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Preface

This volume is dedicated to Henry McKean, on the occasion of his seventy-fifth birthday. His wide spectrum of interests within mathematics is reflected in the variety of theory and applications in these papers, discussed in the Tribute on page xv. Here we comment briefly on the papers that make up this volume, grouping them by topic. (The papers appear in the book alphabetically by first author.)

Since the early 1970s, the subject of completely integrable systems has grown beyond all expectations. The discovery that the Kortweg – de Vries equation, which governs shallow-water waves, has a complete system of integrals of motion has given rise to a search for other such evolution equations. Two of the papers in this volume, one by **Boutet de Monvel and Shepelsky** and the other by **Loubet**, deal with the completely integrable system discovered by Camassa and Holm. This equation provides a model describing the shallow-water approximation in inviscid hydrodynamics. The unknown function $u(x,t)$ refers to the horizontal fluid velocity along the x-direction at time t. The first authors show that the solution of the CH equation in the case of no breaking waves can be expressed in parametric form in terms of the solution of an associated Riemann–Hilbert problem. This analysis allows one to conclude that each solution within this class develops asymptotically into a train of solitons.

Loubet provides a technical *tour de force*, in extending previous results of McKean on the Camassa–Holm equation. More specifically, he gives an explicit formula for the velocity profile in terms of its initial value, when the dynamics are defined by a Hamiltonian which is the sum of the squares of the reciprocals of a pair of eigenvalues of an associated acoustic equation. The proof depends on the analysis of a simpler system, whose Hamiltonian is defined by the reciprocal of a *single* eigenvalue of the acoustic equation. This tool can also solve more complex dynamics, associated to several eigenvalues, which eventually leads to a new proof of McKean's formula for the Fredholm determinant. The paper concludes with an asymptotic analysis (in both past and future directions), which allows partial confirmation of statements about soliton genesis and interaction which were raised in an earlier CPAM paper.

Meanwhile we have a contribution from **Gibbons, Holm and Tronci** of a more geometric nature. This deals with the Vlaslov equation, which describes the evolution of the single-particle probability density in the evolution of N point particles. Specifically, they study the evolution of the p-th moments when the dynamics is governed by a quadratic Hamiltonian. The resulting motion takes place on the manifold of symplectomorphisms, which are smooth invertible maps acting on the phase space. The singular solutions turn out to be closely related to integrable systems governing shallow-water wave theory. In fact, when these equations are "closed" at level p, one retrieves the peaked solitons of the integrable Camassa–Holm equation for shallow-water waves!

Segur's paper provides an excellent overview of the development of our understanding of integrable partial differential equations, from the Boussinesq equation (1871) to the Camassa–Holm equation (1993) and their relations to shallow-water wave theory. In physical terms, an integrable system is equivalent to the existence of *action-angle variables*, where the *action variables* are the integrals of motion and the angle variables evolve according to simple ordinary differential equations. Since each of these PDEs also describes waves in shallow water, it is natural to ask the question: *Does the extra mathematical structure of complete integrability provide useful information about the behavior of actual physical waves in shallow water?* The body of the paper takes up this question in detail with many illustrations of real cases, including the tsunami of December 26, 2004. A video link is provided, for further documentation.

Previato's paper contains a lucid account of the use of theta functions to characterize lines in abelian varieties. Strictly speaking, "line" is short for linear flow, since an abelian variety cannot properly contain a (projective) line. More than twenty years ago Barsotti had proved that, on any abelian variety, there exists a direction such that the derivatives of sufficiently high order of the logarithim of the theta function generates the function field of the abelian variety. The purpose of the present paper is to use the resulting differential equations to characterize theta functions, a generalization of the KP equations, introduced by Kadomtsev and Petviashvili in 1971, as well as to study spectra of commutative rings of partial differential operators.

Arov and **Dym** summarize their recent work on inverse problems for matrix-valued ordinary differential equations. This is related to the notion of reproducing kernel Hilbert spaces and the theory of J-inner matrix functions. A time-independent Schrödinger equation is written as a system of first-order equations, which permits application of the basic results.

Cruzeiro and **Malliavin** study a first-order Burgers equation in the context of flows on the group of diffeomorphisms of the circle, an infinite-dimensional Riemannian manifold. The L^2 norm on the circle defines the Riemannian metric, so that the Burgers equation defines a geodesic flow by means of an ordinary differential equation which flows along the Burgers trajectories. The authors compute the connection coefficients for both the Riemannian connection defined by the parallel transport and the algebraic connection defined by the right-invariant parallelism. These computations are then used to solve a number of problems involving stochastic parallel transport, symmetries of the noise, and control of ultraviolet divergence with the help of an associated Markov jump process.

The paper of **Cambronero**, **Ramírez** and **Rider** describes various links between the spectrum of random Schrödinger operators with particular emphasis on Hill's equation and random matrix theory. The unifying theme is the utility of the Riccati map in converting problems about second-order differential operators and their matrix analogues into questions about one-dimensional diffusions. The paper relies on functional integration to derive many interesting results on the spectral properties of random operators. The paper provides an excellent overview of results on Hill's equation, including the exploitation of low-lying eigenvalues. The penultimate section of the paper describes the recent and exciting developments involving Tracy–Widom distributions and their far-reaching generalizations to all positive values of the inverse temperature β.

Birnir's paper provides an excellent overview of his recent results on unidirectional flows. This is a special chapter in the theory of turbulence, and is not commonly presented. This type of modeling is important in the study of fluvial sedimentation that gives rise to sedimentary rock in petroleum reservoirs. The flow properties of the rock depend strongly on the topological structure of the meandering river channels. The methods developed can also be applied to problems of atmospheric turbulence. Contrary to popular belief, the turbulent temperature variations in the atmosphere may be highly anisotropic, nearly stratified. Thus, the scaling which was first developed in the case of a river or channel may have a close analogue in the turbulent atmosphere.

Halfway between probability theory and classical physics is the subject of statistical mechanics. In the paper of **Costeniuc, Ellis, Touchette** and **Turkington**, the Gaussian ensemble is introduced, to complement the micro-canonical and macro-canonical ensembles which have been known since the time of Gibbs. It is demonstrated that many minimization problems in statistical physics are most effectively expressed in terms of the Gaussian ensemble.

Grinevich and **Novikov** present a lucid overview of their work on finding formulas for the topological charge and other quantities associated with the sine-Gordon equation, which describes immersions of negatively curved surfaces into R^3. Non-singular real periodic finite-gap solutions of the sine-Gordon equation are characterized by a genus g hyperelliptic curve whose branch points are either real positive or form complex conjugate pairs. The authors describe the admissible branch points as zeros of a meromorphic differential of the third kind which, in turn, is defined by a real polynomial $P(\lambda)$ of degree $g-1$. This leads to a formula for the topological charge of these solutions, which was first given in a 1982 paper of Dubrovin and Novikov. The proof relates the topological charge to a set of certain integer characteristics of the polynomial $P(\lambda)$. The methods developed here can be applied, with suitable modifications, to the KdV equation, the defocusing nonlinear Schrödinger equation, and the Kadomtsev–Petvishvilii equation.

The paper of **Ercolani** and **McLaughlin** investigates a system of equations which originate in the physics of two-dimensional quantum gravity, the so-called loop equations of random matrix theory. The analysis depends on an asymptotic formula, of the large deviation type, for the partition function in the Unitary Ensemble of random matrix theory. The loop equations are satisfied by the coefficients in a Laurent expansion expressing certain Cauchy-like transforms in terms of a quadratic expression in the derivatives. The final paragraph of the paper suggests some open problems, such as the following: to use the loop equations to find closed form expressions for the expansion coefficients of the logarithm of the partition function when the dimension $N\to\infty$; it is also anticipated that the loop equations could be used to determine qualitative behavior of these coefficients.

Manna and **Moll** offer a beautiful set of generalizations of the classical Landen transform, which states that a certain elliptic integral of the first kind containing two parameters, when expressed in trigonometric form, is invariant under the transformation defined by replacing the parameters by their arithmetic (resp. geometric) means. Later Gauss used this to prove that the limit of the iterates of this transformation exist and converge to the reciprocal of this elliptic integral, suitably modified. This limit is, by definition, the *arithmetic-geometric mean* of the initial conditions. This idea is generalized in several directions, the first of which is to a five-parameter set of rational integrals, where the numerator is of order four and the denominator of order six. The requisite Landen transformation has a simple geometric interpretation in terms of doubling the angle of the cotangent function in the trigonometric form of the integrand. The remainder of the paper describes generalizations to higher-order rational integrands, where the doubling of the cotangent function is replaced by a magnification of order $m\geq 2$. It is proved that the limit of the Landen transforms exists and can be represented as a suitable integral. All of the models studied in this paper can be considered as discrete-time (partially) integrable dynamical systems where the conserved quantity is the definite integral which is invariant under the change of parameters. The simplest of these is the elliptic integral studied by Landen, where the dynamics is defined by the arithmetic-geometric mean substitution.

Varadhan has contributed a lucid overview of his recent joint work on homogenization. In general, this theory leads to approximations of solutions of a differential equation with rapidly varying coefficients in terms of solutions of a closely related equation with constant coefficients. A model problem is the second-order parabolic equation in one space dimension, where the constant coefficient is the harmonic mean of the given variable coefficient equation. This analytical result can be expressed as a limit theorem in probability, specifically an ergodic theorem for the variable coefficient, when composed with the diffusion process defined by the parabolic equation; the normalized invariant measure is expressed in terms of the harmonic mean. Having moved into the probabilistic realm, one may just as well consider a second-order parabolic equation with random coefficients, where the results are a small variation of those obtained in the non-random case. With this intuitive background, it is natural to expect similar results when one begins with a d-dimensional equation of Hamilton–Jacobi type, defined by a convex function and with a small noise term. In joint work with Kosygina and Rezhakanlou, it is proved that the noise disappears and the HJ solution converges to the solution of a well-determined first-order equation, where the homogenized convex function is determined by a convex duality relation. The reader is invited to pursue the details, which are somewhat parallel to the model case described above.

Camia and **Newman**'s paper relies on the stochastic Schramm–Loewner equation (SLE), which provides a new and powerful tool to study scaling limits of critical lattice models. These ideas have stimulated further progress in understanding the conformally invariant nature of the scaling limits of several such models. The paper reviews some of the recent progress on the scaling limit of two-dimensional critical percolation, in particular the convergence of the exploration path to chordal SLE and the "full" scaling limit of cluster interface loops. The results on the full scaling limit and its conformal invariance are presented here for the first time. For site percolation on the triangular lattice, the results are fully rigorous and the main ideas are explained.

Grünbaum's paper proposes a new spectral theory for a class of discrete-parameter Markov chains, beginning with the case of the birth-death process, studied by Karlin/McGregor in the 1950s. More generally, for each Markov chain there is a system of orthogonal polynomials which define the spectral decomposition. Explicit computation of the relevant orthogonal polynomials is available for other Markov chains, such as random walk on the N-th roots of unity and the processes associated with the names of Ehrenfest and Tchebychev.

A principal emphasis here, due originally to M. G. Krein, is the formulation of *matrix-valued* orthogonal polynomials. Several solved examples are presented, while several natural open problems are suggested for the adventurous reader.

Van Moerbeke provides a masterful account of the close connection between nonintersecting Brownian motions and integrable systems, where the connection is made in terms of the theory of orthogonal polynomials, using previous results of Adler and Van Moerbeke. If N Brownian particles are started at p definite points and required to terminate at q other points in unit time, the object is to study the distribution as $N \to \infty$, especially in the short time limit $t \downarrow 0$ and the unit time limit $t \uparrow 1$. When the supports of these measures merge together, we have a *Markov cloud*, defined as an infinite-dimensional diffusion process which depends only on the nature of the various possible singularities of the equilibrium measure. The connection between non-intersecting Brownian motion and orthogonal polynomials begins with a formula of Karlin and McGregor which expresses the non-intersection probability at a fixed time as the N-dimensional integral of the product of two determinants. Numerous special cases are provided to illustrate the general theory.

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