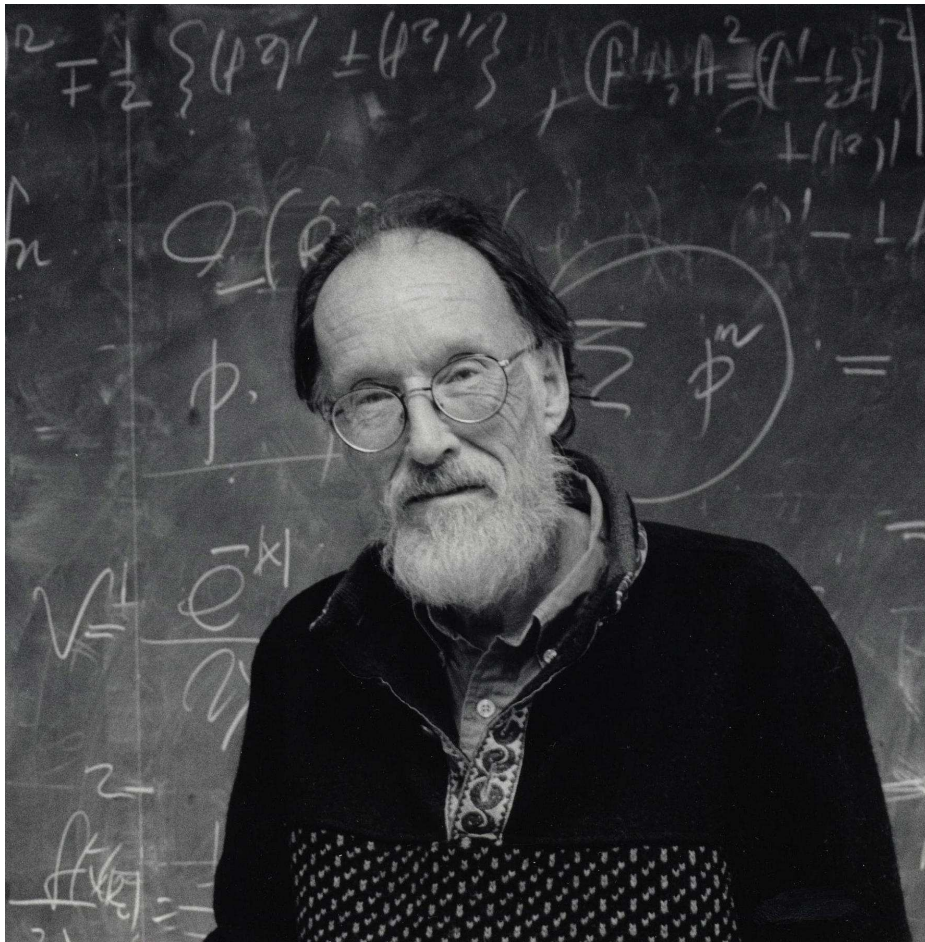


Henry McKean

A TRIBUTE BY THE EDITORS



The significance of McKean's work

Henry McKean has championed a unique viewpoint in mathematics, with good taste and constant care toward a balance between the abstract and the concrete. His great influence has been felt through both his publications and his teaching. His books, all of which have become influential, can be recognized through their concise, elegant and efficient style.

His interests include probability theory, stochastic processes, Brownian motion, stochastic integrals, geometry and analysis of partial differential equations, with emphasis on integrable systems and algebraic curves, theta functions, Hill's equation and nonlinear equations of the KdV type. He also was a pioneer in the area of financial mathematics before it became a household word.

The importance of stochastic models in modern applied mathematics and science cannot be overestimated. This is well documented by the large number of diverse papers in this volume which are formulated in a stochastic context. Henry McKean was one of the early workers in the theory of diffusion processes, as documented in his classic work with K. Itô, *Diffusion Processes and Their Sample Paths* (Springer, 1965). This was followed by his *Stochastic Integrals* (Academic Press, 1969). This book led the way to understanding the close connections between probability and partial differential equations, especially in a geometric setting (Lie groups, Riemannian manifolds).

On January 6, 2007, McKean was awarded the Leroy P. Steele Prize for Lifetime Achievement, presented annually by the American Mathematical Society. The prize citation honors McKean for his "rich and magnificent mathematical career" and for his work in analysis, which has a strong orientation towards probability theory. The prize citation states further that "McKean has had a profound influence on his own and succeeding generations of mathematicians. In addition to the important publications resulting from his collaboration with Itô, McKean has written several books that are simultaneously erudite and gems of mathematical exposition. As his long list of students attest, he has also had enormous impact on the careers of people who have been fortunate enough to study under his direction."

McKean's published work includes five books and more than 120 articles, in such journals as the *Annals of Mathematics*, *Acta Mathematica* and *Inventiones Mathematicae*, to name a few. To illustrate the richness of the mathematics he has been involved with, we take a brief look at published reviews of his books, and then discuss the main threads of his research articles by subject.

1. *Diffusion Processes and Their Sample Paths* (with Kiyosi Itô). *Grundlehren der mathematischen Wissenschaften* **125**, Springer, New York, 1965; second (corrected) printing, 1974; reprinted in *Classics in Mathematics*, 1996.

We adapt the comments by T. Watanabe in *Mathematical Reviews*: Feller's work on linear diffusion was primarily of an analytic character. This spurred some outstanding probabilists (including the authors) to re-establish Feller's results by probabilistic methods, solving some conjectures of Feller and studying profoundly the sample paths of one-dimensional diffusion. Their purpose is to extend the theory of linear diffusion to the same level of understanding which Paul Lévy established for Brownian motion. This is completely realized in this book by combining special tools such as Brownian local time with the general theory of Markov processes. This book is the culmination of a ten-year project to obtain the general linear diffusion from standard Brownian motion by time change and killing involving local times.

On the occasion of the book's republishing in Springer's *Classics in Mathematics* series, the cover blurb could boast without the least exaggeration: "Since its first publication ... this book has had a profound and enduring influence on research into the stochastic processes associated with diffusion phenomena. Generations of mathematicians have appreciated the clarity of the descriptions given of one or more dimensional diffusion processes and the mathematical insight provided into Brownian motion."

2. *Stochastic Integrals*. Academic Press, New York, 1969.

From comments by E. B. Dynkin in *Mathematical Reviews*: "This little book is a brilliant introduction to an important interface between the theory of probability and that of differential equations. The same subject was treated in the recent book of I. I. Gihman and A. V. Skorokhod. The author's book is smaller, contains more examples and applications and is therefore much better suited for beginners. Chapter 1 is devoted to Brownian motion. Chapter 2 deals with stochastic integrals and differentials. Chapter 3 deals with one-dimensional stochastic integral equations. In Chapter 4, stochastic integral equations on smooth manifolds are investigated. Winding properties of planar Brownian motion (about one or two punctures) are deduced from the study of Brownian motion on Riemann surfaces. The last three sections are devoted to constructing Brownian motion on a Lie group, starting from Brownian motion on the Lie algebra. by means of the so-called product integral and the Maljutov–Dynkin results about Brownian motion with oblique reflection. In treating the applications of stochastic integrals, the author frequently explains the main ideas by means of typical examples, thus avoiding exhausting generalities. This remarkable book

will be interesting and useful to physicists and engineers (especially in the field of optimal control) and to experts in stochastic processes.”

3. *Fourier Series and Integrals* (with Harry Dym). Academic Press, New York, 1972.

Elliot Lieb of Princeton University wrote: “In my opinion, the book of Dym and McKean is unique. It is a book on analysis at an intermediate level with a focus on Fourier series and integrals. The reason the book is unique is that books on Fourier analysis tend to be quite abstract, or else they are applied mathematics books which give very little consideration to theory. This book is a solid mathematics book but written with great fluency and many examples. In addition to the above considerations, there is also the fact that the literary style of the authors is excellent, so that the book has a readability that is rarely found in mathematics books, especially in modern texts.”

4. *Stationary Gaussian Processes* (with Harry Dym). Academic Press, New York, 1976.

From comments by S. Kotani in *Mathematical Reviews*: “This is a monograph on stationary Gaussian processes in one dimension. Given a mean zero Gaussian process $x(t)$, $t \in \mathbb{R}$, one may ask the following questions: (i) to predict the future given the past $-\infty < t < 0$; (ii) to predict the future given the finite segment of the past $-2T < t < 0$; (iii) to predict $x(t)$ for $|t| < T$ given $x(t)$ for $|t| > T$; (iv) the degree of dependence of the future on the past; (v) the degree of mixing of the process $x(t)$. This book contains a clear and concise introduction to the subject, often original presentations of known results in addition to several new results. The solution to problem (i) goes back to Kolmogorov, while the solution of (iii) is due to M. G. Krein in 1954. The authors re-work the solution of Krein and from that point they completely solve problem (ii). They also make important contributions to the understanding of problems (iv) and (v).”

5. *Elliptic Curves: Function Theory, Geometry and Arithmetic* (with Victor Moll). Cambridge University Press, Cambridge, 1997.

From W. Kleinert’s review in *Zentralblatt für Mathematik*: “Altogether, this highly non-standard textbook provides the reader with ... a deep insight into historically known mathematical interrelations and references to modern developments in the analysis, geometry and arithmetic of elliptic curves. The book reflects the authors’ profound knowledge and deep devotion to the historical development of the theory of elliptic curves. In these days it is certainly very profitable for the mathematical community to have such a book among the increasing number of others on the subject ... this book is not only a perfect primer for beginners in the field, but also an excellent source for researchers in various areas of mathematics and physics.”

Peter Sarnak agrees: “Very unusual in covering the important aspects of elliptic curves (analytic, geometric and arithmetic) and their applications — in a single reasonably sized volume. This account of the subject, in the style of the original discoverers is, in my opinion, the best way to present the material in an introductory book.”

Works by subject

Given his wide spectrum of interest, it is difficult to summarize his contributions in a few paragraphs. We mention some of the principal themes:

Questions in the theory of probability and stochastic processes, beginning with Brownian motion and leading to the completion of W. Feller’s program, to found the theory of one-dimensional diffusion on a probabilistic basis rather than the analytical foundation (Hille–Yosida theorem) that was standard before the 1950s. One probabilistic approach is to use the natural scale and speed measure to obtain the diffusion process directly from Brownian motion. This approach is well documented in Itô–McKean. The other approach is to develop Itô’s theory of stochastic differential equations (SDE) to obtain the diffusion as the image of a Brownian motion process via a non-linear mapping which defines the solution operator of the SDE. This approach, which works equally well in higher dimensions, is well documented in his *Stochastic Integrals*.

Geometry and analysis of partial differential equations, especially integrable systems and algebraic curves, theta functions and Hill’s equation. Hill’s equation is the one-dimensional Schrödinger equation

$$-y'' + q(x)y = \lambda y$$

where the potential function $q(x)$ is periodic. The spectrum of the operator $L = -d^2/dx^2 + q$ associated to such an equation is generally made up of an infinite number of intervals. But it can happen that, in limiting cases, the spectrum is formed of a finite number of n intervals and an infinite interval $\lambda > \lambda_{2n}$. For which potentials does this occur? Several authors have shown that this phenomenon requires that the potential be a solution of some auxiliary differential equations. Then Peter Lax showed in 1972/1974 the relation with certain solutions of the KdV equation. McKean and van Moerbeke solved the problem by establishing a close relation with the classic theory of hyperelliptic functions. Then McKean and Trubowitz in 1976/1978 extended the results to the case $n = \infty$, showing that the periodic spectrum of the Hill operator is infinite. This work proved to be the starting point of a series of new developments associated with the names of McKean and Trubowitz, Feldman, Knörrer, Krichever and Merkl.

Geometry of KdV equations. The Korteweg–de Vries equation

$$\frac{\partial q}{\partial t} = 3q \frac{\partial q}{\partial x} - \frac{1}{2} \frac{\partial^3 q}{\partial x^3}$$

describes the propagation of a wave $q(x, t)$ in a shallow canal. It has certain common features with other non-linear partial differential equations: KdV has the structure of the equations of Hamiltonian dynamics; in addition it has certain “solitary solutions” rather than wave train solutions; finally it has a rich set of constants of motion related to the spectra of certain associated equations. These equations appear as “isospectral deformations” of some natural operators. This series of articles, devoted to the geometry of KdV, is based on tools from algebraic geometry (in dimension ∞), especially hyperelliptic curves, their Jacobians, theta functions and their connections with the spectral theory of Hill’s equation.

Nonlinear equations. The nonlinear equations that interested McKean are the classical commutation relations

$$\frac{\partial L}{\partial t} = [L, K],$$

where L is a differential operator of first order whose coefficients are 2×2 matrices of class C^∞ and where K is an anti-symmetric operator of the same type. The KdV equation

$$\frac{\partial q}{\partial t} = 3q \frac{\partial q}{\partial x} - \frac{1}{2} \frac{\partial^3 q}{\partial x^3}$$

is of this type where L is the Hill operator

$$\begin{aligned} L &:= -\Delta + q, \\ K &:= 2D^3 - \frac{3}{2}(qD + Dq), \end{aligned}$$

where $D = \partial/\partial x$ and $\Delta = D^2$. In addition to these equations, McKean studied the sine-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial x^2} - \sin q$$

and the equations of Boussinesq and Camassa–Holm, written, respectively, as

$$\begin{aligned} \frac{\partial^2 q}{\partial t^2} &= \frac{\partial^2}{\partial x^2} \left(\frac{4}{3} q^2 + \frac{1}{3} \frac{\partial^2 q}{\partial x^2} \right), \\ \frac{\partial u}{\partial t} + 3u \frac{\partial u}{\partial x} &= \alpha^2 \left(\frac{\partial^3 u}{\partial x^2 \partial t} + 2 \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} + u \frac{\partial^4 u}{\partial x^4} \right). \end{aligned}$$

Financial mathematics. In 1965, following a suggestion of the economist Paul Samuelson, McKean wrote a short article on the problem of “American options”. This was first published as an appendix to a treatise by Samuelson on financial economics. It anticipated by eight years the famous Black–Scholes–Merton formula. In this paper, McKean shows that the price of an American option can be computed by solving a free boundary problem for a parabolic partial differential equation. This work was the first application of nonlinear partial differential equations in financial mathematics.

Some personal tributes

Paul Malliavin: In 1972 I was working in the theory of functions of several complex variables, more specifically on the characterization of the set of zeros of a function of the Nevanlinna class in a strictly pseudo-convex domain of \mathbb{C}^n . In the special case of the ball, by using semi-simple harmonic analysis, I computed exactly the Green function and obtained the desired characterization. For the general case I was completely stuck; I looked at the canonical heat equation associated to an exhaustion function; a constructive tool to evaluate the Green function near the boundary was urgently needed; then I started to study Itô’s papers on SDEs; their constructive essence seemed to me quite appropriate for this estimation problem.

Then McKean’s *Stochastic Integrals* appeared: I was fully rescued in my efforts to grasp from scratch the theory of SDEs. McKean’s book is short, with carefully written concrete estimates; it presents with sparkling clarity a conceptual vision of the theory. In a note in the *Proceedings of the National Academy* I found the needed estimate for the Green function; from there I started to study the case of weakly pseudoconvex domains, where hypoelliptic operators of Hörmander type appeared; from that time onward (for the last thirty years) I became fully involved in probability theory. So Henry has had a key influence on this turning point of my scientific interests.

From the middle seventies Henry has kindly followed the different steps of my career, supporting me at every occasion. For instance, when I was in the process of being fired from Université Paris VI in 1995, he did not hesitate to urgently cross the Atlantic in order to sit on a special Committee, specially constituted by the President of Université Paris VI, in order to judge if my current work was then so obsolete that all the grants for my research associates had to be immediately eliminated.

With gratitude I dedicate our paper to Henry; also with admiration, discovering every day more and more the breadth and the depth of his scientific impact.

Daniel Stroock: I have known Henry since 1964, the year he visited Rockefeller Institute and decided to leave MIT and New England for the city of New York.

I do not know any details of Henry's childhood, but I have a few impressions which I believe bear some reasonable resemblance to the truth. Henry was the youngest child of an old New England family and its only son. He grew up in the North Shore town of Beverly, Massachusetts, where he developed a lasting love for the land, its inhabitants and the way they pronounced the English language. Except for the fact that he was taught to ski by his eldest sister, who was a superb athlete, I gather that Henry was younger than his siblings so that, from an early age, he learned to fend for himself. He was sent to St. Pauls for high school where, so far as I know, his greatest distinction was getting himself kicked out for smoking. Nonetheless, he was accepted to Dartmouth College where, under the influence of Bruce Knight, he began to develop a taste for mathematics. His interests in mathematics were further developed by a course which he took one summer from Mark Kac, who remained a lasting influence on Henry.

In fact, it was Kac who invited Henry to visit and then, in 1966, move to Rockefeller Institute. Since I finished my PhD the spring before Henry joined the faculty, I do not know many details of life in the Rockefeller mathematics-physics group during the late 1960s. However I do know one story from that period which, even if it's not totally true, nicely portrays Henry's affection for Kac, the founding father of the Rockefeller Mathematics Department. As such, it was his job to make it grow. For various reasons, not the least of which was his own frequent absence, Kac was having limited success in this enterprise; at one faculty meeting Kac solicited suggestions from his younger colleagues. Henry's suggestion was that they double Kac's salary in order to have him there on a full-time basis.

Henry's mathematical achievements may be familiar to anyone who is likely to be reading this book, with one proviso: most mathematicians do not delve into the variety of topics to which Henry has contributed. Aside from the constant evidence of his formidable skills, the property shared by all of Henry's mathematics is a strong sense of *taste*. Whether it is his early collaboration with Itô, his excursion into Gaussian prediction theory, or his interest in completely integrable systems and spectral invariance, Henry has chosen problems because they interest him and please his sense of aesthetics. As a result, his mathematics possesses originality which is his own, and a beauty which the rest of us can appreciate.

Srinivasa Varadhan, former director of the Courant Institute: Henry has been my colleague for nearly thirty-five years. I have always been impressed by the number of students Henry has produced. I checked the Genealogy project. It lists him with nearly fifty students. It is even more impressive that he has nearly three hundred descendants, which means he has taught his students how to teach.

Henry is known for meticulous attention to detail. When Dan Stroock and I wrote our first paper on the martingale problem we gave the manuscript to Henry for comments. The paper was typewritten, before the days of word processors and xerox machines. Henry gave it back to us within days and the comments filled out all the empty space between the lines on every page. The typist was not amused.

Henry's own work drifted in the seventies out of probability theory into integrable systems and then back again at some point into probability. Talking to Henry about any aspect of mathematics is always fun. He has many interesting but unsolvable problems and will happily share them with you. If I come across a cute proof of some thing Henry is the first one I will think of telling.

Charles Newman, member and former Director of the Courant Institute: Henry's earliest papers (*Ann. Math.* 1955) concerned sample functions of stable processes, together with several papers on Brownian local time (*J. Math. Kyoto* 1962, *Adv. Math.* 1975). These topics and the general subject of one-dimensional diffusions were of course analyzed in great detail in Henry's celebrated 1965 book with Itô. I mention these all because they play a major role in my own much later work (jointly with Fontes and Isopi, *Ann. Prob.* 2002) on scaling limits of random walks with random rates to singular diffusions (with random speed measures).

There are a number of other connections between papers of Henry's and my own—typically with a multi-decade gap. For example, there are close connections between Henry's paper "Geometry of differential space" (*Ann. Prob.* 1973) and my 2003 paper with D'Aristotile and Diaconis, "Brownian motion and the classical groups".

The paper by Camia and myself in this volume is less directly related to Henry's work. However, the general subject of Schramm–Loewner evolutions, in which our paper belongs, does combine many of the same themes that have permeated Henry's work: Brownian motions and related processes, complex variable theory, and statistical mechanics.