Recent Progress in Dynamics
MSRI Publications
Volume **54**, 2007

An aperiodic tiling using a dynamical system and Beatty sequences

STANLEY EIGEN, JORGE NAVARRO, AND VIDHU S. PRASAD

ABSTRACT. Wang tiles are square unit tiles with colored edges. A finite set of Wang tiles is a valid tile set if the collection tiles the plane (using an unlimited number of copies of each tile), the only requirements being that adjacent tiles must have common edges with matching colors and each tile can be put in place only by translation. In 1995 Kari and Culik gave examples of tile sets with 14 and 13 Wang tiles respectively, which only tiled the plane aperiodically. Their tile sets were constructed using a piecewise multiplicative function of an interval. The fact the sets tile only aperiodically is derived from properties of the function.

1. Introduction

There is a vast literature connecting dynamical systems and tilings of the plane. In this paper, we give an exposition of the work of Kari [7] and Culik [3] to show how by starting with a piecewise multiplicative function f, with rational multiplicands defined on a finite interval, we can produce a finite set of Wang tiles which tiles the plane. Further, a choice of multiplicands and interval, so that the dynamical system f has no periodic points, results in a set of Wang tiles that can only tile the plane aperiodically. In this manner, Kari and Culik produce a set of 13 Wang tiles. This is currently, the smallest known set of Wang tiles which only tiles the plane aperiodically.

The Kari–Culik construction is different from earlier constructions of aperiodic tilings — see Grunbaum and Shephard's book [5, Chapt 11] for a survey of these earlier results. Johnson and Madden [6], provide an accessible presentation

Mathematics Subject Classification: Primary 52C20, 05B45, 52C23; Secondary 37E05.

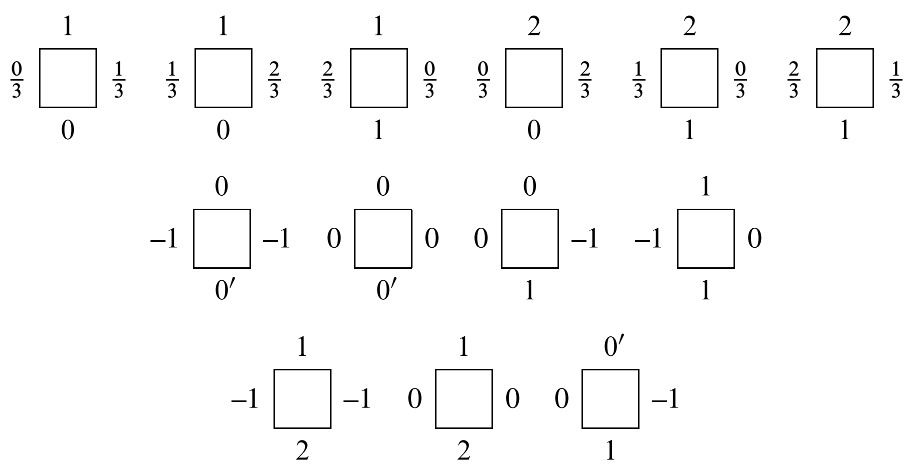
Keywords: Wang tiles, aperiodic tiling of the plane, piecewise multiplicative, dynamical system, Beatty difference sequence.

of Robinson's 1971 [11] example of 6 polygonal tiles which force aperiodicity (allowing rotation and reflection). These 6 tiles convert to a set of 56 Wang tiles which allow only aperiodic tilings of the plane. Kari and Culik's construction uses a dynamical system and Beatty sequences to label the sides of the Wang tiles. The properties of the dynamical system are used to conclude the collection tiles the plane and does so only aperiodically.

1.1. The Kari–Culik tile set. Consider the dynamical system given by the function f defined on the interval $[\frac{1}{3}, 2)$,

$$f(x) = \begin{cases} 2x, & \frac{1}{3} \le x < 1 \\ \frac{1}{3}x, & 1 \le x < 2\,. \end{cases}$$

This gives rise (Section 5 shows how) to a set $\mathcal{T}$ of thirteen Wang tiles, which we call the K-C tile set (see figure below). These thirteen tiles do tile the plane, but only aperiodically.



K-C Tile Set.

The proof that the tile set tiles the plane will follow from the existence of infinite orbits for f. The proof that the tile set tiles only aperiodically relies on the fact that f has no periodic points on the interval $[\frac{1}{3}, 2)$. We note that Kari and Culik [3; 7] use Mealy machines describe these tile sets. We give their description at the end of this paper. In the language of computer science a Mealy machine is a finite state machine where the output is associated with a transition; in symbolic dynamics a Mealy machine is referred to as a finite-state code [8].

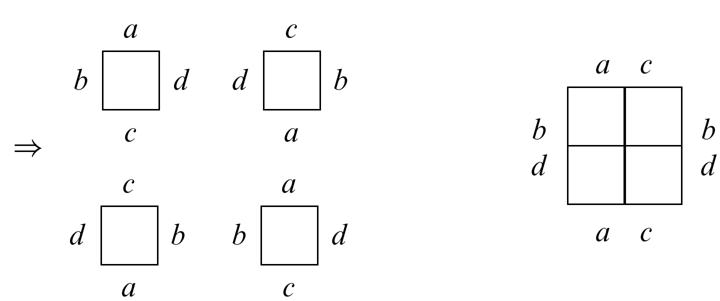
2. Wang tiles: definitions and history

Wang tiles are square unit tiles with colored edges. All tiles in this paper are assumed to be Wang tiles. In Kari and Culik's tile set, numbers are used to color the edges: edges will have a color and a numerical value. Thus, the colored edges 0, $0'$ and $\frac{0}{3}$ are considered different colors, but these edges have a numerical value, which in this case is zero.

A *tiling set* $\mathcal{T}$ consists of a collection of finitely many Wang tiles $T \in \mathcal{T}$, each of which may be copied as much as needed. When used to tile the plane, the tiles must be placed edge-to-edge with common edges having matching colors. Rotations and flips (reflections) of the tiles are not permitted.

A tiling set $\mathcal{T}$ which can tile the plane is said to have a *valid tiling*, and $\mathcal{T}$ is called a *valid tile set*. A valid tiling is a map on the integer lattice, $\tau : \mathbb{Z} \times \mathbb{Z} \to \mathcal{T}$ such that, at each lattice point (i, j) we have a tile $\tau(i, j) = T_{i,j} \in \mathcal{T}$ whose neighboring tiles have matching colors along common edges.

If rotations were permitted, then any tile and its 180 degree rotation forms a valid tile set for the plane, as the following argument shows. Label the four colors of a tile a, b, c, d, (not necessarily distinct). Take two copies of the tile and two copies of its rotation through 180 degrees and construct the following two-by-two block.



Rotation Example.

The two-by-two block has the same colors on the top as the bottom, and the same colors on the left as the right. The two-by-two block tiles the plane.

2.1. Periodicity. A valid tiling is *periodic* with period $(h, v) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ if the tile at position (i, j) is the same as the tile at position $(i + h, j + v)$ for all $(i, j) \in \mathbb{Z}^2$. That is, $\tau(i, j) = T_{i,j} = T_{i+h,j+v} = \tau(i + h, j + v)$.

Needless to say a tile set may have more than one valid tiling; some of which may be periodic and some of which may not. A tile set is called *aperiodic* if it has at least one valid tiling, but does not have a valid tiling which is periodic. The K-C tile set is aperiodic (Theorem 10).

Hao Wang [14] conjectured in 1961 that if a set of tiles has a valid tiling τ then it has a valid tiling τ' which is periodic. However, in 1966 R. Berger showed that there exists a tile set which only tiles aperiodically, and this aperiodic tile set contained 20,426 tiles. Since that time, the size of the smallest known set of aperiodic Wang tiles has been reduced considerably. By 1995, J. Kari [7] and K. Culik [3] constructed a set of 14 and 13 Wang tiles respectively that tiles only aperiodically.

An open problem is to determine W such that any set of Wang tiles $\mathcal{T}$ of size $w \leq W$ which has a valid tiling must also have a periodic tiling. As far as the authors are aware, $4 \leq W < 13$.

2.2. One-dimensional result. In one dimension, Wang's conjecture that any valid tile set for the line must have a periodic tiling, is true. In one dimension the tiles are unit intervals colored on the left and right. A valid tiling is a map $\tau : \mathbb{Z} \to \mathcal{T}$ with adjoining left right edges having the same color. Periodicity of $\mathcal{T}$, in this case, means there exists a $p > 0$ so that $\tau(i) = \tau(i + p)$ for all i.

THEOREM 1 (WANG). *If a set of one-dimensional tiles $\mathcal{T}$ has a valid tiling of the line, then $\mathcal{T}$ has a periodic tiling of the line.*

Let τ be a valid tiling for $\mathcal{T}$, $\tau : \mathbb{Z} \to \mathcal{T}$. Since there are only a finite number of tiles in $\mathcal{T}$, there must be an $n > 0$ such that $\tau(0) = \tau(n)$. Hence the block of tiles $\tau(0)\tau(1)\cdots\tau(n-1)$, endlessly repeated, tiles the line.

A slight strengthening of the hypotheses yields one-dimensional tiling sets that tile only periodically — this shows how different the two-dimensional aperiodic tiling sets are.

PROPOSITION 2. *If $\mathcal{T}$ is valid tile set of one-dimensional tiles and no proper subset of $\mathcal{T}$ is a valid tile set of the line, then the tiles can tile only periodically.*

The proof follows the previous argument. Let m be the shortest length from any tile to its first repetition in a valid tiling τ of the line. Clearly the pigeonhole principle implies $m \leq |\mathcal{T}| + 1$, where $|\mathcal{T}|$ is the cardinality of $\mathcal{T}$. The hypothesis that no proper subset is a valid tile set implies that $m = |\mathcal{T}| + 1$. Let $\tau(1)\tau(2)\cdots\tau(m) = \tau(1)$, be a shortest repeated block. Note that the right colors of all of these tiles in the block must be distinct. Indeed, suppose that two tiles $\tau(i)$ and $\tau(j)$ were the same, so that $\tau(1)\cdots\tau(i)\cdots\tau(j)\tau(k)\cdots\tau(1)$ could be replaced by $\tau(1)\cdots\tau(i)\tau(k)\cdots\tau(1)$, where the tile $\tau(j)$ does not appear. But then $\tau(j)$ is not needed for a valid tiling. Hence all right hand colors are distinct, and similarly we can show all left colors are distinct. Hence, there is exactly one way for these tiles to fit together, and that is with the block $\tau(1)\tau(2)\cdots\tau(m-1)$ endlessly repeated.

A *minimal* tiling set is one that is a valid tile set but no proper subset is a valid tile set. It is an open question whether the K-C tile set is minimal.

2.3. Rectangular tilings. In the Rotation Example given in Section 1.1, the constructed two-by-two block extends to a valid tiling of the plane which has the two linearly independent periods $(2, 0)$ and $(0, 2)$. A *rectangular* tiling of the plane is a valid tiling τ which has two periods $(n, 0)$, $(0, m)$, $n, m > 0$, that is, $\tau(i, j) = \tau(i + n, j)$ and $\tau(i, j) = \tau(i, j + m)$. In other words, it has a rectangular block of size $n \times m$ which tiles the plane.

It is well known that having a rectangular tiling is not stronger than having a periodic tiling [5].

PROPOSITION 3. *If a set of tiles admits a periodic tiling τ of the plane, then it also admits a rectangular tiling.*

We propose the following higher dimensional result (which may be already be known): *If a set of n-dimensional Wang cubes has a valid tiling τ of n-dimensional space and this tiling has $n-1$ linearly independent periods, then*

(i) *there is another tiling with n linearly independent periods, and*
(ii) *there is another tiling which is rectangular, in the sense that there are n periods, $(p_1, 0, \ldots, 0)$, $(0, p_2, 0, \ldots, 0), \ldots, (0, \ldots, 0, p_n)$.*

3. Aperiodicity

The aperiodicity of the K-C tile set is easy to see and does not require understanding how the tiles are derived from the dynamical system. It follows the same reasoning as the following proof that f has no periodic points.

LEMMA 4. *The dynamical system f has no periodic points.*

Suppose $f^n(x) = x$ for $n > 0$. From the definition of f as a piecewise multiplicative function, it follows that $f^n(x) = q_n \cdot q_{n-1} \cdots q_1 \cdot x$ where $q_i \in \{\frac{1}{3}, 2\}$. Hence

$$f^n(x) = \frac{2^{n-k}}{3^k} \cdot x = x$$

for some $0 \le k \le n$. Dividing by $x \in [\frac{1}{3}, 2)$ gives $2^{n-k}/3^k = 1$, a contradiction.

To understand how this applies to the tiles, we consider the notion of a multiplier tile.

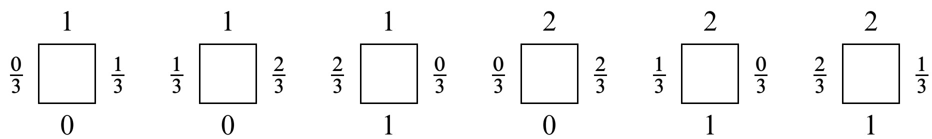
3.1. Multiplier tiles. A tile $\begin{array}{ccc} & a & \\ b & \square & d \\ & c & \end{array}$ is a *multiplier tile* with *multiplier* $q > 0$ if

$$q \cdot a + b - d = c \tag{3–1}$$

Note that this notion requires only the numerical value of the edges. The multiplier for a tile is unique if $a \neq 0$. If $a = 0$ then every real q is a multiplier for the tile when $b - d = c$.

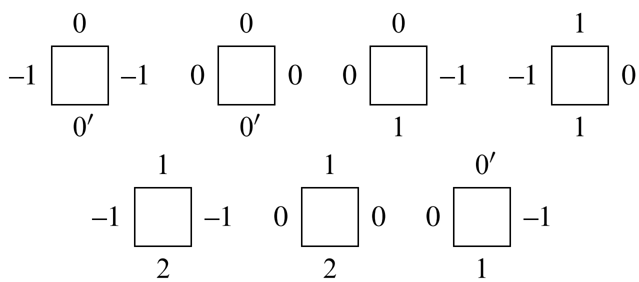
A direct examination of the thirteen tiles in the K-C tile set reveals two facts:

LEMMA 5. *The first six tiles all have multiplier* $\frac{1}{3}$. *We call these Tile Set* $\frac{1}{3}$.



Tile Set $\frac{1}{3}$.

LEMMA 6. *The last seven tiles all have multiplier* 2. *We call these Tile Set* $2'$.



Tile Set $2'$.

Observe that the six tiles in Tile Set $\frac{1}{3}$ have side colors $\{\frac{0}{3}, \frac{1}{3}, \frac{2}{3}\}$ while the seven tiles in Tile Set $2'$ have side colors $\{0, -1\}$. Since these two sets of side colors are disjoint the next lemma is immediate (and is the reason why the two zeros $\{\frac{0}{3}, 0\}$ are defined to be different colors).

LEMMA 7. *If τ is a valid tiling for the tiles in the K-C tile set, then each horizontal row $\{\tau(i, j) : i \in \mathbb{Z}\}$, for j fixed, consists either exclusively of the tiles in Tile Set $\frac{1}{3}$ or exclusively of the tiles in Tile Set $2'$.*

Next, consider the row directly below a given row in a valid tiling. This requires the bottom colors of the higher row to match exactly the colors on the top of the lower row. There are restrictions on the tiles that can appear in the lower row.

LEMMA 8. *Let τ be a valid tiling for the tiles in the K-C tile set. If a horizontal row consists exclusively of tiles from Tile Set $\frac{1}{3}$ then the row immediately below it consists exclusively of tiles from Tile Set $2'$.*

The proof is simply a matter of inspecting the colors on the tiles. Suppose there are two consecutive rows of tiles from Tile Set $\frac{1}{3}$. We examine the colors along the common edge between the two rows. Since the colors along the top of tiles from Tile Set $\frac{1}{3}$ are $\{1, 2\}$ and the colors on the bottom of these tiles are $\{0, 1\}$, the only way the colors along the common edge can match is if they are all 1. However the tiles in Tile Set $\frac{1}{3}$ cannot produce a complete row with all 1's along the bottom, and so there cannot be two consecutive rows of tiles from Tile Set $\frac{1}{3}$. This lemma is related to the dynamics of f in the following manner: if $f(x) = y = \frac{1}{3}x$, then $f(y) = 2y$.

LEMMA 9. *Let τ be a valid tiling for the tiles in the K-C tile set. Then there must exist rows with tiles exclusively from Tile Set $\frac{1}{3}$.*

Lemma 9 is related to the dynamics of f in the following way: given x, $f(x)$, $f^2(x)$, at least one of these three terms must be in the interval $[1, 2)$. Any point in $[1, 2)$ will be mapped by multiplying by $1/3$. This can be used to prove the Lemma. However, we prove the lemma by directly analyzing the tiles.

Assume there are three consecutive rows of tiles from Tile Set $2'$. First consider the common edge between the highest row and the middle row. In particular, observe that the colors along the top of Tile Set $2'$ are $\{0, 0', 1\}$ while the numbers along the bottom of Tile Set $2'$ are $\{0', 1, 2\}$. To match, the common colors must be $\{0', 1\}$. The same argument shows that the colors along the common edge between the middle and lowest row must also be $\{0', 1\}$. This forces the middle row to be restricted to the two tiles

$$\begin{array}{ccccccc} & 0' & & & & 1 & \\ 0 & \boxed{} & -1 & \quad & -1 & \boxed{} & 0 \\ & 1 & & & & 1 & \end{array}$$

from Tile Set $2'$, which means the middle row has only 1 as a bottom color and the pattern $(0', 1)$ repeated as the top colors.

The only way the third row can have a top row of all 1's is if it uses one of the two tiles

$$\begin{array}{ccccccc} & 1 & & & & 1 & \\ -1 & \boxed{} & -1 & \quad & 0 & \boxed{} & 0 \\ & 2 & & & & 2 & \end{array}$$

This forces the fourth row to be restricted to tiles in Tile Set $\frac{1}{3}$.

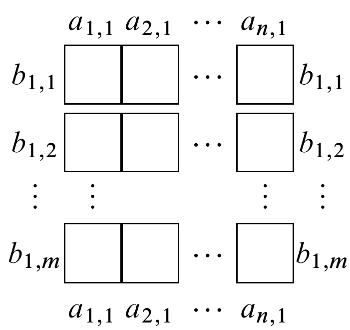
We are now able to show:

THEOREM 10. *The K-C tile set does not have a valid periodic tiling of the plane.*

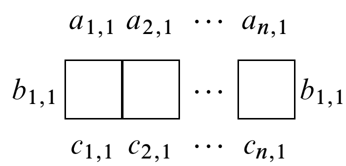
The proof is by contradiction and follows the reasoning that shows the function f has no periodic points (Lemma 4).

Let τ be a periodic tiling. From Proposition 3 we can assume that τ has two periods $(n, 0)$ and $(0, m)$ with $n, m > 0$, and there is an $n \times m$ block with the same colors on both the top and bottom and the same colors on the left and right. For convenience we refer to this block as B.

Denote the top colors of Block B by $a_{i,1}$, $1 \le i \le n$ and the colors along the left side by $b_{1,j}$, $1 \le j \le m$. By the periodicity assumption, the colors along the bottom are also $\{a_{i,1}\}$ and the colors along the right side are $\{b_{1,j}\}$.

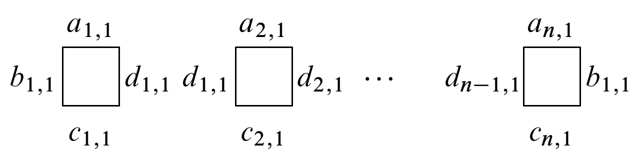


Block B.



First row of Block B.

Consider the first row of Block B. Each edge common to two tiles has the same color for the left tile and the right tile.



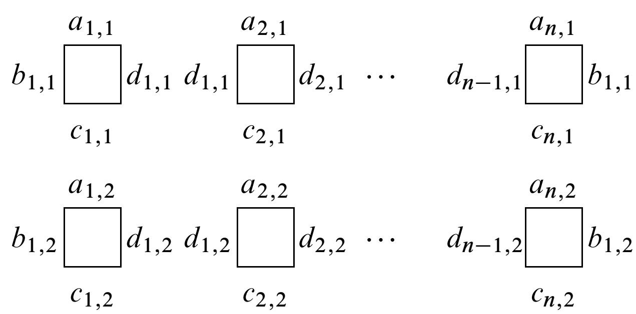
First Row of Block B Expanded.

From Lemma 7, all the tiles in a row have the same multiplier q_1. Apply the multiplier rule (3–1) to each tile in the row.

$$
\begin{aligned}
q_1 a_{1,1} + b_{1,1} - d_{1,1} &= c_{1,1} \\
q_1 a_{2,1} + d_{1,1} - d_{2,1} &= c_{2,1} \\
q_1 a_{3,1} + d_{2,1} - d_{3,1} &= c_{3,1} \\
&\vdots \\
q_1 a_{n,1} + d_{n-1,1} - b_{1,1} &= c_{n,1}
\end{aligned}
$$

Summing results in

$$q_1 \sum_{i=1}^{n} a_{i,1} = \sum_{i=1}^{n} c_{i,1}.$$



First Two Rows of Block B Expanded.

Similarly, all the tiles in the second row of Block B have a common multiplier q_2 giving

$$q_2 \cdot \sum_{i}^{n} a_{i,2} = \sum_{i=1}^{n} c_{i,2}.$$

Combining these two equations and using $c_{i,1} = a_{i,2}$ yields

$$q_2 q_1 \sum_{i=1}^{n} a_{i,1} = \sum_{i=1}^{n} c_{i,2}.$$

Repeating for the rest of the rows in Block B results in

$$q_m \cdots q_2 q_1 \sum_{i=1}^{n} a_{i,1} = \sum_{i=1}^{n} a_{i,1}.$$

By Lemma 9 and the periodicity of the tiling, we can assume the very top row of the block B consists of tiles exclusively from Tile Set $\frac{1}{3}$. Since the top colors of the tiles in Tile Set $\frac{1}{3}$ are $\{1, 2\}$, we can divide by $\sum_{i=1}^{n} a_{i,1}$ getting $\prod_{j=1}^{m} q_j = 1$. As the $q_j \in \{2, \frac{1}{3}\}$ we have a contradiction and conclude that no periodicity can occur.

4. Existence of a valid tiling

In this section we show how to construct the tile set $\mathcal{T}$, and prove that the tile set thus constructed has valid tilings.

The K-C Tile Set is derived from the Basic Tile Construction (given in the next section) resulting in a tile set $\mathcal{T}_f$. The tile colors in $\mathcal{T}_f$ are “tweaked”, to give the K-C tile set. This refers to the fact that the zeros $0, 0', \frac{0}{3}$ are considered different colors. We have already seen the reason $\frac{0}{3}$ is not the same color as 0, namely Lemma 7, which ensures that each row of tiles consists of tiles with the same multiplier. The second "tweaking" concerns $0'$ and will be explained in Section 5.2. We will see the property that $\mathcal{T}_f$ is a valid tile set, is preserved even after the colors are "tweaked".

4.1. The basic tile. All the tiles in the example are constructed as follows. We refer to this as the *Basic Tile Construction*, and it gives the values of the edges of a Basic Tile which we call $B(x, q, n)$.

$$\begin{array}{ccc} & \lfloor nx \rfloor - \lfloor (n-1)x \rfloor & \\ q\lfloor (n-1)x \rfloor - \lfloor (n-1)qx \rfloor & \boxed{} & q\lfloor nx \rfloor - \lfloor nqx \rfloor \\ & \lfloor nqx \rfloor - \lfloor (n-1)qx \rfloor & \end{array}$$

Basic Tile $B(x, q, n)$.

Here, $x > 0$ is a real number, $q > 0$ is a rational, n is an integer and $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

A straightforward calculation gives:

LEMMA 11. *The Basic Tile $B(x, q, n)$, is a multiplier tile with multiplier q.*

Recall a tile $\begin{array}{ccc} & a & \\ b & \square & d \\ & c & \end{array}$ has multiplier q if $q \cdot a + b - d = c$. For the Basic Tile we have

$$\begin{aligned} q\left(\lfloor nx \rfloor - \lfloor (n-1)x \rfloor\right) + \left(q\lfloor (n-1)x \rfloor - \lfloor (n-1)qx \rfloor\right) - \left(q\lfloor nx \rfloor - \lfloor nqx \rfloor\right) \\ = \lfloor nqx \rfloor - \lfloor (n-1)qx \rfloor. \end{aligned}$$

4.2. A finite number of tiles. Clearly when x, q, n are fixed, one gets a single tile. Surprisingly for q rational and x in a bounded interval one gets only a finite number of tiles.

For example, if we set $q = \frac{1}{3}$ and bound $x \in [1, 2)$ then there are only six tiles resulting from the above Basic Tile construction (see Tile Set $\frac{1}{3}$)—despite the fact that x is ranging over all reals in the interval $[1, 2)$ and n is ranging over all integers.

THEOREM 12. *Let q be a rational number and $k > 0$ an integer. If we restrict $x \in [k, k+1)$ then there are only a finite set of tiles derived from the Basic Tile construction*

To prove this, we simply show that the four sides of the Basic Tile can assume only a finite number of values. We use this simple fact:

LEMMA 13. *For all $n \in \mathbb{Z}$ and for all $x \in [k, k+1)$,*

$$\lfloor nx \rfloor - \lfloor (n-1)x \rfloor \in \{k, k+1\}$$

Lemma 13 applies to both the bottom and top of the Basic Tile. The bottom uses the real qx which is bounded by $[qk, q(k+1))$. For example with $q = \frac{1}{3}$ and $x \in [1, 2) \Rightarrow qx \in [\frac{1}{3}, \frac{2}{3}) \subset [0, 1)$. Hence the top of the tiles take values in $\{1, 2\}$ while the bottom of the tiles have values in $\{0, 1\}$.

LEMMA 14. *For $q > 0$ rational, $q\lfloor nx \rfloor - \lfloor nqx \rfloor$ takes on only a finite number of values. To be more precise,*

- *if q is an integer then $q\lfloor nx \rfloor - \lfloor nqx \rfloor \in \{1-q, 2-q, \ldots, 0\}$;*
- *if $q = \frac{r}{s}$, in reduced form, then $q\lfloor nx \rfloor - \lfloor nqx \rfloor \in \{-\frac{1-r}{s}, -\frac{2-r}{s}, \ldots, \frac{s-1}{s}\}$.*

First observe that if q is an integer then clearly $q\lfloor nx \rfloor - \lfloor nqx \rfloor$ is an integer and if $q = \frac{r}{s}$ is rational then $q\lfloor nx \rfloor - \lfloor nqx \rfloor$ is limited to rational numbers of the form $\frac{i}{s}$.

It remains to show that $q\lfloor nx \rfloor - \lfloor nqx \rfloor$ is bounded above and below. From the definition of the greatest integer function $\lfloor \cdot \rfloor$,

$$q\lfloor nx \rfloor \leq qnx < \lfloor qnx \rfloor + 1$$

Subtracting $\lfloor qnx \rfloor$ gives the upper bound

$$q\lfloor nx \rfloor - \lfloor qnx \rfloor < 1$$

Again, from the definition of $\lfloor \cdot \rfloor$,

$$\lfloor qnx \rfloor \leq qnx = q(nx) < q(\lfloor nx \rfloor + 1).$$

Multiplying by -1 and adding $q\lfloor nx \rfloor$ gives the lower bound

$$q\lfloor nx \rfloor - \lfloor qnx \rfloor > -q$$

These bounds clearly place the value of $q\lfloor nx \rfloor - \lfloor qnx \rfloor$ in the ranges listed in the lemma.

4.3. Applying the basic tile construction using f. For x in the domain of f, set

$$q(x) = \begin{cases} 2, & \frac{1}{3} \leq x < 1 \\ \frac{1}{3}, & 1 \leq x < 2 \end{cases}$$

Denote by $\mathcal{T}_f = \{B(x, q(x), n)\}$ the set of tiles constructed for $\{x, q(x), n\}$ with x in the domain of f. Note that this is not yet the K-C tile set $\mathcal{T}$ because there has been no color tweaking yet, *i.e.*, there is only one 0 at this stage.

By Lemma 12, this is a finite set of tiles.

It can be seen quite easily that the tiles for a specific $\{x, q(x)\}$ fit together, with a "natural order", to form a row denoted by $\mathcal{R}(x)$.

LEMMA 15. *Fix x in the domain of f and let n range through the integers to produce a row of valid tiles $\mathcal{R}(x)$.*

$\dots$	$B(x,q(x),n-1)$	$B(x,q(x),n)$	$B(x,q(x),n+1)$	$\dots$

By "natural order" we mean that the tile constructed using $n+1$ is to the immediate right of the tile constructed using n. The tile constructed for n, $B(x, q(x), n)$, has the right side color $\big(q(x) \cdot \lfloor nx \rfloor - \lfloor n \cdot q(x) \cdot x \rfloor\big)$ which is the same as the left side color of the tile constructed for $n+1$, $B(x, q(x), n+1)$, $\big(q(x) \cdot \lfloor (n+1-1)x \rfloor - \lfloor (n+1-1) \cdot q(x) \cdot x \rfloor\big)$.

4.4. Beatty difference sequences. To complete the proof of the existence of valid tilings we use the notion of a Beatty difference sequence. For any real number x, the *Beatty difference sequence* of x is the two-sided sequence $\{\lfloor nx \rfloor - \lfloor (n-1)x \rfloor : n \in \mathbb{Z}\}$. Recalling Lemma 13, if $x \in [k, k+1)$ then the Beatty difference sequence for x belongs to $\prod_{-\infty}^{\infty} \{k, k+1\}$.

Beatty difference sequences and Beatty sequences $\{\lfloor nx \rfloor : n \in \mathbb{Z}\}$ (see [1]) are related to the continued fraction expansion of x. There is a vast literature on Beatty sequences and their applications; see [4] and references therein.

By using the Beatty difference sequence, we see how the rows fit together. That is, the n-th tile in row $\mathcal{R}(x)$ has top $\lfloor nx \rfloor - \lfloor (n-1)x \rfloor$ which is the n-th term in the Beatty difference sequence of x.

The bottoms of the tiles in this row give the Beatty difference sequence for $q(x) \cdot x$, *i.e.*, $\{\lfloor n \cdot q(x) \cdot x \rfloor - \lfloor (n-1) \cdot q(x) \cdot x \rfloor$. But this is also the top of the row of tiles $\mathcal{R}(f(x))$ and the two rows fit together.

THEOREM 16. *Every infinite orbit of the dynamical system f corresponds to a valid tiling of the plane using the tiles in the tile set $\mathcal{T}_f$.*

5. Tweaking the colors

Referring back to the K-C tile set there are two color changes for $\mathcal{T}_f$ that will be incorporated to get $\mathcal{T}$. That is, there are the three "zeros" $\{\frac{0}{3}, 0', 0\}$ — two of which are color changes from the original 0. The first, $\frac{0}{3}$, is concerned with side colors.

5.1. Side color changes. The purpose of changing the color 0 to color $\frac{0}{3}$ is to ensure that each row corresponds to a single multiplicand.

The function f is defined in two pieces:

$$f(x) = \begin{cases} f_1(x) = 2x & \text{if } \frac{1}{3} \le x < 1, \\ f_2(x) = \frac{1}{3}x & \text{if } 1 \le x < 2, \end{cases}$$

with two different multiplicands $\{\frac{1}{3}, 2\}$. When the side colors are calculated, for the two pieces in the Basic Tile Construction one gets

$$\tfrac{1}{3}\lfloor nx \rfloor - \lfloor \tfrac{1}{3}nx \rfloor \in \{0, \tfrac{1}{3}, \tfrac{2}{3}\} \text{ for all } n \quad \text{and} \quad 2\lfloor nx \rfloor - \lfloor 2nx \rfloor \in \{0, -1\} \text{ for all } n.$$

The problem is that 0 appears as a side color for both pieces. This would allow tiles with a multiplier of $\frac{1}{3}$ to appear on the same row as tiles with multiplier 2. The solution is to change one of the 0's to a different color which explains the new color $\frac{0}{3}$ (see also Section 6).

5.2. Top-bottom color changes. In this section, we will change some of the top and bottom 0's to $0'$ in the tile set $\mathcal{T}_f$: such changes are called top-bottom changes. This is necessary because the tile set $\mathcal{T}_f$ (without top-bottom changes) is not aperiodic. By introducing these top-bottom color changes (and possibly additional tiles) periodicity may be avoided.

Note that the top-bottom color changes will not affect the multiplier property of any tile (since the numerical value of an edge will not be changed) but will only be concerned with the "colors" of the tiles. Thus the existence of valid tilings will not be affected.

The final K-C tile set is obtained from $\mathcal{T}_f$ by incorporating both the side color changes and the top-bottom color changes.

Consider the piece f_1 of f. Recall that $f_1(x) = 2x$ has domain $[\frac{1}{3}, 1)$ and range $[\frac{2}{3}, 2)$. The Basic Tile Construction for $x \in [\frac{1}{3}, 1)$ yields the six tiles

$$\begin{array}{ccc} & 0 & \\ 0 & \square & 0 \\ & 0 & \end{array} \quad \begin{array}{ccc} & 0 & \\ -1 & \square & -1 \\ & 0 & \end{array} \quad \begin{array}{ccc} & 0 & \\ 0 & \square & -1 \\ & 1 & \end{array} \quad \begin{array}{ccc} & 1 & \\ -1 & \square & 0 \\ & 1 & \end{array} \quad \begin{array}{ccc} & 1 & \\ 0 & \square & 0 \\ & 2 & \end{array} \quad \begin{array}{ccc} & 1 & \\ -1 & \square & -1 \\ & 2 & \end{array}$$

Tile Set 2.

Unfortunately, this tile set is not aperiodic. The first tile (and the second tile) tiles the plane periodically.

The reason is that the tile set has lost the information that the domain of the piece $f(x) = 2x$ is restricted to $[\frac{1}{3}, 1)$. More specifically, the Basic Tile Construction for $x \in [0, 1)$ yields exactly the same 6 tiles (recall Lemmas 13 and 14) and enlarging the interval would add more tiles. Hence, Tile Set 2 is really the tile set for $f'(x) = 2x$ with domain $[0, 1)$ and range $[0, 2)$. The periodic tiling τ of the plane given by the single tile

$$\tau(i, j) = 0 \underset{0}{\overset{0}{\Box}} 0 \ , \quad -\infty < i, j < \infty,$$

corresponds to the fixed point $f'(0) = 2 \cdot 0 = 0$.

More generally, any tile of the form $a \underset{0}{\overset{0}{\Box}} a$ can tile the plane periodically. Such tiles arise when there are points $x \in [0, 1)$ in the domain of f' such that $f'(x) \in [0, 1)$; Lemma 13 shows that these points may give rise to tiles having 0 on both the top and bottom. It is to avoid such tiles that the additional color changes are made (and additional tiles added to the set).

Examining a portion of a typical orbit for the function f, for example

$$1 \Rightarrow \frac{1}{3} \Rightarrow \frac{2}{3} \Rightarrow \frac{4}{3} \Rightarrow \cdots$$

reveals immediately that the function f has at most two consecutive images in the interval $[\frac{1}{3}, 1) \subset [0, 1)$.

Rewrite the function $f =: F$ in four pieces as

$$F_1(x) = 2x, \quad [\tfrac{1}{3}, \tfrac{1}{2}) \to [\tfrac{2}{3}, 1), \qquad F_2(x) = 2x, \quad [\tfrac{1}{2}, \tfrac{2}{3}) \to [1, \tfrac{4}{3}),$$

$$F_3(x) = 2x, \quad [\tfrac{2}{3}, 1) \to [\tfrac{4}{3}, 2), \qquad F_4(x) = \tfrac{1}{3}x, \quad [1, 2) \to [\tfrac{1}{3}, \tfrac{2}{3}).$$

Only piece F_1 has points with $x, F(x) \in [0, 1)$. Consequently it is only this piece that gives rise to tiles with 0 on both the top and bottom.

In this case, we will make only one color change and that is on the interval $[\frac{2}{3}, 1)$ which is the range of F_1. Specifically, any $x \in [\frac{2}{3}, 1)$ has a Beatty difference sequence using just $0, 1$. We change this 0 to $0'$. That is, for any point $x \in [\frac{2}{3}, 1)$, $\lfloor nx \rfloor - \lfloor (n-1)x \rfloor \in \{0', 1\}$.

This color change will also change the colors for tiles constructed from F_3 because the domain of F_3 is the interval $[\frac{2}{3}, 1)$ where the color change was performed.

Hence the tiles constructed for F_1 via the Basic Tile Construction with multiplier 2, will have top colors $\{0, 1\}$ and bottom colors $\{0', 1\}$. This results in the following four tiles.

$$\begin{array}{ccc} & 0 & \\ 0 & \square & 0 \\ & 0' & \end{array} \quad \begin{array}{ccc} & 0 & \\ -1 & \square & -1 \\ & 0' & \end{array} \quad \begin{array}{ccc} & 1 & \\ -1 & \square & 0 \\ & 1 & \end{array} \quad \begin{array}{ccc} & 0 & \\ 0 & \square & -1 \\ & 1 & \end{array}$$

The piece F_2 will have tiles with top colors $\{0, 1\}$ and bottom colors $\{1, 2\}$. This gives the following four tiles.

$$\begin{array}{ccc} & 1 & \\ 0 & \square & 0 \\ & 2 & \end{array} \quad \begin{array}{ccc} & 1 & \\ -1 & \square & -1 \\ & 2 & \end{array} \quad \begin{array}{ccc} & 1 & \\ -1 & \square & 0 \\ & 1 & \end{array} \quad \begin{array}{ccc} & 0 & \\ 0 & \square & -1 \\ & 1 & \end{array}$$

The third and fourth of these two tiles are already in the tile set for F_1, so the combined tile set for F_1 and F_2 is only six tiles.

The piece F_3 has tiles with top colors $\{0', 1\}$ and bottom colors $\{1, 2\}$. This gives the following four tiles.

$$\begin{array}{ccc} & 1 & \\ 0 & \square & 0 \\ & 2 & \end{array} \quad \begin{array}{ccc} & 1 & \\ -1 & \square & -1 \\ & 2 & \end{array} \quad \begin{array}{ccc} & 1 & \\ -1 & \square & 0 \\ & 1 & \end{array} \quad \begin{array}{ccc} & 0' & \\ 0 & \square & -1 \\ & 1 & \end{array}$$

The first three of these tiles are already in the set of tiles for F_1 and F_2. The combined set of tiles for F_1, F_2, F_3 consists of only seven tiles and these are the tiles given in Tile Set $2'$ (see figure for Lemma 6).

Finally we examine piece F_4. This will have tiles with top colors $\{1, 2\}$ and bottom colors $\{0, 1\}$, and will give the 6 tiles in Tile Set $\frac{1}{3}$ (the side color change has already been incorporated).

Together these result in the thirteen tiles for the K-C tile set.

6. Generalization

In this section we present generalizations of the previous work. Detailed proofs are omitted as the essential ideas have already been given.

Consider a function

$$g(x) = \begin{cases} q_1 x & \text{if } x_0 \le x < x_1, \\ q_2 x & \text{if } x_1 \le x < x_2, \\ & \vdots \\ q_k x & \text{if } x_{k-1} \le x < x_k, \end{cases}$$

defined on a finite interval $[x_0, x_k)$ where the $\{q_1, \ldots, q_k\}$ are positive, rational numbers chosen so that g is an invertible bijection of $[x_0, x_k)$ onto itself.

THEOREM 17. *For g as above, the Basic Tile Construction defines a finite set of tiles $\mathcal{T}_g$, and every infinite two-sided orbit of g gives a valid tiling of the plane.*

An obvious question, which we do not pursue at this time, is whether every valid tiling corresponds to a two-sided orbit or if the tile set can be modified to have this property.

However, we remark that one-to-oneness is not precisely necessary for the existence of valid tilings. If g were defined as above but was only required to be *onto* $[x_0, x_k)$, it would still have a tiling set which has valid tilings; however these valid tilings need not correspond to two-sided orbits of g. Under additional assumptions they will correspond to the two-sided orbit of the Rokhlin invertible extension of g.

Side-color tweaking is always possible.

LEMMA 18. *Given g with pieces g_i defined for $x_{i-1} \leq x < x_i$ it is always possible to change the side colors so that the tiles for each piece have disjoint side colors. These color changes will not affect the existence of valid tilings nor the number of tiles in the tile set $\mathcal{T}_g$.*

THEOREM 19. *Let g be a piecewise, rationally multiplicative, invertible function such that*

(i) $1 \leq x_0$,
(ii) $q_1^{n_1} q_2^{n_2} \cdots q_k^{n_k} = 1$ *for* $n_i \geq 0$ *only if* $n_i = 0$ *for all* $i = 1, \ldots, k$.

If $\mathcal{T}_g$ is the tile set constructed for g with side color changes incorporated then $\mathcal{T}_g$ is aperiodic.

PROOF. Same as that of Theorem 10 — that is, the arguments about the colors of the periodic block B are exactly the same. The assumption $1 \leq x_0$ means that there are no zeros in the Beatty sequence for any x in the domain of g (Lemma 13). Which in turn means the tops of all the tiles are nonzero, and this allows the division by $\sum a_{i,1} \neq 0$. $\square$

This theorem does not apply to f in the K-C tile set because $x_0 = \frac{1}{3} < 1$. This required the additional Top-Bottom color tweaking.

The function f has a maximum consecutive orbit of length 2 wholly contained within the interval $[0, 1)$. Because of this, we used two top-bottom colors $\{0, 0'\}$.

Suppose g has a maximum consecutive orbit of length $0 \leq M < \infty$ wholly contained within $[0, 1)$. We then use M different 0's for the top and bottom colors, $0, 0', 0'', \ldots, 0^{(M-1)}$.

Define

$$I_0 = \{x \in [0,1) : \ g^{-1}(x) \notin [0,1)\},$$
$$I_1 = \{x \in [0,1) : \ g^{-1}(x) \in [0,1), \ g^{-2}(x) \notin [0,1)\},$$
$$\vdots$$
$$I_{M-1} = \{x \in [0,1) : \ g^{-i}(x) \in [0,1), i = 1, \cdots M-1, \ g^{-(M)}(x) \notin [0,1)\}.$$

Then, for $x \in I_j$, use the colors $\{0^{(j)}, 1\}$ when calculating the colors

$$\lfloor nx \rfloor - \lfloor (n-1)x \rfloor$$

in the Basic Tile Construction.

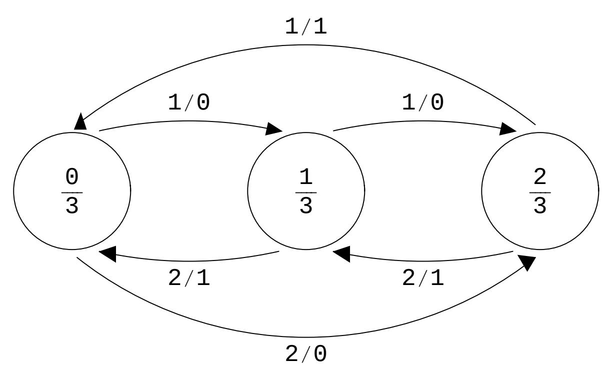
THEOREM 20. *Assume for g as above that*

(i) $q_1^{n_1} q_2^{n_2} \cdots q_k^{n_k} = 1$ *for* $n_i \geq 0$ *only if* $n_i = 0$ *for all* $i = 1, \ldots, k$;
(ii) *there is an* $M \geq 0$ *such that the longest consecutive orbit wholly contained in* $[0,1)$ *is of length* M.

Then by incorporating both side and top-bottom color changes the resulting tile set $\mathcal{T}_g$, *is aperiodic.*

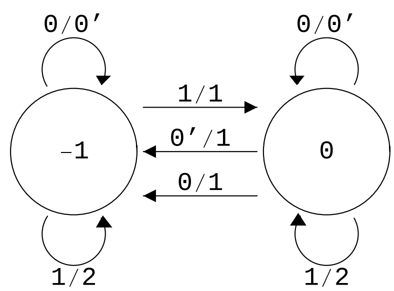
7. Mealy machine representation

Kari and Culik present their tile set using Mealy machines, a type of finite-state automaton where the output is associated with a transition. The K-C tile set can be represented by a pair of Mealy machines, the first of which describes Tile Set $\frac{1}{3}$:



Each edge of the graph represents a tile. The label "i/j" gives the bottom and top numbers of the tile respectively. The tail state (vertex) of the transition arrow is the label of the left side of the tile. The head state (vertex) of the transition arrow is the label of the right side of the tile. This Mealy machine has six edges and these edges correspond to the first six tiles of the K-C tile set.

The following Mealy machine has seven edges which in turn correspond to the last seven tiles of the K-C tile set, namely Tile set 2′.



An infinite 2-sided path through either machine defines an infinite row of tiles. The labels along the tops of this infinite row give an admissible input sequence to the machine. The bottom labels of the row give an admissible output sequence.

Our analysis is essentially a discussion of when an infinite 2-sided output sequence of either machine can be admissible as an input sequence to either machine.

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STANLEY EIGEN
NORTHEASTERN UNIVERSITY
BOSTON, MA 02115
UNITED STATES
eigen@neu.edu

JORGE NAVARRO
UNIVERSITY OF TEXAS
BROWNSVILLE, TX 78520
UNITED STATES
jorge.navarro@utb.edu

VIDHU S. PRASAD
UNIVERSITY OF MASSACHUSETTS
LOWELL, MA 01854
UNITED STATES
vidhu_prasad@uml.edu