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Bitangential direct and inverse problems for systems of differential equations

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ABSTRACT. A number of results obtained by the authors on direct and inverse problems for canonical systems of differential equations, and their implications for certain classes of systems of Schrödinger equations and systems with potential are surveyed. Connections with the theory of J-inner matrix valued and reproducing kernel Hilbert spaces, which play a basic role in the original developments, are discussed.

1. Introduction

In this paper we shall present a brief survey of a number of results on direct and inverse problems for canonical integral and differential systems that have been obtained by the authors over the past several years. We shall not attempt to survey the literature, which is vast, or to compare the methods surveyed here with other approaches. The references in [Arov and Dym 2004; 2005b; 2005c] (the last of which is a survey article) may serve at least as a starting point for those who wish to explore the literature.

The differential systems under consideration are of the form

$$y'(t,\lambda) = i\lambda y(t,\lambda)H(t)J, \quad 0 \leqslant t < d\ , \tag{1.1}$$

where $H(t)$ is an $m \times m$ locally summable mvf (matrix valued function) that is positive semidefinite a.e. on the interval $[0,d)$, J is an $m \times m$ signature matrix, i.e., $J = J^*$ and $J^*J = I_m$, and $y(t,\lambda)$ is a $k \times m$ mvf.

The matrizant or fundamental solution, $Y_t(\lambda) = Y(t,\lambda)$, of (1.1) is the unique locally absolutely continuous $m \times m$ solution of (1.1) that meets the initial condition $Y_0(\lambda) = I_m$, i.e.,

$$Y_t(\lambda) = I_m + i\lambda \int_0^t Y_s(\lambda)H(s)\,ds\,J \quad \text{for } 0 \le t < d\ . \tag{1.2}$$

Standard estimates yield the following properties:

(1) $Y_t(\lambda)$ is an entire mvf that is of exponential type in the variable λ for each fixed $t \in [0, d)$.

(2) The identity

$$\frac{J - Y_t(\lambda) J Y_t(\omega)^*}{-2\pi i(\lambda - \overline{\omega})} = \frac{1}{2\pi} \int_0^t Y_s(\lambda) H(s) Y_s(\omega)^* ds \tag{1.3}$$

holds for each $t \in [0, d)$ and for every pair of points $\lambda, \omega \in \mathbb{C}$.

(3) $Y_t(\lambda)$ is J-inner (in the variable λ) in the open upper half plane $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \lambda + \overline{\lambda} > 0\}$ for each fixed $t \in [0, d)$. This means that

$$J - Y_t(\omega) J Y_t(\omega)^* \geq 0 \quad \text{for } \omega \in \mathbb{C}_+ ,$$

with equality if $\omega \in \mathbb{R}$.

(4) $J - Y_t(\omega) J Y_t(\overline{\omega})^* = 0 \quad \text{for } \omega \in \mathbb{C}$.

(5) The kernel

$$K_\omega^t(\lambda) = \frac{J - Y_t(\lambda) J Y_t(\omega)^*}{-2\pi i(\lambda - \overline{\omega})}$$

is positive in the sense that

$$\sum_{i,j=1}^{n} u_i^* K_{\omega_j}^t(\omega_i) u_j \geqslant 0$$

for every choice of the points $\omega_1, \ldots, \omega_n$, vectors $u_1, \ldots, u_n$ and every positive integer n.

(6) $Y_{t_1}^{-1} Y_{t_2}$ is also an entire J-inner mvf for $0 \leq t_1 \leq t_2 < d$.

Every $m \times m$ signature matrix $J \neq \pm I_m$ is unitarily equivalent to the matrix

$$j_{pq} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}, \ p + q = m,$$

with

$$p = \text{rank}\,(I_m + J) \geq 1 \quad \text{and} \quad q = \text{rank}\,(I_m - J) \geq 1 .$$

The main examples of signature matrices, apart from $\pm j_{pq}$, are $\pm j_p$, $\pm J_p$ and $\pm \mathcal{J}_p$, where

$$J_p = \begin{bmatrix} 0 & -I_p \\ -I_p & 0 \end{bmatrix}, \quad j_p = \begin{bmatrix} I_p & 0 \\ 0 & -I_p \end{bmatrix}, \quad \mathcal{J}_p = \begin{bmatrix} 0 & -iI_p \\ iI_p & 0 \end{bmatrix}.$$

In the last three examples $q = p$ (so that $2p = m$). The signature matrices J_p and j_p are connected by the signature matrix

$$\mathfrak{V} = \frac{1}{\sqrt{2}} \begin{bmatrix} -I_p & I_p \\ I_p & I_p \end{bmatrix}, \quad \text{i.e.,} \quad \mathfrak{V} J_p \mathfrak{V} = j_p \quad \text{and} \quad \mathfrak{V} j_p \mathfrak{V} = J_p \,.$$

There is a natural link between each of the three principal signature matrices and each of the following three classes of mvf's that are holomorphic in $\mathbb{C}_+$, the open upper half plane:

(1) The Schur class $\mathcal{S}^{p\times q}$ of $p \times q$ mvf's $\varepsilon(\lambda)$ that are holomorphic in $\mathbb{C}_+$ and satisfy the constraint $I_q - \varepsilon(\lambda)^*\varepsilon(\lambda) \geq 0$, since

$$I_q - \varepsilon(\lambda)^*\varepsilon(\lambda) \geq 0 \Longleftrightarrow [\, \varepsilon(\lambda)^* \quad I_q \,]\, j_{pq} \begin{bmatrix} \varepsilon(\lambda) \\ I_q \end{bmatrix} \leq 0 \,. \tag{1.4}$$

(2) The Carathéodory class $\mathcal{C}^{p\times p}$ of $p \times p$ mvf's $\tau(\lambda)$ that are holomorphic in $\mathbb{C}_+$ and satisfy the constraint $\tau(\lambda) + \tau(\lambda)^* \geq 0$, since

$$\tau(\lambda) + \tau(\lambda)^* \geq 0 \Longleftrightarrow [\, \tau(\lambda)^* \quad I_p \,]\, J_p \begin{bmatrix} \tau(\lambda) \\ I_p \end{bmatrix} \leq 0 \,. \tag{1.5}$$

(3) The Nevanlinna class $\mathcal{R}^{p\times p}$ of $p \times p$ mvf's $\tau(\lambda)$ that are holomorphic in $\mathbb{C}_+$ and satisfy the constraint $(\tau(\lambda) - \tau(\lambda)^*)/i \geq 0$, since

$$(\tau(\lambda) - \tau(\lambda)^*)/i \geq 0 \Longleftrightarrow [\, \tau(\lambda)^* \quad I_p \,]\, \mathcal{J}_p \begin{bmatrix} \tau(\lambda) \\ I_p \end{bmatrix} \leq 0 \,, \tag{1.6}$$

A general $m \times m$ mvf $U(\lambda)$ is said to be J-inner with respect to the open upper half plane $\mathbb{C}_+$ if it is meromorphic in $\mathbb{C}_+$ and if

(1) $J - U(\lambda)^* J U(\lambda) \geq 0$ for every point $\lambda \in \mathfrak{h}_U^+$ and
(2) $J - U(\mu)^* J U(\mu) = 0$ a.e. on $\mathbb{R}$,

in which $\mathfrak{h}_U^+$ denotes the set of points in $\mathbb{C}_+$ at which U is holomorphic. This definition is meaningful because every mvf $U(\lambda)$ that is meromorphic in $\mathbb{C}_+$ and satisfies the first constraint automatically has nontangential boundary values. The second condition guarantees that $\det U(\lambda) \not\equiv 0$ in $\mathfrak{h}_U^+$ and hence permits a pseudo-continuation of $U(\lambda)$ to the open lower half plane $\mathbb{C}_-$ by the symmetry principle

$$U(\lambda) = J\{U^\#(\lambda)\}^{-1} J \quad \text{for } \lambda \in \mathbb{C}_- \,,$$

where $f^\#(\lambda) = f(\overline{\lambda})^*$.

The symbol $\mathcal{U}(J)$ will denote the class of J-inner mvf's considered on the set $\mathfrak{h}_U$ of points of holomorphy of $U(\lambda)$ in the full complex plane $\mathbb{C}$ and $\mathcal{E} \cap \mathcal{U}(J)$ will denote the class of entire J-inner mvf's.

If $U \in \mathcal{U}(I_m)$, then $U \in \mathcal{S}^{m\times m}$ and $U(\lambda)$ is said to be an $m \times m$ inner mvf. The set of $m \times m$ inner mvf's will be denoted $\mathcal{S}_{in}^{m\times m}$ and the set of outer $m \times m$

mvf's in $\mathcal{S}^{m\times m}$ will be denoted $\mathcal{S}^{m\times m}_{out}$. (It is perhaps useful to recall that if $s\in\mathcal{S}^{m\times m}$, then $s\in\mathcal{S}^{m\times m}_{out}$ if and only if det $s\in\mathcal{S}^{1\times 1}_{out}$, and that if $s\in\mathcal{S}^{1\times 1}$, then

$$s\in\mathcal{S}^{1\times 1}_{out} \Longleftrightarrow \ln\,|s(i)| = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\ln|s(\mu)|}{1+\mu^2}d\mu\ .)$$

If $A\in\mathcal{U}(J_p)$, there exists a pair of $p\times p$ inner mvf's $b_3(\lambda)$ and $b_4(\lambda)$ that are uniquely characterized in terms of the blocks of $B(\lambda)=A(\lambda)\mathfrak{V}$ and the set

$$\mathcal{N}^{p\times p}_{out} = \left\{\frac{g}{h} : \ g\in\mathcal{S}^{p\times p}_{out} \quad \text{and} \quad h\in\mathcal{S}^{1\times 1}_{out}\right\}$$

by the constraints

$$b_{21}^{\#}b_3\in\mathcal{N}^{p\times p}_{out} \quad \text{and} \quad b_4b_{22}\in\mathcal{N}^{p\times p}_{out}\ ,$$

up to a constant $p\times p$ unitary multiplier on the left of $b_3(\lambda)$ and a constant $p\times p$ unitary multiplier on the right of $b_4(\lambda)$. Such a pair will be referred to as an *associated pair of the second kind* for $A(\lambda)$ and denoted

$$\{b_3,b_4\}\in ap_{II}(A)\,.$$

(There is also a set of associated pairs $\{b_1,b_2\}$ of the first kind that is more convenient to use in some other classes of problems that will not be discussed here.) The pairs $\{b_3^t,b_4^t\}\in ap_{II}(A_t)$ that are associated with the matrizant $A_t(\lambda)$, $0\le t<d$, of a canonical system of the form (1.1) with $J=J_p$ are entire $p\times p$ inner mvf's that are *monotonic* in the sense that

$$(b_3^{t_1})^{-1}b_3^{t_2} \quad \text{and} \quad b_4^{t_2}(b_4^{t_1})^{-1}$$

are $p\times p$ entire inner mvf's for $0\le t_1\le t_2<d$. Moreover, they are uniquely specified by imposing the *normalization* conditions $b_3^t(0)=b_4^t(0)=I_p$ for $0\le t<d$.

2. Reproducing kernel Hilbert spaces

If $U\in\mathcal{U}(J)$ and

$$\rho_\omega(\lambda) \ = \ -2\pi i(\lambda-\overline{\omega})\,,$$

then the kernel

$$K^U_\omega(\lambda \ = \ \frac{J-U(\lambda)JU(\omega)^*}{\rho_\omega(\lambda)}$$

is positive on $\mathfrak{h}_U\times\mathfrak{h}_U$ in the sense that $\sum_{i,j=1}^n u_i^*K^U_{\omega_j}(\omega_i)u_j\ge 0$ for every set of vectors $u_1,\dots,u_n\in\mathbb{C}^m$ and points $\omega_1,\dots,\omega_n\in\mathfrak{h}_U$; see [Dym 1989], for example. Therefore, by the matrix version of a theorem of Aronszajn [1950], there is an associated RKHS (reproducing kernel Hilbert space) $\mathcal{H}(U)$ of $m\times 1$ mvf's defined and holomorphic in $\mathfrak{h}_U$ with RK (reproducing kernel) $K^U_\omega(\lambda)$. This means that for every choice of $\omega\in\mathfrak{h}_U$, $u\in\mathbb{C}^m$ and $f\in\mathcal{H}(U)$,

(1) $K_\omega u \in \mathcal{H}(U)$ and
(2) $\langle f, K_\omega u\rangle_{\mathcal{H}(U)} = u^* f(\omega)$.

Thus, item (5) in the list of properties of the matrizant implies that there is a RKHS $\mathcal{H}(Y_t)$ of entire $m \times 1$ mvf's with RK $K_\omega^t(\lambda)$ for each $t \in [0, d)$, i.e., for each choice of $\omega \in \mathbb{C}$, $u \in \mathbb{C}^m$ and $f \in \mathcal{H}(Y_t)$,

(a) $K_\omega^t u \in \mathcal{H}(Y_t)$ and
(b) $\langle f, K_\omega^t u\rangle_{\mathcal{H}(Y_t)} = u^* f(\omega)$.

Moreover, property (6) of the matrizant Y_t implies that if $0 \le t_1 \le t_2 < d$, then $\mathcal{H}(Y_{t_1}) \subseteq \mathcal{H}(Y_{t_2})$ as sets and

$$\|f\|_{\mathcal{H}(Y_{t_2})} \leqslant \|f\|_{\mathcal{H}(Y_{t_1})}$$

for every $f \in \mathcal{H}(U_{t_1})$.

In this short review we shall restrict attention to canonical systems with signature matrices $J = J_p$ and shall denote the matrizant of such a system by $A_t(\lambda)$ and the corresponding RK by $K_\omega^{A_t}(\lambda)$. Thus,

$$K_\omega^{A_t}(\lambda) = \frac{J_p - A_t(\lambda) J_p A_t(\omega)^*}{\rho_\omega(\lambda)}.$$

Let

$$N_2^* = \sqrt{2}\,[\,0 \quad I_p\,], \qquad B_t(\lambda) = A_t(\lambda)\mathfrak{V}$$

and

$$\mathcal{E}_t(\lambda) = N_2^* B_t(\lambda) = [\,E_-(t,\lambda) \quad E_+(t,\lambda)\,]$$

with $p \times p$ components $E_\pm(t,\lambda)$. Then, since

$$N_2^* K_\omega^{A_t}(\lambda) N_2 = -\frac{\mathcal{E}_t(\lambda) j_p \mathcal{E}_t(\omega)^*}{\rho_\omega(\lambda)},$$

the kernel

$$K_\omega^{\mathcal{E}_t}(\lambda) = -\frac{\mathcal{E}_t(\lambda) j_p \mathcal{E}_t(\omega)^*}{\rho_\omega(\lambda)} = \frac{E_+(t,\lambda)E_+(t,\omega)^* - E_-(t,\lambda)E_-(t,\omega)^*}{\rho_\omega(\lambda)}$$

is also positive and defines a RKHS of entire $p \times 1$ entire mvf's that we shall denote $\mathcal{B}(\mathcal{E}_t)$. These spaces will be called *de Branges spaces*, since they were introduced and extensively studied by L. de Branges; see e.g., [de Branges 1968b; 1968a] and, for additional applications and expository material, [Dym and McKean 1976; Dym 1970; Dym and Iacob 1984]. They can be characterized in terms of the blocks $E_\pm(t,\lambda)$ by the following criteria:

$$f \in \mathcal{B}(\mathcal{E}_t) \iff (E_+^t)^{-1} f \in H_2^p \quad \text{and} \quad (E_-^t)^{-1} f \in K_2^p,$$

where H_2^p denotes the vector Hardy space of order 2 and K_2^p denotes its orthogonal complement with respect to the standard inner product

$$\langle g, h\rangle_{st} = \int_{-\infty}^{\infty} h(\mu)^* g(\mu)\, d\mu \tag{2.1}$$

in $L_2^p(\mathbb{R})$. Moreover, if $f \in \mathcal{B}(\mathcal{E}_t)$, then

$$\|f\|^2_{\mathcal{B}(\mathcal{E}_t)} = \langle (E_+^t)^{-1} f, (E_+^t)^{-1} f\rangle_{st}\,.$$

It turns out that with each matrizant $A_t(\lambda)$, there is a unique associated pair $b_3^t(\lambda)$ and $b_4^t(\lambda)$ of $p\times p$ entire inner mvf's that meet the normalization conditions $b_3^t(0) = I_p$ and $b_4^t(0) = I_p$ and corresponding sets of RKHS's $\mathcal{H}(b_3^t)$ and $\mathcal{H}_*(b_4^t)$ with RK's

$$k_\omega^{b_3^t}(\lambda) = \frac{I_p - b_3^t(\lambda) b_3^t(\omega)^*}{\rho_\omega(\lambda)} \quad \text{and} \quad \ell_\omega^{b_4^t}(\lambda) = \frac{b_4^t(\lambda) b_4^t(\omega)^* - I_p}{\rho_\omega(\lambda)}\,,$$

respectively.

3. Linear fractional transformations

The linear fractional transformation T_U based on the four block decomposition

$$U(\lambda) = \begin{bmatrix} u_{11}(\lambda) & u_{12}(\lambda) \\ u_{12}(\lambda) & u_{22}(\lambda) \end{bmatrix},$$

of an $m\times m$ mvf $U(\lambda)$ that is meromorphic in $\mathbb{C}_+$ with diagonal blocks $u_{11}(\lambda)$ of size $p\times p$ and $u_{22}(\lambda)$ of size $q\times q$ is defined on the set

$$\mathfrak{D}(T_U) = \{p\times q \text{ meromorphic mvf's } \varepsilon(\lambda) \text{ in } \mathbb{C}_+ \\ \text{such that } \det\{u_{21}(\lambda)\varepsilon(\lambda) + u_{22}(\lambda)\} \not\equiv 0 \text{ in } \mathbb{C}_+\}$$

by the formula

$$T_U[\varepsilon] = (u_{11}\varepsilon + u_{12})(u_{21}\varepsilon + u_{22})^{-1}.$$

If $U_1, U_2 \in \mathcal{U}(J)$ and if $\varepsilon \in \mathfrak{D}(T_{U_2})$ and $T_{U_2}[\varepsilon] \in \mathfrak{D}(T_{U_1})$ then

$$T_{U_1U_2}[\varepsilon] = T_{U_1}[T_{U_2}[\varepsilon]]\,.$$

The notation

$$T_U[E] = \{T_U[\varepsilon] : \varepsilon \in E\} \quad \text{for } E \subseteq \mathfrak{D}(T_U)$$

will be useful.

The principal facts are that

(1) $U \in \mathcal{U}(j_{pq}) \Longrightarrow \mathcal{S}^{p\times q} \subseteq \mathfrak{D}_{T_U}$ and $T_U[\mathcal{S}^{p\times q}] \subseteq \mathcal{S}^{p\times q}$.
(2) $U \in \mathcal{U}(J_p) \Longrightarrow T_U[\mathcal{C}^{p\times p} \cap \mathfrak{D}_{T_U}] \subseteq \mathcal{C}^{p\times p}$.

Moreover, if

$$B(\lambda) = A(\lambda)\mathfrak{V},$$

then

$$T_A[\mathcal{C}^{p\times p} \cap \mathcal{D}(T_A)] \subseteq T_B[\mathcal{S}^{p\times p} \cap \mathcal{D}(T_B)] \subseteq \mathcal{C}^{p\times p},$$

where the first inclusion may be proper. The set

$$\mathcal{C}(A) = T_B[\mathcal{S}^{p\times p} \cap \mathcal{D}(T_B)].$$

is more useful than the set $T_A[\mathcal{C}^{p\times p} \cap \mathcal{D}(T_A)]$.

We remark that

$$\mathcal{S}^{p\times p} \subseteq \mathcal{D}(T_B) \iff b_{22}(\omega)b_{22}(\omega)^* > b_{21}(\omega)b_{21}(\omega)^* \tag{3.1}$$

for some (and hence every) point $\omega \in \mathfrak{h}_A^+$; see Theorem 2.7 in [Arov and Dym 2003a].

4. Restrictions and consequences

In addition to fixing the signature matrix $J = J_p$ in the canonical system (1.1), we shall assume that $\mathcal{H}(A_t) \subset L_2^m$ for every $t \in [0, d)$, i.e., (in our current terminology) A_t belongs to the class $\mathcal{U}_{rsR}(J_p)$ of *right strongly regular J-inner mvf's*. (In our earlier papers the set $\mathcal{U}_{rsR}(J_p)$ was designated $\mathcal{U}_{sR}(J_p)$.) One of the important consequences of this assumption rests on the fact that

$$\mathcal{H}(A_t) \subset L_2^m \iff \mathcal{C}(A_t) \cap \mathring{\mathcal{C}}^{p\times p} \neq \varnothing, \tag{4.1}$$

where

$$\mathring{\mathcal{C}}^{p\times p} = \{c \in \mathcal{C}^{p\times p} : c \in H_\infty^{p\times p} \quad \text{and} \quad \Re c(\mu) \geq \delta_c I_p \quad \text{a.e. on } \mathbb{R}\} \tag{4.2}$$

and $\delta_c > 0$. Other characterizations of the class $\mathcal{U}_{rsR}(J)$ in terms of the Treil–Volberg matrix version of the Muckenhoupt $(A)_2$ condition presented in [Treil and Volberg 1997] are furnished in [Arov and Dym 2001] and [Arov and Dym 2003b].

If $A_t \in \mathcal{U}_{rsR}(J_p)$ for every $t \in [0, d)$, the following conclusions are in force:

(1) The unique normalized monotonic chain of $p \times p$ entire inner mvf's

$$\{b_3^t, b_4^t\} \in ap_{II}(A_t)$$

consists of continuous functions of t on the interval $0 \leq t < d$ for each fixed point $\lambda \in \mathbb{C}$.

(2) The RKHS $\mathcal{H}(A_{t_1})$ is isometrically included in the RKHS $\mathcal{H}(A_{t_2})$ for $0 \leq t_1 \leq t_2 < d$.

(3) The de Branges spaces $\mathcal{B}(\mathfrak{E}_t)$ based on the de Branges matrix

$$\mathfrak{E}_t(\lambda) = \sqrt{2}\,[0 \quad I_p]\,A_t(\lambda)\mathfrak{V}, \quad \text{for } 0 \le t < d$$

are nested by isometric inclusion, i.e., $\mathcal{B}(\mathfrak{E}_{t_1})$ is isometrically included in $\mathcal{B}(\mathfrak{E}_{t_2})$ for $0 \le t_1 \le t_2 < d$.

(4) $\mathcal{B}(\mathfrak{E}_t) = \mathcal{H}(b_3^t) \oplus \mathcal{H}_*(b_4^t)$ as Hilbert spaces with equivalent norms.

(5) The mapping $N_2^* : f \in \mathcal{H}(A_t) \longrightarrow N_2^* f \in \mathcal{B}(\mathfrak{E}_t)$ is unitary for every $t \in [0, d)$.

(6) If $\mathcal{S}^{p\times p} \subseteq \mathfrak{D}(T_{B_{t_0}})$ for some $t_0 \in [0, d)$, then:

(a) $\mathcal{S}^{p\times p} \subseteq \mathfrak{D}(T_{B_t})$ for every $t \in [t_0, d)$.

(b) $\mathcal{C}(A_{t_2}) \subseteq \mathcal{C}(A_{t_1})$ for $t_0 \le t_1 \le t_2 < d$.

(c) $\bigcap_{t_0 \le t < d} \mathcal{C}(A_t) \neq \varnothing$.

(d) $\{c(\omega) : \omega \in \mathbb{C}_+ \quad \text{and} \quad c \in \bigcap_{t_0\le t<d} \mathcal{C}(A_t)\}$ is a (Weyl–Titchmarsh) matrix ball with left and right semiradii $R_\ell(\omega)$ and $R_r(\omega)$ with

$$\operatorname{rank} R_\ell(\omega) = \operatorname{rank}\{\lim_{t\uparrow\infty} b_3^t(\omega) b_3^t(\omega)^*\} \tag{4.3}$$

and

$$\operatorname{rank} R_r(\omega) = \operatorname{rank}\{\lim_{t\uparrow\infty} b_4^t(\omega)^* b_4^t(\omega)\}\,. \tag{4.4}$$

Moreover, these ranks are independent of the choice of the point $\omega \in \mathbb{C}_+$.

An mvf $c(\lambda)$ that belongs to the set

$$\mathcal{C}_{\mathrm{imp}}(H) = \bigcap_{t_0\le t<d} \mathcal{C}(A_t)$$

is called an input impedance (or Weyl function) of the system (1.1). If $H \in L_1^{m\times m}$, then, without loss of generality, it may be assumed that $d < \infty$. In this case, $A_d(\lambda)$ is the monodromy matrix of the system (1.1), $\mathcal{C}_{\mathrm{imp}}(H) = \mathcal{C}(A_d)$ and the semiradii $R_\ell(\omega)$ and $R_r(\omega)$ are both positive definite.

5. Inverse problems for canonical systems

Inverse problems for the canonical system (1.1) aim to recover $H(t)$, given some information about the solution of the system. In this direction it is convenient to first consider inverse problems for the canonical integral system

$$y(t,\lambda) = y(0,\lambda) + i\lambda \int_0^t y(s,\lambda)\, dM(s)\, J_p \quad \text{for } 0 \le t < d\,, \tag{5.1}$$

in which the mass function $M(t)$, $0 \le t < d$ is a continuous nondecreasing $m \times m$ mvf on the interval $[0, d)$ with $M(0) = 0$. Then the matrizant $A_t(\lambda) = A(t, \lambda)$ of this system is a continuous solution of the equation

$$A_t(\lambda) = I_m + i\lambda \int_0^t A_s(\lambda)\, dM(s)\, J_p \quad \text{for } 0 \le t < d \tag{5.2}$$

and consequently,

$$\frac{\partial A_t}{\partial \lambda}(\lambda)\Big|_{\lambda=0} = \lim_{\lambda \to 0} \frac{A_t(\lambda) - I_m}{\lambda} = iM(t)J_p\,. \tag{5.3}$$

Thus, $M(t)$, $0 \le t < d$, can be recovered from the matrizant $A_t(\lambda)$, $0 \le t < d$. The main tool is the following result, which, for a given triple $b_3 \in \mathcal{S}_{in}^{p\times p}$, $b_4 \in \mathcal{S}_{in}^{p\times p}$ and $c \in \mathcal{C}^{p\times p}$, is formulated in terms of the sets

$$\mathcal{C}(b_3, b_4; c) = \{\widetilde{c} \in \mathcal{C}^{p\times p} : b_3^{-1}(\widetilde{c} - c)b_4^{-1} \in \mathcal{N}_+^{p\times p}\}$$

and

$$\mathcal{N}_+^{p\times p} = \left\{\frac{g}{h} : g \in \mathcal{S}^{p\times p} \quad \text{and} \quad h \in \mathcal{S}_{out}^{1\times 1}\right\}.$$

The set $\mathcal{C}(b_3, b_4; c)$ was identified as the set of solutions of a generalized Carathéodory interpolation problem that is formulated in terms of the three given mvf's b_3, b_4 and c in [Arov 1993] and connections with the class $\mathcal{U}(J_p)$ were studied there. These results were developed further in [Arov and Dym 1998] in the case that $b_3(\lambda)$ and $b_4(\lambda)$ are also entire mvf's. In that special case, the interpolation problem is equivalent to a bitangential Krein extension problem in a class of helical mvf's. Krein understood the deep connections between such extension problems and inverse problems for canonical systems. Theorem 5.2, below, illustrates the Krein strategy of identifying the solution of an inverse problem with an appropriately defined chain of extension problems; see [Arov and Dym 2005b] for more details.

THEOREM 5.1. *Let $b_3(\lambda)$, $b_4(\lambda)$ be a pair of entire $p \times p$ inner mvf's and let $c \in \mathcal{C}^{p\times p}$. Then there exists at most one mvf $A \in \mathcal{E} \cap \mathcal{U}(J_p)$ such that*

(1) $\mathcal{C}(A) = \mathcal{C}(b_3, b_4; c)$.
(2) $\{b_3, b_4\} \in ap_{II}(A)$.
(3) $A(0) = I_m$.

Moreover, if

$$\mathcal{C}(b_3, b_4; c) \cap \mathring{\mathcal{C}}^{p\times p} \neq \varnothing\,,$$

there exists exactly one mvf $A \in \mathcal{E} \cap \mathcal{U}(J_p)$ for which these three conditions are met and it is automatically right strongly regular.

Correspondingly, in our formulation of the inverse impedance problem (inverse spectral problem) for the canonical integral system (5.1) we shall specify a continuous monotonic normalized chain of entire inner $p \times p$ entire inner mvf's $\{b_3^t(\lambda), b_4^t(\lambda)\}$, in addition to an mvf $c \in \mathcal{C}^{p\times p}$ (or a spectral function $\sigma(\mu)$). Spectral functions and the inverse spectral problem are introduced in Section 9.

THEOREM 5.2. *Let $\{b_3^t(\lambda), b_4^t(\lambda)\}, 0 \le t < d$, be a normalized monotonic continuous chain of pairs of entire inner $p \times p$ mvf's and let $c \in \mathcal{C}^{p\times p}$. Then there exists at most one Hamiltonian $H(t)$, $0 \le t < d$, such that the matrizant $A_t(\lambda)$ of the corresponding canonical system meets the following conditions for every $t \in [0, d)$:*

(1) $\mathcal{C}(A_t) = \mathcal{C}(b_3^t, b_4^t; c)$.
(2) $\{b_3^t, b_4^t\} \in ap_{II}(A_t)$.
(3) $A_t(0) = I_m$.

There exists exactly one continuous nondecreasing mvf $M(t)$ on the interval $[0, d)$ with $M(0) = 0$ such that the matrizant A_t of the integral system (5.1) *meets these conditions if*

$$\mathcal{C}(b_3^t, b_4^t; c) \cap \mathring{\mathcal{C}}^{p\times p} \neq \varnothing \text{ for every } t \in [0, d)\ . \tag{5.4}$$

PROOF. See Theorem 7.9 in [Arov and Dym 2003a]. □

6. Description of the RKHS's $\mathcal{H}(A)$ and $\mathcal{B}(\mathfrak{E})$ for $A \in \mathcal{U}_{rsR}(J_p)$

THEOREM 6.1. *If $A \in \mathcal{U}_{rsR}(J_p)$, $\{b_3(\lambda), b_4(\lambda)\} \in ap_{II}(A)$ and $c \in \mathcal{C}(A) \cap H_\infty^{p\times p}$, then*

$$\mathcal{H}(A) = \left\{ \begin{bmatrix} -\Pi_+ c^* g + \Pi_- ch \\ g + h \end{bmatrix} : g \in \mathcal{H}(b_3) \text{ and } h \in \mathcal{H}_*(b_4) \right\},$$

where Π_+ denotes the orthogonal projection of L_2^p onto the Hardy space H_2^p, $\Pi_- = I - \Pi_+$ denotes the orthogonal projection of L_2^p onto $K_2^p = L_2^p \ominus H_2^p$,

$$\mathcal{H}(b_3) = H_2^p \ominus b_3 H_2^p \quad \text{and} \quad \mathcal{H}_*(b_4) = K_2^p \ominus b_4^* K_2^p\ .$$

Moreover,

$$f = \begin{bmatrix} -\Pi_+ c^* g + \Pi_- ch \\ g + h \end{bmatrix} \Longrightarrow \langle f, f\rangle_{\mathcal{H}(A)} = \langle (c + c^*)(g + h), g + h\rangle_{st}\ ,$$

where $g \in \mathcal{H}(b_3)$, $h \in \mathcal{H}_*(b_4)$ and $\langle \cdot, \cdot \rangle_{st}$ denotes the standard inner product (2.1) *in L_2^p.*

PROOF. See Theorem 3.8 in [Arov and Dym 2005a]. □

THEOREM 6.2. *If* $A \in \mathcal{U}_{rsR}(J_p)$, $\{b_3, b_4\} \in ap_{II}(A)$ *and* $\mathfrak{E}(\lambda) = N_2^* B(\lambda)$, *then*

$$\langle f, f\rangle_{\mathcal{H}(A)} = 2\|[0 \quad I_p] f\|^2_{\mathcal{B}(\mathfrak{E})}$$

for every $f \in \mathcal{H}(A)$ *and*

$$\mathcal{B}(\mathfrak{E}) = \mathcal{H}(b_3) \oplus \mathcal{H}_*(b_4) \quad \textit{as Hilbert spaces with equivalent norms}.$$

PROOF. See Theorem 3.8 in [Arov and Dym 2005a]. □

REMARK 6.3. If $\mathcal{B}(\mathfrak{E}) = \mathcal{H}(b_3) \oplus \mathcal{H}_*(b_4)$ as linear spaces, the two norms in these spaces are equivalent, i.e., there exist a pair of positive constants γ_1, γ_2 such that

$$\gamma_1 \|f\|_{st} \le \|f\|_{\mathcal{B}(\mathfrak{E})} \le \gamma_2 \|f\|_{st}$$

for every $f \in \mathcal{B}(\mathfrak{E})$. This follows from the closed graph theorem and the fact that $\mathcal{B}(\mathfrak{E})$ and $\mathcal{H}(b_3) \oplus \mathcal{H}_*(b_4)$ are both RKHS's.

7. A basic conclusion

In order to apply Theorem 5.2, we need to know when condition (5.4) is in force. In particular, the condition (5.4) is satisfied if $c \in \mathring{\mathcal{C}}^{p\times p}$. However, if the given matrix $c \in \mathcal{C}^{p\times p} \cap \mathcal{W}_+^{p\times p}(\gamma)$, i.e., if $c(\lambda)$ admits a representation of the form

$$c(\lambda) = \gamma_c + \int_0^\infty e^{i\lambda t} h_c(t)\,dt \quad \text{with } \gamma_c \in \mathbb{C}^{p\times p} \quad \text{and} \quad h_c \in L_1^{p\times p}([0,\infty)), \tag{7.1}$$

then condition (5.4) will be in force if $\gamma_c + \gamma_c^* > 0$, even if $\det \Re\, c(\mu) = 0$ at some points $\mu \in \mathbb{R}$; see Theorem 5.2 in [Arov and Dym 2005a]. Moreover, if either

$$\lim_{\nu\uparrow\infty} b_3^{t_0}(i\nu) = 0 \quad \text{or} \quad \lim_{\nu\uparrow\infty} b_4^{t_0}(i\nu) = 0$$

for some point $t_0 \in [0, d)$, the condition $\gamma + \gamma^* > 0$ is necessary for (5.4) to be in force and hence for the existence of a canonical system (1.1) with a matrizant $A_t(\lambda)$, $0 \le t < d$, that meets the conditions (1) (2) and (3) in Theorem 5.2; see Theorem 5.4 in [Arov and Dym 2005a].

REMARK 7.1. The method of solution depends upon the interplay between the RKHS's that play a role in the parametrization formulas presented in Theorem 6.1 and their corresponding RK's. This method also yields the formulas for $M(t)$ and the corresponding matrizant $A_t(\lambda)$ that are discussed in the next section. It differs from the known methods of Gelfand–Levitan, Marchenko and Krein, which are not directly applicable to the bitangential problems under consideration.

8. An algorithm for solving the inverse impedance problem

In this section we shall assume that an mvf $c \in \mathcal{C}^{p\times p}$ and a normalized monotonic continuous chain of pairs $\{b_3^t(\lambda), b_4^t(\lambda)\}$, $0 \le t < d$, of entire inner $p\times p$ mvf's that meet the condition (5.4) have been specified. Then there exists an mvf

$$c^t \in \mathcal{C}(b_3^t, b_4^t; c) \cap H_\infty^{p\times p} \tag{8.1}$$

for every $t \in [0, d)$ and hence, the operators

$$\Phi_{11}^t = \Pi_{\mathcal{H}(b_3^t)} M_{c^t}\Big|_{H_2^p}, \quad \Phi_{22}^t = \Pi_- M_{c^t}\Big|_{\mathcal{H}_*(b_4^t)}, \quad \Phi_{12}^t = \Pi_{\mathcal{H}(b_3^t)} M_{c^t}\Big|_{\mathcal{H}_*(b_4^t)}, \tag{8.2}$$

$$Y_1^t = \Pi_{\mathcal{H}(b_3^t)}\{M_{c^t} + (M_{c^t})^*\}|_{\mathcal{H}(b_3^t)} = 2\Re\big(\Phi_{11}^t|_{\mathcal{H}(b_3^t)}\big) \tag{8.3}$$

and

$$Y_2^t = \Pi_{\mathcal{H}_*(b_4^t)}\{M_{c^t} + (M_{c^t})^*\}|_{\mathcal{H}_*(b_4^t)} = 2\Re\big(\Pi_{\mathcal{H}_*(b_4^t)}\Phi_{22}^t\big) \tag{8.4}$$

are well defined. Moreover, they do not depend upon the specific choice of the mvf c^t in the set indicated in formula (8.1). In order to keep the notation relatively simple, an operator T that acts in the space of $p \times 1$ vvf's will be applied to $p \times p$ mvf's with columns $f_1, \dots, f_p$ column by column: $T[f_1 \cdots f_p] = [Tf_1 \cdots Tf_p]$.

We define three sets of $p\times p$ mvf's $\widehat{y}_{ij}^t(\lambda)$, $\widehat{u}_{ij}^t(\lambda)$ and $\widehat{x}_{ij}^t(\lambda)$ by the following system of equations, in which $\tau_3(t)$ and $\tau_4(t)$ denote the exponential types of $b_3^t(\lambda)$ and $b_4^t(\lambda)$, respectively,

$$e_t(\lambda) = e^{i\lambda t} \quad \text{and} \quad (R_0 f)(\lambda) = \{f(\lambda) - f(0)\}/\lambda \; :$$

$$\begin{aligned} \widehat{y}_{11}^t(\lambda) &= i\big(\Phi_{11}^t(R_0 e_{\tau_3(t)} I_p)\big)(\lambda)\,, & \widehat{y}_{12}^t(\lambda) &= -i(R_0 b_3^t)(\lambda), \\ \widehat{y}_{21}^t(\lambda) &= i\big((\Phi_{22}^t)^*(R_0 e_{-\tau_4(t)} I_p)\big)(\lambda)\,, & \widehat{y}_{22}^t(\lambda) &= i(R_0 (b_4^t)^{-1})(\lambda), \end{aligned} \tag{8.5}$$

THEOREM 8.1. *In the setting of this section, there exists exactly one mvf* $A_t \in \mathcal{E} \cap \mathcal{U}(J_p)$ *for each* $t \in [0, d)$ *such that*:

(1) $\mathcal{C}(A_t) = \mathcal{C}(b_3^t, b_4^t; c)$.
(2) $\{b_3^t, b_4^t\} \in ap_{II}(A_t)$.
(3) $A_t(0) = I_m$.

The RK $K_\omega^t(\lambda)$ *of the RKHS* $\mathcal{H}(A_t)$ *evaluated at* $\omega = 0$ *is given by the formula*

$$K_0^t(\lambda) = \frac{1}{2\pi}\begin{bmatrix} \widehat{x}_{11}^t(\lambda) + \widehat{x}_{21}^t(\lambda) & \widehat{x}_{12}^t(\lambda) + \widehat{x}_{22}^t(\lambda) \\ \widehat{u}_{11}^t(\lambda) + \widehat{u}_{21}^t(\lambda) & \widehat{u}_{12}^t(\lambda) + \widehat{u}_{22}^t(\lambda) \end{bmatrix}, \tag{8.6}$$

where:

(1) *The* $\widehat{u}^t_{ij}(\lambda)$ *are* $p\times p$ *mvf's such that the columns of* $\widehat{u}^t_{1j}(\lambda)$ *belong to* $\mathcal{H}(b^t_3)$ *and the columns of* $\widehat{u}^t_{2j}(\lambda)$ *belong to* $\mathcal{H}_*(b^t_4)$. *The* $\widehat{u}^t_{ij}(\lambda)$ *may be defined as the solutions of the systems of equations*:

$$\begin{aligned} Y^t_1\widehat{u}^t_{1j}+\Phi^t_{12}\widehat{u}^t_{2j} &= \widehat{y}^t_{1j}(\lambda)\\ (\Phi^t_{12})^*\widehat{u}^t_{1j}+Y^t_2\widehat{u}^t_{2j} &= \widehat{y}^t_{2j}(\lambda)\ ,\ \ j=1,2\,. \end{aligned} \tag{8.7}$$

(2) *The mvf's* $\widehat{x}^t_{ij}(\lambda)$ *are defined by the formulas*

$$\begin{aligned} \widehat{x}^t_{1j}(\lambda) &= -(\Phi^t_{11})^*\widehat{u}^t_{1j}\ ,\\ \widehat{x}^t_{2j}(\lambda) &= \Phi^t_{22}\widehat{u}^t_{2j}\ ,\ \ j=1,2\,. \end{aligned} \tag{8.8}$$

PROOF. This theorem is Theorem 4.2 of [Arov and Dym 2005a]. □

REMARK 8.2. In the one-sided cases when either $b^t_4(\lambda)=I_p$ or $b^t_3(\lambda)=I_p$, the formulas for recovering $M(t)$ are simpler:

If, for example, $b^t_4(\lambda)=I_p$, then $\tau_4(t)=0$ and $\mathcal{H}_*(b^t_4)=\{0\}$ and hence equations (8.7) and (8.8) simplify to

$$Y^t_1\widehat{u}^t_{1j}=\widehat{y}^t_{1j}(\lambda)\quad\text{and}\quad \widehat{x}^t_{1j}=-(\Phi^t_{11})^*\widehat{u}^t_{1j}\quad\text{for } j=1,2\,, \tag{8.9}$$

and

$$\widehat{u}^t_{2j}=0\quad\text{and}\quad\widehat{x}^t_{2j}=0\quad\text{for } j=1,2\,.$$

THEOREM 8.3. *Let* $\{c(\lambda);b^t_3(\lambda),b^t_4(\lambda),\,0\le t<d\}$ *be given where* $c\in\mathcal{C}^{p\times p}$, $\{b^t_3(\lambda),b^t_4(\lambda)\}$, $0\le t<d$, *is a normalized monotonic continuous chain of pairs of entire inner* $p\times p$ *mvf's and let assumption* (5.4) *be in force. Then the unique solution* $M(t)$ *of the inverse input impedance problem considered in Theorem* 5.2 *is given by the formula*

$$\begin{aligned} M(t) &= 2\pi K^t_0(0)\\ &= \int_0^{\tau_3(t)}\begin{bmatrix} x^t_{11}(a) & x^t_{12}(a)\\ u^t_{11}(a) & u^t_{12}(a)\end{bmatrix}da+\int_{-\tau_4(t)}^0\begin{bmatrix} x^t_{21}(a) & x^t_{22}(a)\\ u^t_{21}(a) & u^t_{22}(a)\end{bmatrix}da \end{aligned} \tag{8.10}$$

and the corresponding matrizant may be defined by the formula

$$A_t(\lambda)=I_m+2\pi i\lambda K^t_0(\lambda)J_p\ , \tag{8.11}$$

where $K^t_0(\lambda)$ *is specified by formula* (8.6) *and* $x^t_{ij}(a)$ *and* $u^t_{ij}(a)$ *designate the inverse Fourier transforms of* $\widehat{x}^t_{ij}(\lambda)$ *and* $\widehat{u}^t_{ij}(\lambda)$, *respectively.*

PROOF. Formula (8.11) follows from the definition of the RK $K^t_0(\lambda)=K^{A_t}_0(\lambda)$ and the fact that $A_t(0)=I_m$. Formula (8.10) follows from (5.3), (8.6) and (8.11). □

9. Spectral functions

The term *spectral function* is defined in two different ways: The first definition is in terms of the generalized Fourier transform

$$(\mathscr{F}_2 f)(\lambda) = [0 \quad I_p]\frac{1}{\sqrt{\pi}}\int_0^d A(s,\lambda)\,dM(s)\,f(s) \tag{9.1}$$

based on the matrizant of the canonical system (5.1) applied initially to the set of $f \in L_2^m(dMs;[0,d))$ with compact support inside the interval $[0,d)$.

A nondecreasing $p \times p$ mvf $\sigma(\mu)$ on $\mathbb{R}$ is said to be a *spectral function* for the system (5.1) if the Parseval equality

$$\int_{-\infty}^{\infty}(\mathscr{F}_2 f)(\mu)^* d\sigma(\mu)(\mathscr{F}_2 f)(\mu) = \int_0^d f(t)^* dM(t) f(t) \tag{9.2}$$

holds for every $f \in L_2^m(dMs;[0,d))$ with compact support. The notation $\Sigma_{sf}^d(M)$ will be used to denote the set of spectral functions of a canonical system of the form (5.1).

REMARK 9.1. The generalized Fourier transform introduced in formula (9.1) is a special case of the transform

$$(\mathscr{F}^L f)(\lambda) = L^*\frac{1}{\sqrt{\pi}}\int_0^d A(s,\lambda)\,dM(s)\,f(s) \tag{9.3}$$

that is based on a fixed $m \times p$ matrix L that meets the conditions $L^* J_p L = 0$ and $L^* L = I_p$. The mvf $y(t,\lambda) = L^* A_t(\lambda)$ is the unique solution of the system (5.1) that satisfies the initial condition $y(0,\lambda) = L^*$. Spectral functions may be defined relative to the transform $\mathscr{F}^L$ in just the same way that they were defined for the transform $\mathscr{F}_2$. Direct and inverse spectral problems for these spectral functions are easily reduced to the corresponding problems based on $\mathscr{F}_2$.; see Sections 4 and 5 of [Arov and Dym 2004] and Section 16 of [Arov and Dym 2005c] for additional discussion.

The second definition of spectral function is based on the Riesz–Herglotz representation

$$c(\lambda) = i\alpha - i\beta\lambda + \frac{1}{\pi i}\int_{-\infty}^{\infty}\left\{\frac{1}{\mu-\lambda} - \frac{\mu}{1+\mu^2}\right\}d\sigma(\mu)\,, \quad \lambda \in \mathbb{C}_+\,, \tag{9.4}$$

which defines a correspondence between $p \times p$ mvf's $c \in \mathscr{C}^{p\times p}$ and a set $\{\alpha, \beta, \sigma\}$, in which $\sigma(\mu)$ is a nondecreasing $p \times p$ mvf on $\mathbb{R}$ that is normalized to be left continuous with $\sigma(0) = 0$ and is subject to the constraint

$$\int_{-\infty}^{\infty}\frac{d\,\mathrm{trace}\,\sigma(\mu)}{1+\mu^2} < \infty\,, \tag{9.5}$$

and α and β are constant $p \times p$ matrices such that $\alpha = \alpha^*$ and $\beta \geq 0$.

The mvf $\sigma(\mu)$ in the representation (9.4) will be referred to as the *spectral function* of $c(\lambda)$. Correspondingly, if $\mathcal{F} \subseteq \mathcal{C}^{p\times p}$, then

$$(\mathcal{F})_{sf} = \{\sigma : \sigma \text{ is the spectral function of some } c \in \mathcal{F}\}.$$

If $A \in \mathcal{U}_{rsR}(J_p)$ and $c(\lambda) = T_A[I_p]$, then $\beta = 0$ and $\sigma(\mu)$ is absolutely continuous with $\sigma'(\mu) = \Re c(\mu)$ a.e. on $\mathbb{R}$; see Lemma 2.2 and the discussion following Lemma 2.3 in [Arov and Dym 2005a]. Moreover, if $A \in \mathcal{U}_{rsR}(J_p)$ and $\mathcal{S}^{p\times p} \subseteq \mathfrak{D}(T_{A\mathfrak{V}})$, then for each $\sigma \in (\mathcal{C}(A))_{sf}$, there exists at least one $p \times p$ Hermitian matrix α such that

$$c^{(\alpha)}(\lambda) = i\alpha + \frac{1}{\pi i}\int_{-\infty}^{\infty}\left\{\frac{1}{\mu-\lambda} - \frac{\mu}{1+\mu^2}\right\} d\sigma(\mu) \tag{9.6}$$

belongs to $\mathcal{C}(A)$; see Theorem 2.14 in [Arov and Dym 2004].

We shall also make use of the following condition on the growth of the mvf $\chi_1^t(\lambda) = b_4^t(\lambda)b_3^t(\lambda)$:

$$\|\chi_1^a(re^{i\theta} + \omega)\| \leq \gamma < 1 \quad \text{on the indicated ray } r \geq 0 \text{ in } \mathbb{C}_+, \tag{9.7}$$

i.e., the inequality holds for some fixed choice of $\theta \in [0, \pi]$, $\omega \in \mathbb{C}_+$, $a \in (0, d)$ and all $r \geq 0$. It is readily checked that if this inequality is in force for some point $a \in (0, d)$, then it holds for all $t \in [a, d)$.

REMARK 9.2. The condition (9.7) will be in force if

$$e_{-a}\chi_1^{t_0}(\lambda) \in \mathcal{S}_{in}^{p\times p}$$

for some choice of $a > 0$ and $t_0 \in (0, d)$.

These observations leads to the following conclusion:

LEMMA 9.3. *If the matrizant $A_t(\lambda)$ of the canonical differential system* (1.1) *with $J = J_p$ satisfies the condition*

$$\mathcal{C}(A_t) \cap \mathring{\mathcal{C}}^{p\times p} \neq \varnothing \quad \textit{for every } t \in [0, d) \tag{9.8}$$

and if condition (9.7) *is in force for some $a \in [0, d)$ when $\{b_3^t, b_4^t\} \in ap_{II}(A_t)$ for $t \in [0, d)$ and if $c \in \mathcal{C}(A_a)$, then $\beta = 0$ in the representation* (9.4).

In view of the fact that

$$A \in \mathcal{U}_{rsR}(J_p) \iff \mathcal{C}(A) \cap \mathring{\mathcal{C}}^{p\times p} \neq \varnothing, \tag{9.9}$$

the conditions (5.4) and (9.8) are equivalent if $\mathcal{C}(A_t) = \mathcal{C}(b_3^t, b_4^t; c)$ for every $0 \leq t < d$. In particular, these conditions are satisfied if $\mathcal{C}_{\text{imp}}^d(M) \cap \mathring{\mathcal{C}}^{p\times p} \neq \varnothing$. They are also satisfied if there exists an mvf $c \in \mathcal{C}_{\text{imp}}^d(M)$ of the form (7.1) with $\gamma + \gamma^* > 0$, by Theorem 5.2 in [Arov and Dym 2005a].

Direct problems. Two direct problems for a given canonical system with mass function $M(t)$ on the interval $[0,d)$ are to describe the set of input impedances $\mathscr{C}^d_{\mathrm{imp}}(M)$ and the set $\Sigma^d_{sf}(M)$ of spectral functions of the system.

THEOREM 9.4. *Let $A_t(\lambda)$ denote the matrizant of a canonical integral system* (5.1) *and suppose that the two conditions* (5.4) *and* (9.7) *are met. Then*

(1) $\Sigma^d_{sf}(M) = (\mathscr{C}^d_{\mathrm{imp}}(M))_{sf}$.

(2) *To each $\sigma \in \Sigma^d_{sf}(M)$ there exists exactly one mvf $c(\lambda) \in \mathscr{C}^d_{\mathrm{imp}}(M)$ with spectral function $\sigma(\mu)$. Moreover, this mvf $c(\lambda)$ is equal to one of the mvf's $c^{(\alpha)}(\lambda)$ defined by formula* (9.6) *for some Hermitian matrix α.*

(3) *If $d < \infty$ and* trace $M(t) < \delta < \infty$ *for every $t \in [0,d)$, equation* (5.1) *and the matrizant $A_t(\lambda)$ may be considered on the closed interval $[0,d]$ and $\mathscr{C}^d_{\mathrm{imp}}(M) = \mathscr{C}(A_d)$.*

PROOF. See Theorem 2.21 in [Arov and Dym 2004]. □

A spectral function $\sigma \in \Sigma^d_{sf}(M)$ of the canonical integral system (5.1) with $J = J_p$ is said to be *orthogonal* if the isometric operator that extends the generalized Fourier transform $\mathscr{F}_2$ defined by formula (9.1) maps $L^m_2(dM;[0,d))$ onto $L^p_2(d\sigma;\mathbb{R})$.

THEOREM 9.5. *Let the canonical integral system* (5.1) *with mass function $M(t)$ and matrizant $A_t(\lambda)$ be considered on a finite closed interval $[0,d]$* (*so that* trace $M(d) < \infty$) *and let $A(\lambda) = A_d(\lambda)$, $B(\lambda) = A(\lambda)\mathfrak{V}$ and $\mathfrak{E}(\lambda) = \sqrt{2}[0 \quad I_p]B(\lambda)$. Suppose further that*

(a) $(\mathscr{C}(A))_{sf} = \Sigma^d_{sf}(M)$ *and*

(b) $K^{\mathfrak{E}}_\omega(\omega) > 0$ *for at least one* (*and hence every*) *point $\omega \in \mathbb{C}_+$.*

Then:

(1) $\mathscr{S}^{p\times p} \subseteq \mathfrak{D}(T_B)$.

(2) *The spectral function $\sigma(\mu)$ of the mvf $c(\lambda) = T_B[\varepsilon]$ is an orthogonal spectral function of the given canonical system if ε is a constant $p\times p$ unitary matrix.*

PROOF. The first assertion is equivalent to condition (b); see (3.1). The proof of assertion (2) will be given elsewhere. □

10. The bitangential inverse spectral problem

In our formulation of the *bitangential inverse spectral problem* the given data $\{\sigma; b^t_3, b^t_4,\ 0 \le t < d\}$ is a $p\times p$ nondecreasing mvf $\sigma(\mu)$ on $\mathbb{R}$ that meets the constraint (9.5) and a normalized monotonic continuous chain $\{b^t_3, b^t_4\}$, $0 \le t < d$, of pairs of entire inner $p\times p$ mvf's. An $m\times m$ mvf $M(t)$ on the interval

$[0, d)$ is said to be a solution of the bitangential inverse spectral problem with data $\{\sigma(\mu); b_3^t(\lambda), b_4^t(\lambda),\ 0 \le t < d\}$ if $M(t)$ is a continuous nondecreasing $m \times m$ mvf on the interval $[0, d)$ with $M(0) = 0$ such that the matrizant $A_t(\lambda)$ of the corresponding canonical integral system (5.1) meets the following three conditions:

(i) $\sigma(\mu)$ is a spectral function for this system, i.e., $\sigma \in \Sigma_{sf}^d(M)$.

(ii) $\{b_3^t, b_4^t\} \in ap_{II}(A_t)$ for every $t \in [0, d)$.

(iii) $A_t \in \mathcal{U}_{rsR}(J_p)$ for every $t \in [0, d)$.

The constraint (ii) defines the class of canonical integral systems in which we look for a solution of the inverse problem for the given spectral function $\sigma(\mu)$. Subsequently, the condition (iii) guarantees that in this class there is at most one solution.

The solution of this problem rests on the preceding analysis of the bitangential inverse input impedance problem with data $\{c^{(\alpha)};\ b_3^t, b_4^t,\ 0 \le t < d\}$, where $c^{(\alpha)}(\lambda)$ is given by formula (9.6).

THEOREM 10.1. *If the data* $\{\sigma; b_3^t, b_4^t,\ 0 \le t < d\}$ *for a bitangential inverse spectral problem meets the conditions* (9.5) *and* (9.7) *and the mvf* $c(\lambda) = c^{(0)}(\lambda)$ *satisfies the constraint* (5.4), *the following conclusions hold*:

(1) *For each Hermitian matrix* $\alpha \in \mathbb{C}^{p\times p}$, *there exists exactly one solution* $M^{(\alpha)}(t)$ *of the bitangential inverse input spectral problem such that* $c^{(\alpha)}(\lambda)$ *is an input impedance for the corresponding canonical integral system* (5.1) *with* $J = J_p$ *based on the mass function* $M^{(\alpha)}(t)$.

(2) *The solutions* $M^{(\alpha)}(t)$ *are related to* $M^{(0)}(t)$ *by the formula*

$$M^{(\alpha)}(t) = \begin{bmatrix} I_p & i\alpha \\ 0 & I_p \end{bmatrix} M^{(0)}(t) \begin{bmatrix} I_p & 0 \\ -i\alpha & I_p \end{bmatrix}. \tag{10.1}$$

The corresponding matrizants are related by the formula

$$A_t^{(\alpha)}(\lambda) = \begin{bmatrix} I_p & i\alpha \\ 0 & I_p \end{bmatrix} A_t^{(0)}(\lambda) \begin{bmatrix} I_p & -i\alpha \\ 0 & I_p \end{bmatrix}. \tag{10.2}$$

(3) *If* $M(t)$ *is a solution of the bitangential inverse spectral problem, then* $M(t) = M^{(\alpha)}(t)$ *for exactly one Hermitian matrix* $\alpha \in \mathbb{C}^{p\times p}$.

(4) *The solution* $M^{(0)}(t)$ *and matrizant* $A_t^{(0)}(\lambda)$ *may be obtained from the formulas for the solution of the bitangential inverse input impedance problem with data* $\{c^{(0)}; b_3^t, b_4^t, 0 \le t < d\}$ *that are given in Theorem* 8.3.

PROOF. See Theorem 2.20 in [Arov and Dym 2004]. □

The condition (5.4) is clearly satisfied if $c^{(0)} \in \mathring{\mathcal{C}}^{p\times p}$. However, this condition is far from necessary. If, for example, $c^{(0)}$ is of the form (7.1) with $\gamma+\gamma^*>0$, then, as noted earlier, condition (5.4) holds if $\gamma+\gamma^*>0$, even if $\det\{\Re c(\mu)\}=0$ on some set of points $\mu\in\mathbb{R}$.

EXAMPLE. If $c^\circ(\lambda)=I_p$, i.e., if $\sigma(\mu)=\mu$, then the unique solution of the inverse input impedance problem based given data $\{b_3^t, b_4^t; c^\circ\}$ is

$$M^{(0)}(t)=\mathfrak{V}\begin{bmatrix} m_3(t) & 0\\ 0 & m_4(t)\end{bmatrix}\mathfrak{V},$$

where

$$m_3(t)=-i\frac{\partial b_3^t}{\partial\lambda}|_{\lambda=0} \quad\text{and}\quad m_4(t)=-i\frac{\partial b_4^t}{\partial\lambda}|_{\lambda=0}. \tag{10.3}$$

Moreover, in this case

$$A_t^{(0)}(\lambda)=\mathfrak{V}\begin{bmatrix} b_3^t(\lambda) & 0\\ 0 & (b_4^t)^\#(\lambda)\end{bmatrix}\mathfrak{V},\quad B_t^{(0)}(\lambda)=\frac{1}{\sqrt{2}}\begin{bmatrix} -b_3^t(\lambda) & (b_4^t)^\#(\lambda)\\ b_3^t(\lambda) & (b_4^t)^\#(\lambda)\end{bmatrix},$$

$\mathfrak{E}^t(\lambda)=[\,b_3^t(\lambda)\ (b_4^t)^\#(\lambda)\,]$ and $\mathcal{B}(\mathfrak{E}^t)=\mathcal{H}(b_3^t)\oplus\mathcal{H}_*(b_4^t)$ as equivalent RKHS's. If $\|b_4^s(\omega)b_3^s(\omega)\|<1$ for some $\omega\in\mathbb{C}_+$ and $s\in(0,d)$, then

$$\mathcal{C}(A_t^{(0)})=\{T_{\mathfrak{V}}[b_3^t\varepsilon b_4^t]:\ \varepsilon\in\mathcal{S}^{p\times p}\}\quad\text{for } t\ge s.$$

Although the choice $c^\circ(\lambda)=I_p$ in the preceding example is very special, the exhibited one-to-one correspondence between monotonic normalized continuous chains of $p\times p$ entire inner mvf's $\{b_3^t(\lambda), b_4^t(\lambda)\}$ and pairs $\{m_3(t), m_4(t)\}$ of continuous nondecreasing $p\times p$ mvf's on the interval $[0,d)$ with $m_3(0)=m_4(0)=0$ exhibited in (10.3) is completely general. Moreover, the mvf's $b_3^t(\lambda)$ and $b_4^t(\lambda)$ are the unique continuous solutions of the integral equations

$$b_3^t(\lambda)=I_p+i\lambda\int_0^t b_3^s(\lambda)\,dm_3(s)\quad\text{and}\quad b_4^t(\lambda)=I_p+i\lambda\int_0^t b_4^s(\lambda)\,dm_4(s),$$

respectively, for $0\le t<d$.

11. Differential systems with potential

The results referred to above have implications for differential systems of the form

$$u'(t,\lambda)=i\lambda u(t,\lambda)NJ+u(t,\lambda)\mathcal{V}(t),\quad 0\le t<d, \tag{11.1}$$

with an $m\times m$ signature matrix J, a constant $m\times m$ matrix N such that

$$N\ge 0 \tag{11.2}$$

and an $m \times m$ matrix valued potential $\mathcal{V}(t)$ such that

$$\mathcal{V} \in L_{1,loc}^{m\times m}([0,d)) \quad \text{and} \quad \mathcal{V}(t)J + J\mathcal{V}(t)^* = 0 \quad \text{a.e. on } [0,d)\,. \tag{11.3}$$

It is readily checked that the matrizant $U_t(\lambda) = U(t,\lambda)$, $0 \le t < d$, of this system satisfies the identity

$$\{U_t(\lambda)JU_t(\omega)^*\}' = i(\lambda - \overline{\omega})U_t(\lambda)NU_t(\omega)^* \quad \text{for } 0 \le t < d, \tag{11.4}$$

and hence that

$$\frac{J - U_t(\lambda)JU_t(\omega)^*}{\rho_\omega(\lambda)} = \frac{1}{2\pi}\int_0^t U_s(\lambda)NU_s(\omega)^* ds \quad \text{for } 0 \le t < d. \tag{11.5}$$

This in turn leads easily to the conclusion that

$$U_t \in \mathcal{U}(J) \quad \text{for every } t \in [0,d)\,. \tag{11.6}$$

In particular, $U_t(0)$ is J-unitary and so invertible. Moreover, the mvf

$$Y_t(\lambda) = U_t(\lambda)U_t(0)^{-1} \quad \text{for } 0 \le t < d,$$

is the matrizant of the canonical differential system

$$y'(t,\lambda) = i\lambda y(t,\lambda)H(t)J, \quad 0 \le t < d, \tag{11.7}$$

with Hamiltonian

$$H(t) = U_t(0)NU_t(0)^*, \quad 0 \le t < d. \tag{11.8}$$

THEOREM 11.1. *If*

$$NJ = JN\,, \tag{11.9}$$

then the matrizants $U_t(\lambda)$ *and* $Y_t(\lambda)$ *of the systems* (11.1) *and* (11.7) *are both right strongly regular*:

$$U_t \in \mathcal{U}_{rsR}(J) \quad \textit{and} \quad Y_t \in \mathcal{U}_{rsR}(J) \quad \textit{for every } t \in [0,d)\,, \tag{11.10}$$

and

$$\bigcap_{0\le t<d} \mathcal{C}(Y_t) = \bigcap_{0\le t<d} \mathcal{C}(U_t)\,.$$

PROOF. See Section 3 in [Arov and Dym 2005b]. □

In particular, the condition $NJ = JN$ is met if N is a convex combination of the orthogonal projections

$$P_J = \frac{I_m + J}{2} \quad \text{and} \quad Q_J = \frac{I_m - J}{2}\,,$$

or, even more generally, if $N = N_{\gamma,\delta}$, where

$$N_{\gamma,\delta} = \gamma P_J + \delta Q_J \quad \text{with} \quad \gamma \ge 0\,,\ \delta \ge 0 \quad \text{and} \quad \kappa = \gamma + \delta > 0\,. \tag{11.11}$$

In [Arov and Dym 2004; 2005c; 2005b], the direct and inverse impedance spectral problems are considered under the assumption that

$$\operatorname{rank} P_J = \operatorname{rank} Q_J = p\,,$$

i.e., that J is unitarily equivalent to J_p. If $\gamma = \delta$, then the system (11.1) is called a Dirac system: if $\gamma = 0$ or $\delta = 0$, it is called a Krein system. In the sequel we shall take $J = J_p$ in order to simplify the exposition.

The *generalized Fourier transform* for the system (11.1) with $J = J_p$ is defined by the formula

$$g^{\triangle}(\lambda) = [0 \quad I_p]\frac{1}{\sqrt{\pi}}\int_0^d U(s,\lambda)Ng(s)\,ds \tag{11.12}$$

for every $g \in L_2^m(Nds;[0,d))$ with compact support in $[0,d)$. Correspondingly, a nondecreasing $p \times p$ mvf $\sigma(\mu)$ on $\mathbb{R}$ for which the Parseval equality

$$\int_{\infty}^{\infty} g^{\triangle}(\mu)^* d\sigma(\mu) g^{\triangle}(\mu) = \int_0^d g(s)^* Ng(s)\,ds \tag{11.13}$$

holds for every $g \in L_2^m(Nds;[0,d))$ with compact support in $[0,d)$ is called a *spectral function* for the system (11.1), and the symbol $\Sigma_{sf}^d(\mathcal{V})$ denotes the set of spectral functions for this system. The generalized Fourier transform (9.1) for the canonical system with $H(t) = U(t)NU(t)^*$ for $0 \le t < d$, is related to the transform (11.12):

$$(\mathcal{F}_2 f)(\lambda) = [0 \quad I_p]\frac{1}{\sqrt{\pi}}\int_0^d U(s,\lambda)U(s,0)^{-1}H(s)f(s)\,ds \tag{11.14}$$

$$= [0 \quad I_p]\frac{1}{\sqrt{\pi}}\int_0^d U(s,\lambda)NU(s,0)^* f(s)\,ds \tag{11.15}$$

for $f \in L_2^m(H(s)\,ds;[0,d))$ with compact support in $[0,d)$.

The direct problem. The following results on the direct problem are established in [Arov and Dym 2005b]:

THEOREM 11.2. *Let $A_t(\lambda) = A(t,\lambda)$, $0 \le t < d$, be the matrizant of the system* (11.1) *with $J = J_p$ and $N = N_{\gamma,\delta}$ and assume that the potential $\mathcal{V}(t)$ meets the conditions in* (11.3) *and that $\mathcal{V}(t) = \mathcal{V}(t)^*$ a.e. on the interval* $[0,d)$. *Then*:

(1) $A_t \in \mathcal{U}_{rsR}(J_p)$ *for every* $t \in [0,d)$.
(2) $\{e_{\gamma t}I_p, e_{\delta t}I_p\} \in ap_{II}(A_t)$ *for every* $t \in [0,d)$.
(3) *The de Branges spaces $\mathcal{B}(\mathfrak{E}_t)$ based on $\mathfrak{E}_t(\lambda) = \sqrt{2}A_t(\lambda)\mathfrak{V}$ are independent of the potential $\mathcal{V}(t)$ as linear topological spaces*, *i.e.*,

$$\mathcal{B}(\mathfrak{E}_t) = \left\{\int_{-\delta t}^{\gamma t} e^{i\lambda s}h(s)\,ds : h \in L_2^p([-\delta t,\gamma t])\right\}$$

as linear spaces and for each $t \in [0, d)$, *there exist a pair of positive constants* $k_1 = k_1(t)$ *and* $k_2 = k_2(t)$ *such that*

$$k_1 \|f\|_{st} \leq \|f\|_{\mathcal{B}(\mathfrak{E}_t)} \leq k_2 \|f\|_{st}$$

for every $f \in \mathcal{B}(\mathfrak{E}_t)$.

(4) $\mathcal{S}^{p\times p} \subseteq \mathfrak{D}(T_{A_t\mathfrak{V}})$ *for every* $t \in (0, d)$.

(5) $\mathcal{C}(A_t) = T_{A_t\mathfrak{V}}[\mathcal{S}^{p\times p}]$ *for every* $t \in (0, d)$ *and* $\beta = 0$ *in the integral representation* (9.4) *of every mvf* $c \in \mathcal{C}(A_t)$, $0 < t < d$.

(6) *The set of input impedances* $\mathcal{C}^d_{\text{imp}}(\mathcal{V}) = \bigcap_{0\leq t<d} \mathcal{C}(A_t)$ *is not empty.*

(7) $\Sigma^d_{sf}(\mathcal{V}) = (\mathcal{C}^d_{\text{imp}}(\mathcal{V}))_{sf}$ *and the integral representation* (9.4) *defines a one-to-one correspondence between these two sets.*

(8) *If* $d < \infty$ *and* $\mathcal{V} \in L_1([0, d])$, *then* $\mathcal{C}^d_{\text{imp}}(\mathcal{V}) = \mathcal{C}(A_d)$.

(9) *If* $d = \infty$, *the set* $\mathcal{C}^d_{\text{imp}}(V)$ *contains exactly one mvf* $c(\lambda)$. (*This is the Weyl limit point case.*) *If also* $V \in L_1^{m\times m}([0, \infty))$, *then this mvf* $c(\lambda)$ *admits a representation of the form* (7.1) *with* $\gamma_c = I_p$.

PROOF. A proof is supplied in [Arov and Dym 2005b]. □

REMARK 11.3. In the preceding theorem, the sets $\mathcal{C}(A_t)$ depend only upon the potential $\mathcal{V}(t)$ and the positive number $\kappa = \gamma + \delta$ and not on the particular choices $\gamma \geq 0$ and $\delta \geq 0$. This follows from the fact that the mvf $e^{i\delta_0\lambda t}U_t(\lambda)$ is the matrizant of the system (11.1) with potential $\mathcal{V}(t)$ that is independent of δ_0 and with $N = (\gamma + \delta_0)P_J + (\delta - \delta_0)Q_J$ for every number δ_0 in the interval $-\gamma \leq \delta_0 \leq \delta$. Consequently, the sets $\mathcal{C}^d_{\text{imp}}(\mathcal{V})$ and $\Sigma^d_{sf}(\mathcal{V})$ depend only upon the potential $\mathcal{V}(t)$ and the number κ. Thus, any system of the form (11.1) with $N = N_{\gamma,\delta}$ may be reduced to a Dirac system as well as to a Krein system.

The inverse input impedance problem. The data for the inverse input impedance problem for differential systems of the form (11.1) on an interval $[0, d)$ is an mvf $c \in \mathcal{C}^{p\times p}$ and the right hand endpoint d, $0 < d \leq \infty$, of the interval and the problem is to find a locally summable potential $\mathcal{V}(t)$ of the prescribed form on $[0, d)$ such that $c \in \mathcal{C}^d_{\text{imp}}(\mathcal{V})$, the class of input impedances of the system. In the setting of Theorem 11.2, it is not necessary to specify a chain $\{b_3^t, b_4^t\}$, $0 \leq t < d$, to solve this inverse problem, because, as noted in (2) of that theorem, it is automatically prescribed by the choice $N = N_{\gamma,\delta}$.

THEOREM 11.4. *Let an mvf* $c \in \mathcal{C}^{p\times p}$, *a number* d, $0 < d < \infty$ *and an* $m \times m$ *matrix* N *of the form* (11.11) *be given. Then*:

(1) *There exists at most one differential system of the form* (11.1) *with the given* N, $J = J_p$, *and potential* $\mathcal{V}(t) = \mathcal{V}(t)^*$ *a.e. on* $[0, d]$ *that meets the condition* (11.3) *such that* $c \in \mathcal{C}^d_{\text{imp}}(\mathcal{V})$.

(2) *If $c \in \mathscr{C}^{p\times p}$ admits a representation of the form*

$$c(\lambda) = \gamma_c + \int_0^\infty e^{i\lambda t} h_c(t)\, dt\,, \qquad (11.16)$$

in which $h_c \in L_1^{p\times p}([0,\infty))$ and $h_c(t)$ is continuous on the interval $[0, \kappa d]$ and if $\Re\gamma_c$ is positive definite and $\kappa > 0$, there exists exactly one locally summable potential $\mathscr{V}(t)$, $0 \le t \le d$, such that

$$\mathscr{V}(t) = \mathscr{V}(t)^* \quad \text{and} \quad \mathscr{V}(t)J_p + J_p\mathscr{V}(t)^* = 0 \quad \text{a.e. on the interval } [0,d] \qquad (11.17)$$

and $c \in \mathscr{C}_{\rm imp}^d(\mathscr{V})$. Moreover, this potential $\mathscr{V}(t)$ is continuous on the interval $[0,d]$ and is of the form

$$\mathscr{V}(t) = \mathfrak{V}\begin{bmatrix} 0 & a(t) \\ a(t)^* & 0 \end{bmatrix}\mathfrak{V} \quad \text{for } 0 \le t \le d\,. \qquad (11.18)$$

(3) *If $\kappa = 1$ and $c(\lambda)$ is given by formula* (11.16) *with $\gamma_c = I_p$, then $a(t) = \gamma^t(t,0)$, where $\gamma^t(a,b)$ is the unique solution of the integral equation*

$$\gamma^t(a,b) - \int_0^t h_c(a-c)\gamma^t(c,b)\, dc = h_c(a-b) \quad \text{for } 0 \le a, b \le t \le d\,. \qquad (11.19)$$

If $d = \infty$, analogous conclusions hold on the open interval $[0,\infty)$, if $\gamma_c = I_p$ in formula (11.16).

(The modifications needed for general γ_c are discussed in Theorem 5.5 of [Arov and Dym 2005a].)

PROOF. Assertion (1) follows from (1) and (2) of Theorem 11.2, Theorem 7.9 of [Arov and Dym 2003a] and the fact that the set $\mathscr{C}_{\rm imp}^d(\mathscr{V})$ for the system (11.1) coincides with the set $\mathscr{C}_{\rm imp}^d(H)$ for the corresponding cannonical differential system (11.7) with Hamiltonian (11.8). Assertion (2) follows from Theorem 5.13 of [Arov and Dym 2005a], Remark 11.3 and the connection between the systems (11.1) and (11.7). □

If $c \in H_\infty^{p\times p} \cap \mathscr{C}_{\rm imp}^d(\mathscr{V})$ and $d = \infty$, then $\mathscr{C}_{\rm imp}^d(\mathscr{V}) = \{c\}$ for every N of the form (11.11) with $\gamma + \delta > 0$, i.e., the limit point case prevails for all such $\kappa = \gamma + \delta$. This follows from the formulas for the ranks of the left and right semiradii of the Weyl balls that are given in formulas (4.3) and (4.4). In this case, $\mathscr{V} \notin L_1^{m\times m}([0,\infty))$ if c does not admit a representation of the form (7.1). Moreover, if $\gamma > 0$, the values of the input impedance $c(\lambda)$ may be characterized by the Weyl–Titchmarsh property:

$$[\xi^* \quad \eta^*]VU_t(\overline{\lambda})V^* \in L_2^{m\times m} \Longleftrightarrow \eta = c(\lambda)\xi$$

for every point $\lambda \in \mathbb{C}_+$.

The inverse spectral problem. The data for the inverse spectral problem is a $p\times p$ nondecreasing mvf $\sigma(\mu)$ on $\mathbb{R}$ that meets the condition (9.5). The special form of N in (11.11) automatically insures that the matrizant will be strongly regular and prescribes the associated pair of the matrizant in accordance with (1) and (2) of Theorem 11.2. Moreover, for a fixed pair of nonnegative numbers $\gamma\geq 0, \delta\geq 0$ with $\kappa=\gamma+\delta>0$, there is at most one mvf $c\in\mathcal{C}^d_{\rm imp}(\mathcal{V})$ with the spectral function in its Riesz–Herglotz representation (9.4) equal to the given spectral function $\sigma(\mu)$. This is established in Theorem 23.4 of [Arov and Dym 2005c] for the case $\gamma=\delta$. The case $\gamma\neq\delta$ may be reduced to the case $\gamma=\delta$ by invoking Remark 11.3. Thus, Theorem 11.4 yields exactly one solution for the inverse spectral problem for a system of the form (11.1) with $N=N_{\gamma,\delta}$ as in (11.11).

12. Spectral problems for the Schrödinger equation

The matrizant (or fundamental matrix) $U_t(\lambda)=U(t,\lambda)$, $0\leq t<d$, of the Schrödinger equation

$$-u''(t,\lambda)+u(t,\lambda)q(t)=\lambda u(t,\lambda)\,,\quad 0\leq t<d\,, \tag{12.1}$$

with a $p\times p$ matrix valued potential $q(t)$ of the form

$$q(t)=v'(t)+v(t)^2\quad\text{for every } t\in[0,d)\,, \tag{12.2}$$

where

$$v(t)=v(t)^*\quad\text{is locally absolutely continuous on the interval }\ [0,d)\,, \tag{12.3}$$

enjoys the following properties:

(1) $U_t\in\mathcal{U}(-\mathcal{J}_p)$ for every $t\in[0,d)$.

(2) $\limsup_{r\uparrow\infty}\dfrac{\ln\max\{\|U_t(\lambda)\|:\ |z|\leq r\}}{r^{1/2}}=\limsup_{\mu\downarrow-\infty}\dfrac{\ln\|U_t(\mu)\|}{|\mu|^{1/2}}=t\,.$

In particular, $U_t\notin\mathcal{U}_{rsR}(-\mathcal{J}_p)$ and therefore, the results discussed in the preceding sections are not directly applicable. Nevertheless, it turns out that for Schrödinger equations with potential of the given form, the mvf $A_t(\lambda)$ that is defined by the formulas

$$A_t(\lambda)=L_\lambda Y_t(\lambda^2)L_\lambda^{-1}\quad\text{for } 0\leq t<d\,,\quad\text{where}\quad L_\lambda=\begin{bmatrix} I_p & 0\\ 0 & \lambda I_p\end{bmatrix} \tag{12.4}$$

and

$$Y(t,\lambda)=\begin{bmatrix} I_p & v(0)\\ 0 & -iI_p\end{bmatrix}U(t,\lambda)\begin{bmatrix} I_p & -iv(t)\\ 0 & iI_p\end{bmatrix}\quad\text{for } 0\leq t<d\,, \tag{12.5}$$

is a solution of the Cauchy problem

$$A_t'(\lambda) = i\lambda A_t(\lambda)J_p + A_t(\lambda)\mathcal{V}(t), \quad 0 \le t < d,$$

and

$$A_0(\lambda) = I_m,$$

with potential

$$\mathcal{V}(t) = \begin{bmatrix} v(t) & 0 \\ 0 & -v(t) \end{bmatrix}, \quad 0 \le t < d.$$

Thus, $A_t(\lambda)$ is the matrizant of a differential system of the form (11.1), with $N = I_m$, $J = J_p$ and a potential

$$\mathcal{V}(t) = \mathcal{V}(t)^*$$

that meets the constraints (11.3). Therefore, by Theorem 11.2, $A_t \in \mathcal{U}_{rsR}(J_p)$ for every $t \in [0, d)$ and hence, Theorem 11.4 is applicable to A_t.

Let $\psi(t, \lambda)$ and $\varphi(t, \lambda)$ be the unique solutions of equation (12.1) that meet the initial conditions

$$\psi(0, \lambda) = I_p, \quad \psi'(0, \lambda) = 0, \quad \varphi(0, \lambda) = 0 \quad \text{and} \quad \varphi'(0, \lambda) = I_p,$$

respectively, and let

$$U(t, \lambda) = \begin{bmatrix} \psi(t, \lambda) & \psi'(t, \lambda) \\ \varphi(t, \lambda) & \varphi'(t, \lambda) \end{bmatrix} \tag{12.6}$$

be the matrizant (fundamental matrix) of equation (12.1).

A nondecreasing $p \times p$ mvf $\sigma(\mu)$ on $\mathbb{R}$ is said to be a spectral function of (12.1) with respect to the transform

$$g^{\triangle}(\lambda) = \frac{1}{\sqrt{\pi}} \int_0^d \varphi(s, \lambda) g(s)\, ds \tag{12.7}$$

(of vvf's $g \in L_2^p([0, d))$ with compact support in $[0, d)$), if the Parseval equality

$$\int_{\infty}^{\infty} g^{\triangle}(\mu)^* d\sigma(\mu) g^{\triangle}(\mu) = \int_0^d g(s)^* g(s)\, ds \tag{12.8}$$

holds for every $g \in L_2^p([0, d))$ with compact support in $[0, d)$. The symbol $\Sigma_{sf}^d(q)$ will be used to denote the set of all spectral functions of (12.1) with respect to this transform.

Formulas (12.4) and (12.5) imply that

$$-iu_{21}(s, \lambda) = \frac{a_{21}(s, \sqrt{\lambda})}{\sqrt{\lambda}} = -i\varphi(s, \lambda)$$

for $s \in [0, d)$. This connection permits one to reduce the spectral problem for the Schrödinger equation (12.1) to a spectral problem for Dirac systems. This strategy was initiated by M. G. Krein in [Kreĭn 1955].

de Branges spaces. Let

$$\begin{aligned}\mathfrak{E}_t^A(\lambda) &= \sqrt{2}\,[0 \quad I_p]\, A_t(\lambda)\mathfrak{V} \\ &= [a_{22}(t,\lambda) - a_{21}(t,\lambda) \quad a_{22}(t,\lambda) + a_{21}(t,\lambda)]\end{aligned}$$

denote the de Branges matrix based on the matrizant $A_t(\lambda)$ of equation that was defined by formulas (12.4) and (12.5). Then the corresponding de Branges space

$$\mathcal{B}(\mathfrak{E}_t^A) = \left\{ \frac{1}{\sqrt{\pi}} \int_0^t [a_{21}(s,\lambda) \quad a_{22}(s,\lambda)] f(s)\, ds : f \in L_2^m([0,t]) \text{ for every } t \in [0,d) \right\}$$

with norm

$$\langle f^\triangle, f^\triangle \rangle_{\mathcal{B}(\mathfrak{E}_t^A)} = \int_{-\infty}^{\infty} f^\triangle(\mu)^* \Delta_t^A(\mu) f^\triangle(\mu)\, d\mu\,,$$

where, upon writing $f = \text{col}[g\, h]$, with components $g, h \in L_2^p([0,t])$,

$$\begin{aligned} f^\triangle(\mu) &= \frac{1}{\sqrt{\pi}} \int_0^t [a_{21}(s,\lambda) \quad a_{22}(s,\lambda) f(s)\, ds \\ &= \frac{1}{\sqrt{\pi}} \int_0^t a_{21}(s,\lambda) g(s) + a_{22}(s,\lambda) h(s)\, ds \end{aligned}$$

and

$$\begin{aligned} \Delta_t^A(\mu)^{-1} &= (a_{22}(t,\mu) + a_{21}(t,\mu))(a_{22}(t,\mu) + a_{21}(t,\mu))^* \\ &= a_{22}(t,\mu)a_{22}(t,\mu)^* + a_{21}(t,\mu)a_{21}(t,\mu)^*\,, \end{aligned}$$

since $A_t \in \mathcal{U}(J_p)$. Moreover, formula (12.5) implies that $a_{21}(t,\lambda)$ is an odd function of λ, whereas $a_{22}(t,\lambda)$ is an even function of λ. Thus,

$$\mathcal{B}(\mathfrak{E}_t^A) = \mathcal{B}(\mathfrak{E}_t^A)_{\text{odd}} \oplus \mathcal{B}(\mathfrak{E}_t^A)_{\text{ev}}\,,$$

where

$$\begin{aligned} \mathcal{B}(\mathfrak{E}_t^A)_{\text{odd}} &= \left\{ \int_0^t a_{21}(s,\lambda) g(s)\, ds :\ g \in L_2^p([0,t]) \right\}, \\ \mathcal{B}(\mathfrak{E}_t^A)_{\text{ev}} &= \left\{ \int_0^t a_{22}(s,\lambda) g(s)\, ds :\ g \in L_2^p([0,t]) \right\}. \end{aligned}$$

At the same time, Theorem 23.1 of [Arov and Dym 2005c] implies that

$$\mathcal{B}(\mathfrak{E}_t^A) = \left\{ \int_{-t}^{t} e^{i\lambda s} g(s)\, ds : \ g \in L_2^p([-t,t]) \right\}$$

and hence that

$$\mathcal{B}(\mathfrak{E}_t^A)_{\text{odd}} = \left\{ \int_0^t \sin(s\lambda) g(s)\, ds : \ g \in L_2^p([0,t]) \right\},$$

$$\mathcal{B}(\mathfrak{E}_t^A)_{\text{ev}} = \left\{ \int_0^t \cos(s\lambda) g(s)\, ds : \ g \in L_2^p([0,t]) \right\}.$$

Thus, as

$$y_{21}(s,\lambda) = \frac{a_{21}(s,\sqrt{\lambda})}{\sqrt{\lambda}} = -i\varphi(s,\lambda),$$

we obtain the following conclusion:

THEOREM 12.1. *If the potential* $q(t) = v'(t) + v^2(t)$ *of the Schrödinger equation* (12.1) *is subject to the constraints* (12.3), *the de Branges space* $\mathcal{B}(\mathfrak{E}_t^A)$ *equals*

$$\left\{ \frac{1}{\sqrt{\pi}} \int_0^t \frac{\sin\sqrt{\lambda}s}{\sqrt{\lambda}} g(s)\, ds : \ g \in L_2^p([0,t]) \quad \textit{for every } t \in [0,d) \right\}, \tag{12.9}$$

as linear spaces and hence these spaces do not depend upon the potential.

In view of the indicated connection between Dirac systems and Schrödinger equations, Theorems 23.2 and 23.4 of [Arov and Dym 2005c] may be applied to yield existence and uniqueness theorems for the inverse input impedance problem and the inverse spectral problem for the latter when the potential is of the form (12.2), as well as recipes for the solution. A detailed analysis will be presented elsewhere.

REMARK 12.2. The identification (12.9) for the scalar case $p = 1$ was obtained in [Remling 2002] under less restrictive assumptions on the potential $q(t)$ of the Schrödinger equation than are imposed here.

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