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SLE_6 and CLE_6 from critical percolation

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ABSTRACT. We review some of the recent progress on the scaling limit of two-dimensional critical percolation; in particular, the convergence of the exploration path to chordal SLE_6 and the full scaling limit of cluster interface loops. The results given here on the full scaling limit and its conformal invariance extend those presented previously. For site percolation on the triangular lattice, the results are fully rigorous. We explain some of the main ideas, skipping most technical details.

1. Introduction

In the theory of critical phenomena it is usually assumed that a physical system near a continuous phase transition is characterized by a single length scale (the correlation length) in terms of which all other lengths should be measured. When combined with the experimental observation that the correlation length diverges at the phase transition, this simple but strong assumption, known as the scaling hypothesis, leads to the belief that at criticality the system has no characteristic length, and is therefore invariant under scale transformations. This suggests that all thermodynamic functions at criticality are homogeneous functions, and predicts the appearance of power laws.

It also implies that if one rescales appropriately a critical lattice model, shrinking the lattice spacing to zero, it should be possible to obtain a continuum model, known as the scaling limit. The scaling limit is not restricted to a lattice and may possess more symmetries than the original model. Indeed, the scaling limits

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of many critical lattice models are believed to be conformally invariant and to correspond to Conformal Field Theories (CFTs). But until recently, such a correspondence was at most heuristic, and was assumed as a starting point by physicists working in CFT. The methods of CFT themselves proved hard to put into a rigorous mathematical formulation.

The introduction by Oded Schramm [2000] of Stochastic/Schramm Loewner Evolution (SLE) has provided a new powerful and mathematically rigorous tool to study scaling limits of critical lattice models. Thanks to this, in recent years tremendous progress has been made in understanding the conformally invariant nature of the scaling limits of several such models.

While CFT focuses on correlation functions of local operators (e.g., spin variables in the Ising model), SLE describes the behavior of macroscopic random curves present in these models, such as percolation cluster boundaries. In the scaling limit, the distribution of such random curves can be uniquely identified thanks to their conformal invariance and a certain "Markovian" property. There is a one-parameter family of SLEs, indexed by a positive real number κ, and they appear to be essentially the only possible candidates for the scaling limits of interfaces of two-dimensional critical systems that are believed to be conformally invariant.

The main power of SLE stems from the fact that it allows to compute different quantities; for example, percolation crossing probabilities and various percolation critical exponents. Therefore, relating the scaling limit of a critical lattice model to SLE allows for a rigorous determination of some aspects of the large scale behavior of the lattice model.

In the context of the Ising, Potts and $O(n)$ models, an SLE curve is believed to describe the scaling limit of a single interface, which can be obtained by imposing special boundary conditions. A single SLE curve is therefore not in itself sufficient to immediately describe the scaling limit of the unconstrained model without boundary conditions in the whole plane (or in domains with boundary conditions that do not determine a single interface), and contains only limited information concerning the connectivity properties of the model.

A more complete description can be obtained in terms of loops, corresponding to the scaling limits of cluster boundaries. Such loops should also be random and have a conformally invariant distribution. This approach led Wendelin Werner [2005b; 2005a] (see also [Werner 2003]) to the definition of Conformal Loop Ensembles (CLEs), which are, roughly speaking, random collections of fractal loops with a certain conformal restriction/renewal property.

For percolation, a complete proof of the connection with SLE, first conjectured in [Schramm 2000], has recently been given in [Camia and Newman 2007]. The proof relies heavily on the ground breaking result of Stas Smirnov [2001] about the existence and conformal invariance of the scaling limit of crossing probabilities (see [Cardy 1992]). The last section of this paper explains the main ideas of that proof, highlighting the role of conformal invariance, but without dwelling on the heavy technical details.

As for the Ising, Potts and $O(n)$ models, the scaling limit of percolation in the whole plane should be described by a measure on loops, where the loops are closely related to SLE curves. Such a description in the case of percolation was presented in [Camia and Newman 2004], where the authors of the present paper constructed a probability measure on collections of fractal conformally invariant loops in the plane (closely related to a CLE), arguing that it corresponds to the full scaling limit of critical two-dimensional percolation. A proof of that statement was subsequently provided in [Camia and Newman 2006].

Here, we will briefly explain how to go from a single SLE curve to the full scaling limit, again skipping the technical details, for the case of a Jordan domain with monochromatic boundary conditions (see Theorem 2). This extends the results presented in [Camia and Newman 2006], where the scaling limit was first taken in the unit disc and then an infinite volume limit was taken in order to obtain the full scaling limit in the whole plane. Moving from the unit disc (or any convex domain) to a general Jordan domain introduces extra complications that are dealt with using a new argument, developed in [Camia and Newman 2007], that exploits the continuity of Cardy's formula [1992] with respect to changes in the shape of the domain (see the discussion in Section 5). Taking scaling limits in general Jordan domains is a necessary step in order to consider conformal restriction/renewal properties as in Theorem 4 below.

Using the full scaling limit, one can attempt to understand the geometry of the near-critical scaling limit, where the percolation density tends to the critical one in an appropriate way as the lattice spacing tends to zero. A heuristic analysis [Camia et al. 2006a; 2006b] based on a natural ansatz leads to a one-parameter family of loop models (i.e., probability measures on random collections of loops), with the critical full scaling limit corresponding to a particular choice of the parameter. Except for the latter case, these measures are not scale invariant, but are mapped into one another by scale transformations. This framework can be used to define a renormalization group flow (under the action of dilations), and to describe the scaling limit of related models, such as invasion and dynamical percolation and the minimal spanning tree. In particular, this analysis helps explain why the scaling limit of the minimal spanning tree may be scale invariant but *not* conformally invariant, as first observed numerically by Wilson [2004].

2. SLE and CLE

The Stochastic/Schramm Loewner Evolution with parameter $\kappa > 0$ (SLE$_\kappa$) was introduced by Schramm [2000] as a tool for studying the scaling limit of two-dimensional discrete (defined on a lattice) probabilistic models whose scaling limits are expected to be conformally invariant. In this section we define the chordal version of SLE$_\kappa$; for more on the subject, the interested reader can consult Schramm's paper as well as the fine reviews by Lawler [2004], Kager and Nienhuis [2004], and Werner [2004], and Lawler's book [2005].

Let $\mathbb{H}$ denote the upper half-plane. For a given continuous real function U_t with $U_0 = 0$, define, for each $z \in \overline{\mathbb{H}}$, the function $g_t(z)$ as the solution to the ODE

$$\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \tag{2-1}$$

with $g_0(z) = z$. This is well defined as long as $g_t(z) - U_t \neq 0$, i.e., for all $t < T(z)$, where

$$T(z) \equiv \sup\{t \geq 0 : \min_{s \in [0,t]} |g_s(z) - U_s| > 0\}. \tag{2-2}$$

Let $K_t \equiv \{z \in \overline{\mathbb{H}} : T(z) \leq t\}$ and let $\mathbb{H}_t$ be the unbounded component of $\mathbb{H} \setminus K_t$; it can be shown that K_t is bounded and that g_t is a conformal map from $\mathbb{H}_t$ onto $\mathbb{H}$. For each t, it is possible to write $g_t(z)$ as

$$g_t(z) = z + \frac{2t}{z} + O\left(\frac{1}{z^2}\right), \tag{2-3}$$

when $z \to \infty$. The family $(K_t, t \geq 0)$ is called the *Loewner chain* associated to the driving function $(U_t, t \geq 0)$.

DEFINITION 2.1. *Chordal SLE$_\kappa$* is the Loewner chain $(K_t, t \geq 0)$ that is obtained when the driving function $U_t = \sqrt{\kappa} B_t$ is $\sqrt{\kappa}$ times a standard real-valued Brownian motion $(B_t, t \geq 0)$ with $B_0 = 0$.

For all $\kappa \geq 0$, chordal SLE$_\kappa$ is almost surely generated by a continuous random curve γ in the sense that, for all $t \geq 0$, $\mathbb{H}_t \equiv \mathbb{H} \setminus K_t$ is the unbounded connected component of $\mathbb{H} \setminus \gamma[0,t]$; γ is called the *trace* of chordal SLE$_\kappa$.

It is not hard to see, as argued by Schramm, that any continuous random curve γ in the upper half-plane starting at the origin and going to infinity must be an SLE curve if it possesses the following *conformal Markov property*. For any fixed $T \in \mathbb{R}$, conditioning on $\gamma[0,T]$, the image under g_T of $\gamma[T,\infty)$ is distributed like an independent copy of γ, up to a time reparametrization. This implies that the driving function U_t in the Loewner chain associated to the curve γ is continuous and has stationary and independent increments. If the time parametrization implicit in Definition 2.1 and the discussion preceding it is chosen for γ, then scale invariance also implies that the law of U_t is the same as the law of $\lambda^{-1/2}U_{\lambda t}$ when $\lambda > 0$. These properties together imply that U_t must be a constant multiple of standard Brownian motion.

Now let $D \subset \mathbb{C}$ ($D \neq \mathbb{C}$) be a simply connected domain whose boundary is a continuous curve. By Riemann's mapping theorem, there are (many) conformal maps from the upper half-plane $\mathbb{H}$ onto D. In particular, given two distinct points $a, b \in \partial D$ (or more accurately, two distinct prime ends), there exists a conformal map f from $\mathbb{H}$ onto D such that $f(0) = a$ and $f(\infty) \equiv \lim_{|z|\to\infty} f(z) = b$. In fact, the choice of the points a and b on the boundary of D only characterizes $f(\cdot)$ up to a scaling factor $\lambda > 0$, since $f(\lambda\cdot)$ would also do.

Suppose that $(K_t, t \geq 0)$ is a chordal SLE_κ in $\mathbb{H}$ as defined above; we define chordal SLE_κ $(\tilde{K}_t, t \geq 0)$ in D from a to b as the image of the Loewner chain $(K_t, t \geq 0)$ under f. It is possible to show, using scaling properties of SLE_κ, that the law of $(\tilde{K}_t, t \geq 0)$ is unchanged, up to a linear time-change, if we replace $f(\cdot)$ by $f(\lambda\cdot)$. This makes it natural to consider $(\tilde{K}_t, t \geq 0)$ as a process from a to b in D, ignoring the role of f. The trace of chordal SLE in D from a to b will be denoted by $\gamma_{D,a,b}$.

We now move from the conformally invariant random curves of SLE to collections of conformally invariant random loops and introduce the concept of Conformal Loop Ensemble (CLE — see [Werner 2003; 2005b; 2005a; Sheffield 2006]). The key feature of a CLE is a sort of conformal restriction/renewal property. Roughly speaking, a CLE in D is a random collection $\mathcal{L}_D$ of loops such that if all the loops intersecting a (closed) subset D' of D or of its boundary are removed, the loops in any one of the various remaining (disjoint) subdomains of D form a random collection of loops distributed as an independent copy of $\mathcal{L}_D$ conformally mapped to that subdomain (see Theorem 4). We will not attempt to be more precise here since somewhat different definitions (although, in the end, substantially equivalent) have appeared in the literature, but the meaning of the conformal restriction/renewal property should be clear from Theorem 4.

For formal definitions and more discussion on the properties of a CLE, see the original literature on the subject [Werner 2005b; 2005a; Sheffield 2006], where it is shown that there is a one-parameter family CLE_κ of conformal loop ensembles with the above conformal restriction/renewal property and that for $\kappa \in (8/3, 8]$, the CLE_κ loops locally look like SLE_κ curves.

There are numerous lattice models that can be described in terms of random curves and whose scaling limits are assumed (and in a few cases proved) to be conformally invariant. These include the Loop Erased Random Walk, the Self-Avoiding Walk and the Harmonic Explorer, all of which can be defined as polygonal paths along the edges of a lattice. The Ising, Potts and percolation models instead are naturally defined in terms of clusters, and the interfaces between different clusters form random loops. In the $O(n)$ model, configurations of loops along the edges of the hexagonal lattice are weighted according to the total number and length of the loops. All of these models are supposed to have scaling limits described by SLE_κ or CLE_κ for some value of κ between 2 and 8. For more information on these lattice models and their scaling limits, the interested reader can consult [Cardy 2001; 2005; Kager and Nienhuis 2004; Werner 2005b; Sheffield 2006].

In the rest of the paper we will restrict attention to percolation, where the connection with SLE_6 and CLE_6 has been made rigorous [Smirnov 2001; Camia and Newman 2006; 2007].

3. Conformal invariance of critical percolation

In this section we will consider critical site percolation on the triangular lattice, for which conformal invariance in the scaling limit was rigorously proved [Smirnov 2001]. A precise formulation of conformal invariance, attributed to Michael Aizenman, is that the probability that a percolation cluster crosses between two disjoint segments of the boundary of some simply connected domain should converge to a conformally invariant function of the domain and the two segments of the boundary. This conjecture is connected with the extensive numerical investigations reported in [Langlands et al. 1994]. A formula for the purposed limit was then derived by John Cardy [1992] using (nonrigorous) field theoretical methods. The interest of mathematicians was already evident in [Langlands et al. 1994], but a proof of the conjecture [Smirnov 2001] (and of Cardy's formula) did not come until 2001.

We will denote by $\mathcal{T}$ the two-dimensional triangular lattice, whose sites are identified with the elementary cells of a regular hexagonal lattice $\mathcal{H}$ embedded in the plane as in Figure 1. We say that two hexagons are neighbors (or that they are adjacent) if they have a common edge. A sequence $(\xi_0, \dots, \xi_n)$ of hexagons of $\mathcal{H}$ such that ξ_{i-1} and ξ_i are neighbors for all $i = 1, \dots, n$ and $\xi_i \neq \xi_j$ whenever $i \neq j$ will be called a $\mathcal{T}$-*path*. If the first and last hexagons of the path are neighbors, the path will be called a $\mathcal{T}$-*loop*.

Let D be a bounded simply connected domain containing the origin whose boundary ∂D is a continuous curve. Let $\phi : \overline{\mathbb{D}} \to D$ be the (unique) continuous function that maps $\mathbb{D}$ onto D conformally and such that $\phi(0) = 0$ and $\phi'(0) > 0$. Let z_1, z_2, z_3, z_4 be four points of ∂D in counterclockwise order — i.e., such that $z_j = \phi(w_j)$, $j = 1, 2, 3, 4$, with $w_1, \dots, w_4$ in counterclockwise order. Also, let

$$\eta = \frac{(w_1 - w_2)(w_3 - w_4)}{(w_1 - w_3)(w_2 - w_4)}.$$

Cardy's formula [1992] for the probability $\Phi_D(z_1, z_2; z_3, z_4)$ of a crossing inside D from the counterclockwise arc $\overline{z_1 z_2}$ to the counterclockwise arc $\overline{z_3 z_4}$ is

$$\Phi_D(z_1, z_2; z_3, z_4) = \frac{\Gamma(2/3)}{\Gamma(4/3)\Gamma(1/3)} \eta^{1/3} {}_2F_1(1/3, 2/3; 4/3; \eta), \tag{3-1}$$

where ${}_2F_1$ is a hypergeometric function.

For a given mesh $\delta > 0$, the probability of a blue crossing inside D from the counterclockwise arc $\overline{z_1 z_2}$ to the counterclockwise arc $\overline{z_3 z_4}$ is the probability of the existence of a blue $\mathcal{T}$-path $(\xi_0, \dots, \xi_n)$ such that ξ_0 intersects the counterclockwise arc $\overline{z_1 z_2}$, ξ_n intersects the counterclockwise arc $\overline{z_3 z_4}$, and $\xi_1, \dots, \xi_{n-1}$ are all contained in D. Smirnov [2001] proved that crossing probabilities converge in the scaling limit to conformally invariant functions of the domain and the four points on its boundary, and identified the limit with Cardy's formula (3-1).

The proof of Smirnov's theorem is based on the identification of certain generalized crossing probabilities that are almost discrete harmonic functions and whose scaling limits converge to harmonic functions. The behavior on the boundary of such functions is easy to determine and is sufficient to specify them uniquely. The relevant crossing probabilities can be expressed in terms of the boundary values of such harmonic functions, and as a consequence are invariant under conformal transformations of the domain and the two segments of its boundary.

The presence of a blue crossing in D from the counterclockwise boundary arc $\overline{z_1 z_2}$ to the counterclockwise boundary arc $\overline{z_3 z_4}$ can be determined using a clever algorithm that explores the percolation configuration inside D starting at, say, z_1 and assumes that the hexagons just outside $\overline{z_1 z_2}$ are all blue and those just outside $\overline{z_4 z_1}$ are all yellow. The exploration proceeds following the interface between the blue cluster adjacent to $\overline{z_1 z_2}$ and the yellow cluster adjacent to $\overline{z_4 z_1}$. A blue crossing is present if the exploration process reaches $\overline{z_3 z_4}$ before $\overline{z_2 z_3}$. This *exploration process* and the *exploration path* (see Figure 1) associated to it were introduced in [Schramm 2000].

The exploration process can be carried out in $\mathbb{H} \cap \mathcal{H}$, where the hexagons in the lowest row and to the left of a chosen hexagon have been colored yellow and the remaining hexagons in the lowest row have been colored blue. This produces an infinite exploration path, whose scaling limit was conjectured [Schramm 2000] by Schramm to converge to SLE$_6$.

It is easy to see that the exploration process is Markovian in the sense that, conditioned on the exploration up to a certain (stopping) time, the future of the exploration evolves in the same way as the past except that it is now performed in a different domain, where some of the explored hexagons have become part of the boundary (see, e.g., Figure 1).

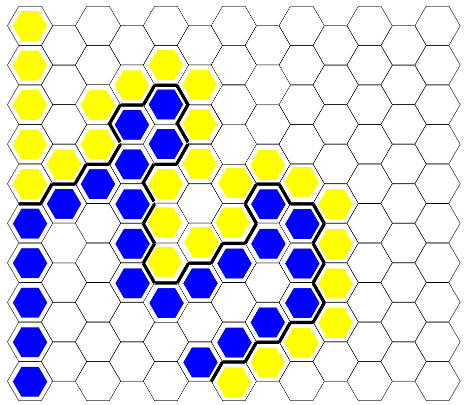

Figure 1. Percolation exploration path in a portion of the hexagonal lattice with blue/yellow boundary conditions on the first column, corresponding to the boundary of the region where the exploration is carried out. The colored hexagons that do not belong to the first column have been explored during the exploration process. The heavy line between yellow (light) and blue (dark) hexagons is the exploration path produced by the exploration process.

This observation, together with the connection between the exploration process and crossing probabilities, Smirnov's theorem about the conformal invariance of crossing probabilities in the scaling limit, and Schramm's characterization of SLE via the conformal Markov property discussed in Section 2, strongly support the above conjecture.

As we now explain, the natural setting to define the exploration process is that of *lattice domains*, i.e., sets D^δ of hexagons of $\delta\mathcal{H}$ that are *connected* in the sense that any two hexagons in D^δ can be joined by a ($\delta\mathcal{T}$)-path contained in D^δ. We say that a bounded lattice domain D^δ is *simply connected* if both D^δ and $\delta\mathcal{T} \setminus D^\delta$ are connected. A *lattice-Jordan* domain D^δ is a bounded simply connected lattice domain such that the set of hexagons adjacent to D^δ is a ($\delta\mathcal{T}$)-loop.

Given a lattice-Jordan domain D^δ, the set of hexagons adjacent to D^δ can be partitioned into two (lattice-)connected sets. If those two sets of hexagons are assigned different colors, for any coloring of the hexagons inside D^δ, there is an interface between two clusters of different colors starting and ending at two boundary points, a^δ and b^δ, corresponding to the locations on the boundary of D^δ where the color changes. If one performs an exploration process in D^δ starting at a^δ, one ends at b^δ, producing an exploration path γ^δ that traces the entire interface from a^δ to b^δ.

Given a planar domain D, we denote by ∂D its topological boundary. Let ∂D be locally connected (i.e., a continuous curve), and assume that D contains the origin. Then one can parametrize ∂D by $\varphi : S^1 \to \partial D$, where φ is the restriction to the unit circle S^1 of the continuous map $\phi : \overline{\mathbb{D}} \to \overline{D}$ that is conformal in $\mathbb{D}$ and satisfies $\phi(0) = 0$, $\phi'(0) > 0$. With this notation, we say that D^δ converges to D as $\delta \to 0$ if

$$\lim_{\delta \to 0} \inf_h \sup_{z \in S^1} |\varphi(z) - \varphi^\delta(h(z))| = 0, \tag{3-2}$$

where the infimum is over monotonic functions $h : S^1 \to S^1$ (and the objects with the superscript δ refer to D^δ — for simplicity we are assuming that all domains contain the origin). If moreover two points, $a^\delta, b^\delta \in \partial D^\delta$, converge respectively to $a, b \in \partial D$ as $\delta \to 0$, we write $(D^\delta, a^\delta, b^\delta) \to (D, a, b)$. In the following theorem the topology on curves is that induced by the supremum norm, but with monotonic reparametrizations of the curves allowed (see [Aizenman and Burchard 1999; Camia and Newman 2006; 2007]), i.e., the distance between curves is

$$\mathrm{d}(\gamma, \gamma^\delta) = \inf_h \sup_{t \in [0,\infty)} |\gamma(t) - \gamma^\delta(h(t))|, \tag{3-3}$$

where $\gamma(t), \gamma^\delta(t), t \in [0, \infty)$, are parametrizations of $\gamma_{D,a,b}$ and $\gamma^\delta_{D,a,b}$ respectively, and the infimum is over monotonic functions $h : [0, \infty) \to [0, \infty)$. A proof of the theorem can be found in [Camia and Newman 2007] and a detailed sketch is presented in Section 6 below.

THEOREM 1. *Let (D, a, b) be a Jordan domain with two distinct selected points on its boundary ∂D. Then, for lattice-Jordan domains D^δ from $\delta\mathcal{H}$ with $a^\delta, b^\delta \in \partial D^\delta$ such that $(D^\delta, a^\delta, b^\delta) \to (D, a, b)$ as $\delta \to 0$, the percolation exploration path $\gamma^\delta_{D,a,b}$ in D^δ from a^δ to b^δ converges in distribution to the trace $\gamma_{D,a,b}$ of chordal* SLE$_6$ *in D from a to b, as $\delta \to 0$.*

4. The full scaling limit in a Jordan domain

In this section we define the *Continuum Nonsimple Loop (CNL) process* in a Jordan domain D, a random collection of countably many nonsimple fractal loops in D which corresponds to the full scaling limit of percolation in D with monochromatic boundary conditions. This refers to the collection of all cluster boundaries of percolation configurations in D with the hexagons at the boundary of D all blue (obviously, one could as well choose yellow boundary conditions). The algorithmic construction that we present below is analogous to that of [Camia and Newman 2004; 2006] for the unit disc $\mathbb{D}$, but here we perform it in a general Jordan domain.

The CNL process on the full plane can be obtained by taking a sequence of domains D tending to $\mathbb{C}$. This was done in the two works just cited, and for that purpose, discs of radius R with $R \to \infty$ suffice. This full plane CNL process is the scaling limit of the collection of all cluster boundaries in the full lattice (without boundary conditions). In order to consider conformal restriction/renewal properties (as we do in Theorem 4 below), one needs to consider the CNL process in fairly general bounded domains D. There are extra complications in taking the scaling limit when D is nonconvex, as discussed in Section 5.

The basic ingredient in our algorithmic construction consists of a chordal SLE_6 path between two points on the boundary of a Jordan domain. As we will explain soon, sometimes the two boundary points are naturally determined as a product of the construction itself, and sometimes they are given as an input to the construction. In the second case, there are various procedures which would yield the "correct" distribution for the resulting CNL process; one possibility is as follows. Given a domain D, choose a and b so that, of all points in ∂D, they have maximal x-distance or maximal y-distance, whichever is greater. It is important to stress that in the end, the CNL process will turn out to be independent of the actual choice of boundary points, as is evident in Theorem 2. (One caveat is that one should avoid malicious choices of the boundary points for which the entire original domain would not be explored asymptotically.)

The first step of our construction is a chordal SLE_6, $\gamma \equiv \gamma_{D,a,b}$, between two boundary points $a, b \in \partial D$ chosen according to the above rule (see Figure 2). The set $D \setminus \gamma_{D,a,b}[0,\infty)$ is a countable union of its connected components, which are open and simply connected. If z is a deterministic point in D, then with probability one, z is not touched by γ [Rohde and Schramm 2005] and so belongs to a unique one of these, that we denote $D_{a,b}(z)$. There are four kinds of components which may be usefully thought of in terms of how a point z in the interior of the component was first trapped at some time t_1 by $\gamma[0,t_1]$ perhaps together with either the counterclockwise arc $\partial_{a,b}D$ of ∂D between a and b or the counterclockwise arc $\partial_{b,a}D$ of ∂D between b and a: (1) those components whose boundary contains a segment of $\partial_{b,a}D$ between two successive visits at $\gamma(t_0)$ and $\gamma(t_1)$ to $\partial_{b,a}D$ (where here and below $t_0 < t_1$), (2) the analogous components with $\partial_{b,a}D$ replaced by the other part of the boundary $\partial_{a,b}D$, (3) those components formed when $\gamma(t_0) = \gamma(t_1)$ with γ winding about z in a counterclockwise direction between t_0 and t_1, and finally (4) the analogous clockwise components.

To conclude the first step, we consider all domains of type (1), corresponding to excursions of the SLE_6 path from the portion $\partial_{b,a}D$ of ∂D. For each such domain D', the points a' and b' on its boundary are chosen to be respectively those points where the excursion ends and where it begins, that is, for $D_{a,b}(z)$ we set $a' = \gamma((t_1(z))$ and $b' = \gamma(t_0(z))$. We then run a chordal SLE_6 from a' to b'. The loop obtained by pasting together the excursion from b' to a' followed

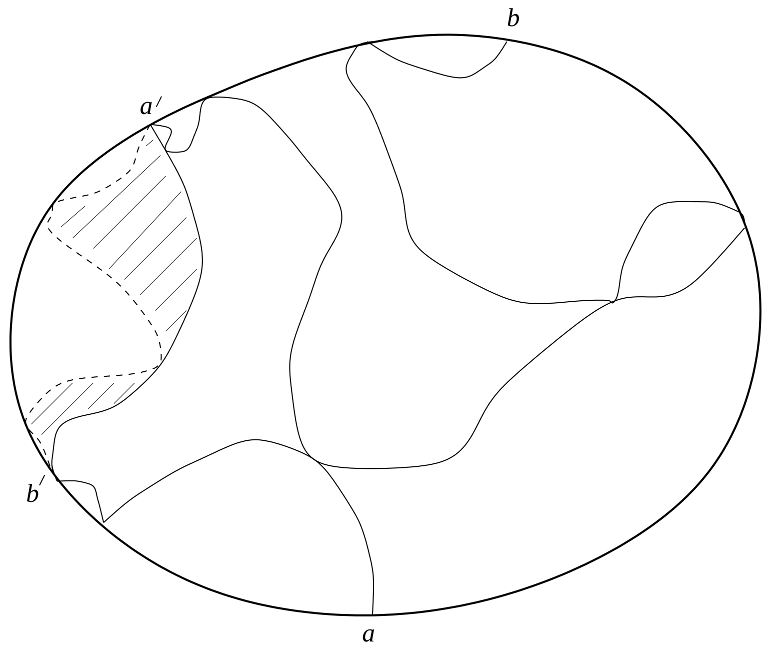

Figure 2. Schematic drawing of the construction of continuum nonsimple loops inside a Jordan domain D. The construction starts with a chordal SLE$_6$ (full line) between two points, a and b, on ∂D. To obtain loops, other chordal SLE$_6$s (e.g., dashed line) are run (e.g., from a' to b') between where an excursion from the counterclockwise arc $\partial_{b,a} D$ of ∂D of the first SLE$_6$ respectively ends and starts. The inside of one such loop is shaded.

by the new SLE$_6$ path from a' to b' is one of our continuum loops (see Figure 2). At the end of the first step, then, the procedure has generated countably many loops that touch $\partial_{b,a} D$; each of these loops touches $\partial_{b,a} D$ but may or may not touch $\partial_{a,b} D$.

The last part of the first step also produces new domains, corresponding to the connected components of $D' \setminus \gamma_{D',a',b'}[0,\infty)$ for all domains D' of type (1). Each one of these components, together with all the domains of type (2), (3) and (4) previously generated, is to be used in the next step of the construction, playing the role of the original domain D. For each one of these domains, we choose the new a and new b on the boundary as explained before, and then continue with the construction. Note that the new a and new b are chosen according to the rule explained at the beginning of this section also for domains of type (2), even though they are generated by excursions like the domains of type (1).

This iterative procedure produces at each step a countable set of loops. The limiting object, corresponding to the collection of all such loops, is our basic process. (Technically speaking, we should include also trivial loops fixed at each $z \in D$ so that the collection of loops is closed in an appropriate sense [Aizenman and Burchard 1999].)

As explained, the construction is carried out iteratively and can be performed simultaneously on all the domains that are generated at each step. We wish to emphasize, though, that the obvious monotonicity of the procedure, where at each step new paths are added independently in different domains, and new domains are formed from the existing ones, implies that any other choice of the order in which the domains are used would give the same result (i.e., produce the same limiting distribution), provided that every domain that is formed during the construction is eventually used.

The main interest of the loop process defined above is in the following theorem, where the topology on collections of loops is that of [Aizenman and Burchard 1999] (see also [Camia and Newman 2006]).

THEOREM 2. *In the scaling limit*, $\delta \to 0$, *the collection of all cluster boundaries of critical site percolation on the triangular lattice in a Jordan domain D with monochromatic boundary conditions converges in distribution to the Continuum Nonsimple Loop process in D.*

A key property of the CNL process is conformal invariance.

THEOREM 3. *Let D, D' be Jordan domains and $f : \overline{D} \to \overline{D}'$ a continuous function that maps D conformally onto D'. Then the CNL process in D' is distributed like the image under f of the CNL process in D.*

Moreover, as shown in the next theorem, the outermost loops of the CNL process in a Jordan domain satisfy a conformal restriction/renewal property, as in the definitions of the Conformal Loop Ensembles of Werner [2005b] and Sheffield [2006].

THEOREM 4. *Let D be a Jordan domain and $\mathcal{L}_D$ be the collection of CN loops in $\overline{D}$ that are not surrounded by any other loop. Consider an arc Γ of ∂D and let $\mathcal{L}_{D,\Gamma}$ be the set of loops of $\mathcal{L}_D$ that touch Γ. Then, conditioned on $\mathcal{L}_{D,\Gamma}$, for any connected component D' of $D \setminus \overline{\bigcup\{L : L \in \mathcal{L}_{D,\Gamma}\}}$, the loops in $\overline{D}'$ form a random collection of loops distributed as an independent copy of $\mathcal{L}_D$ conformally mapped to D'.*

Yet another form of conformal invariance is illustrated by showing how to obtain a (conformally invariant) SLE_6 curve from the CNL process. Given a Jordan domain D and two points $a, b \in \partial D$, let $\Gamma = \overline{ba}$ be the counterclockwise closed arc $\overline{ba}$ of ∂D. Define $\mathcal{L}_D$ and $\mathcal{L}_{D,\Gamma}$ as in Theorem 4. For each $L \in \mathcal{L}_{D,\Gamma}$, going from a to b clockwise, there are a first and a last point, x and y respectively, where L intersects Γ. We call the counterclockwise arc $\overline{xy}(L)$ of L between

x and y a (counterclockwise) *excursion* from $\overline{ba}$. We call such an $\overline{xy}(L)$ a *maximal* excursion if there is no other excursion from $\overline{ba}$ in (the closure of) the domain created by $\overline{xy}(L)$ and the counterclockwise arc $\overline{yx}$ of ∂D. The random curve obtained by pasting together (in the order in which they are encountered going from a to b clockwise) all such maximal excursions from $\overline{ba}$ is distributed like a chordal SLE$_6$ in D from a to b.

The procedure described above obviously requires some care, since there are countably many such excursions and there is no such thing as the first excursion encountered from a, or the next excursion. What this means is that in order to properly define the curve, one needs to use a limiting procedure. Since it is quite obvious how to do it but tedious to explain, we leave the details to the interested reader; see [Camia and Newman 2006].

5. Convergence and conformal invariance of the full scaling limit

SKETCH OF THE PROOF OF THEOREM 2. It follows directly from [Aizenman and Burchard 1999] that the family of distributions of the collections of cluster boundaries in D with monochromatic boundary conditions is tight, as $\delta \to 0$, in the sense of the induced Hausdorff metric on closed sets of curves based on the metric (3-3) for single curves (see [Aizenman and Burchard 1999] and [Camia and Newman 2006]), and so there is convergence along subsequences $\delta_k \to 0$. What needs to be proved is that the limiting distribution is that of the CNL process, independently of the subsequence δ_k.

The key to the proof is an algorithmic construction on the lattice which parallels the continuum construction of Section 4 used to define the CNL process in D. The construction takes place in a lattice-domain $D_k \equiv D^{\delta_k}$ that converges to D in the sense of (3-2) as $k \to \infty$ ($\delta_k \to 0$) and is essentially the same as the continuum one but with exploration paths instead of the SLE$_6$ curves.

This raises the question of how to define an exploration process and obtain an exploration path in a lattice-domain with monochromatic boundary conditions. The basic idea is that away from the boundary, the exploration process does not know the boundary conditions. For two given points x and y on the boundary of a lattice-domain with, say, blue boundary conditions, split the boundary into two arcs, the counterclockwise arc $\overline{xy}$ and the counterclockwise arc $\overline{yx}$. Then, one can run an exploration process from x to y with the usual rule inside the domain and on the counterclockwise arc $\overline{xy}$, while pretending that the counterclockwise arc $\overline{yx}$ is colored yellow (see Figure 3).

If we run such an exploration process in D_k and then look at the hexagons that have not yet been explored, we will see several disjoint lattice subdomains, all of which are lattice-Jordan. This amounts to removing the fattened exploration

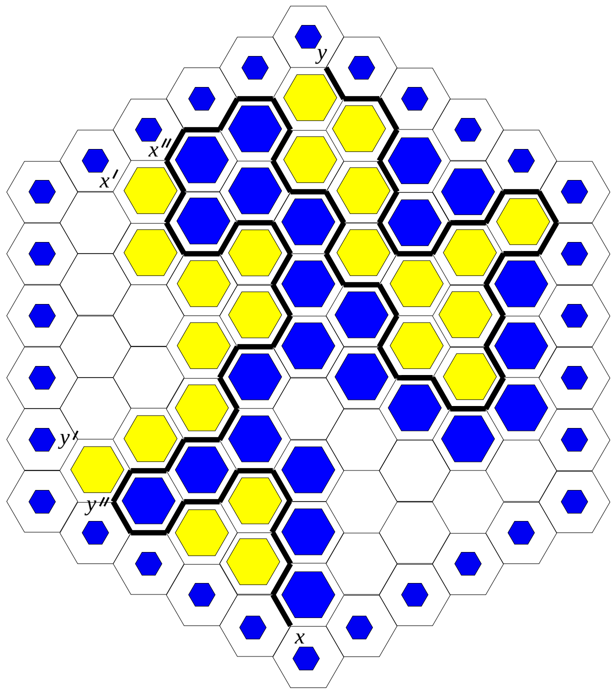

Figure 3. First step of the construction of the outer contour of a cluster of yellow (light in the figure) hexagons consisting of an exploration (heavy line) from x to y. The outer layer of hexagons does not belong to the domain where the explorations are carried out, but represents its monochromatic blue external boundary. x'' and y'' are the ending and starting points of an excursion that determines a new domain D', and x' and y' are the vertices where the edges that separate the yellow and blue portions of the external boundary of D' intersect $\partial D'$. The second step will consist of an exploration process in D' from x' to y'.

path consisting of the exploration path $\gamma_k \equiv \gamma^{\delta_k}_{D_k,x,y}$ itself and the hexagons immediately to its right and to its left.

The resulting lattice-Jordan subdomains are of four types, which may be usefully thought of in terms of their external boundaries: (1) those components whose boundary contains both sites in the fattened exploration path and in ∂^k_{yx}, the counterclockwise portion between y and x of the boundary of D_k, (2) the analogous components with ∂^k_{yx} replaced by the other boundary portion ∂^k_{xy}, (3) those components whose boundary only contains yellow hexagons from the fattened exploration path and finally (4) the analogous components whose boundary only contains blue hexagons from the fattened exploration path.

Notice that the components of type 1 are the only ones with mixed (partly blue and partly yellow) boundary conditions, while all other components have

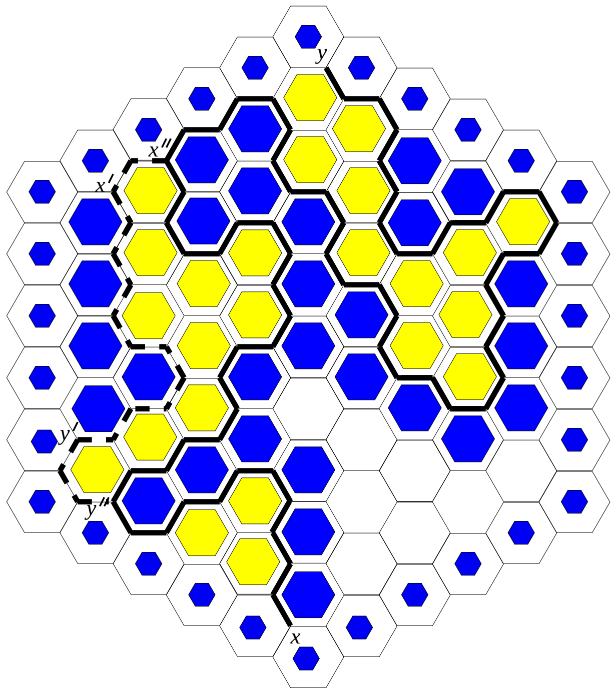

Figure 4. Second step of the construction of the outer contour of a cluster of yellow (light in the figure) hexagons consisting of an exploration from x' to y' whose resulting path (heavy broken line) is pasted to a portion of the previous exploration path with the help of the edges (indicated again by a heavy broken line) between x' and x'' and between y' and y'' in such a way as to obtain a loop around a yellow cluster (light in the figure) touching the boundary portion ∂^k_{yx}.

monochromatic (blue or yellow) boundary conditions; type 1 components are special because we have taken blue boundary conditions on D_k while the exploration path has yellow on its left and blue on its right. Because of the mixed boundary conditions, each lattice subdomain of type 1 must contain an interface between the two boundary points where the color changes. It is also clear that to find such an interface one has to start an exploration process at one of the two boundary points where the color changes (the two choices give the same exploration path).

If we run such an exploration process inside a lattice subdomain D'_k of type 1 and paste it to a portion of γ_k as in Figure 4, we obtain a loop corresponding to the interface surrounding a yellow cluster that touches ∂^k_{yx}. If we then again remove the fattened exploration path, D'_k is split into various components, but this time those lattice subdomains all have monochromatic boundary conditions.

If we do the same in each subdomain of type 1, we obtain a collection of loops. Moreover, all the lattice subdomains of D_k of nonexplored hexagons then have monochromatic boundary conditions. Thus we can iterate the whole procedure inside each of those lattice subdomains, until we have found all the interfaces contained in D_k.

The similarity between this construction and the continuum one of the CNL process should be apparent. To continue the proof one needs first to show that the exploration paths used in the lattice construction converge to chordal SLE_6 curves. The first step is a simple application of Theorem 1 to the first exploration path

$$\gamma_k = \gamma^{\delta_k}_{D_k,x_k,y_k},$$

where D_k, x_k, y_k are chosen so that D_k converges to D and x_k and y_k converge to the a and b of the continuum construction. However, in order to iterate this step and apply Theorem 1 again, we need to also show that the subdomains of the lattice construction converge to those of the continuum construction.

The convergence in distribution of γ_k to $\gamma = \gamma_{D,a,b}$ implies that we can find versions of γ_k and γ on some probability space $(\Omega, \mathcal{B}, \mathbb{P})$ such that $\gamma_k(\omega)$ converges to $\gamma(\omega)$ for all $\omega \in \Omega$. Using the coupling, γ_k and γ, for δ_k small, are close in the sense of (3-3). This is, however, not sufficient. If we want to conclude convergence of the subdomains, we need that wherever γ touches the boundary of D, γ_k touches the boundary of D_k nearby. Closeness in the sense of (3-3) does not ensure this but only that γ_k gets close to the boundary ∂D_k.

Note that, if γ_k gets within distance R_1 of some point z on ∂D_k without touching ∂D_k within distance R_2 of z, with $R_2 > R_1 > \delta_k$, considering the fattened version of γ_k shows the existence of two $(\delta_k \mathcal{T})$-paths of one color, say yellow, and one $(\delta_k \mathcal{T})$-path of the other color, blue, crossing the annulus of inner radius R_1 and outer radius R_2 centered at z.

In [Camia and Newman 2006], where the construction of the CN loops is carried out in the unit disc $\mathbb{D}$, the problem is solved by using the fact that $\mathbb{D}$ is convex and resorting to an upper bound (see, e.g., [Lawler et al. 2002]) on the probability that three disjoint monochromatic $\mathcal{T}$-paths cross a semiannulus in a half-plane. More precisely, the probability that the upper half-plane $\mathbb{H}$ contains three disjoint monochromatic $(\delta\mathcal{T})$-paths crossing the annulus of inner radius R_1 and outer radius R_2 centered at a point z of the real axis is bounded above by a constant times $(R_1/R_2)^{1+\varepsilon}$ for some $\varepsilon > 0$ (for all $\delta < R_1 < R_2$). Since $\varepsilon > 0$, if we let δ, $R_1 \to 0$ and cover any finite part of $\partial\mathbb{H}$ by $O(R_1)$ such annuli, the bound shows that such three-arm events with $R_1 \to 0$ do not occur in the scaling limit $\delta \to 0$ near $\partial\mathbb{H}$. For a domain D with a locally flat boundary or for a convex domain, this implies that, as $k \to \infty$ ($\delta_k \to 0$), the (lim sup of the) probability that γ_k gets within distance R_1 of *any* $z \in \partial D_k$ without touching the boundary within distance R_2 of z goes to zero as $R_1 \to 0$ for all (fixed) $R_2 > 0$. (In the case of a convex D this follows from the fact that the intersection of an annulus centered at the origin of the real axis with an appropriate translation and rotation of D is smaller than the intersection of the same annulus with $\mathbb{H}$, thus making the probability of three arms even smaller than in the case of the upper half-plane.)

We cannot use that bound here, since D is not necessarily convex (and even if it were, the D' domains of Theorems 3 and 4 will not generally be convex). Instead, we will use the continuity of Cardy's formula with respect to small changes in the shape of the domain. We postpone this issue until later and proceed with the sketch of the proof assuming that γ_k does not get close to the boundary of the domain without touching it nearby (probably).

Then the boundaries of the lattice/continuum subdomains obtained after running the first (coupled) exploration path/SLE$_6$ curve are close to each other in the metric (3-3). I.e., we can match lattice and continuum subdomains, at least for those whose diameter is larger than some ε_k which depends on δ_k. It is important that, as $k \to \infty$ (and $\delta_k \to 0$), we can let $\varepsilon_k \to 0$.

If we run an exploration process inside a (large) lattice subdomain D'_k converging to a continuum subdomain D', Theorem 1 allows us to conclude that the exploration path γ'_k in D'_k converges to the SLE$_6$ curve γ' in D' from a' to b', provided that the starting and ending points x'_k and y'_k of the exploration process are chosen so that they converge to a' and b' respectively as $k \to \infty$. We can now work with coupled versions of γ'_k and γ' and repeat the above argument with the new subdomains that they produce, obtaining again a match (with high probability).

This allows us to keep the lattice and continuum constructions coupled, which ensures in particular that the $(\delta_k \mathcal{T})$-loops obtained in the lattice construction converge, as $\delta_k \to 0$, to the loops obtained in the continuum construction.

For any fixed δ_k, it is clear that the lattice construction eventually finds all the boundary loops. However, to conclude that the CNL process is indeed the scaling limit of the collection of all interfaces, we need to show that, for any $\varepsilon > 0$, the number of steps of the discrete construction needed to find all the loops of diameter at least ε does not diverge as $k \to \infty$ (otherwise some loops would never be found in the scaling limit).

In [Camia and Newman 2006], this is resolved using percolation arguments (that make use of the RSW theorem [Russo 1978; Seymour and Welsh 1978] and FKG inequalities) to show that the size of the subdomains has a bounded away from zero probability of decreasing significantly at each iteration. We point out that the argument used in [Camia and Newman 2006], where the construction of the CN loops in carried out in the unit disc, is independent of the actual shape of

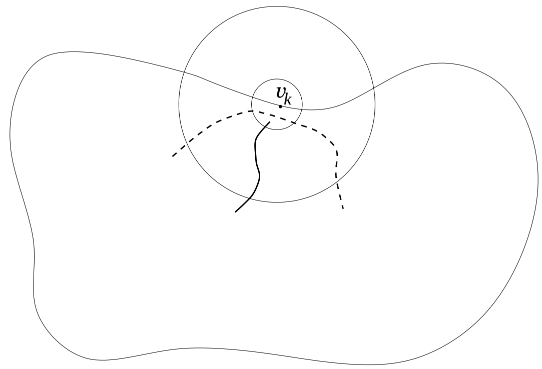

Figure 5. The figure shows a blue $(\delta_k \mathcal{T})$-path (heavy full line) crossing the partial annulus $D_k \cap \{B(v_k, R) \setminus B(v_k, r)\}$ that fails to connect to ∂D_k near v_k because it is blocked by a yellow $(\delta_k \mathcal{T})$-path (heavy dashed line) that twice crosses the annulus $B(v_k, R) \setminus B(v_k, r)$.

the domain so that it can be applied to the present situation. Since that argument is long, we will not repeat it here.

Returning to the problem of close encounters of γ_k with ∂D_k, we will try to provide the intuition on which the proof of touching is based. Suppose, by contradiction, that γ_k enters the disc $B(v_k, \varepsilon_k)$ of radius ε_k centered at $v_k \in \partial D_k$ without touching ∂D_k inside the disc $B(v_k, r)$ of radius r, and that $\varepsilon_k \to 0$. As $k \to \infty$, $D_k \to D$ and we can assume by compactness that v_k converges to some $v \in \partial D$. Considering the fattened version of γ_k shows the existence of two $(\delta_k \mathcal{T})$-paths of one color, say yellow, and one $(\delta_k \mathcal{T})$-path of the other color, blue, crossing the annulus $B(v_k, r) \setminus B(v_k, \varepsilon_k)$ (see Figure 5).

Assume for simplicity that v is far enough from a and b so that $a, b \notin \overline{B(v, R)}$ for some $R > r$, and consequently $x_k, y_k \notin B(v_k, R)$ for k large enough. Then, in the domain $D_k \cap \{B(v_k, R) \setminus B(v_k, r)\}$ there is a blue crossing between a certain portion J_k of the circle of radius R centered at v_k and a certain portion J'_k of the circle of radius r centered at v_k. If we consider instead the domain $D_k \cap B(v_k, R)$, there is no blue crossing between J_k and the portion of $\partial D_k \cap B(v_k, r)$ containing v_k (see Figure 5). If this discrepancy persists as $k \to \infty$, it must show up in the scaling limit of crossing probabilities for the domains $D \cap \{B(v, R) \setminus B(v, r)\}$ and $D \cap B(v, R)$. On the other hand, since $\varepsilon_k \to 0$, we can take r very small, and so $D \cap \{B(v, R) \setminus B(v, r)\}$ is very close to $D \cap B(v, R)$ so that the crossing probabilities in the two domains between the corresponding arcs, given in the continuum by Cardy's formula, should be very close. This follows from the continuity of Cardy's formula with respect to the shape of the domain and the positions of the boundary arcs (see, e.g., Lemma A.2 of [Camia and Newman 2007]).

Using this idea, one can show that the assumption that γ_k comes close to ∂D_k without touching it nearby produces a contradiction. Although the idea outlined above is relatively simple, the arguments needed to obtain a contradiction are rather involved (see Lemmas 7.1, 7.2, 7.3 and 7.4 of [Camia and Newman 2007]), so we will not present them here, except for a brief discussion in the proof of Lemma 6.2 below. □

SKETCH OF THE PROOF OF THEOREM 3. In order to prove the claim, we will define a lattice construction inside D' coupled to the continuum construction inside D, by means of the conformal map f from D to D'. Roughly speaking, this new lattice construction for D' is one in which the (x, y) pairs at each step are chosen to be close to the $(f(a), f(b))$ points in D' mapped from D via f, where the pairs (a, b) are those that appear at the corresponding steps of the continuum construction inside D.

More precisely, let $\gamma_{(1)}$ be the first SLE_6 curve in D from $a_{(1)}$ to $b_{(1)}$. Because of the conformal invariance of SLE_6, the image $f(\gamma_{(1)})$ of $\gamma_{(1)}$ under f is a curve distributed as the trace of chordal SLE_6 in D' from $f(a_{(1)})$ to $f(b_{(1)})$. Therefore, the exploration path $\gamma^{\delta}_{(1)}$ inside D' from $x_{(1)}$ to $y_{(1)}$, chosen so that they converge to $f(a_{(1)})$ and $f(b_{(1)})$ respectively as $\delta \to 0$, converges in distribution to $f(\gamma_{(1)})$, as $\delta \to 0$, which means that there exists a coupling between $\gamma^{\delta}_{(1)}$ and $f(\gamma_{(1)})$ such that the curves stay close for δ small.

We see that one can use the same strategy as in the sketch of the proof of Theorem 1, and obtain a lattice construction whose exploration paths are coupled to the SLE_6 curves in D' that are the images under f of the SLE_6 curves in D. Then, for this discrete construction, the scaling limits of the exploration paths will be distributed as the images of the SLE_6 curves in D.

To conclude the proof, we should show that the lattice construction inside D' defined above finds all the boundaries in a number of steps that is bounded in probability as $\delta \to 0$. But this is essentially equivalent to the analogous claim in the sketch of the proof of Theorem 1. Thus the scaling limit, as $\delta \to 0$, of this new lattice construction for D' gives the CNL process in D', which by construction is distributed like the image under f of the CNL process in D. □

SKETCH OF THE PROOF OF THEOREM 4. Let $a, b \in \partial D$ be the endpoints of Γ in clockwise order, i.e., $\Gamma = \overline{ba}$ is the counterclockwise arc of ∂D from b to a. As explained at the end of Section 4, the random curve γ obtained by pasting together the maximal excursions $\overline{xy}(L)$ from $\overline{ba}$, for $L \in \mathcal{L}_{D,\Gamma}$, is distributed like chordal SLE_6 in D from a to b. Indeed, removing γ from D is equivalent (in distribution) to the first step of the algorithmic construction presented in Section 4 to produce a realization of the CNL process, if we choose

a and b with $\overline{ba} = \Gamma$ as starting and ending points of the first SLE_6 curve of the construction.

Note that γ is in $\mathcal{L}^*_{D,\Gamma} \equiv \overline{\bigcup\{L : L \in \mathcal{L}_{D,\Gamma}\}}$, and the remaining pieces of $\mathcal{L}^*_{D,\Gamma}$ are all in (the closures of) subdomains of $D \setminus \gamma$ of type 1. If we condition on γ and run the algorithmic construction described in Section 4 inside a subdomain of $D \setminus \gamma$ of type 2, 3 or 4, we get an independent CNL process or, by Theorem 3, an independent copy of $\mathcal{L}_D$ conformally mapped to that domain. This already proves part of the claim.

Consider now a subdomain D' of $D \setminus \gamma$ of type 1 and let a', b' be the endpoints of the excursion that generated D'. Part of $\partial D'$ is in ∂D and we choose a', b' so that the counterclockwise arc $\Gamma' = \overline{b'a'} \subset \partial D$ is that part of $\partial D'$. The excursion that generated D' is part of a loop L' whose other "half" is in D' and runs from b' to a'. We know from the construction of Section 4 that if we trace the "half" of L' contained in D' from b' to a' we get a curve γ' distributed like chordal SLE_6 in D' from b' to a'. Note that γ' is contained in $\mathcal{L}^*_{D,\Gamma}$.

The subdomains of $D' \setminus \gamma'$ are of two types: (I) those whose boundary does not contain a portion of ∂D and (II) those whose boundary does contain a portion, $\Gamma'' = \overline{b''a''} \subset \partial D$, of Γ. If we condition on γ and γ' and run the algorithmic construction described in Section 4 inside a subdomain of $D' \setminus \gamma'$ of type I, we get an independent CNL process or, by Theorem 3, an independent copy of $\mathcal{L}_D$ conformally mapped to that domain.

The remaining pieces of $\mathcal{L}^*_{D,\Gamma}$ are all contained inside the (closures of) domains of type II (for all the subdomains of $D \setminus \gamma$ of type 1). Inside each subdomain D'' of type II, the CN loops that touch Γ'' are contained in $\mathcal{L}^*_{D,\Gamma}$ and can be used to obtain a curve γ'' distributed like chordal SLE_6 in D'' from a'' to b'' by pasting together maximal excursions as above (and at the end of Section 4). It should now be clear how to complete the argument by iterating the steps described above inside each subdomain D''. □

6. Convergence of exploration path to SLE_6

SKETCH OF THE PROOF OF THEOREM 1. We begin discussing the proof of Theorem 1 by noting, as in the proof of Theorem 2 discussed in Section 5, that it follows from [Aizenman and Burchard 1999] that the family of distributions of $\gamma^{\delta}_{D,a,b}$ is tight (as $\delta \to 0$, in the sense of the metric (3-3)) and so there is convergence along subsequences $\delta_k \to 0$. We write, in simplified notation, $\gamma_k \to \tilde{\gamma}$ along such a convergent subsequence. What needs to be proved is that the distribution $\tilde{\mu}$ of $\tilde{\gamma}$ is that of γ^{SLE_6}, the trace of chordal SLE_6 in D from a to b.

We next discuss how much information about $\tilde{\mu}$ can be extracted from Cardy's formula for crossing probabilities. We note that there are versions of Smirnov's

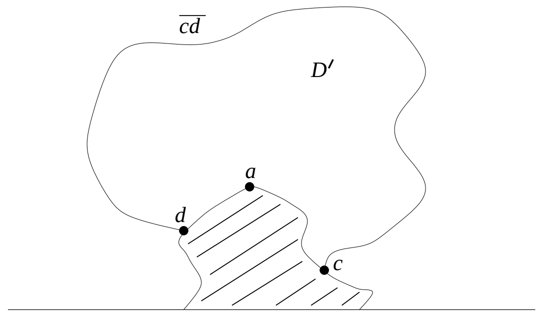

Figure 6. D is the upper half-plane $\mathbb{H}$ with the shaded portion removed, $b = \infty$, C' is an unbounded subdomain, and $D' = D \setminus C'$ is indicated in the figure. The counterclockwise arc $\overline{cd}$ indicated in the figure belongs to $\partial D'$.

result on convergence of crossing probabilities to Cardy's formula that allow the domains being crossed and the target boundary arcs to vary as $\delta \to 0$. Theorem 3 of [Camia and Newman 2007] is such a version that suffices for our purposes. Let $D_t \equiv D \setminus \tilde{K}_t$ denote the (unique) connected component of $D \setminus \tilde{\gamma}[0,t]$ whose closure contains b, where $\tilde{K}_t$, the *filling* of $\tilde{\gamma}[0,t]$, is a closed connected subset of $\overline{D}$. $\tilde{K}_t$ is called a *hull* if it satisfies the condition

$$\overline{\tilde{K}_t \cap D} = \tilde{K}_t. \tag{6-1}$$

We will consider curves $\tilde{\gamma}$ such that $\tilde{K}_t$ is a hull for each t, although here we only consider $\tilde{K}_T$ at certain stopping times T.

Let $C' \subset D$ be a closed subset of $\overline{D}$ such that $a \notin C'$, $b \in C'$, and $D' = D \setminus C'$ is a bounded simply connected domain whose boundary contains the counterclockwise arc $\overline{cd}$ that does not belong to ∂D (except for its endpoints c and d – see Figure 6).

Let $T' = \inf\{t : \tilde{K}_t \cap C' \neq \varnothing\}$ be the first time that $\tilde{\gamma}(t)$ hits C' and assume that the filling $\tilde{K}_{T'}$ of $\tilde{\gamma}[0,T']$ is a hull. We say that the hitting distribution of $\tilde{\gamma}(t)$ at the stopping time T' is determined by Cardy's formula (see (3-1)) if, for any C' and any counterclockwise arc $\overline{xy}$ of $\overline{cd}$, the probability that $\tilde{\gamma}$ hits C' at time T' on $\overline{xy}$ is given by

$$\mathbb{P}(\tilde{\gamma}(T') \in \overline{xy}) = \Phi_{D'}(a,c;x,d) - \Phi_{D'}(a,c;y,d). \tag{6-2}$$

We want to relate the distribution of $\tilde{K}_{T'}$ to the distribution of hitting *locations* for a family of C'''s related to C'. To explain, consider the set $\tilde{\mathcal{A}}$ of closed subsets $\tilde{A}$ of $\overline{D'}$ that do not contain a and such that $\partial \tilde{A} \setminus \partial D'$ is a simple (continuous) curve contained in D' except for its endpoints, one of which is on $\partial D' \cap D$ and the other is on ∂D (see Figure 7). Let $\mathcal{A}$ be the set of closed subsets of $\overline{D'}$ of the form $\tilde{A}_1 \cup \tilde{A}_2$, where $\tilde{A}_1, \tilde{A}_2 \in \tilde{\mathcal{A}}$ and $\tilde{A}_1 \cap \tilde{A}_2 = \varnothing$.

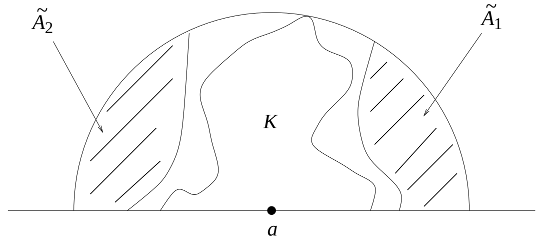

Figure 7. Example of a hull K and a set $\tilde{A}_1 \cup \tilde{A}_2$ (shaded regions) in $\mathcal{A}$. Here, $D = \mathbb{H}$ and D' is the semidisc centered at a.

It is easy to see that if the hitting distribution of $\tilde{\gamma}(T')$ is determined by Cardy's formula, then the probabilities of events of the form $\{\tilde{K}_{T'} \cap A = \varnothing\}$ for $A \in \mathcal{A}$ are also determined by Cardy's formula in the following way. Let $A \in \mathcal{A}$ be the union of $\tilde{A}_1, \tilde{A}_2 \in \tilde{\mathcal{A}}$, with $\partial \tilde{A}_1 \setminus \partial D'$ given by a curve from $u_1 \in \partial D' \cap D$ to $v_1 \in \partial D$ and $\partial \tilde{A}_2 \setminus \partial D'$ given by a curve from $u_2 \in \partial D' \cap D$ to $v_2 \in \partial D$; then, assuming that a, v_1, u_1, u_2, v_2 are ordered counterclockwise around $\partial D'$,

$$\mathbb{P}(\tilde{K}_{T'} \cap A = \varnothing) = \Phi_{D' \setminus A}(a, v_1; u_1, v_2,) - \Phi_{D' \setminus A}(a, v_1; u_2, v_2). \qquad \text{(6-3)}$$

The probabilities of such events determine uniquely the distribution of the hull (for more detail, see Section 5 of [Camia and Newman 2007]). Thus we have the following useful lemma, since the hitting distribution for SLE_6 is determined by Cardy's formula [Lawler et al. 2001].

LEMMA 6.1. *If $\tilde{K}_{T'}$ is a hull and the hitting distribution of $\tilde{\gamma}$ at the stopping time T' is determined by Cardy's formula, then $\tilde{K}_{T'}$ is distributed like the corresponding hull of γ^{SLE_6}.*

We next define the sequence of hitting times for $\tilde{\gamma}$ that will be used to compare it to γ^{SLE_6}. They involve conformal maps of semiballs (i.e., half-disks) in the upper half-plane. Let $\tilde{f}_0$ be a conformal map from the upper half-plane $\mathbb{H}$ to D such that $\tilde{f}_0^{-1}(a) = 0$ and $\tilde{f}_0^{-1}(b) = \infty$. (Since ∂D is a continuous curve, the map $\tilde{f}_0^{-1}$ has a continuous extension from D to $D \cup \partial D$ and, by a slight abuse of notation, we do not distinguish between $\tilde{f}_0^{-1}$ and its extension; the same applies to $\tilde{f}_0$.) These two conditions determine $\tilde{f}_0$ only up to a scaling factor. For $\varepsilon > 0$ fixed, let $C(u, \varepsilon) = \{z : |u - z| < \varepsilon\} \cap \mathbb{H}$ denote the semiball of radius ε centered at u on the real line and let $\tilde{T}_1 = \tilde{T}_1(\varepsilon)$ denote the first time $\tilde{\gamma}(t)$ hits $D \setminus \tilde{G}_1$, where $\tilde{G}_1 \equiv \tilde{f}_0(C(0, \varepsilon))$. Define recursively $\tilde{T}_{j+1}$ as the first time $\tilde{\gamma}[\tilde{T}_j, \infty)$ hits $\tilde{D}_{\tilde{T}_j} \setminus \tilde{G}_{j+1}$, where $\tilde{D}_{\tilde{T}_j} \equiv D \setminus \tilde{K}_{\tilde{T}_j}$, $\tilde{G}_{j+1} \equiv \tilde{f}_{\tilde{T}_j}(C(0, \varepsilon))$, and $\tilde{f}_{\tilde{T}_j}$ is a conformal map from $\mathbb{H}$ to $\tilde{D}_{\tilde{T}_j}$ whose inverse maps $\tilde{\gamma}(\tilde{T}_j)$ to 0 and b to ∞. We also define $\tilde{\tau}_{j+1} \equiv \tilde{T}_{j+1} - \tilde{T}_j$, so that $\tilde{T}_j = \tilde{\tau}_1 + \ldots + \tilde{\tau}_j$. We choose

$\tilde{f}_{\tilde{T}_j}$ so that its inverse is the composition of the restriction of $\tilde{f}_0^{-1}$ to $\tilde{D}_{\tilde{T}_j}$ with $\tilde{\varphi}_{\tilde{T}_j}$, where $\tilde{\varphi}_{\tilde{T}_j}$ is the unique conformal transformation from $\mathbb{H} \setminus \tilde{f}_0^{-1}(\tilde{K}_{\tilde{T}_j})$ to $\mathbb{H}$ that maps ∞ to ∞ and $\tilde{f}_0^{-1}(\tilde{\gamma}(\tilde{T}_j))$ to the origin of the real axis, and has derivative at ∞ equal to 1.

Notice that $\tilde{G}_{j+1}$ is a bounded simply connected domain chosen so that the conformal transformation which maps $\tilde{D}_{\tilde{T}_j}$ to $\mathbb{H}$ maps $\tilde{G}_{j+1}$ to the semiball $C(0, \varepsilon)$ centered at the origin on the real line. With these definitions, we consider the (discrete-time) stochastic process $\tilde{X}_j \equiv (\tilde{K}_{\tilde{T}_j}, \tilde{\gamma}(\tilde{T}_j))$ for $j = 1, 2, \dots$. Analogous quantities can be defined for the trace of chordal SLE$_6$. They are indicated by the superscript SLE$_6$; we choose $f_0^{\mathrm{SLE}_6} = \tilde{f}_0$, so that $G_1^{\mathrm{SLE}_6} = \tilde{G}_1$. Our aim is to prove that the variables $\tilde{X}_1, \tilde{X}_2, \dots$ are (jointly) equidistributed with the corresponding SLE$_6$ hull and tip variables $X_1^{\mathrm{SLE}_6}, X_2^{\mathrm{SLE}_6}, \dots$. By letting $\varepsilon \to 0$, this will directly yield that $\tilde{\gamma}$ is equidistributed with γ^{SLE_6} as desired. Since γ_k converges in distribution to $\tilde{\gamma}$, we can find coupled versions of γ_k and $\tilde{\gamma}$ on some probability space $(\Omega, \mathcal{B}, \mathbb{P})$ such that γ_k converges to $\tilde{\gamma}$ for all $\omega \in \Omega$; in the rest of the proof we work with these new versions which, with a slight abuse of notation, we denote with the same names as the original ones.

For each k, let K_t^k denote the filling (or *lattice hull*) at time t of γ_k, i.e., the set of hexagons that at time t have been explored or have been disconnected from b by the exploration path. Let now f_0^k be a conformal transformation that maps $\mathbb{H}$ to $D_k \equiv D^{\delta_k}$ such that $(f_0^k)^{-1}(a_k) = 0$ and $(f_0^k)^{-1}(b_k) = \infty$ and let $T_1^k = T_1^k(\varepsilon)$ denote the first exit time of $\gamma_k^{\delta_k}(t)$ from $G_1^k \equiv f_0^k(C(0, \varepsilon))$ defined as the first time that γ_k intersects the image under f_0^k of the semicircle $\{z : |z| = \varepsilon\} \cap \mathbb{H}$. Define recursively T_{j+1}^k as the first exit time of $\gamma_k^{\delta_k}[T_j^k, \infty)$ from $G_{j+1}^k \equiv f_{T_j^k}^k(C(0, \varepsilon))$, where $f_{T_j^k}^k$ is a conformal map from $\mathbb{H}$ to $D_k \setminus K_{T_j^k}^k$ whose inverse maps $\gamma_k(T_j^k)$ to 0 and b_k to ∞. Each of the maps $f_{T_j^k}^k$, where $j \geq 1$, is defined only up to a scaling factor. We also set $\tau_{j+1}^k \equiv T_{j+1}^k - T_j^k$, so $T_j^k = \tau_1^k + \dots + \tau_j^k$, and define the (discrete-time) stochastic process

$$X_j^k \equiv (K_{T_j^k}^k, \gamma_k^{\delta_k}(T_j^k)) \quad \text{for } j = 1, 2, \dots.$$

We want to show recursively that, for any j, as $k \to \infty$, $\{X_1^k, \dots, X_j^k\}$ converge jointly in distribution to $\{\tilde{X}_1, \dots, \tilde{X}_j\}$. By recursively applying the convergence of crossing probabilities to Cardy's formula (i.e., Theorem 3 of [Camia and Newman 2007]) and Lemma 6.1, we will then be able to conclude, as explained in more detail below, that $\{\tilde{X}_1, \tilde{X}_2, \dots\}$ are jointly equidistributed with the corresponding SLE_6 hull variables (at the corresponding stopping times) $\{X_1^{\text{SLE}_6}, X_2^{\text{SLE}_6}, \dots\}$.

The zeroth step consists in noticing that the convergence of (D_k, a_k, b_k) to (D, a, b) as $k \to \infty$ allows us to select a sequence of conformal maps f_0^k that converge to $f_0^{\text{SLE}_6} = \tilde{f}_0$ uniformly in $\overline{\mathbb{H}}$ as $k \to \infty$, which implies that the boundary ∂G_1^k of $G_1^k = f_0^k(C(0,\varepsilon))$ converges to the boundary $\partial \tilde{G}_1$ of $\tilde{G}_1 = \tilde{f}_0(C(0,\varepsilon))$ in the uniform metric on continuous curves (see Corollary A.2 of [Camia and Newman 2007]).

The next lemma is the technical heart of the proof. It basically allows us to interchange the scaling limit $\delta \to 0$ and the process of filling (which generates hulls) by declaring that the hull of the limiting curve is the limit of the (lattice) hulls. The proof of the lemma involves extensive use of nontrivial results from percolation theory. Although the lemma is stated here in the framework of the first step of the proof where we are analyzing convergence of X_1^k to $\tilde{X}_1$, essentially the same lemma can be applied sequentially to the convergence of X_j^k conditioned on $\{X_1^k, \dots, X_{j-1}^k\}$.

LEMMA 6.2. *$(\gamma_k, K^k_{T_1^k})$ converges in distribution to $(\tilde{\gamma}, \tilde{K}_{\tilde{T}_1})$ as $k \to \infty$. Furthermore $\tilde{K}_{\tilde{T}_1}$ is almost surely a hull equidistributed with the hull $K_{T_1}^{\text{SLE}_6}$ of* SLE_6 *at the corresponding stopping time T_1.*

PROOF. Proving the first claim, that for the exploration path γ_k in G_1^k one can interchange the limit $k \to \infty$ $(\delta_k \to 0)$ with the process of filling, requires showing two things about the exploration path: (1) the return of a (macroscopic) segment of the path close to an earlier segment (and away from ∂G_1^k) without nearby (microscopic) touching does not occur (probably), and (2) the close approach of a (macroscopic) segment of the path to ∂G_1^k without nearby (microscopic) touching either of ∂G_1^k itself or else of another segment of the path that touches ∂G_1^k does not occur (probably). If G_1^k (or more accurately, its limit $\tilde{G}_1$) were replaced by a convex domain like the unit disk, these could be controlled by known estimates on probabilities of six-arm events in the full plane for (1) and of three-arm events in the half-plane for (2). But $\tilde{G}_1$ is not in general convex and then the three-arm event argument for (2) appears to break down. The replacement in [Camia and Newman 2007] is the use of several lemmas in Section 7 there. Basically, these control (2) by a novel argument about "mushroom events" in $\tilde{G}_1$, which is based on continuity of Cardy's formula with respect to changes in $\partial\tilde{G}_1$. Roughly speaking, mushroom events are ones where (in the limit $k \to \infty$) there is a macroscopic monochromatic path in $\tilde{G}_1$ just reaching to $\partial\tilde{G}_1$, but blocked from it by a macroscopic path in $\tilde{G}_1$ of the other color (see Figure 5). It is shown in [Camia and Newman 2007] (see Lemma 7.4 there) that mushroom events cannot occur with positive probability while on the other hand they would occur if (2) were not the case. The second claim of Lemma 6.2 now follows from Smirnov's result [2001] on convergence to Cardy's formula (see also Theorem 3 of [Camia and Newman 2007]) and Lemma 6.1. □

Using Lemma 6.2, the first step of our recursion argument is organized as follows, where all limits and equalities are in distribution:

(i) $K^k_{T^k_1} \to \tilde{K}_{\tilde{T}_1} = K^{\mathrm{SLE}_6}_{T_1}$ by Lemma 6.2.

(ii) By (i), $D_k \setminus K^k_{T^k_1} \to D \setminus \tilde{K}_{\tilde{T}_1} = D \setminus K^{\mathrm{SLE}_6}_{T_1}$.

(iii) By (ii), $f^{\mathrm{SLE}_6}_{T_1} = \tilde{f}_{\tilde{T}_1}$, and we can select a sequence $f^k_{T^k_1} \to \tilde{f}_{\tilde{T}_1} = f^{\mathrm{SLE}_6}_{T_1}$.

(iv) By (iii), $G^k_2 \to \tilde{G}_2 = G^{\mathrm{SLE}_6}_2$.

At this point, we are in the same situation as at the zeroth step, but with G^k_1, $\tilde{G}_1$ and $G^{\mathrm{SLE}_6}_1$ replaced by G^k_2, $\tilde{G}_2$ and $G^{\mathrm{SLE}_6}_2$, and we proceed by induction, as follows.

The next step consists in proving that

$$\big((K^k_{T^k_1}, \gamma^{\delta_k}_k(T^k_1)), (K^k_{T^k_2}, \gamma^{\delta_k}_k(T^k_2))\big)$$

converges in distribution to $\big((\tilde{K}_{\tilde{T}_1}, \tilde{\gamma}(\tilde{T}_1)), (\tilde{K}_{\tilde{T}_2}, \tilde{\gamma}(\tilde{T}_2))\big)$. Since we have already proved the convergence of $(K^k_{T^k_1}, \gamma^{\delta_k}_k(T^k_1))$ to $(\tilde{K}_{\tilde{T}_1}, \tilde{\gamma}(\tilde{T}_1))$, all we need to prove is the convergence of $(K^k_{T^k_2} \setminus K^k_{T^k_1}, \gamma^{\delta_k}_k(T^k_2))$ to $(\tilde{K}_{\tilde{T}_2} \setminus \tilde{K}_{\tilde{T}_1}, \tilde{\gamma}(\tilde{T}_2))$. To do this, notice that $K^k_{T^k_2} \setminus K^k_{T^k_1}$ is distributed like the lattice hull of a percolation exploration path inside $D_k \setminus K^k_{T^k_1}$. Besides, the convergence in distribution of $(K^k_{T^k_1}, \gamma^{\delta_k}_k(T^k_1))$ to $(\tilde{K}_{\tilde{T}_1}, \tilde{\gamma}(\tilde{T}_1))$ implies that we can find versions of $(\gamma^{\delta_k}_k, K^k_{T^k_1})$ and $(\tilde{\gamma}, \tilde{K}_{\tilde{T}_1})$ on some probability space $(\Omega, \mathcal{B}, \mathbb{P})$ such that $\gamma^{\delta_k}_k(\omega)$ converges to $\tilde{\gamma}(\omega)$ and $(K^k_{T^k_1}, \gamma^{\delta_k}_k(T^k_1))$ converges to $(\tilde{K}_{\tilde{T}_1}, \tilde{\gamma}(\tilde{T}_1))$ for all $\omega \in \Omega$. These two observations imply that, if we work with the coupled versions of $(\gamma^{\delta_k}_k, K^k_{T^k_1})$ and $(\tilde{\gamma}, \tilde{K}_{\tilde{T}_1})$, we are in the same situation as before, but with D_k and D replaced by $D_k \setminus K^k_{T^k_1}$ and $D \setminus \tilde{K}_{\tilde{T}_1}$, and a_k and a replaced by $\gamma^{\delta_k}_k(T^k_1)$ and $\tilde{\gamma}(\tilde{T}_1)$, respectively. Then, the conclusion that $(K^k_{T^k_2} \setminus K^k_{T^k_1}, \gamma^{\delta_k}_k(T^k_2))$ converges in distribution to $(\tilde{K}_{\tilde{T}_2} \setminus \tilde{K}_{\tilde{T}_1}, \tilde{\gamma}(\tilde{T}_2))$ follows, as before, by arguments like those used for Lemma 6.2. We can now iterate the above arguments j times, for any $j > 1$. If we keep track at each step of the previous ones, this provides the *joint* convergence of all the curves and lattice hulls involved at each step.

The proof of Theorem 1 is concluded by letting $\varepsilon \to 0$. We note that in this paper we circumvent the use of a "spatial Markov property" that played a role in [Camia and Newman 2007] in the $\varepsilon \to 0$ limit. The point is that that property was proved as a consequence of the equidistribution of $\tilde{X}_1, \tilde{X}_2, \dots$ with $X_1^{\mathrm{SLE}_6}, X_2^{\mathrm{SLE}_6}, \dots$ and here we apply the equidistribution directly. It should be noted however that there needs to be some a priori information about $\tilde{\gamma}$ to insure that this equidistribution for each $\varepsilon > 0$ implies equidistribution of $\tilde{\gamma}$ with γ^{SLE_6}. For example, one could create by hand a process $\hat{\gamma}$ which behaved like γ^{SLE_6} except that at random times it retraced back and forth part of its previous path. Such a $\hat{\gamma}$ would have its $\hat{X}_j$ variables equidistributed with those of SLE_6 but as a random curve (modulo monotonic reparametrizations) would not be equidistributed with γ^{SLE_6}; it would also not be describable by a Loewner chain. Such possibilities can be ruled out by the same arguments as those used in proving Lemma 6.2; see Lemma 6.4 of [Camia and Newman 2007]. □

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