

Riemann–Hilbert problem in the inverse scattering for the Camassa–Holm equation on the line

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Dedicated to Henry McKean in deep admiration and friendship

ABSTRACT. We present a Riemann–Hilbert problem formalism for the initial value problem for the Camassa–Holm equation $u_t - u_{txx} + 2\omega u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}$ on the line (CH), where ω is a positive parameter. We show that, for all $\omega > 0$, the solution of this initial value problem can be obtained in a parametric form from the solution of some associated Riemann–Hilbert problem; that for large time, it develops into a train of smooth solitons; and that for small ω , this soliton train is close to a train of peakons, which are piecewise smooth solutions of the CH equation for $\omega = 0$.

1. Introduction

The main purpose of this paper is to develop an inverse scattering approach, based on an appropriate Riemann–Hilbert problem formulation, for the initial value problem for the Camassa–Holm (CH) equation [Camassa and Holm 1993] on the line, whose form is

$$u_t - u_{txx} + 2\omega u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad -\infty < x < \infty, \quad t > 0, \quad (1-1a)$$

$$u(x, 0) = u_0(x), \quad (1-1b)$$

where ω is a positive parameter. The CH equation is a model equation describing the shallow-water approximation in inviscid hydrodynamics. In this equation $u = u(x, t)$ is a real-valued function that refers to the horizontal fluid velocity along the x direction (or equivalently, the height of the water's free surface above a flat bottom) as measured at time t . The constant ω is related to the

critical shallow water wave speed $\sqrt{gh_0}$, where g is the acceleration of gravity and h_0 is the undisturbed water depth; hence, the case $\omega > 0$ is physically more relevant than the case $\omega = 0$, though the latter has attracted more attention in the mathematical studies due to interesting specific features such as the existence of peaked (nonanalytic) solitons.

In terms of the “momentum” variable

$$m := u - u_{xx}, \quad (1-2)$$

the CH equation (1-1a) reads

$$m_t + 2\omega u_x + u m_x + 2m u_x = 0. \quad (1-3)$$

Assuming further that $m + \omega \geq 0$, equation (1-1a) can be equivalently expressed as

$$(\sqrt{m + \omega})_t = - (u \sqrt{m + \omega})_x. \quad (1-4)$$

The function $u_0(x)$ in (1-1b), as well as

$$m_0(x) := u_0(x) - u_{0xx}(x)$$

is assumed to be sufficiently smooth and to decay fast as $|x| \rightarrow \infty$. It is known (see, e.g., [Constantin 2001]) that if $m_0(x) + \omega > 0$ for all x then the solution $m(x, t)$ to (1-3) exists for all t ; moreover, $m(x, t) + \omega > 0$ for all $x \in \mathbb{R}$ and all $t > 0$. This justifies the form (1-4) of the CH equation, which will be used in our constructions below.

Our goal is to develop the inverse scattering approach to the CH equation, in view of its further application for studying the long-time asymptotics. The starting point of the approach is the Lax pair representation: the CH equation is indeed the compatibility condition of two linear equations [Camassa and Holm 1993]

$$\psi_{xx} = \frac{1}{4}\psi + \lambda(m + \omega)\psi, \quad (1-5a)$$

$$\psi_t = \left(\frac{1}{2\lambda} - u\right)\psi_x + \frac{1}{2}u_x\psi. \quad (1-5b)$$

Together with the fact that the x -equation of the Lax pair can be transformed to the spectral problem for the one-dimensional Schrödinger equation, by means of the Liouville transformation, this allows using the inverse scattering transform method to study the initial value problem for the CH equation with $\omega > 0$; see [Constantin 2001; Lenells 2002; Constantin and Lenells 2003; Johnson 2003; Constantin et al. 2006].

In the present paper, we propose a “scattering – inverse scattering” formalism, in which the Lax pair is used in the form of a system of first order matrix-valued linear equations. Then dedicated solutions of this system are defined and used to construct a multiplicative Riemann–Hilbert (RH) problem in the complex

plane of the spectral parameter. The main advantage of the representation of a solution of the CH equation in terms of the solution of a RH problem is that it allows applying the nonlinear steepest descent method by Deift and Zhou [1993] in order to obtain rigorous results on the long-time asymptotic behavior of the solution.

An alternative inverse scattering method based on an additive RH problem formulation for the associated eigenfunctions is given in [Constantin et al. 2006].

In Section 2, we define appropriate eigenfunctions and spectral functions, which are used in Section 3 in the reformulation of the scattering problem as a Riemann–Hilbert problem of analytic conjugation in the complex plane of the spectral parameter. We also introduce a scale in which the Riemann–Hilbert problem becomes explicitly given. In Sections 4 and 5, we briefly discuss the soliton solutions and the soliton asymptotics of the solution of a general initial value problem. Finally, Section 6 deals with the small- ω analysis, the main result of which can be formulated as follows: for sufficiently small ω , the solution of the initial value problem for the Camassa–Holm equation is seen, for large time, as a train of “almost” peakons, which are piecewise smooth weak solutions of the Camassa–Holm equation with $\omega = 0$, the parameters of which are determined by the spectrum of the associated linear problem (1-5a) with $\omega = 0$.

The results of this paper were announced in [Boutet de Monvel and Shepelsky 2006b]. For their application to long-time asymptotics, see [Boutet de Monvel and Shepelsky 2007].

2. Eigenfunctions and spectral functions

2.1. Eigenfunctions. We present the general formalism scaling out the parameter ω in the CH equation and thus assuming that $\omega = 1$ and $m + 1 > 0$ (in Section 6 we will return to arbitrary positive ω when studying the small- ω limit of solutions).

Let $u(x, t)$ be a solution to (1-1a) with $\omega = 1$ such that $u(x, t) \rightarrow 0$ as $|x| \rightarrow \infty$ for all t . First, we rewrite the Lax pair in vector form. Let $\Phi_{\text{init}} := \begin{pmatrix} \psi \\ \psi_x \end{pmatrix}$; then (1-5) is equivalent to

$$(\Phi_{\text{init}})_x = \begin{pmatrix} 0 & 1 \\ \lambda(m+1) + \frac{1}{4} & 0 \end{pmatrix} \Phi_{\text{init}} =: U_{\text{init}} \Phi_{\text{init}}, \quad (2.1a)$$

$$(\Phi_{\text{init}})_t = \begin{pmatrix} \frac{1}{2}u_x & \frac{1}{2\lambda} - u \\ \frac{1}{8\lambda} + \frac{1}{2} + \frac{1}{4}u - \lambda u(m+1) & -\frac{1}{2}u_x \end{pmatrix} \Phi_{\text{init}} =: V_{\text{init}} \Phi_{\text{init}}. \quad (2.1b)$$

Our aim is to introduce special solutions to this system that are well-controlled as functions of the spectral parameter.

In order to control the behavior of solutions of this system for $\lambda \rightarrow \infty$, it is convenient to transform it in such a way that

- (i) the principal terms for $\lambda \rightarrow \infty$ in the Lax equations be diagonal and the terms of order λ^0 be off-diagonal; and
(ii) all the lower order terms vanish as $|x| \rightarrow \infty$.

Writing U_{init} in the form

$$U_{\text{init}} = \begin{pmatrix} 0 & 1 \\ (\lambda + \frac{1}{4})(m+1) & 0 \end{pmatrix} - \frac{m}{4} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

suggests introducing the spectral parameter k and the transformation matrix G_∞ by

$$k^2 = -\lambda - \frac{1}{4}, \quad G_\infty(x, t; k) := \frac{1}{2} \begin{pmatrix} 1 & -\frac{1}{ik} \\ 1 & \frac{1}{ik} \end{pmatrix} \begin{pmatrix} (m+1)^{1/4} & 0 \\ 0 & (m+1)^{-1/4} \end{pmatrix}.$$

Defining $\tilde{\Phi} := G_\infty \Phi_{\text{init}}$ transforms (2-1) into

$$\tilde{\Phi}_x + ik\sqrt{m+1}\sigma_3\tilde{\Phi} = U\tilde{\Phi}, \quad (2-2a)$$

$$\tilde{\Phi}_t + ik\left(\frac{1}{2\lambda} - u\sqrt{m+1}\right)\sigma_3\tilde{\Phi} = V\tilde{\Phi}, \quad (2-2b)$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

$$U(x, t; k) = \frac{1}{4} \frac{m_x}{m+1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{1}{8ik} \frac{m}{\sqrt{m+1}} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \quad (2-3)$$

and

$$\begin{aligned} V(x, t; k) &= -\frac{u}{4} \frac{m_x}{m+1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{8ik} \frac{u(m+2)}{\sqrt{m+1}} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \\ &+ \frac{ik}{4\lambda} \left\{ \sqrt{m+1} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} + \frac{1}{\sqrt{m+1}} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \right\} + \frac{ik}{2\lambda} \sigma_3. \end{aligned} \quad (2-4)$$

It is clear that $U(x, t; k) \rightarrow 0$ as $|x| \rightarrow \infty$. As for the t -equation (2-2b), the term $\frac{ik}{2\lambda}\sigma_3$ has been introduced into the r.h.s. of (2-4) in order to provide $V(x, t; k) \rightarrow 0$ as $|x| \rightarrow \infty$.

Now the equations (2-2) suggest introducing a scalar function $p(x, t; k)$ in such a way that

$$p_x = \sqrt{m+1}, \quad p_t = \frac{1}{2\lambda} - u\sqrt{m+1}.$$

Indeed, due to (1-4), one can define such p (normalized by $p(x, 0; k) \sim x$ as $x \rightarrow +\infty$) as follows:

$$\begin{aligned} p(x, t; k) &:= x - \int_x^\infty \left(\sqrt{m(\xi, t) + 1} - 1 \right) d\xi + \frac{t}{2\lambda(k)} \\ &= x - \int_x^\infty \left(\sqrt{m(\xi, t) + 1} - 1 \right) d\xi - \frac{2}{1+4k^2} t. \end{aligned} \quad (2-5)$$

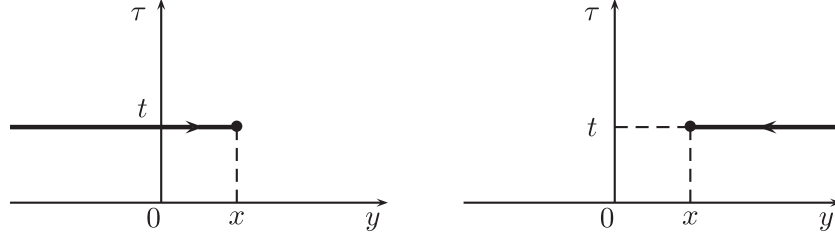


Figure 1. Paths of integration for Φ_- and Φ_+ .

Finally, assuming $\tilde{\Phi}$ to be a 2×2 matrix-valued function, let $\Phi(x, t; k) := \tilde{\Phi} e^{ikp(x, t; k)\sigma_3}$. Then (2-2) becomes

$$\Phi_x + ikp_x[\sigma_3, \Phi] = U\Phi, \quad \Phi_t + ikp_t[\sigma_3, \Phi] = V\Phi, \quad (2-6)$$

where $[a, b] := ab - ba$.

The Lax pair in the form (2-6) is very convenient for defining dedicated solutions via integral Volterra equations by specifying the initial point of integration (x^*, t^*) in the (x, t) -plane:

$$\begin{aligned} \Phi(x, t; k) = I + \int_{(x^*, t^*)}^{(x, t)} e^{-ik(p(x, t; k) - p(y, \tau; k))\hat{\sigma}_3} \\ \times \{U(y, \tau; k)\Phi(y, \tau; k)dy + V(y, \tau; k)\Phi(y, \tau; k)d\tau\}, \end{aligned} \quad (2-7)$$

where I is the 2×2 identity matrix, $e^{\hat{\sigma}_3 A} := e^{\sigma_3 A} e^{-\sigma_3}$ for any 2×2 matrix A , and the r.h.s. is independent of the integration path. Then the special structure of U and V (recall that their main terms as $k \rightarrow \infty$ are off-diagonal) provides a well-controlled behavior of solutions of (2-7) for large k .

Choosing the initial points of integration to be $(-\infty, t)$ and $(+\infty, t)$ and the paths of integration to be parallel to the x -axis (see Figure 1) leads to the integral equations for Φ_+ and Φ_- :

$$\begin{aligned} \Phi_-(x, t; k) &= I + \int_{-\infty}^x e^{-ik \int_y^x \sqrt{m(\xi, t)+1} d\xi \hat{\sigma}_3} (U\Phi_-)(y, t; k) dy, \\ \Phi_+(x, t; k) &= I - \int_x^{\infty} e^{ik \int_x^y \sqrt{m(\xi, t)+1} d\xi \hat{\sigma}_3} (U\Phi_+)(y, t; k) dy. \end{aligned} \quad (2-8)$$

It is due to the condition $V \rightarrow 0$ as $|x| \rightarrow \infty$ as well as the compatibility of the two equations in (2-6) that the solutions of (2-8) satisfy the t -equation in (2-6). Since U and V are traceless, it follows that $\det \Phi_{\pm} \equiv 1$.

The structure of the integral equations (2-8) provides the following analytic properties of the eigenfunctions Φ_{\pm} as functions of k . We denote $\mu^{(1)}$ and $\mu^{(2)}$ the columns of a 2×2 matrix $\mu = (\mu^{(1)} \quad \mu^{(2)})$. Then, for all (x, t) , the following conditions are satisfied:

- (a) $\Phi_-^{(1)}$ and $\Phi_+^{(2)}$ are analytic in $\{k \mid \text{Im } k > 0\}$ and continuous in $\{k \mid \text{Im } k \geq 0, k \neq 0\}$;
- (b) $\Phi_+^{(1)}$ and $\Phi_-^{(2)}$ are analytic in $\{k \mid \text{Im } k < 0\}$ and continuous in $\{k \mid \text{Im } k \leq 0, k \neq 0\}$;
- (c) as $k \rightarrow \infty$ in $\{k \mid \text{Im } k \geq 0\}$, $(\Phi_-^{(1)} \quad \Phi_+^{(2)}) \rightarrow I$;
- (d) as $k \rightarrow \infty$ in $\{k \mid \text{Im } k \leq 0\}$, $(\Phi_+^{(1)} \quad \Phi_-^{(2)}) \rightarrow I$;
- (e) as $k \rightarrow 0$, $\Phi_{\pm} = \frac{\alpha_{\pm}(x,t)}{ik} \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} + O(1)$ with $\alpha_{\pm} \in \mathbb{R}$.

REMARK 1. The Lax pair in the form (2-6) (see also (3-9) below) and the associated integral equations (2-7) turn out to be useful also in studying initial boundary value problems for the CH equation, see [Boutet de Monvel and Shepelsky 2006a]. Indeed, the structure of the t -equation is as “good” as the structure of the x -equation for controlling the properties of the appropriate eigenfunctions in the k -plane even in the case where the integration paths in (2-7) are not parallel to the x -axis; such paths (parallel to the t -axis) are needed in order to relate the eigenfunctions to the boundary values (i.e., at $x = 0$) of a solution of the nonlinear equation in question. These eigenfunctions can be viewed as coming from the simultaneous spectral analysis (of the x - and t -equations) of the Lax pair.

2.2. Spectral functions. For $k \in \mathbb{R}$, the eigenfunctions Φ_- and Φ_+ , being the solutions of the system of differential equations (2-6), are related by a matrix independent of (x, t) ; this allows introducing the scattering matrix $s(k)$ by

$$\Phi_+(x, t; k) = \Phi_-(x, t; k) e^{-ikp(x,t;k)\hat{\sigma}_3} s(k), \quad k \in \mathbb{R}, k \neq 0. \quad (2-9)$$

Since the matrix U satisfies the symmetry relations

$$\overline{U(\cdot, \cdot, \bar{k})} = U(\cdot, \cdot, -k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U(\cdot, \cdot, k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (2-10)$$

the spectral matrix $s(k)$ can be written as

$$s(k) = \begin{pmatrix} \overline{a(k)} & b(k) \\ \overline{b(k)} & a(k) \end{pmatrix}, \quad k \in \mathbb{R}, \quad (2-11)$$

where $\overline{a(k)} = a(-k)$ and $\overline{b(k)} = b(-k)$.

Writing the spectral functions $a(k)$ and $b(k)$ in terms of determinants, namely

$$\begin{aligned} a(k) &= \det(\Phi_-^{(1)} \quad \Phi_+^{(2)}), \\ b(k) &= e^{2ikp} \det(\Phi_+^{(2)} \quad \Phi_-^{(2)}), \end{aligned}$$

allows us to establish the following properties:

- (i) $a(k)$ and $b(k)$ are determined by Φ_{\pm} for $t = 0$ and thus by $m(x, 0)$ (or, equivalently, by $u(x, 0)$).
- (ii) $a(k)$ is analytic in $\{k \mid \operatorname{Im} k > 0\}$ and continuous in $\{k \mid \operatorname{Im} k \geq 0, k \neq 0\}$; moreover, $a(k) \rightarrow 1$ as $k \rightarrow \infty$.
- (iii) $b(k)$ is continuous for $k \in \mathbb{R}, k \neq 0$ and $b(k) \rightarrow 0$ as $|k| \rightarrow \infty$.
- (iv) as $k \rightarrow 0$, $a(k) = \alpha_0/(ik) + O(1)$ and $b(k) = -\alpha_0/(ik) + O(1)$ with $\alpha_0 \in \mathbb{R}$.
- (v) $|a(k)|^2 - |b(k)|^2 = 1$ for $k \in \mathbb{R}, k \neq 0$.
- (vi) Let $\{k_j\}_{j=1}^N$ be the set of zeros of $a(k)$: $a(k_j) = 0$. Then $N < \infty$ and the zeros are simple with $(da/dk)(k_j) \in i\mathbb{R}$; moreover, $k_j = iv_j$ with $0 < v_j < \frac{1}{2}$ for all $1 \leq j \leq N$; and the eigenvectors are related by

$$\Phi_{-}^{(1)}(x, t; iv_j) = \chi_j e^{-2v_j p(x, t; iv_j)} \Phi_{+}^{(2)}(x, t; iv_j) \quad (2-12)$$

with $\chi_j \in \mathbb{R}$.

The statements in the last item follow from the fact that $\lambda_j = v_j^2 - \frac{1}{4}$, $j = 1, 2, \dots, N$ are the eigenvalues of the x -equation (1-5a) with $m = m(x, 0)$: they are known (see, e.g., [Constantin 2001]) to be simple and to satisfy $-\frac{1}{4} < \lambda_j < 0$.

REMARK 2. It follows from the construction of the eigenfunctions Φ_{\pm} that $s(k)$ is the scattering matrix for the one-dimensional Schrödinger equation $-\varphi_{yy} + Q(y)\varphi = k^2\varphi$ associated to (1-5a) with $m = m(x, 0)$ via the Liouville transformation:

$$y = x - \int_x^{\infty} \left(\sqrt{m(\xi, 0) + 1} - 1 \right) d\xi, \quad q(y) = m(x, 0) + 1,$$

$$\varphi(y, k) = \psi(x, k)q(y)^{1/4}, \quad Q(y) = \frac{q_{yy}}{4q} - \frac{3(q_y)^2}{16q^2} + \frac{1-q}{4q}.$$

3. Riemann–Hilbert problem

3.1. Scattering problem as a Riemann–Hilbert problem in the (x, t) scale. Regrouping the columns of the scattering relation (2-9) in accordance with their analyticity properties allows rewriting (2-9) in the form of a conjugation of piecewise meromorphic, matrix-valued functions. Let us define a 2×2 matrix function $M(x, t; k)$ by

$$M(x, t; k) = \begin{cases} \left(\begin{array}{cc} \frac{\Phi_{-}^{(1)}(x, t; k)}{a(k)} & \Phi_{+}^{(2)}(x, t; k) \end{array} \right) & \text{if } \operatorname{Im} k > 0, \\ \left(\begin{array}{cc} \Phi_{+}^{(1)}(x, t; k) & \frac{\Phi_{-}^{(2)}(x, t; k)}{a(\bar{k})} \end{array} \right) & \text{if } \operatorname{Im} k < 0. \end{cases} \quad (3-1)$$

Then the limiting values $M_{\pm}(x, t; \zeta)$ of $M(x, t; k)$ as k approaches the real axis ($k = \zeta \pm i\varepsilon$, $\varepsilon > 0$ and $\varepsilon \rightarrow 0$) are related as follows:

$$M_{-}(x, t; k) = M_{+}(x, t; k)e^{-ikp(x, t; k)\sigma_3} J_0(k)e^{ikp(x, t; k)\sigma_3}, \quad k \in \mathbb{R}, \quad (3-2)$$

where

$$J_0(k) = \begin{pmatrix} 1 & -r(k) \\ r(k) & 1 - |r(k)|^2 \end{pmatrix} \quad (3-3)$$

with $r(k) = b(k)/\overline{a(k)}$.

By construction, $M(x, t; k)$ also satisfies the following properties:

- (i) $M \rightarrow I$ as $k \rightarrow \infty$.
- (ii) $M = \frac{\alpha_{+}(x, t)}{ik} \begin{pmatrix} -c & -1 \\ c & 1 \end{pmatrix} + O(1)$ as $k \rightarrow 0$ in $\text{Im } k \geq 0$, where $c = 0$ if $\lim_{k \rightarrow 0} ka(k) \neq 0$.
- (iii) Symmetry properties:

$$\overline{M(\cdot, \cdot, \bar{k})} = M(\cdot, \cdot, -k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M(\cdot, \cdot, k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

- (iv) M has poles at the zeros $k_j = iv_j$ of $a(k)$ (in the upper half-plane $\text{Im } k > 0$) and at $\bar{k}_j = -iv_j$ (in the lower half-plane $\text{Im } k < 0$), $j = 1, 2, \dots, N$, with the following residue conditions:

$$\begin{aligned} \text{Res}_{k=iv_j} M^{(1)}(x, t; k) &= i\gamma_j e^{-2v_j p(x, t; iv_j)} M^{(2)}(x, t; iv_j), \\ \text{Res}_{k=-iv_j} M^{(2)}(x, t; k) &= -i\gamma_j e^{-2v_j p(x, t; iv_j)} M^{(1)}(x, t; -iv_j) \end{aligned} \quad (3-4)$$

$$\text{with } \gamma_j = -i \frac{\chi_j}{(da/dk)(iv_j)} \in \mathbb{R}.$$

In the Riemann–Hilbert approach to nonlinear evolution equations, one tries to interpret a jump relation (of type (3-2)) across a contour in the k -plane, together with residue conditions (of type (3-4)) and certain normalization conditions, as a Riemann–Hilbert problem, the data for which are the jump matrix and the residue parameters (which can be obtained by solving the direct scattering problem for an operator with coefficients determined by the initial data for the nonlinear problem), and the solution of which gives the solution to the nonlinear equation in question.

In the case of the Camassa–Holm equation, the jump relation (3-2) cannot be used immediately for this purpose. In the construction of the jump matrix $e^{-ikp} J_0(k) e^{ikp}$ the factor $J_0(k)$ is indeed given in terms of the known initial data but $p(x, t; k)$ is not: it involves $m(x, t)$ which is unknown (and, in fact, is to be reconstructed) in the framework of the inverse problem.

To remedy this, we introduce the new (time-dependent) scale (cf. the Liouville transformation)

$$y(x, t) = x - \int_x^\infty \left(\sqrt{m(\xi, t) + 1} - 1 \right) d\xi, \quad (3-5)$$

in terms of which the jump matrix and the residue conditions become explicit. The price to pay for this, however, is that the solution to the nonlinear problem can be given only implicitly, or parametrically: it will be given in terms of functions in the new scale, whereas the original scale will also be given in terms of functions in the new scale.

In order to achieve this program, we use a particular feature of the Lax pair for the Camassa–Holm equation, namely, the fact that the x -equation (1-5a) becomes trivial (independent of the “momentum” m) for $\lambda = 0$, which corresponds to $k = \pm \frac{i}{2}$. In order to translate this into the properties of the eigenfunctions involved in the construction of the analytic conjugation problem (3-2), it is convenient to transform the Lax pair equations in such a way that the main terms become diagonal as $\lambda \rightarrow 0$.

3.2. Eigenfunctions near $\lambda = 0$. Setting $\tilde{\Phi}^0 := \frac{1}{2} \begin{pmatrix} 1 & -\frac{1}{ik} \\ 1 & \frac{1}{ik} \end{pmatrix} \Phi_{\text{init}}$ transforms (2-1) into

$$\tilde{\Phi}_x^0 + ik\sigma_3 \tilde{\Phi}^0 = U_0 \tilde{\Phi}^0, \quad \tilde{\Phi}_t^0 + \frac{ik}{2\lambda} \sigma_3 \tilde{\Phi}^0 = V_0 \tilde{\Phi}^0, \quad (3-6)$$

where

$$U_0(x, t; k) = \frac{\lambda}{2ik} m(x, t) \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, \quad (3-7)$$

and

$$V_0(x, t; k) = \frac{u_x}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{u}{4ik} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{\lambda u}{2ik} \left\{ 2\sigma_3 - m \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \right\}. \quad (3-8)$$

The eigenfunctions Φ_\pm^0 are defined similarly to Φ_\pm : setting

$$\Phi^0 := \tilde{\Phi}^0 \exp \left\{ \left(ikx + \frac{ik}{2\lambda} t \right) \sigma_3 \right\}$$

transforms (3-6) into

$$\Phi_x^0 + ik[\sigma_3, \Phi^0] = U_0 \Phi^0, \quad \Phi_t^0 + \frac{ik}{2\lambda} [\sigma_3, \Phi^0] = V_0 \Phi^0. \quad (3-9)$$

Now the eigenfunctions Φ_{\pm}^0 are defined as solutions of the integral equations

$$\Phi_{\pm}^0(x, t; k) = I + \int_{\pm\infty}^x e^{-ik(x-y)\hat{\sigma}_3} (U_0 \Phi_{\pm}^0)(y, t; k) dy \quad (3-10)$$

(notice that the fact that $V_0 \rightarrow 0$ as $|x| \rightarrow \infty$ is again of importance here). Since $U_0(x, t; \pm \frac{i}{2}) \equiv 0$ in (3-10), we have that $\Phi_{\pm}^0(x, t; \pm \frac{i}{2}) \equiv I$.

Since Φ_{\pm} and Φ_{\pm}^0 solve a system of differential equations which are transformations of the same system (2-1), they are related by matrices $C_{\pm}(k)$ independent of (x, t) :

$$\Phi_{\pm}(x, t; k) = F(x, t) \Phi_{\pm}^0(x, t; k) e^{-ik(x + \frac{t}{2x})\sigma_3} C_{\pm}(k) e^{ikp(x, t; k)\sigma_3}, \quad (3-11)$$

where

$$F = \frac{1}{2} \begin{pmatrix} q + q^{-1} & q - q^{-1} \\ q - q^{-1} & q + q^{-1} \end{pmatrix}, \quad q(x, t) = (m(x, t) + 1)^{1/4},$$

and $C_{\pm}(k)$ are determined by the boundary conditions

$$\Phi_{\pm}(\pm\infty, t; k) = \Phi_{\pm}^0(\pm\infty, t; k) = I;$$

this gives $C_+(k) = I$ and $C_-(k) = e^{ik\kappa\sigma_3}$ with $\kappa = \int_{-\infty}^{\infty} (\sqrt{m(\xi, t) + 1} - 1) d\xi$ independent of t (conservation law).

In particular, evaluating (3-11) at $k = \pm \frac{i}{2}$ we have

$$\begin{aligned} \Phi_+^{(2)}(x, t; \frac{i}{2}) &= F^{(2)}(x, t) e^{-\frac{1}{2} \int_x^{\infty} (\sqrt{m(\xi, t) + 1} - 1) d\xi}, \\ \Phi_-^{(1)}(x, t; \frac{i}{2}) &= F^{(1)}(x, t) e^{-\frac{1}{2} \int_{-\infty}^x (\sqrt{m(\xi, t) + 1} - 1) d\xi}. \end{aligned}$$

Calculating $a(\frac{i}{2})$ using the determinant formula gives

$$a(\frac{i}{2}) = \det(\Phi_-^{(1)} \quad \Phi_+^{(2)})|_{k=\frac{i}{2}} = e^{-\frac{1}{2} \int_{-\infty}^{\infty} (\sqrt{m(\xi, t) + 1} - 1) d\xi} = e^{-\kappa/2},$$

which finally yields

$$M(x, t; \frac{i}{2}) = F(x, t) \begin{pmatrix} e^{\frac{1}{2} \int_x^{\infty} (\sqrt{m(\xi, t) + 1} - 1) d\xi} & 0 \\ 0 & e^{-\frac{1}{2} \int_x^{\infty} (\sqrt{m(\xi, t) + 1} - 1) d\xi} \end{pmatrix}. \quad (3-12)$$

Equation (3-12) allows relating the original scale x and the new scale y , see (3-5), in terms of M evaluated at $k = \frac{i}{2}$. Indeed, let

$$\begin{aligned} \tilde{\mu}_1(x, t) &:= M_{11}(x, t; \frac{i}{2}) + M_{21}(x, t; \frac{i}{2}), \\ \tilde{\mu}_2(x, t) &:= M_{12}(x, t; \frac{i}{2}) + M_{22}(x, t; \frac{i}{2}). \end{aligned}$$

Then from (3-12) and (3-5) we have

$$\frac{\tilde{\mu}_1(x, t)}{\tilde{\mu}_2(x, t)} = e^{\int_x^{\infty} (\sqrt{m(\xi, t) + 1} - 1) d\xi} = e^{x-y(x, t)} \quad (3-13)$$

and

$$[\tilde{\mu}_1(x, t)\tilde{\mu}_2(x, t)]^2 = m(x, t) + 1. \quad (3-14)$$

3.3. Riemann–Hilbert problem in the (y, t) scale. We observe that the jump conditions (3-2) as well as the residue conditions (3-4) become explicit in the variables y and t . This, together with the considerations above concerning the relations involving the x and y scales, suggest introducing the vector Riemann–Hilbert problem, parametrized by (y, t) , as follows (cf. (3-2), (3-4)):

RH-PROBLEM. Given $r(k)$ for $k \in \mathbb{R}$, $\{v_j\}_{j=1}^N$ ($0 < v_j < \frac{1}{2}$), and $\{\gamma_j\}_{j=1}^N$ ($\gamma_j > 0$), find a row function $\mu(y, t; k) = (\mu_1(y, t; k) \ \mu_2(y, t; k))$ such that:

- (i) $\mu(\cdot, \cdot; k)$ is analytic in $\{k \mid \text{Im } k > 0\}$ and in $\{k \mid \text{Im } k < 0\}$.
- (ii) The limits $\mu_{\pm}(\cdot, \cdot, \zeta) = \lim_{\varepsilon \rightarrow +0} \mu(\cdot, \cdot; \zeta \pm i\varepsilon)$, $\zeta \in \mathbb{R}$ are related by

$$\mu_{-}(y, t; \zeta) = \mu_{+}(y, t; \zeta)J(y, t; \zeta), \quad \zeta \in \mathbb{R}, \quad (3-15)$$

where the jump matrix is

$$J(y, t; k) = e^{-ik\left(y - \frac{2}{1+4k^2}t\right)\sigma_3} J_0(k) e^{ik\left(y - \frac{2}{1+4k^2}t\right)\sigma_3} \quad (3-16)$$

with

$$J_0(k) = \begin{pmatrix} 1 & -r(k) \\ \frac{1}{r(k)} & 1 - |r(k)|^2 \end{pmatrix}.$$

- (iii) Normalization at infinity:

$$\mu(y, t; k) \rightarrow (1 \ 1) \quad \text{as } k \rightarrow \infty. \quad (3-17)$$

- (iv) Residue conditions:

$$\begin{aligned} \text{Res}_{k=iv_j} \mu_1(y, t; k) &= i\gamma_j e^{-2v_j\left(y - \frac{2}{1-4v_j^2}t\right)} \mu_2(y, t; iv_j), \\ \text{Res}_{k=-iv_j} \mu_2(y, t; k) &= -i\gamma_j e^{-2v_j\left(y - \frac{2}{1-4v_j^2}t\right)} \mu_1(y, t; -iv_j). \end{aligned} \quad (3-18)$$

REMARKS. The symmetry properties of the jump matrix imply that

$$\overline{\mu_1(\cdot, \cdot; \bar{k})} = \mu_1(\cdot, \cdot; -k) = \mu_2(\cdot, \cdot; k). \quad (3-19)$$

Now we notice the following:

- ▷ The data for the Riemann–Hilbert problem (3-15)–(3-18) are determined in terms of the scattering data $r(k)$, $\{v_j\}_{j=1}^N$, $\{\gamma_j\}_{j=1}^N$, which, in turn, are determined by $m(x, 0)$, the initial value of the solution of the Camassa–Holm equation, via the solutions Φ_{\pm} of the direct scattering problem at $t = 0$, see (2-8), (2-9), (2-12).

▷ This RH problem has the same structure as that for the Korteweg–de Vries equation (except for the k -dependence of the velocity in the phase factors, see (3-16) and (3-18)), which implies that there exists a vanishing lemma [Beals et al. 1988] guarantying that the RH problem has a unique solution $\mu(y, t; k)$ for all $y \in (-\infty, \infty)$ and $t > 0$.

Let us evaluate this solution at $k = \frac{i}{2}$. Then, the relations (3-13) and (3-14) allow us to:

(a) Determine the function $y = y(x, t)$ as the inverse to the function

$$x(y, t) = y + \ln \frac{\mu_1(y, t; \frac{i}{2})}{\mu_2(y, t; \frac{i}{2})}. \quad (3-20)$$

(b) Determine the momentum $m(x, t)$ of the Camassa–Holm equation by

$$m(x, t) = \left(\mu_1(y, t; \frac{i}{2}) \mu_2(y, t; \frac{i}{2}) \right)^2 \Big|_{y=y(x, t)} - 1. \quad (3-21)$$

Now the solution $u(x, t)$ of the Camassa–Holm equation can be determined from $m(x, t)$ by

$$u(x, t) = \frac{1}{2} \left(\int_{-\infty}^x e^{y-x} m(y, t) dy + \int_x^{\infty} e^{x-y} m(y, t) dy \right). \quad (3-22)$$

Alternatively, and more directly, u can be determined in terms of μ_1 and μ_2 using the equality

$$\frac{\partial x}{\partial t}(y, t) = u(x, t), \quad (3-23)$$

which follows from the definition (3-5) of the function $y = y(x, t)$, in which m satisfies (1-4). In view of (3-20) one has

$$u(x, t) = \left(\frac{\partial}{\partial t} \ln \frac{\mu_1}{\mu_2}(y, t; \frac{i}{2}) \right) \Big|_{y=y(x, t)}. \quad (3-24)$$

Equation (3-12) provides also alternative (nonlocal) ways for determining $x = x(y, t)$. Indeed,

$$\frac{\partial x}{\partial y}(y, t) = (m(x(y, t), t) + 1)^{-\frac{1}{2}} = \frac{1}{\mu_1(y, t) \mu_2(y, t)} = \frac{e^{y-x}}{\mu_2^2(y, t)} = \frac{e^{x-y}}{\mu_1^2(y, t)},$$

and the integral formulae for $x = x(y, t)$ emerge.

The discussion above is summarized as follows:

PROPOSITION 1. *The solution $u(x, t)$ of the initial value problem for the Camassa–Holm equation (1-1) with $\omega = 1$, where the initial data $u_0(x)$ is rapidly decreasing as $|x| \rightarrow \infty$ and such that $u_{0xx}(x) - u_0(x) + 1 > 0$, can be expressed parametrically, by (3-20), (3-24), in terms of the solution of the Riemann–Hilbert problem (3-15)–(3-18). \square*

4. Soliton solutions

Equations (3-20) and (3-24) give a parametric representation for the solution of the initial value problem for the Camassa–Holm equation for general initial data. They have the same structure as the parametric formulae representing pure multisoliton solutions [Matsuno 2005] in terms of two determinants (at the places of μ_1 and μ_2). Therefore, the multisoliton solutions [Matsuno 2005] are “embedded” into our scheme for the solution of the initial value problem: they correspond to reflectionless ($r(k) \equiv 0$) initial data, for which the solution of the RH problem is reduced to solving linear algebraic equations.

Notice also that the formulae by McKean [McKean 2003] for the solution of the Camassa–Holm equation with $\omega = 0$ in terms of the associated theta functions have a similar structure.

Since the algebraic structure of the RH problem is exactly the same as in the case of the KdV equation, its solution (for all k) can be obtained by solving the same linear algebraic equations. Then, comparing to the KdV, the difference in the construction of the solitons is threefold:

- (i) the solution of the RH problem is to be evaluated at $k = \frac{i}{2}$ (rather than as $k \rightarrow \infty$ for the KdV);
- (ii) the phases $yk + 4tk^3$ for $k = k_j$, $j = 1, 2, \dots, N$ in the case of the KdV equation are to be replaced by $yk - t \frac{2k}{1+4k^2}$ for the Camassa–Holm equation;
- (iii) the original scale x is to be related to the y -scale (again by using the solutions of the RH problem evaluated at $k = \frac{i}{2}$).

If $N = 1$, $k_1 = iv_1 \equiv iv$ then the solution of the corresponding RH problem (3-15) normalized by (3-17) and having the trivial jump $J(y, t; \xi) \equiv I$ is a row-valued rational function with poles at $k = \pm iv$ and thus has the form

$$(\mu_1(y, t; k) \quad \mu_2(y, t; k)) = \left(\frac{k - B(y, t)}{k - iv} \quad \frac{k + B(y, t)}{k + iv} \right). \quad (4-1)$$

Here B can be calculated using the residue conditions (3-18); this gives $B = iv(1 - g)/(1 + g)$ with

$$g(y, t) = \begin{cases} \exp\{-2v(y - 4v^2t - y_0)\} & \text{for KdV,} \\ \exp\left\{-2v\left(y - \frac{2}{1-4v^2}t - y_0\right)\right\} & \text{for CH,} \end{cases} \quad (4-2)$$

where y_0 is the phase shift determined by the norming constant $\gamma > 0$ in the residue relation: $y_0 = \frac{1}{2v} \ln \frac{\gamma}{2v}$. Then the 1-soliton solution for the KdV equation is given in terms of $\mu_1^0(y, t)$, where $\mu_1(y, t; k) = 1 + \mu_1^0(y, t)/k + o(1/k)$ as $k \rightarrow \infty$, by

$$u_{\text{KdV}}(y, t) = -2i \frac{\partial}{\partial x} \mu_1^0(y, t) = -8v \frac{g_{\text{KdV}}}{(1 + g_{\text{KdV}})^2}(y, t), \quad (4-3)$$

whereas the 1-soliton solution for the CH equation (in the y scale) is given in terms of $\mu_j(y, t; \frac{i}{2})$, $j = 1, 2$ by (in a form comparable with that of (4-3))

$$u_{\text{CH}}(y, t) = \frac{\partial}{\partial t} \ln \frac{\mu_1}{\mu_2}(y, t; \frac{i}{2}) = \frac{32v^2}{(1-4v^2)^2} \frac{g_{\text{CH}}}{(1+g_{\text{CH}})^2 + \frac{16v^2}{1-4v^2} g_{\text{CH}}}(y, t). \quad (4-4)$$

The associated relation between the scales (3-20) becomes

$$x(y, t) = y + \ln \frac{1 + g \frac{1+2v}{1-2v}}{1 + g \frac{1-2v}{1+2v}}. \quad (4-5)$$

Introducing

$$v := \frac{2}{1-4v^2}, \quad \phi := -2v(y - vt - y_0)$$

allows rewriting (4-4) as

$$u_{\text{CH}}(y, t) = \frac{16v^2}{1-4v^2} \times \frac{1}{1+4v^2 + (1-4v^2) \cosh \phi}. \quad (4-6)$$

Since $0 < v < 1/2$, it follows that the soliton velocity v is greater than 2. Similarly, the velocities of the N solitons appearing asymptotically, as $t \rightarrow \infty$, from the N -soliton solution (associated with N residue conditions of type (3-18) at the poles $k = \pm iv_j$, $j = 1, 2, \dots, N$) are all greater than 2.

5. Long-time asymptotics

The representation of the solution of a nonlinear equation in terms of the solution of the associated Riemann–Hilbert problem has proved to be crucial in studying its long-time behavior using the nonlinear steepest descent method by Deift and Zhou [Deift and Zhou 1993]. The solution of the RH problem with poles (3-18) can be represented as

$$\mu(y, t; k) = \tilde{\mu}(y, t; k) M_r(y, t; k) D(k),$$

where

$$D = \begin{pmatrix} \prod_{j=1}^N (k - iv_j)^{-1} & 0 \\ 0 & \prod_{j=1}^N (k + iv_j)^{-1} \end{pmatrix},$$

M_r is the solution of the 2×2 regular (i.e., without residue conditions) RH problem with the jump matrix $\tilde{J} = D J D^{-1}$, and $\tilde{\mu}(y, t; k)$ is a row polynomial in k with coefficients determined by the residue conditions.

For the convenience of the reader, we present here a scheme for studying the large- t behavior of solutions of Riemann–Hilbert problems with rapidly oscillating jump data (cf. [Deift et al. 1993]).

PROPOSITION 2. *In the soliton region, $y > (2 + \delta)t$ with any $\delta > 0$, we have*

$$M_r(y, t; k) = I + o(1), \quad t \rightarrow +\infty.$$

PROOF. In the spirit of the nonlinear steepest descent method, the RH problem (3-15) is to be deformed in a way that its jump matrix would approach the identity matrix. In the original setting, J in (3-16) is rapidly oscillating with t , with the exponential factors

$$J(y, t; k) = \begin{pmatrix} 1 & -r(k)e^{-2it\theta} \\ \frac{1}{r(k)}e^{2it\theta} & 1 - |r(k)|^2 \end{pmatrix},$$

where

$$\theta(y, t; k) = \frac{y}{t}k - \frac{2k}{1 + 4k^2}. \quad (5-1)$$

The deformation of the original contour (real axis) is guided by the “signature table” i.e., the decomposition of the k -plane into domains where $\text{Im } \theta$ keeps its sign. In the domain $y/t > 2$, the signature table is shown in Figure 2. Therefore, in this case the whole real axis is the boundary of the domains where $\text{Im } \theta > 0$ (for $\text{Im } k > 0$) and $\text{Im } \theta < 0$ (for $\text{Im } k < 0$). This suggests using the factorization of the jump matrix

$$J = \begin{pmatrix} 1 & 0 \\ \frac{1}{r(k)}e^{2it\theta} & 1 \end{pmatrix} \begin{pmatrix} 1 & -r(k)e^{-2it\theta} \\ 0 & 1 \end{pmatrix}, \quad k \in \mathbb{R}.$$

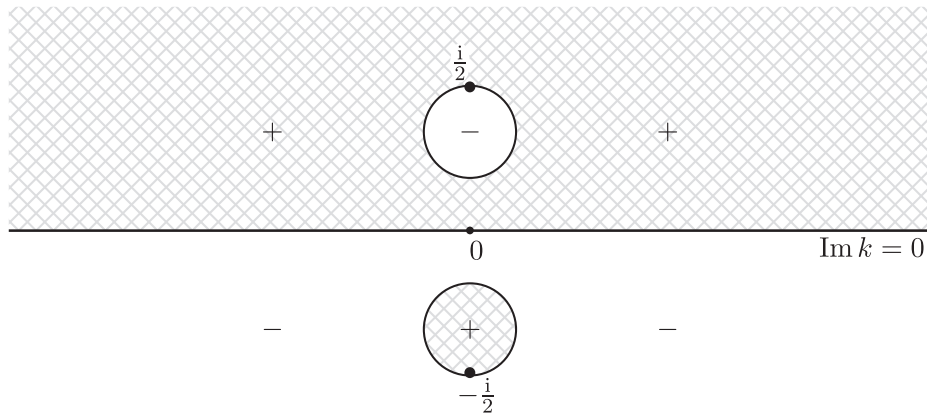


Figure 2. Signature table for $\text{Im } \theta(y, t; k)$ when $\frac{y}{t} > 2$.

Then appropriate rational approximations of $r(k)$ and $\overline{r(k)}$ are used in order to deform the contour into two lines $\text{Im } k = \pm \varepsilon$ and to absorb the relevant triangular factors (near the real axis) into the new function

$$\hat{M}_r = \begin{cases} M_r & \text{if } |\text{Im } k| > \varepsilon, \\ M_r \begin{pmatrix} 1 & 0 \\ r(k)e^{2it\theta} & 1 \end{pmatrix} & \text{if } 0 < \text{Im } k < \varepsilon, \\ M_r \begin{pmatrix} 1 & r(k)e^{-2it\theta} \\ 0 & 1 \end{pmatrix} & \text{if } 0 > \text{Im } k > -\varepsilon. \end{cases}$$

Now the Riemann–Hilbert problem for \hat{M}_r becomes

$$\hat{M}_{r-} = \hat{M}_{r+} \hat{J} \quad \text{for } |\text{Im } k| = \varepsilon,$$

where

$$\hat{J} = \begin{cases} \begin{pmatrix} 1 & 0 \\ r(k)e^{2it\theta} & 1 \end{pmatrix} & \text{if } \text{Im } k = \varepsilon, \\ \begin{pmatrix} 1 & -r(k)e^{-2it\theta} \\ 0 & 1 \end{pmatrix} & \text{if } \text{Im } k = -\varepsilon. \end{cases}$$

Since the jump matrix \hat{J} approaches I , as $t \rightarrow \infty$, exponentially fast, this implies $\hat{M}_r \rightarrow I$ and thus $M_r \rightarrow I$ for $\{k \mid |\text{Im } k| > \varepsilon\}$. \square

As a consequence of Proposition 2 and the fact that $x - y = o(1)$ as $y \rightarrow +\infty$, see (3-5), we have that

$$u(x, t) = u_{\text{solit}}(x, t) + o(1), \quad t \rightarrow \infty, \quad (5-2)$$

where $u_{\text{solit}}(x, t)$ is the pure N -soliton solution of the CH equation [Matsuno 2005], which corresponds to the Riemann–Hilbert problem with $r(k) \equiv 0$ and with residue parameters $\{\nu_j\}_{j=1}^N$ and $\{\gamma_j\}_{j=1}^N$. In turn, $u_{\text{solit}}(x, t)$ develops, for large t , into a superposition of 1-solitons of type (4-6).

REMARK 3. The nonlinear steepest descent method allows rigorous studying the asymptotics in other domains of the (y, t) plane; see [Boutet de Monvel and Shepelsky 2007].

Now let us return to the CH equation in the form (1-1a) (see also (1-3) and (1-4) depending on the parameter $\omega > 0$). Observe that (1-4) and the x -equation of the Lax pair (1-5a) take their forms corresponding to $\omega = 1$ if we replace

$$m \mapsto \frac{m}{\omega}, \quad \lambda \mapsto \lambda\omega. \quad (5-3)$$

Accordingly, the spectral parameter k is introduced by

$$k^2 = -\lambda\omega - \frac{1}{4},$$

(2-5) becomes

$$p(x, t; k) = x - \int_x^\infty \left(\sqrt{\frac{m(\xi, t)}{\omega} + 1} - 1 \right) d\xi - \frac{2\omega}{1 + 4k^2} t, \quad (5-4)$$

and the scale

$$y(x, t) = x - \int_x^\infty \left(\sqrt{\frac{m(\xi, t)}{\omega} + 1} - 1 \right) d\xi \quad (5-5)$$

is ω -dependent as well. The coefficient matrices in the Lax pair equations are also to be modified in accordance with (5-3).

6. The limit $\omega \rightarrow 0$

Now consider a family of initial value problems (1-1) parametrized by ω , where the initial function $u_0(x)$ in (1-1b) is the same for all the differential equations (1-1a). Then the spectral functions (2-11) as well as the parameters of the discrete spectrum $\{v_j^\omega\}, \{\gamma_j^\omega\}$ become ω -dependent.

The results of the Section 5 show that for every fixed ω , the observer will see a train of solitons, parameters of which are determined by $\{v_j^\omega\}_{j=1}^{N_\omega}$ and $\{\gamma_j^\omega\}_{j=1}^{N_\omega}$. Then an interesting question is as follows:

QUESTION. What happens with the solution of the initial value problem as $\omega \rightarrow 0$? More precisely, what will we see in the long-time asymptotics?

It has been observed (see, e.g., [Matsuno 2005; Parker 2004]) that if the parameters of an ω -soliton (i.e., a one-soliton solution of (1-1a)) are changing appropriately with ω , then this soliton approaches, as $\omega \rightarrow 0$, a peakon, which is a piecewise smooth (weak), stable (cf. [Beals et al. 1999; Constantin and Strauss 2000]) solution of (1-1a) with $\omega = 0$ having a peak at its maximum point:

$$u^0(x, t) = v^0 e^{-|x - v^0 t - x^0|}, \quad v^0 > 0. \quad (6-1)$$

We will show that the solutions of the initial value problem for the CH equations with varying ω but with the same initial data approach, as $\omega \rightarrow 0$, a train of peakons with parameters determined by the spectrum of (1-5a) for $\omega = 0$.

It is known (see [Constantin 2001]) that

- for any $\omega > 0$ fixed, the spectrum of (1-5a) consists of
 - (i) a continuous part $\lambda \in (-\infty, -\frac{1}{4\omega})$ and
 - (ii) a finite set of simple eigenvalues $\{\lambda_j^\omega\}_{j=1}^{N_\omega}$.
- For $\omega = 0$, the spectrum is discrete and consists of

- ▷ simple eigenvalues $\{\lambda_j^0\}_{j=1}^\infty$ which accumulate at $-\infty$ only: if they are enumerated in decreasing order $0 > \lambda_1^0 > \lambda_2^0 > \dots$, then $\lambda_N^0 \rightarrow -\infty$ as $N \rightarrow \infty$.

Let $\Psi_\pm^\omega(x, \lambda_j^\omega)$, $\omega \geq 0$ be the eigenfunctions of (1-5a) associated with the eigenvalues λ_j^ω , and let $\hat{\chi}_j^\omega$ be the corresponding norming constants: $\Psi_\pm^\omega(x, \lambda_j^\omega)$ are normalized by the limit conditions

- (a) $\Psi_\pm^\omega(x, \lambda_j^\omega) \rightarrow e^{\mp v_j^\omega x}$ as $x \rightarrow \pm\infty$ with $v_j^\omega = \sqrt{\omega\lambda_j^\omega + \frac{1}{4}}$ for $\omega > 0$,
 (b) $\Psi_\pm^0(x, \lambda_j^0) \sim e^{\mp x/2}$ as $x \rightarrow \pm\infty$ for $\omega = 0$.

Then $\Psi_-^\omega(x, \lambda_j^\omega) = \hat{\chi}_j^\omega \Psi_+^\omega(x, \lambda_j^\omega)$. Passing from (1-5a) to the spectral problem for the operators $K^*(m + \omega)Kf = \frac{1}{\lambda}f$ with $(Kg)(x) = \int_x^\infty e^{(x-y)/2}g(y)dy$ [Constantin and McKean 1999; McKean 2003] and considering ω as the perturbation parameter, the following properties are not hard to obtain.

PROPOSITION 3. *As $\omega \rightarrow 0$, we have that $N_\omega \rightarrow \infty$, $\lambda_j^\omega \rightarrow \lambda_j^0$, and $\Psi_\pm^\omega(x, \lambda_j^\omega) \rightarrow \Psi_\pm^0(x, \lambda_j^0)$ in $L^2(-\infty, \infty)$, $j = 1, 2, \dots$ in the sense that as $\omega \rightarrow 0$, new eigenvalues are “escaping”, one by one, from the continuous spectrum whereas the already existing ω -eigenvalues and the associated ω -eigenfunctions are approaching respectively the corresponding eigenvalues and eigenfunctions of (1-5a) with $\omega = 0$. \square*

PROPOSITION 4. *Let $u_0(x)$ be a smooth, rapidly decreasing, as $|x| \rightarrow \infty$, function such that $m_0(x) := u_{0xx}(x) - u_0(x) > 0$ for all $x \in \mathbb{R}$. Let $\{v_j^\omega\}_{j=1}^{N_\omega}$, $\{\gamma_j^\omega\}_{j=1}^{N_\omega}$ be the residue parameters associated with $u_0(x)$ viewed as fixed initial data in the initial value problems (1-1) parametrized by $\omega > 0$.*

Then, as $\omega \rightarrow 0$, we have the following asymptotic behavior of these parameters:

$$v_j^\omega = \frac{1}{2} + \omega\lambda_j^0 + o(\omega), \quad (6-2)$$

$$\gamma_j^\omega = \omega\Gamma_j^0 + o(\omega), \quad (6-3)$$

where $\Gamma_j^0 = \frac{1}{\int_{-\infty}^\infty m_0(x)|\Psi_+^0(x, \lambda_j^0)|^2 dx}$.

PROOF. Taking into account the relation between v_j^ω and λ_j^ω , (6-2) follows immediately from Proposition 3.

Differentiating (1-5a) with respect to k (the derivative with respect to k is denoted by dot) and combining the resulting equation with (1-5a), after some manipulations similar to those for the Sturm-Liouville equation [Marchenko 1986] leading to the expression relating the derivative of the spectral function $a(k)$ at the spectrum points with the norm of the corresponding eigenfunctions

(the details for the CH equation are given in [Constantin 2001]) we arrive at the following expression

$$\frac{1}{\omega} \int_{-\infty}^{\infty} (m^0(x) + \omega) |\Psi_{-}^{\omega}(x, \lambda_j^{\omega})|^2 dx = \frac{\hat{\chi}_j^{\omega}}{2iv_j^{\omega}} \frac{\partial}{\partial k} W\{\Psi_{+}^{\omega}, \Psi_{-}^{\omega}\}|_{k=iv_j^{\omega}}, \quad (6-4)$$

where W is the Wronskian bilinear form $W\{f, g\} := f'g - fg'$. Comparing the asymptotics of $\Psi_{\pm}^{\omega}(x, \lambda)$ with those of $\Phi_{\pm}(x, 0; k)$ and taking into account that $x - y \rightarrow 0$ as $x \rightarrow +\infty$ and $x - y \rightarrow \exp\left\{\frac{1}{\omega} \int_{-\infty}^{\infty} (\sqrt{m + \omega} - \omega) dx\right\} =: \mathcal{E}(\omega)$ as $x \rightarrow -\infty$ we have that

$$a^{\omega}(k) = W\{\Psi_{+}^{\omega}, \Psi_{-}^{\omega}\} \cdot \mathcal{E}, \quad \chi_j^{\omega} = \hat{\chi}_j^{\omega} \cdot \mathcal{E},$$

where χ_j^{ω} are the norming constants in (2-12). Therefore, (6-4) can be written as

$$\begin{aligned} \frac{1}{\omega} \int_{-\infty}^{\infty} (m^0(x) + \omega) |\Psi_{-}^{\omega}(x, \lambda_j^{\omega})|^2 dx \\ = \frac{1}{\mathcal{E}} \frac{\chi_j^{\omega}}{2iv_j^{\omega}} \frac{1}{\mathcal{E}} (-2v_j^{\omega}) \dot{a}^{\omega}(iv_j^{\omega}) = \frac{i\chi_j^{\omega}}{\mathcal{E}^2} \dot{a}^{\omega}(iv_j^{\omega}). \end{aligned} \quad (6-5)$$

Now recall that γ_j^{ω} in the residue relations (3-18) are related to χ_j^{ω} by $i\gamma_j^{\omega} = \chi_j^{\omega} / \dot{a}^{\omega}(iv_j^{\omega})$. Hence, (6-5) gives for γ_j^{ω} the expression

$$\begin{aligned} \gamma_j^{\omega} &= \frac{-i\chi_j^{\omega}}{\dot{a}^{\omega}(iv_j^{\omega})} = \omega \frac{(\hat{\chi}_j^{\omega})^2}{\int_{-\infty}^{\infty} (m^0(x) + \omega) |\Psi_{-}^{\omega}(x, \lambda_j^{\omega})|^2 dx} \\ &= \frac{\omega}{\int_{-\infty}^{\infty} (m^0(x) + \omega) |\Psi_{+}^{\omega}(x, \lambda_j^{\omega})|^2 dx} \end{aligned} \quad (6-6)$$

and, by Proposition 3, (6-3) follows. \square

Now, as we have established the behavior of the soliton parameters as $\omega \rightarrow 0$, we are able to study the limiting behavior of ω -solitons in the original scale; since, as $t \rightarrow +\infty$, a multisoliton solution behaves as a superposition of one-soliton solutions [Matsuno 2005], it is enough to see what happens with a one-soliton solution.

An ω -soliton (with parameters v^{ω} and γ^{ω}) is given parametrically by equations of type (4-5), (4-6) appropriately modified in order to take into account the dependence on ω :

$$u(y, t) = \frac{16\omega(v^{\omega})^2}{1 - 4(v^{\omega})^2} \times \frac{1}{1 + 4(v^{\omega})^2 + (1 - 4(v^{\omega})^2) \cosh \phi(y, t)}, \quad (6-7)$$

where

$$\phi(y, t) = -2v^\omega \left(y - \frac{2\omega}{1-4(v^\omega)^2} t - \frac{1}{2v^\omega} \ln \frac{\gamma^\omega}{2v^\omega} \right), \quad (6-8)$$

and

$$x(y, t) = y + \ln \frac{1 + g(y, t) \frac{1+2v^\omega}{1-2v^\omega}}{1 + g(y, t) \frac{1-2v^\omega}{1+2v^\omega}} \quad (6-9)$$

with $g = e^\phi$.

Rewrite (6-9) as

$$x(y, t) = y + \ln \frac{1+2v^\omega}{1-2v^\omega} + \ln \frac{1-2v^\omega + (1+2v^\omega)g}{1+2v^\omega + (1-2v^\omega)g} \quad (6-10)$$

and introduce the new (moving) variables

$$X = x - v^\omega t - x_0^\omega, \quad Y = y - v^\omega t - y_0^\omega, \quad (6-11)$$

where

$$v^\omega = \frac{2\omega}{1-4(v^\omega)^2}, \quad x_0^\omega = \frac{1}{2v^\omega} \ln \frac{\gamma^\omega}{2v^\omega} + \ln \frac{1+2v^\omega}{1-2v^\omega}, \quad y_0^\omega = \frac{1}{2v^\omega} \ln \frac{\gamma^\omega}{2v^\omega}. \quad (6-12)$$

Then the one-soliton is given parametrically by

$$u(x, t) = U(Y(X)) \Big|_{X=x-v^\omega t-x_0^\omega} \equiv U^\omega(X) \Big|_{X=x-v^\omega t-x_0^\omega}, \quad (6-13)$$

where

$$U(Y) = \frac{16\omega(v^\omega)^2}{1-4(v^\omega)^2} \times \frac{1}{1+4(v^\omega)^2 + (1-4(v^\omega)^2) \cosh(2v^\omega Y)} \quad (6-14)$$

and $Y(X)$ is inverse to

$$X(Y) = Y + \ln \frac{1-2v^\omega + (1+2v^\omega)e^{-2v^\omega Y}}{1+2v^\omega + (1-2v^\omega)e^{-2v^\omega Y}}. \quad (6-15)$$

Applying Proposition 4 to (6-12)–(6-15), we see that as $\omega \rightarrow 0$,

- (i) The soliton velocity v^ω approaches the finite limit associated with the corresponding eigenvalue of (1-5a) with $\omega = 0$:

$$v^\omega \rightarrow -\frac{1}{2\lambda^0}.$$

- (ii) The phase shift x_0^ω also approaches a finite value:

$$x_0^\omega \sim \ln \omega + \ln \Gamma^0 + \ln \frac{2}{-2\omega\lambda^0} \rightarrow \ln \frac{\Gamma^0}{-\lambda^0}.$$

(iii) For all $X > \omega^\alpha > 0$ with $\alpha \in (0, 1)$, $Y(X) \rightarrow +\infty$. Moreover,

$$X(Y) \sim Y + \ln \frac{2 - 2\omega\lambda^0 e^Y}{2e^Y} = \ln(1 - \omega\lambda^0 e^Y)$$

and thus

$$U(Y) \sim -\frac{1}{2\lambda^0} \frac{1}{1 - \omega\lambda^0 e^Y} \sim -\frac{1}{2\lambda^0} e^{-X}.$$

Since $X(-Y) = -X(Y)$ in (6-15) and $U(-Y) = U(Y)$ in (6-14), it finally follows that

$$U(Y(X)) \sim -\frac{1}{2\lambda^0} e^{-|X|}, \quad |X| > \omega^\alpha. \quad (6-16)$$

The right-hand side of (6-16) is nothing but the peakon solution (6-1) of (1-5a) with $\omega = 0$ having the velocity (= amplitude) $v^0 = -1/(2\lambda^0)$ associated with the corresponding eigenvalue λ^0 . Now taking into account the phase shift when passing from a multisoliton solution to a superposition of one-solitons [Matsuno 2005], we arrive at the following proposition (see Figure 3).

PROPOSITION 5. *Let $u_0(x)$ be a smooth function, fast decreasing as $|x| \rightarrow \infty$, and such that $m_0(x) := u_{0xx}(x) - u_0(x) > 0$ for all $x \in \mathbb{R}$.*

- ▷ Let $\{\lambda_j^0\}_{j=1}^\infty$ be the eigenvalues of the spectral problem (1-5a) with $\omega = 0$ and $m = m_0$.
- ▷ For $\omega > 0$, let $u^\omega(x, t)$ be the solution of the initial value problem for the Camassa–Holm equation (1-1).
- ▷ Fix $C > 0$, $\delta > 0$, and $\varepsilon > 0$.
- ▷ Let $\{\lambda_j^0\}_{j=1}^{N(C)}$ be those λ_j^0 satisfying $0 > \lambda_1^0 > \dots > \lambda_{N(C)}^0 > -\frac{1}{2C}$.

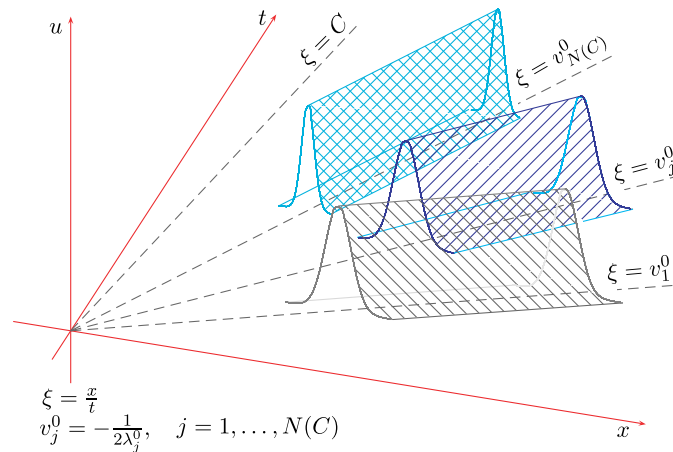


Figure 3. Long-time asymptotics of $u = u^\omega(x, t)$ for small ω .

Then there exists $\tilde{\omega} = \tilde{\omega}(C, \delta, \varepsilon)$ such that for all $0 < \omega < \tilde{\omega}$ the asymptotics of $u^\omega(x, t)$ in the domain $x > Ct$ is given by $N(C)$ one-solitons of type (6-13)–(6-15) with velocities and forms close to those of the corresponding peakons:

$$u^\omega(x, t) = U_j^\omega(X) + o(1) \text{ as } t \rightarrow \infty, \quad |X| = O(1) \text{ with } X = x - v_j^\omega t - x_{0j}^\omega,$$

where

- $o(1)$ depends on ω ,
- $|v_j^\omega - v_j^0| < \varepsilon$ with $v_j^0 = -\frac{1}{2\lambda_j^0}$,
- $|x_{0j}^\omega - x_{0j}^0| < \varepsilon$ with

$$x_{0j}^0 = \ln\left(\frac{\Gamma_j^0}{-\lambda_j^0} \prod_{l=1}^{j-1} \left(\frac{\lambda_j^0}{\lambda_l^0} - 1\right)^2\right)$$

$$\text{and } \left|U_j^\omega(X) - v_j^0 e^{-|X|}\right| < \varepsilon \text{ for } |X| > \delta.$$

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