $d_{\mathrm{sol}^\iota} : \mathrm{sol}^\iota \times \mathrm{sol}^\iota \to \mathbb{R}^+$ on the set sol^ι by

$$d_{\mathrm{sol}^\iota}\left((I, J), \left(I', J'\right)\right) = \max\left\{d_\Lambda\left(I, I'\right), d_\Lambda\left(J, J'\right)\right\} .$$

Let $g \in \mathcal{T}(f, \Lambda)$. For $\iota = s$ and u, we call the restriction of an ι-ratio function $r_{\iota,g}$ to sol^ι a *realised solenoid function* $\sigma_{\iota,g}$. By construction, for $\iota = s$ and u, the restriction of an ι-ratio function to sol^ι gives an Hölder continuous function satisfying the matching condition, the boundary condition and the cylinder-gap condition as we pass to describe.

3.5. Hölder continuity of solenoid functions. This means that for $t = (I, J)$ and $t' = (I', J')$ in sol^ι, $|\sigma_\iota(t) - \sigma_\iota(t')| \le \mathcal{O}\left((d_{\mathrm{sol}^\iota}(t, t'))^\alpha\right)$. The Hölder continuity of $\sigma_{g,\iota}$ and the compactness of its domain imply that $\sigma_{g,\iota}$ is bounded away from zero and infinity.

3.6. Matching condition. Let $(I, J) \in \mathrm{sol}^\iota$ be a pair of primary cylinders and suppose that we have pairs

$$(I_0, I_1), (I_1, I_2), \ldots, (I_{n-2}, I_{n-1}) \in \mathrm{sol}^\iota$$

of primary cylinders such that $f_\iota I = \bigcup_{j=0}^{k-1} I_j$ and $f_\iota J = \bigcup_{j=k}^{n-1} I_j$. Then

$$\frac{|f_\iota I|}{|f_\iota J|} = \frac{\sum_{j=0}^{k-1} |I_j|}{\sum_{j=k}^{n-1} |I_j|} = \frac{1 + \sum_{j=1}^{k-1} \prod_{i=1}^{j} |I_i|/|I_{i-1}|}{\sum_{j=k}^{n-1} \prod_{i=1}^{j} |I_i|/|I_{i-1}|} .$$

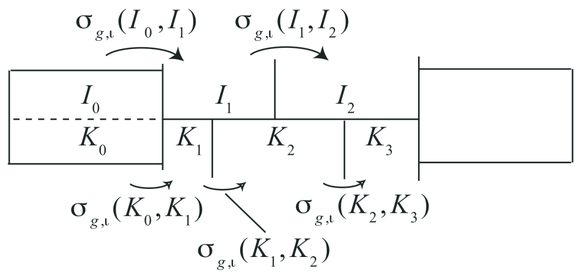

Figure 5. The boundary condition for ι-leaf segments.

Hence, noting that $g|\Lambda = f|\Lambda$, the realised solenoid function $\sigma_{\iota,g}$ must satisfy the *matching condition* (see Figure 4) for all such leaf segments:

$$\sigma_{\iota,g}(I : J) = \frac{1 + \sum_{j=1}^{k-1} \prod_{i=1}^{j} \sigma_{\iota,g}(I_i : I_{i-1})}{\sum_{j=k}^{n-1} \prod_{i=1}^{j} \sigma_{\iota,g}(I_i : I_{i-1})} \,. \tag{3–4}$$

3.7. Boundary condition. If the stable and unstable leaf segments have Hausdorff dimension equal to 1, then leaf segments I in the boundaries of Markov rectangles can sometimes be written as the union of primary cylinders in more than one way. This gives rise to the existence of a boundary condition that the realised solenoid functions have to satisfy.

If J is another leaf segment adjacent to the leaf segment I then the value of $|I|/|J|$ must be the same whichever decomposition we use. If we write $J = I_0 = K_0$ and I as $\bigcup_{i=1}^{m} I_i$ and $\bigcup_{j=1}^{n} K_j$ where the I_i and K_j are primary cylinders with $I_i \neq K_j$ for all i and j, then the above two ratios are

$$\sum_{i=1}^{m} \prod_{j=1}^{i} \frac{|I_j|}{|I_{j-1}|} = \frac{|I|}{|J|} = \sum_{i=1}^{n} \prod_{j=1}^{i} \frac{|K_j|}{|K_{j-1}|} \,.$$

Thus, noting that $g|\Lambda = f|\Lambda$, a realised solenoid function $\sigma_{\iota,g}$ must satisfy the following *boundary condition* (see Figure 5) for all such leaf segments:

$$\sum_{i=1}^{m} \prod_{j=1}^{i} \sigma_{\iota,g}\left(I_j : I_{j-1}\right) = \sum_{i=1}^{n} \prod_{j=1}^{i} \sigma_{\iota,g}\left(K_j : K_{j-1}\right) \,. \tag{3–5}$$

3.8. Cylinder-gap condition. If the ι-leaf segments have Hausdorff dimension less than one and the ι'-leaf segments have Hausdorff dimension equal to 1, then a primary cylinder I in the ι-boundary of a Markov rectangle can also be written as the union of gaps and cylinders of other Markov rectangles. This gives rise to the existence of a cylinder-gap condition that the ι-realised solenoid functions have to satisfy.

Before defining the cylinder-gap condition, we will introduce the scaling function that will be useful to express the cylinder-gap condition, and also, in Definition 3.2, the bounded equivalence classes of solenoid functions and, in Definition 5.3, the δ-bounded solenoid equivalence classes of a Gibbs measure.

Let scl^ι be the set of all pairs (K, J) of ι-leaf segments with the following properties:

(i) K is a leaf n_1-cylinder or an n_1-gap segment for some $n_1 > 1$;
(ii) J is a leaf n_2-cylinder or an n_2-gap segment for some $n_2 > 1$;
(iii) $m^{n_1-1}K$ and $m^{n_2-1}J$ are the same primary cylinder.

LEMMA 3.2. *Every function* $\sigma_\iota : \mathrm{sol}^\iota \to \mathbb{R}^+$ *has a canonical extension* s_ι *to* scl^ι. *Furthermore, if* σ_ι *is the restriction of a ratio function* $r_\iota|\mathrm{sol}^\iota$ *to* sol^ι *then*

$$s_\iota = r_\iota|\mathrm{scl}^\iota.$$

The above map $s_\iota : \mathrm{scl}^\iota \to \mathbb{R}^+$ is the *scaling function* determined by the solenoid function $\sigma_\iota : \mathrm{sol}^\iota \to \mathbb{R}^+$. Lemma 3.2 is proved in Section 3.8 in [42].

Let (I, K) be a leaf-gap pair such that the primary cylinder I is the ι-boundary of a Markov rectangle R_1. Then the primary cylinder I intersects another Markov rectangle R_2 giving rise to the existence of a cylinder-gap condition that the realised solenoid functions have to satisfy as we pass to explain. Take the smallest $l \geq 0$ such that $f_{\iota'}^l I \cup f_{\iota'}^l K$ is contained in the intersection of the boundaries of two Markov rectangles M_1 and M_2. Let M_1 be the Markov rectangle with the property that $M_1 \cap f_{\iota'}^l R_1$ is a rectangle with nonempty interior (and so $M_2 \cap f_{\iota'}^l R_2$ also has nonempty interior). Then, for some positive n, there are distinct n-cylinder and gap leaf segments $J_1, \ldots, J_m$ contained in a primary cylinder of M_2 such that $f_{\iota'}^l K = J_m$ and the smallest full ι-leaf segment containing $f_{\iota'}^l I$ is equal to the union $\bigcup_{i=1}^{m-1} \hat{J}_i$, where $\hat{J}_i$ is the smallest full ι-leaf segment containing J_i. Hence,

$$\frac{|f_{\iota'}^l I|}{|f_{\iota'}^l K|} = \sum_{i=1}^{m-1} \frac{|J_i|}{|J_m|} \ .$$

Hence, noting that $g|\Lambda = f|\Lambda$, a realised solenoid function $\sigma_{g,\iota}$ must satisfy the *cylinder-gap condition* for all such leaf segments:

$$\sigma_{g,\iota}(I, K) = \sum_{i=1}^{m-1} s_{g,\iota}(J_i, J_m),$$

where $s_{g,\iota}$ is the scaling function determined by the solenoid function $\sigma_{g,\iota}$. See Figure 6.

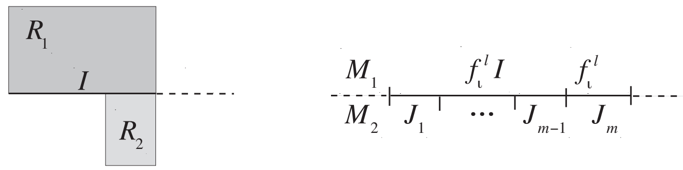

Figure 6. The cylinder-gap condition for ι-leaf segments.

3.9. Solenoid functions. Now, we are ready to present the definition of an ι-solenoid function.

DEFINITION 3.1. A Hölder continuous function $\sigma_\iota : \mathrm{sol}^\iota \to \mathbb{R}^+$ is an *ι-solenoid function* if it satisfies the matching condition, the boundary condition and the cylinder-gap condition.

We denote by $\mathcal{PS}(f)$ the set of pairs (σ_s, σ_u) of stable and unstable solenoid functions.

REMARK 3.3. Let $\sigma_\iota : \mathrm{sol}^\iota \to \mathbb{R}^+$ be an ι-solenoid function. The matching, the boundary and the cylinder-gap conditions are trivially satisfied except in the following cases:

(i) The matching condition if $\delta_{\iota,f} = 1$.
(ii) The boundary condition if $\delta_{s,f} = \delta_{u,f} = 1$.
(iii) The cylinder-gap condition if $\delta_{\iota,f} < 1$ and $\delta_{\iota',f} = 1$.

THEOREM 3.4. *The map $r_\iota \to r_\iota|\mathrm{sol}^\iota$ gives a one-to-one correspondence between ι-ratio functions and ι-solenoid functions.*

PROOF. Every ι-ratio function restricted to the set sol^ι determines an ι-solenoid function $r_\iota|\mathrm{sol}^\iota$. Now we prove the converse. Since the solenoid functions are continuous and their domains are compact they are bounded away from 0 and ∞. By this boundedness and the f-matching condition of the solenoid functions and by iterating the domains sol^s and sol^u of the solenoid functions backward and forward by f, we determine the ratio functions r^s and r^u at very small (and large) scales, such that f leaves the ratios invariant. Then, using the boundedness again, we extend the ratio functions to all pairs of small adjacent leaf segments by continuity. By the boundary condition and the cylinder-gap condition of the solenoid functions, the ratio functions are well determined at the boundaries of the Markov rectangles. Using the Hölder continuity of the solenoid function, we deduce inequality (3–2). □

The set $\mathcal{PS}(f)$ of all pairs (σ_s, σ_u) has a natural metric. Combining Theorem 3.1 with Theorem 3.4, we obtain that the set $\mathcal{PS}(f)$ forms a moduli space for the C^{1+} conjugacy classes of C^{1+} hyperbolic diffeomorphisms $g \in \mathcal{T}(f, \Lambda)$:

COROLLARY 3.5. *The map* $g \to (r_{s,g}|\mathrm{sol}^s, r_{u,g}|\mathrm{sol}^u)$ *determines a one-to-one correspondence between* C^{1+} *conjugacy classes of* $g \in \mathcal{T}(f, \Lambda)$ *and pairs of solenoid functions in* $\mathcal{PS}(f)$.

DEFINITION 3.2. We say that any two ι-solenoid functions $\sigma_1 : \mathrm{sol}^\iota \to \mathbb{R}^+$ and $\sigma_2 : \mathrm{sol}^\iota \to \mathbb{R}^+$ are in the same *bounded equivalence class* if the corresponding scaling functions $s_1 : \mathrm{scl}^\iota \to \mathbb{R}^+$ and $s_2 : \mathrm{scl}^\iota \to \mathbb{R}^+$ satisfy the following property: There is $C > 0$ such that

$$\left|\log s_1(J, m^i J) - \log s_2(J, m^i J)\right| < C \qquad (3\text{–}6)$$

for every ι-leaf $(i+1)$-cylinder or $(i+1)$-gap J.

In Lemma 8.8 in [42], it is proved that two C^{1+} hyperbolic diffeomorphisms g_1 and g_2 are Lipschitz conjugate if, and only if, the solenoid functions $\sigma_{g_1,\iota}$ and $\sigma_{g_2,\iota}$ are in the same bounded equivalence class for ι equal to s and u.

4. Rigidity

If the holonomies are sufficiently smooth then the system is essentially affine. To see that, rather than consider all holonomies, it is enough to consider a C^{1,HD^ι} complete set of holonomies.

4.1. Complete sets of holonomies. Before introducing the notion of a C^{1,α^ι} complete set of holonomies, we define the $C^{1,\alpha}$ regularities, with $0 < \alpha \le 1$, for diffeomorphisms.

DEFINITION 4.1. Let $\theta : I \subset \mathbb{R} \to J \subset \mathbb{R}$ be a diffeomorphism. For $0 < \alpha < 1$, the diffeomorphism θ is $C^{1,\alpha}$ if, for all points $x, y \in I$,

$$|\theta'(y) - \theta'(x)| \le \chi_{\theta,\alpha}(|y - x|) \qquad (4\text{–}1)$$

where the positive function $\chi_{\theta,\alpha}(t)$ is $o(t^\alpha)$ i.e. $\lim_{t\to 0} \chi_{\theta,\alpha}(t)/t^\alpha = 0$.

The map $\theta : I \to J$ is $C^{1,1}$ if, for all points $x, y \in I$,

$$\left|\log \theta'(x) + \log \theta'(y) - 2\log \theta'\left(\frac{x+y}{2}\right)\right| \le \chi_{\theta,1}(|y - x|) \qquad (4\text{–}2)$$

where the positive function $\chi_{\theta,1}(t)$ is $o(t)$, i.e. $\lim_{t\to 0} \chi_{\theta,1}(t)/t = 0$. For $0 < \alpha \le 1$, the functions $\chi_{\theta,\alpha}$ are called the *α-modulus of continuity of θ*.

In particular, for every $\beta > \alpha > 0$, a $C^{1+\beta}$ diffeomorphism is $C^{1,\alpha}$, and, for every $\gamma > 0$, a $C^{2+\gamma}$ diffeomorphism is $C^{1,1}$. We note that the regularity $C^{1,1}$ (also denoted by $C^{1+\mathrm{zygmund}}$) of a diffeomorphism θ used in this paper is stronger than the regularity $C^{1+\mathrm{Zygmund}}$ (see [32]). The importance of these $C^{1,\alpha}$ smoothness classes for a diffeomorphism $\theta : I \to J$ follows from the fact that if $0 < \alpha < 1$ then the map θ will distort ratios of lengths of short intervals

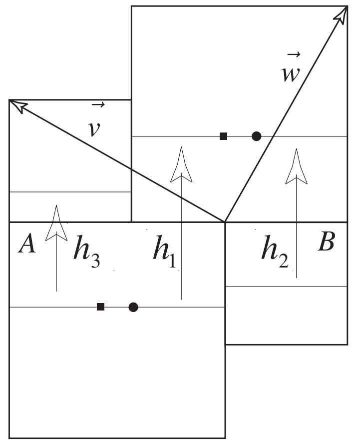

Figure 7. The complete set of holonomies $\mathcal{H} = \{h_1, h_2, h_3, h_1^{-1}, h_2^{-1}, h_3^{-1}\}$ for the Anosov map $g : \mathbf{R}^2 \setminus (\mathbf{Z}\vec{v} \times \mathbf{Z}\vec{w}) \to \mathbf{R}^2 \setminus (\mathbf{Z}\vec{v} \times \mathbf{Z}\vec{w})$ defined by $g(x, y) = (x + y, y)$ and with Markov partition $\mathcal{M} = \{A, B\}$.

in an interval $K \subset I$ by an amount that is $o(|I|^{\alpha})$, and if $\alpha = 1$ the map θ will distort the cross-ratios of quadruples of points in an interval $K \subset I$ by an amount that is $o(|I|)$ (see Appendix in [41]).

Suppose that R_i and R_j are Markov rectangles, $x \in R_i$ and $y \in R_j$. We say that x and y are *ι-holonomically related* if there is an ι'-leaf segment $\ell^{\iota'}(x, y)$ such that $\partial\ell^{\iota'}(x, y) = \{x, y\}$, and there are two distinct spanning ι'-leaf segments $\ell^{\iota'}(x, R_i)$ and $\ell^{\iota'}(y, R_j)$ such that their union contains $\ell^{\iota'}(x, y)$.

For every Markov rectangle $R_i \in \mathcal{R}$, let x_i be a chosen point in R_i. Let $\mathcal{I}^{\iota} = \{I_i = \ell^{\iota}(x_i, R_i) : R_i \in \mathcal{R}\}$. A *complete set of ι-holonomies* $\mathcal{H}^{\iota} = \{h_{\beta}\}$ with respect to $\mathcal{I}^{\iota}$ consists of a minimal set of basic holonomies with the following property: if $x \in I_i$ is holonomically related to $y \in I_j$, where $I_i, I_j \in \mathcal{I}^{\iota}$, then for some β either h_{β} or h_{β}^{-1} is the holonomy from a neighborhood of x in I_i to I_j which sends x to y (see Figure 7). We call $\mathcal{I}^{\iota}$ the *domain of the complete set of ι-holonomies* $\mathcal{H}^{\iota}$. For each Êι-leaf segment I_i in the domain $\mathcal{I}^{\iota}$ of the complete set of holonomies $\mathcal{H}^{\iota}$, let $\hat{I}_i$ be a full ι-leaf segment such that $I_i = \hat{I}_i \cap \Lambda$, and let $u_i : \hat{I}_i \to \mathbb{R}$ be a C^r ι-leaf chart of the submanifold structure of $\hat{I}_i$ given by the Stable Manifold Theorem (for instance, we can consider the charts $u_i \in \mathcal{A}(\rho)$ as defined in Section 2.10).

Definition 4.2. A complete set of holonomies $\mathcal{H}^{\iota}$ is $C^{1,\alpha^{\iota}}$ if for every holonomy $h_{\beta} : I \to J$ in $\mathcal{H}^{\iota}$, the map $u_j \circ h_{\beta} \circ u_i^{-1}$ and its inverse have $C^{1,\alpha^{\iota}}$ diffeomorphic extensions to $\mathbb{R}$ such that the modulus of continuity does not depend upon $h_{\beta} \in \mathcal{H}^{\iota}$.

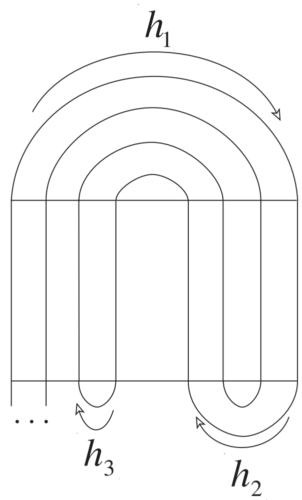

Figure 8. The cardinality of the complete set of holonomies $\mathcal{H} = \{h_1, h_2, h_3, \ldots\}$ is not necessarily finite.

For many systems such as Anosov diffeomorphisms and codimension one attractors there is only a finite number of holonomies in a complete set. In this case the uniformity hypothesis in the modulus of continuity of Definition 4.2 is redundant. However, for Smale horseshoes this is not the case (see Figure 8).

4.2. Hyperbolic affine models. A *hyperbolic affine model for f on Λ* is an *atlas $\mathcal{A}$* with the following properties (see Figure 9):

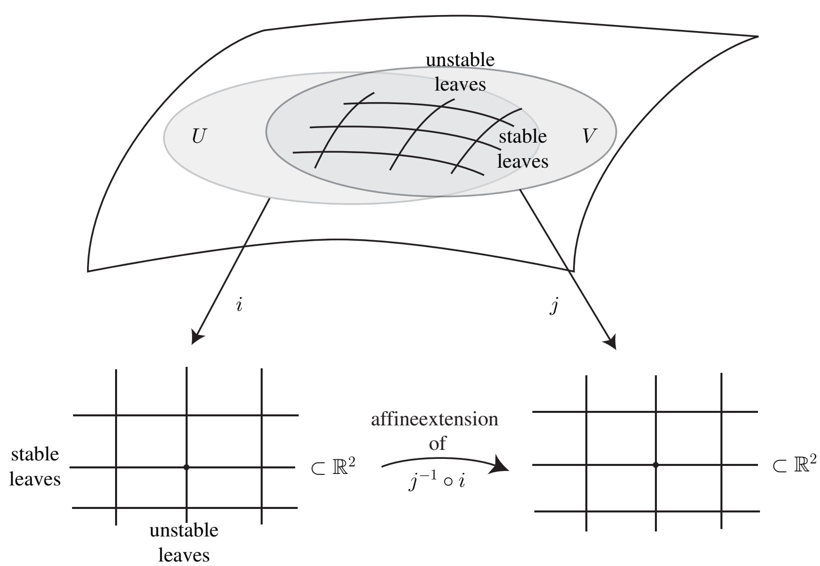

Figure 9. Affine model for f.

(i) the union of the domains of the charts of $\mathscr{A}$ cover an open set of M containing Λ;
(ii) any two charts $i : U \to \mathbb{R}^2$ and $j : V \to \mathbb{R}^2$ in $\mathscr{A}$ have overlap maps $j \circ i^{-1} : i(U \cap V) \to \mathbb{R}^2$ with affine extensions to $\mathbb{R}^2$;
(iii) f is affine with respect to the charts in $\mathscr{A}$;
(iv) Λ is a basic hyperbolic set;
(v) the images of the stable and unstable local leaves under the charts in $\mathscr{A}$ are contained in horizontal and vertical lines; and
(vi) the basic holonomies have affine extensions to the stable and unstable leaves with respect to the charts in $\mathscr{A}$.

THEOREM 4.1. *Let HD^s and HD^u be respectively the Hausdorff dimension of the intersection with Λ of the stable and unstable leaves of f. If f is C^r with $r-1 > \max\{HD^s, HD^u\}$, and there is a complete set of holonomies for f in which the stable holonomies are C^{1,HD^s}, and the unstable holonomies are C^{1,HD^u}, then the map f on Λ is C^{1+} conjugate to a hyperbolic affine model.*

Theorem 4.1 follows from Theorem 1 in [41]. In assuming that f is C^r with $r-1 > \max\{HD^s, HD^u\}$ in the previous theorem, we actually only use the fact that f is C^{1,HD^ι} along ι-leaves for $\iota \in \{s, u\}$.

5. Hausdorff measures

We now introduce the notion of stable and unstable measure solenoid functions. We will use the measure solenoid functions to determine which Gibbs measures are C^{1+}-realisable by C^{1+} hyperbolic diffeomorphisms. We define the δ-bounded solenoid equivalence classes of Gibbs measures which allow us to construct all hyperbolic diffeomorphisms with an f-invariant measure absolutely continuous with respect to the Hausdorff measure.

5.1. Gibbs measures. Let us give the definition of an infinite two-sided subshift of finite type $\Theta = \Theta(A)$. The elements of Θ are all infinite two-sided words $w = \ldots w_{-1} w_0 w_1 \ldots$ in the symbols $1, \ldots, k$ such that $A_{w_i w_{i+1}} = 1$, for all $i \in \mathbb{Z}$. Here $A = (A_{ij})$ is any matrix with entries 0 and 1 such that A^n has all entries positive for some $n \geq 1$. We write $w \overset{n_1,n_2}{\sim} w'$ if $w_j = w'_j$ for every $j = -n_1, \ldots, n_2$. The metric d on Θ is given by $d(w, w') = 2^{-n}$ if $n \geq 0$ is the largest such that $w \overset{n,n}{\sim} w'$. Together with this metric Θ is a compact metric space. The two-sided shift map $\tau : \Theta \to \Theta$ is the mapping which sends $w = \ldots w_{-1} w_0 w_1 \ldots$ to $v = \ldots v_{-1} v_0 v_1 \ldots$ where $v_j = w_{j+1}$ for every $j \in \mathbb{Z}$. An (n_1, n_2)-*cylinder* $\Theta_{w_{-n_1} \ldots w_{n_2}}$, where $w \in \Theta$, consists of all those words w' in Θ such that $w \overset{n_1,n_2}{\sim} w'$. Let Θ^u be the set of all words $w_0 w_1 \ldots$ which extend to words $\ldots w_0 w_1 \ldots$ in Θ, and, similarly, let Θ^s be the set of all words

$\ldots w_{-1} w_0$ which extend to words $\ldots w_{-1} w_0 \ldots$ in Θ. Then $\pi_u : \Theta \to \Theta^u$ and $\pi_s : \Theta \to \Theta^s$ are the natural projection given, respectively, by

$$\pi_u(\ldots w_{-1} w_0 w_1 \ldots) = w_0 w_1 \ldots \text{ and } \pi_s(\ldots w_{-1} w_0 w_1 \ldots) = \ldots w_{-1} w_0 \ .$$

An n-cylinder $\Theta^u_{w_0 \ldots w_{n-1}}$ is equal to $\pi_u(\Theta_{w_0 \ldots w_{n-1}})$, and likewise an n-cylinder $\Theta^s_{w_{-(n-1)} \ldots w_0}$ is equal to $\pi_s(\Theta_{w_{-(n-1)} \ldots w_0})$. Let $\tau_u : \Theta^u \to \Theta^u$ and $\tau_s : \Theta^s \to \Theta^s$ be the corresponding one-sided shifts.

DEFINITION 5.1. For $\iota = s$ and u, we say that $s_\iota : \Theta^\iota \to \mathbb{R}^+$ is an *ι-measure scaling function* if s_ι is a Hölder continuous function, and for every $\xi \in \Theta^\iota$

$$\sum_{\tau_\iota \eta = \xi} s_\iota(\eta) = 1 \ ,$$

where the sum is upon all $\xi \in \Theta^\iota$ such that $\tau_\iota \eta = \xi$.

For $\iota \in \{s, u\}$, a τ-invariant measure ν on Θ determines a unique τ_ι-invariant measure $\nu_\iota = (\pi_\iota)_* \nu$ on Θ^ι. We note that a τ_ι-invariant measure ν_ι on Θ^ι has a unique τ-invariant natural extension to an invariant measure ν on Θ such that $\nu(\Theta_{w_{n_1} \ldots w_{n_2}}) = \nu_\iota(\Theta^\iota_{w_{n_1} \ldots w_{n_2}})$.

DEFINITION 5.2. A τ-invariant measure ν on Θ is a *Gibbs measure*:

(i) if the function $s_{\nu,u} : \Theta^u \to \mathbb{R}^+$ given by

$$s_{\nu,u}(w_0 w_1 \ldots) = \lim_{n \to \infty} \frac{\nu(\Theta_{w_0 \ldots w_n})}{\nu(\Theta_{w_1 \ldots w_n})} \ ,$$

is well-defined and it is an u-measure scaling function; and

(ii) if the function $s_{\nu,s} : \Theta^s \to \mathbb{R}^+$ given by

$$s_{\nu,s}(\ldots w_1 w_0) = \lim_{n \to \infty} \frac{\nu(\Theta_{w_n \ldots w_0})}{\nu(\Theta_{w_n \ldots w_1})} \ ,$$

is well-defined and it is an s-measure scaling function.

The following theorem follows from Corollary 2 in [38].

THEOREM 5.1 (MODULI SPACE FOR GIBBS MEASURES). *If $s_\iota : \Theta^\iota \to \mathbb{R}^+$ is an* ι-measure scaling function*, for $\iota = s$ or u, then there is a unique τ-invariant Gibbs measure ν such that $s_{\nu,\iota} = s_\iota$.*

5.2. Hausdorff realisations of Gibbs measures for Smale horseshoes. The properties of the Markov partition $\mathcal{R} = \{R_1, \ldots, R_k\}$ of f imply the existence of a unique two-sided subshift τ of finite type $\Theta = \Theta_A$ and a continuous surjection $i : \Theta \to \Lambda$ such that (a) $f \circ i = i \circ \tau$ and (b) $i(\Theta_j) = R_j$ for every $j = 1, \ldots, k$. We call such a map $i : \Theta \to \Lambda$ *a marking of a C^{1+} hyperbolic diffeomorphism*

(f, Λ). Since f admits more than one Markov partition, a C^{1+} hyperbolic diffeomorphism (f, Λ) admits always a marking which is not unique.

Recall, from the Introduction, that a Gibbs measure ν on Θ is *C^{1+}-Hausdorff realisable by a hyperbolic diffeomorphism* $g \in \mathcal{T}(f, \Lambda)$ if, for every chart $c : U \to \mathbb{R}^2$ in the C^{1+} structure $\mathcal{C}_g$ of g, the pushforward $(c \circ i)_* \nu$ of ν is absolutely continuous (in fact, equivalent) with respect to the Hausdorff measure on $c(U \cap \Lambda)$.

THEOREM 5.2 (SMALE HORSESHOES). *Let (f, Λ) be a Smale horseshoe. Every Gibbs measure ν is C^{1+}-Hausdorff realisable by a hyperbolic diffeomorphism contained in $\mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$.*

However, there are Gibbs measures that are not C^{1+}-Hausdorff realisable by Anosov diffeomorphisms and codimension one attractors due to the fact that the Markov rectangles have common boundaries.

Theorem 5.2 follows from Theorem 1.6 in [42].

5.3. Hausdorff realisations of Gibbs measures for Anosov diffeomorphisms. We will use stable and unstable measure solenoid functions to present a classification of Gibbs measures C^{1+}-Hausdorff realisable by Anosov diffeomorphisms and codimension one attractors.

Let Msol^ι be the set of all pairs (I, J) with the following properties: (a) If $\delta_\iota = 1$ then $\mathrm{Msol}^\iota = \mathrm{sol}^\iota$. (b) If $\delta_\iota < 1$ then $f_{\iota'} I$ and $f_{\iota'} J$ are ι-leaf 2-cylinders of a Markov rectangle R such that $f_{\iota'} I \cup f_{\iota'} J$ is an ι-leaf segment, i.e. there is a unique ι-leaf 2-gap between them. Let msol^ι be the set of all pairs $(I, J) \in \mathrm{Msol}^\iota$ such that the leaf segments I and J are not contained in an ι-global leaf containing an ι-boundary of a Markov rectangle. By construction, the set msol^ι is dense in Msol^ι, and for every pair $(C, D) \subset \mathrm{msol}^\iota$ there is a unique $\psi \in \Theta^{\iota'}$ and a unique $\xi \in \Theta^{\iota'}$ such that $i(\pi_{\iota'}^{-1}(\psi)) = C$ and $i(\pi_{\iota'}^{-1}(\xi)) = D$. Hence, we will denote the elements of msol^ι by $(\psi_\Lambda, \xi_\Lambda)$, where $\psi_\Lambda = i(\pi_{\iota'}^{-1}(\psi))$ and $\xi_\Lambda = i(\pi_{\iota'}^{-1}(\xi))$.

LEMMA 5.3. *Let ν be a Gibbs measure on Θ. The s-*measure solenoid function $\sigma_{\nu,s} : \mathrm{msol}^s \to \mathbb{R}^+$ *of ν given by*

$$\sigma_{\nu,s}(\psi_\Lambda, \xi_\Lambda) = \lim_{n\to\infty} \frac{\nu(\Theta_{\psi_0 \dots \psi_n})}{\nu(\Theta_{\xi_0 \dots \xi_n})}$$

*is well-defined. The u-*measure solenoid function $\sigma_{\nu,u} : \mathrm{msol}^u \to \mathbb{R}^+$ *of ν given by*

$$\sigma_{\nu,u}(\psi_\Lambda, \xi_\Lambda) = \lim_{n\to\infty} \frac{\nu(\Theta_{\psi_n \dots \psi_0})}{\nu(\Theta_{\xi_n \dots \xi_0})}$$

is well-defined.

Lemma 5.3 follows from Lemma 5.4 in [42].

LEMMA 5.4. *Let* $\delta_{f,\iota} = 1$. *If an* ι*-measure solenoid function* $\sigma_{\nu,\iota} : \mathrm{msol}^\iota \to \mathbb{R}^+$ *has a continuous extension to* sol^ι *then its extension satisfies the matching condition.*

PROOF. Let $(J_0, J_1) \in \mathrm{sol}^\iota$ be a pair of primary cylinders and suppose that we have pairs

$$(I_0, I_1), (I_1, I_2), \ldots, (I_{n-2}, I_{n-1}) \in \mathrm{sol}^\iota$$

of primary cylinders such that $f_\iota J_0 = \bigcup_{j=0}^{k-1} I_j$ and $f_\iota J_1 = \bigcup_{j=k}^{n-1} I_j$. Since the set msol^ι is dense in sol^ι there are pairs $(J_0^l, J_1^l) \in \mathrm{msol}^\iota$ and pairs (I_j^l, I_{j+1}^l) with the following properties:

(i) $f_\iota J_0^l = \bigcup_{j=0}^{k-1} I_j^l$ and $f_\iota J_1^l = \bigcup_{j=k}^{n-1} I_j^i$.
(ii) The pair (J_0^l, J_1^l) converges to (J_0, J_1) when i tends to infinity.

Therefore, for every $j = 0, \ldots, n-2$ the pair (I_j^l, I_{j+1}^l) converges to (I_j, I_{j+1}) when i tends to infinity. Since ν is a τ-invariant measure, we get that the matching condition

$$\sigma_{\nu,\iota}(J_0^l : J_1^l) = \frac{1 + \sum_{j=1}^{k-1} \prod_{i=1}^{j} \sigma_{\nu,\iota}(I_j^l : I_{i-1}^l)}{\sum_{j=k}^{n-1} \prod_{i=1}^{j} \sigma_{\nu,\iota}(I_j^l : I_{i-1}^l)}$$

is satisfied for every $l \geq 1$. Since the extension of $\sigma_{\nu,\iota} : \mathrm{msol}^\iota \to \mathbb{R}^+$ to the set sol^ι is continuous, we get that the matching condition also holds for the pairs (J_0, J_1) and $(I_0, I_1), \ldots, (I_{n-2}, I_{n-1})$. □

THEOREM 5.5 (ANOSOV DIFFEOMORPHISMS). *Suppose that* f *is a* C^{1+} *Anosov diffeomorphism of the torus* Λ. *Fix a Gibbs measure* ν *on* Θ. *Then the following statements are equivalent:*

(i) *The set* ν, $[\nu] \subset \mathcal{T}_{f,\Lambda}(1,1)$ *is nonempty and is precisely the set of* $g \in \mathcal{T}_{f,\Lambda}(1,1)$ *such that* (g, Λ_g, ν) *is a* C^{1+} *Hausdorff realisation. In this case* $\mu = (i_g)_* \nu$ *is absolutely continuous with respect to Lesbegue measure.*
(ii) *The stable measure solenoid function* $\sigma_{\nu,s} : \mathrm{msol}^s \to \mathbb{R}^+$ *has a nonvanishing Hölder continuous extension to the closure of* msol^s *satisfying the boundary condition.*
(iii) *The unstable measure solenoid function* $\sigma_{\nu,u} : \mathrm{msol}^u \to \mathbb{R}^+$ *has a nonvanishing Hölder continuous extension to the closure of* msol^s *satisfying the boundary condition.*

Theorem 5.5 follows from Theorem 1.4 in [42].

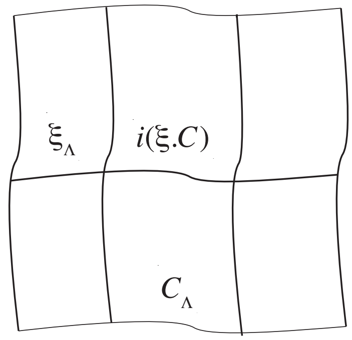

Figure 10. An ι-admissible pair (ξ, C) where $\xi_\Lambda = i(\pi_{\iota'}^{-1}\xi)$ and $C_\Lambda = i(\pi_\iota^{-1}C)$.

5.4. Extended measure scaling functions. To present a classification of Gibbs measures C^{1+}-Hausdorff realisable by codimension one attractors, we have to define the cylinder-cylinder condition. We will express the cylinder-cylinder condition, in Section 5.5, using the extended measure scaling functions which are also useful to present, in Section 5.6, the δ_ι-bounded solenoid equivalence class of a Gibbs measure.

Throughout the paper, if $\xi \in \Theta^{\iota'}$, we denote by ξ_Λ the leaf primary cylinder segment $i(\pi_{\iota'}^{-1}\xi) \subset \Lambda$. Similarly, if C is an n_ι-cylinder of Θ^ι then we denote by C_Λ the $(1, n_\iota)$-rectangle $i(\pi_\iota^{-1}C) \subset \Lambda$.

Given $\xi \in \Theta^{\iota'}$ and an n-cylinder C of Θ^ι, we say that the pair (ξ, C) is *ι-admissible* if the set

$$\xi.C = \pi_\iota^{-1}C \cap \pi_{\iota'}^{-1}\xi$$

is nonempty (see Figure 10). The set of all ι-admissible pairs (ξ, C) is the *ι-measure scaling set* msc^ι. We construct the *extended ι-measure scaling function* $\rho : \mathrm{msc}^\iota \to \mathbb{R}^+$ as follows: If C is a 1-cylinder then we define $\rho_\xi(C) = 1$. If C is an n-cylinder ($\Theta^u_{w_0 \ldots w_{(n-1)}}$ or $\Theta^s_{w_{-(n-1)} \ldots w_0}$), with $n \geq 2$, then we define

$$\rho_\xi(C) = \prod_{j=1}^{n-1} s_{\nu,\iota}(\pi_{\iota'}\tau_\iota^{n-j}(\xi.m_\iota^{j-1}C))$$

(see Figure 11), where (a) $s_{\nu,\iota}$ is the ι-measure scaling function of the Gibbs measure ν and (b) $m_u^{j-1}\Theta^u_{w_0 \ldots w_{(n-1)}} = \Theta^u_{w_0 \ldots w_{(n-j)}}$ and $m_s^{j-1}\Theta^s_{w_{-(n-1)} \ldots w_0} = \Theta^s_{w_{-(n-j)} \ldots w_0}$ (see Section 5.1).

Recall that a τ-invariant measure ν on Θ determines a unique τ_u-invariant measure $\nu_u = (\pi_u)_*\nu$ on Θ^u and a unique τ_s-invariant measure $\nu_s = (\pi_s)_*\nu$ on Θ^s. The following result follows from Theorem 1 in [38].

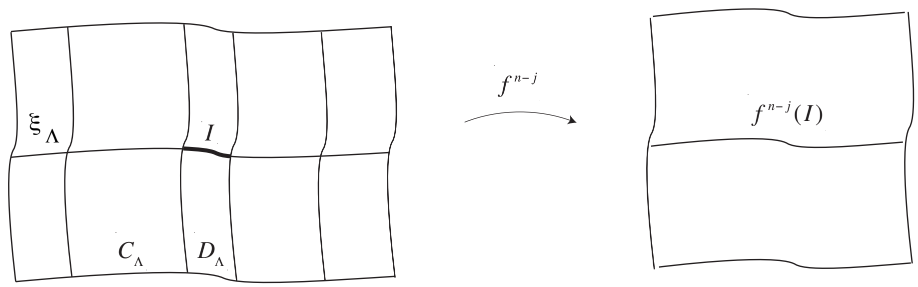

Figure 11. The $(n-j+1)$-cylinder leaf segment $I = \xi_\Lambda \cap D_\Lambda$ and the primary leaf segment $f^{n-j}(I) = i(\pi_{\iota'}\tau_\iota^{n-j}(\xi.D))$, where $D = m_\iota^{j-1}C$.

THEOREM 5.6 (RATIO DECOMPOSITION OF A GIBBS MEASURE). *Let $\rho : \mathrm{msc}_\iota \to \mathbb{R}^+$ be an extended ι-measure scaling function and ν the corresponding τ-invariant Gibbs measure. If C is an n-cylinder of Θ^ι then*

$$\nu_\iota(C) = \int_{\xi \in M} \rho_\xi(C)\nu_{\iota'}(d\xi),$$

where $M = \pi_{\iota'} \circ \pi_\iota^{-1}C$ is a 1-cylinder of $\Theta^{\iota'}$. The ratios $\nu_\iota(C)/\rho_\xi(C)$ are uniformly bounded away from 0 and ∞.

We note that $\rho_\xi(C)$ is the measure of $\xi.C$ with respect to the probability conditional measure of ν on ξ.

5.5. Hausdorff realisations of Gibbs measures for Codimension one attractors. We introduce the cylinder-cylinder condition that we will use to classify all Gibbs measures that are C^{1+}-Hausdorff realizable by codimension one attractors.

Similarly to the cylinder-gap condition given in Section 3.8 for a solenoid function, we are going to construct the cylinder-cylinder condition for a given measure solenoid function $\sigma_{\nu,\iota}$. Let $\delta_\iota < 1$ and $\delta_{\iota'} = 1$. Let $(I, J) \in \mathrm{Msol}^\iota$ be such that the ι-leaf segment $f_{\iota'}I \cup f_{\iota'}J$ is contained in an ι-boundary K of a Markov rectangle R_1. Then $f_{\iota'}I \cup f_{\iota'}J$ intersects another Markov rectangle R_2. Take the smallest $k \geq 0$ such that $f_{\iota'}^k I \cup f_{\iota'}^k J$ is contained in the intersection of the boundaries of two Markov rectangles M_1 and M_2. Let M_1 be the Markov rectangle with the property that $M_1 \cap f_{\iota'}^k R_1$ is a rectangle with nonempty interior, and so $M_2 \cap f_{\iota'}^k R_2$ has also nonempty interior. Then, for some positive n, there are distinct ι-leaf n-cylinders $J_1, \ldots, J_m$ contained in a primary cylinder L of M_2 such that $f_{\iota'}^k I = \bigcup_{i=1}^{p-1} J_i$ and $f_{\iota'}^k J = \bigcup_{i=p}^{m} J_i$. Let $\eta \in \Theta^{\iota'}$ be such that $\eta_\Lambda = L$ and, for every $i = 1, \ldots, m$, let D_i be a cylinder of Θ^ι such that $i(\eta.D_i) = J_i$. Let $\xi \in \Theta^{\iota'}$ be such that $\xi_\Lambda = K$ and C_1 and C_2 cylinders of Θ^ι such that $i(\xi.C_1) = f_{\iota'}I$ and $i(\xi.C_2) = f_{\iota'}J$. We say that the measure solenoid function $\sigma_{\nu,\iota}$ of the Gibbs measure ν satisfies the *cylinder-cylinder condition*

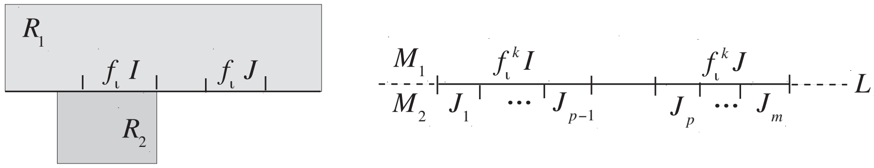

Figure 12. The cylinder-cylinder condition for ι-leaf segments.

(see Figure 12) if, for all such leaf segments,

$$\frac{\rho_\xi(C_2)}{\rho_\xi(C_1)} = \frac{\sum_{i=p}^{m} \rho_\eta(D_i)}{\sum_{i=1}^{p-1} \rho_\eta(D_i)}$$

where ρ is the measure scaling function determined by ν.

REMARK 5.7. A function $\sigma : \mathrm{msol}^\iota \to \mathbb{R}^+$ that has an Hölder continuous extension to Msol^ι determines an extended scaling function, and so we can check if the function σ satisfies or not the cylinder-cylinder condition.

THEOREM 5.8 (CODIMENSION ONE ATTRACTORS). *Suppose that f is a C^{1+} surface diffeomorphism and Λ is a codimension one hyperbolic attractor. Fix a Gibbs measure ν on Θ. Then the following statements are equivalent:*

(i) *For all $0 < \delta_s < 1$, $[\nu] \subset \mathcal{T}_{f,\Lambda}(\delta_s, 1)$ is nonempty and is precisely the set of $g \in \mathcal{T}_{f,\Lambda}(\delta_s, 1)$ such that (g, Λ_g, ν) is a C^{1+} Hausdorff realisation. In this case $\mu = (i_g)_*\nu$ is absolutely continuous with respect to the Hausdorff measure on Λ_g.*
(ii) *The stable measure solenoid function $\sigma_{\nu,s} : \mathrm{msol}^s \to \mathbb{R}^+$ has a nonvanishing Hölder continuous extension to the closure of* msol^s *satisfying the cylinder-cylinder condition.*
(iii) *The unstable measure solenoid function $\sigma_{\nu,u} : \mathrm{msol}^u \to \mathbb{R}^+$ has a nonvanishing Hölder continuous extension to the closure of* msol^u.

Theorem 5.8 follows from Theorem 1.5 in [42].

5.6. The moduli space for hyperbolic realizations of Gibbs measures. Let $\mathcal{SOL}^\iota$ be the space of all Hölder continuous functions $\sigma_\iota : \mathrm{Msol}^\iota \to \mathbb{R}^+$ with the following properties:

(i) If $HD^\iota = 1$ then σ_ι is an ι-solenoid function.
(ii) If $HD^\iota < 1$ and $HD^{\iota'} = 1$ then σ_ι satisfies the cylinder-cylinder condition.
(iii) If $HD^\iota < 1$ and $HD^{\iota'} < 1$ then σ_ι does not have to satisfy any extra property.

We recall that HD^ι is the Hausdorff dimension of the ι-leaf segments intersected with Λ.

By Theorems 5.2, 5.5 and 5.8, for ι equal to s and u, we obtain that the map $\nu \to \sigma_{\nu,\iota}$ gives a one-to-one correspondence between the sets $[\nu]$ contained in

$\mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$ and the space of measure solenoid functions $\sigma_{g,\iota}$ whose continuous extension is contained in $\mathcal{SOL}^\iota$. Hence, the set $\mathcal{SOL}^\iota$ is a moduli space parameterizing all Lipschitz conjugacy classes $[\nu]$ of C^{1+} hyperbolic diffeomorphisms contained in $\mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$ (see Corollary 1.11).

DEFINITION 5.3. The *δ_ι-bounded solenoid equivalence class* of a Gibbs measure ν is the set of all solenoid functions σ_ι with the following properties: There is $K = K(\sigma_\iota) > 0$ such that for every pair $(\xi, C) \in \mathrm{msc}_\iota$

$$\left|\delta_\iota \log s_\iota(C_\Lambda \cap \xi_\Lambda : \xi_\Lambda) - \log \rho_\xi(C)\right| < K ,$$

where (i) ρ is the ι-measure scaling function of ν and (ii) s_ι is the scaling function determined by σ_ι.

Let $\sigma_{1,\iota}$ and $\sigma_{2,\iota}$ be two solenoid functions in the same δ_ι-bounded equivalence class of a Gibbs measure ν. Using that $\sigma_{1,\iota}$ and $\sigma_{2,\iota}$ are bounded away from zero, we obtain that the corresponding scaling functions also satisfy inequality (3–6) for all pairs $(J, m^i J)$ where J is an ι-leaf $(i+1)$-gap. Hence, the solenoid functions $\sigma_{1,\iota}$ and $\sigma_{2,\iota}$ are in the same bounded equivalence class (see Definition 3.2).

THEOREM 5.9. (i) *There is a natural map $g \to (\sigma_s(g), \sigma_u(g))$ which gives a one-to-one correspondence between C^{1+} conjugacy classes of C^{1+} hyperbolic diffeomorphisms $g \in [\nu]$ and pairs $(\sigma_s(g), \sigma_u(g))$ of stable and unstable solenoid functions such that, for ι equal to s and u, $\sigma_\iota(g)$ is contained in the δ_ι-bounded solenoid equivalence class of ν.*

(ii) *Given an ι-solenoid function σ_ι and $0 < \delta_{\iota'} \le 1$, there is a unique Gibbs measure ν and a unique $\delta_{\iota'}$-bounded equivalence class of ν consisting of ι'-solenoid functions $\sigma_{\iota'}$ such that the C^{1+} conjugacy class of hyperbolic diffeomorphisms $g \in \mathcal{T}_{f,\Lambda}(\delta_s, \delta_u)$ determined by the pair (σ_s, σ_u) have an invariant measure $\mu = (i_g)_* \nu$ absolutely continuous with respect to the Hausdorff measure.*

Theorem 5.9 follows from combining Lemmas 8.17 and 8.18 in [42].

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