

A quick derivation of the loop equations for random matrices

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ABSTRACT. The *loop equations* of random matrix theory are a hierarchy of equations born of attempts to obtain explicit formulae for generating functions of map enumeration problems. These equations, originating in the physics of 2-dimensional quantum gravity, have lacked mathematical justification. The goal of this paper is to provide a complete and short proof, relying on a recently established complete asymptotic expansion for the random matrix theory partition function.

1. Background and preliminaries

The study of the unitary ensembles (UE) of random matrices [Mehta 1991], begins with a family of probability measures on the space of $N \times N$ Hermitian matrices. The measures are of the form

$$d\mu_{\mathbf{t}} = \frac{1}{\tilde{Z}_N} \exp \{-N \operatorname{Tr} [V_{\mathbf{t}}(M)]\} dM,$$

where the function $V_{\mathbf{t}}$ is a scalar function, referred to as the potential of the external field, or simply the “external field” for short. Typically it is taken to be a polynomial, and written as follows:

$$V_{\mathbf{t}} = V(\lambda; t_1, \dots, t_\nu) = \frac{1}{2}\lambda^2 + \sum_{j=1}^{\nu} t_j \lambda^j$$

where the parameters $\{t_1, \dots, t_\nu\}$ are assumed to be such that the integral converges. For example, one may suppose that ν is even, and $t_\nu > 0$. The *partition*

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function \tilde{Z}_N , is the normalization factor which makes the UE measures be probability measures.

Expectations of conjugation invariant matrix random variables with respect to these measures can be reduced, via the Weyl integration formula, to an integration against a symmetric density over the eigenvalues which has a form proportional to (1-1), below:

$$\exp \left\{ -N^2 \left(\frac{1}{N} \sum_{j=1}^N V(\lambda_j; t_1, \dots, t_v) - \frac{1}{N^2} \sum_{j \neq \ell} \log |\lambda_j - \lambda_\ell| \right) \right\} d^N \lambda. \quad (1-1)$$

These latter multiple integrals can be more compactly expressed in terms of kernels constructed from polynomials $\{p_\ell(\lambda)\}$ orthogonal with respect to the exponential weight $e^{-NV_t(\lambda)}$ [Mehta 1991]. For instance, the fundamental matrix moments $E(\text{Tr} M^j)$, where E denotes expectation with respect to the measure $d\mu_t$, are expressed as

$$E(\text{Tr} M^j) = N \int_{-\infty}^{\infty} \lambda^j \rho_N^{(1)}(\lambda) d\lambda, \quad (1-2)$$

where $\rho_N^{(1)}(\lambda)$ denotes the so-called *one-point function*

$$\rho_N^{(1)}(x) = \frac{d}{dx} \mathbb{E} \left(\frac{1}{N} \# \{j : \lambda_j \in (-\infty, x)\} \right) = \frac{1}{N} K_N(x, x), \quad (1-3)$$

with the kernel

$$K_N(x, y) = e^{-(N/2)(V_t(x)+V_t(y))} \sum_{\ell=0}^{N-1} p_\ell(x) p_\ell(y).$$

The symbol \mathbb{E} denotes expectation with respect to the normalization of the measure (1-1) which is given by dividing this family of measures by the corresponding family of integrals:

$$Z_N(t_1, t_2, \dots, t_v) = \int \cdots \int \exp \left\{ -N^2 \left(\frac{1}{N} \sum_{j=1}^N V(\lambda_j; t_1, \dots, t_v) - \frac{1}{N^2} \sum_{j \neq \ell} \log |\lambda_j - \lambda_\ell| \right) \right\} d^N \lambda. \quad (1-4)$$

We will sometimes refer to the following set of $\mathbf{t} = (t_1, \dots, t_v)$ for which (1-4) converges. For any given $T > 0$ and $\gamma > 0$, define

$$\mathbb{T}(T, \gamma) = \left\{ \mathbf{t} \in \mathbb{R}^v : |\mathbf{t}| \leq T, t_v > \gamma \sum_{j=1}^{v-1} |t_j| \right\}.$$

The leading order behavior of $Z_N(t_1, t_2, \dots, t_\nu)$ is rather classical, and is known for a very wide class of external fields V (see [Johansson 1998], for example). We will require the following result.

THEOREM 1.1. *There is $T > 0$ and $\gamma > 0$ such that, for all $\mathbf{t} \in \mathbb{T}(T, \gamma)$, the following statements hold:*

$$(i) \quad \lim_{N \rightarrow \infty} \frac{1}{N^2} \log\{Z_N(t_1, t_2, \dots, t_\nu)\} = -I(t_1, \dots, t_\nu), \tag{1-5}$$

where $I(t_1, \dots, t_\nu)$ is the infimum of

$$\int V(\lambda) d\mu(\lambda) - \iint \log|\lambda - \mu| d\mu(\lambda) d\mu(\eta) \tag{1-6}$$

over all positive, normalized Borel measures μ .

(ii) *There is a unique measure μ_V that achieves this infimum. This measure is absolutely continuous with respect to Lebesgue measure, and*

$$d\mu_V = \psi d\lambda, \\ \psi(\lambda) = \frac{1}{2\pi} \chi_{(\alpha, \beta)}(\lambda) \sqrt{(\lambda - \alpha)(\beta - \lambda)} h(\lambda),$$

where $h(\lambda)$ is a polynomial of degree $\nu - 2$, which is strictly positive on the interval $[\alpha, \beta]$ (recall that the external field V is a polynomial of degree ν). The polynomial h is defined by

$$h(z) = \frac{1}{2\pi i} \oint \frac{V'(s)}{\sqrt{(s - \alpha)}\sqrt{(s - \beta)}} \frac{ds}{s - z}$$

where the integral is taken on a circle containing (α, β) and z in the interior, oriented counterclockwise.

(iii) *There exists a constant l , depending on V , such that the following variational equations are satisfied by μ_V :*

$$\int 2 \log|\lambda - \eta|^{-1} d\mu_V(\eta) + V(\lambda) \geq l \text{ for } \lambda \in \mathbb{R} \setminus \text{supp}(\mu_V), \\ \int 2 \log|\lambda - \eta|^{-1} d\mu_V(\eta) + V(\lambda) = l \text{ for } \lambda \in \text{supp}(\mu_V). \tag{1-7}$$

(iv) *The endpoints α and β are determined by the equations*

$$\int_{\alpha}^{\beta} \frac{V'(s)}{\sqrt{(s - \alpha)}(\beta - s)} ds = 0, \quad \int_{\alpha}^{\beta} \frac{sV'(s)}{\sqrt{(s - \alpha)}(\beta - s)} ds = 2\pi.$$

- (v) *The endpoints $\alpha(\mathbf{t})$ and $\beta(\mathbf{t})$ are actually analytic functions of \mathbf{t} , which possess smooth extensions to the closure of $\{\mathbf{t} : \mathbf{t} \in \mathbb{T}(T, \gamma)\}$. They also satisfy $-\alpha(\mathbf{0}) = \beta(\mathbf{0}) = 2$. In addition, the coefficients of the polynomial $h(\lambda)$ are also analytic functions of \mathbf{t} , with smooth extensions to the closure of $\{\mathbf{t} : \mathbf{t} \in \mathbb{T}(T, \gamma)\}$, with*

$$h(\lambda, \mathbf{t} = \mathbf{0}) = 1.$$

REMARKS. (1) The variational problem appearing in 1.1(i) is a fundamental component in the theory of random matrices, as well as integrable systems and approximation theory. It is well known, (see, for example, [Saff and Totik 1997]), that under general assumptions on V , the infimum is achieved at a unique measure μ_V , called the equilibrium measure. For external fields V that are analytic in a neighborhood of the real axis, and with sufficient growth at ∞ , the equilibrium measure is supported on finitely many intervals, with density that is analytic on the interior of each interval, behaving at worst like a square root at each endpoint, (see [Deift et al. 1998] and [Deift et al. 1999]).

(2) The result in (1-5) is commonly known in the approximation theory literature. For a proof, see [Johansson 1998].

(3) It will prove useful to adapt the following alternative presentation for the function ψ :

$$\psi(\lambda) = \frac{1}{2\pi i} R_+(\lambda) h(\lambda), \quad \lambda \in (\alpha, \beta), \quad (1-8)$$

where the function $R(\lambda)$ is defined via $R(\lambda)^2 = (\lambda - \alpha)(\lambda - \beta)$, with $R(\lambda)$ analytic in $\mathbb{C} \setminus [\alpha, \beta]$, and normalized so that $R(\lambda) \sim \lambda$ as $\lambda \rightarrow \infty$. The subscript \pm in $R_{\pm}(\lambda)$ denotes the boundary value obtained from the upper (lower) half-plane.

The goal of this paper is to provide a rigorous justification for the loop equations, which originated in the physics of two-dimensional quantum gravity (see, for example, the survey [Di Francesco et al. 1995] and the references therein). More precisely, this entails

- proving that the quantity

$$\int_{-\infty}^{\infty} \frac{\rho_N^{(1)}(x)}{x-z} dx$$

possesses a complete asymptotic expansion in even powers of N (see Theorem 2.3 and (3-11)), and

- establishing that the coefficients $P_g(z)$ in the asymptotic expansion (3-11) satisfy the hierarchy of nonlocal equations (3-13), which are the loop equations.

Once the coefficients $P_g(z)$ are known to exist as analytic functions of z and the times \mathbf{t} , they may be interpreted as generating functions for a collection of graphical enumeration problems for labelled maps, counted according to vertex valences and the genus of the underlying Riemann surface into which the maps are embedded. (See [Bessis et al. 1980], and also [Ercolani and McLaughlin 2003].) Because of the combinatorial connection and its use in 2-dimensional quantum gravity, obtaining explicit formulae has been a fundamental goal within the physics community of quantum gravity. The loop equations arose as a means to obtain explicit information (and possibly explicit formulae) for these coefficients, although without mathematical justification.

In Section 3 we will need to consider the Cauchy transform of the equilibrium measure

$$F(z) = \int_{-\infty}^{\infty} \frac{\psi(\lambda)}{z-\lambda} d\lambda, \quad z \in \mathbb{C}/\mathbb{R}.$$

It follows from differentiating the variational equations in Theorem 1.1(iii) that $F(z)$ solves the scalar Riemann–Hilbert problem

$$\begin{aligned} F_+(s) + F_-(s) &= V'(s), \quad s \in [\alpha, \beta], \\ F_+(s) - F_-(s) &= 0, \quad s \in \mathbb{R}/[\alpha, \beta], \end{aligned}$$

where the subscript \pm in $F_{\pm}(\lambda)$ denotes the boundary value obtained from the upper (lower) half-plane, and $F(z) = 1/z + \mathcal{O}(z^{-2})$ as $z \rightarrow \infty$. From this it is straightforward to deduce that

$$2F(z) = V'(z) - R(z)h(z). \tag{1-9}$$

2. Large- N expansions

The fundamental theorem for establishing complete large N expansions of expectations of random variables related to eigenvalue statistics was developed in [Ercolani and McLaughlin 2003]. A concise statement of this result is:

THEOREM 2.1 [Ercolani and McLaughlin 2003]. *There exist $T > 0$ and $\gamma > 0$ such that for all $\mathbf{t} \in \mathbb{T}(T, \gamma)$, the expansion*

$$\int_{-\infty}^{\infty} f(\lambda) \rho_N^{(1)}(\lambda) d\lambda = f_0 + N^{-2} f_1 + N^{-4} f_2 + \dots, \tag{2-1}$$

holds, provided the function $f(\lambda)$ is C^∞ smooth and grows no faster than a polynomial for $\lambda \rightarrow \infty$. The coefficients f_j depend analytically on \mathbf{t} for $\mathbf{t} \in \mathbb{T}(T, \gamma)$, and the asymptotic expansion may be differentiated term by term with respect to \mathbf{t} .

The complete details for the derivation of this result are presented in [Ercolani and McLaughlin 2003]; however, there are a few specifics presented there that we repeat here for use in subsequent sections and for general background information:

- The function $\rho_N^{(1)}$ has a full and uniform asymptotic expansion, which starts off as follows:

$$\rho_N^{(1)}(\lambda) = \psi(\lambda) + \mathcal{O}(N^{-1/2}). \quad (2-2)$$

- The specific form that this expansion takes depends very much on where one is looking; for example, for $\lambda \in (\alpha, \beta)$, the expansion takes the form:

$$\begin{aligned} \rho_N^{(1)}(\lambda) = \psi(\lambda) + \frac{1}{4\pi N} \left(\frac{1}{\lambda - \beta} - \frac{1}{\lambda - \alpha} \right) \cos \left(N \int_{\lambda}^{\beta} \psi(s) ds \right) \\ + \frac{1}{N^2} \left[H(\lambda) + G(\lambda) \sin \left(N \int_{\lambda}^{\beta} \psi(s) ds \right) \right] + \dots \end{aligned} \quad (2-3)$$

in which $H(\lambda)$ and $G(\lambda)$ are locally analytic functions which are explicitly computable in terms of the original external field $V(\lambda)$.

In [Ercolani and McLaughlin 2003] the primary application of this theorem was to establish that a complete large N asymptotic expansion of 1-1 exists:

THEOREM 2.2. *There is $T > 0$ and $\gamma > 0$ so that for $\mathbf{t} \in \mathbb{T}(T, \gamma)$, one has the $N \rightarrow \infty$ asymptotic expansion*

$$\log \left(\frac{Z_N(\mathbf{t})}{Z_N(\mathbf{0})} \right) = N^2 e_0(x, \mathbf{t}) + e_1(x, \mathbf{t}) + \frac{1}{N^2} e_2(x, \mathbf{t}) + \dots \quad (2-4)$$

The meaning of this expansion is: if you keep terms up to order N^{-2h} , the error term is bounded by $C N^{-2h-2}$, where the constant C is independent of \mathbf{t} for all $\mathbf{t} \in \mathbb{T}(T, \gamma)$. For each j , the function $e_j(\mathbf{t})$ is an analytic function of the (complex) vector (\mathbf{t}) , in a neighborhood of $(\mathbf{0})$. Moreover, the asymptotic expansion of derivatives of $\log(Z_N)$ may be calculated via term-by-term differentiation of the series above. \square

REMARKS. (1) Bleher and Its [2005] recently carried out a similar asymptotic expansion of the partition function for a 1-parameter family of external fields. A very interesting aspect of their work is that they establish the nature of the asymptotic expansion of the partition through a critical phase transition.

(2) A subsequent application in [Ercolani, McLaughlin and Pierce 2006] is to develop a hierarchy of ordinary differential equations whose solutions determine recursively the coefficients e_g for potentials of the form $V(\lambda) = \lambda^2/2 + t\lambda^{2\nu}$.

(3) The asymptotic results in [Ercolani and McLaughlin 2003] were also used recently in [Gustavsson 2005] to establish that asymptotics of each individual eigenvalue have Gaussian fluctuations, regardless of whether one is in the bulk or near the edge of the spectrum (provided only that the eigenvalue number, when counted from the edge, grows to ∞).

In the present paper we will make use of a mild extension of Theorem 2.1, in which the function f is of the form $w(\lambda)/(\lambda - z)$, with z living outside the interval $[\alpha, \beta]$:

THEOREM 2.3. *For each $\delta > 0$, there exist $T > 0$ and $\gamma > 0$ such that for all $\mathbf{t} \in \mathbb{T}(T, \gamma)$, the expansion*

$$\int_{-\infty}^{\infty} \left(\frac{w(\lambda)}{\lambda - z} \right) \rho_N^{(1)}(\lambda) d\lambda = w_0(z) + N^{-2}w_1(z) + N^{-4}w_2(z) + \dots, \quad (2-5)$$

holds, provided $z \in \mathbb{C} \setminus [\alpha - \delta, \beta + \delta]$ and the function $w(\lambda)$ is analytic in a neighborhood of \mathbb{R} and grows no faster than a polynomial as $\lambda \rightarrow \infty$. The coefficients w_j depend analytically on z and \mathbf{t} for $\mathbf{t} \in \mathbb{T}(T, \gamma)$, and possess convergent Laurent expansions for $z \rightarrow \infty$. Furthermore, the asymptotic expansion may be differentiated term by term.

For z bounded away from the real axis, this follows from Theorem 2.1. The mild extension to the case when z may be near the axis (but bounded away from the support $[\alpha, \beta]$) follows by exploiting analyticity to replace the integral along the real axis near z by a semicircular contour so that λ remains uniformly bounded away from z . Once the contour is such that λ is bounded away from z , the uniform asymptotic expansion for $\rho_N^{(1)}$ may be used. Since z is away from the support $[\alpha, \beta]$, the newly introduced semicircular contour is also bounded away from the support, and one may use arguments similar to those presented in [Ercolani and McLaughlin 2003, Observation 4.2] (where they were used for λ real and outside the interval $[\alpha, \beta]$) to show that $\rho_N^{(1)}(\lambda)$ is uniformly exponentially small on the semicircular contour, and also that the residue term obtained from deforming the contour is also uniformly exponentially small. We will leave these details for the interested reader.

3. Derivation of the loop equations

Denote the Green's function of a random matrix M by

$$G(z, M) = (z - M)^{-1}$$

and its trace by

$$g(z) = \text{Tr } G(z).$$

We evaluate $\partial G_{kl}/\partial M_{ij}$ in two different ways to get a useful relation.

LEMMA 3.1. $\mathbf{E}(G_{ki}G_{jl}) = N \mathbf{E}(G_{kl}V'(M)_{ji})$.

PROOF. We begin by assuming $i \leq j$. Note that when $i < j$, $\partial/\partial M_{ij}$ denotes differentiation with respect to the *complex* variable M_{ij} . Since $G \cdot (z - M) = \mathbf{1}$, we have $\partial/\partial M_{ij} G \cdot (z - M) \equiv \mathbf{0}$; equivalently,

$$\partial G/\partial M_{ij} \cdot (z - M) - G \cdot E_{ij} = 0,$$

where E_{ij} is the elementary matrix with a 1 in the (i, j) entry and all other entries zero (note that $\partial M_{ji}/\partial M_{ij} = 0$ for $i < j$). It follows that

$$\partial G/\partial M_{ij} = G \cdot E_{ij} \cdot G;$$

in particular,

$$\partial G_{kl}/\partial M_{ij} = (G_{ki}G_{jl}),$$

and so

$$\mathbf{E}(\partial G_{kl}/\partial M_{ij}) = \mathbf{E}(G_{ki}G_{jl}).$$

On the other hand, integrating by parts yields

$$\begin{aligned} \mathbf{E}(\partial G_{kl}/\partial M_{ij}) &= \frac{1}{\tilde{Z}_N} \int_{\mathfrak{H}} \frac{\partial G_{kl}}{\partial M_{ij}} \exp(-N \operatorname{Tr}[V_t(M)]) dM \\ &= -\frac{1}{\tilde{Z}_N} \int_{\mathfrak{H}} G_{kl} \frac{\partial}{\partial M_{ij}} \exp(-N \operatorname{Tr}[V_t(M)]) dM \\ &= N \frac{1}{\tilde{Z}_N} \int_{\mathfrak{H}} G_{kl} \operatorname{Tr}(\nabla V(M) \cdot E_{ij}) \exp(-N \operatorname{Tr}[V_t(M)]) dM \\ &= N \frac{1}{\tilde{Z}_N} \int_{\mathfrak{H}} G_{kl} V'(M)_{ji} \exp(-N \operatorname{Tr}[V_t(M)]) dM \\ &= N \mathbf{E}(G_{kl}V'(M)_{ji}). \end{aligned}$$

Combining the two representations above for $\mathbf{E}(\partial G_{kl}/\partial M_{ij})$ gives the result. One proves the analogous relation when $i > j$ in a similar fashion. \square

PROPOSITION 3.2. $\mathbf{E}((g(z))^2) = N \mathbf{E}(\operatorname{Tr}(G \cdot V'(M)))$. (3-1)

PROOF. This follows directly from the lemma by setting $i = k$ and $j = l$, summing over k and l and dividing by $1/N^2$, which yields

$$\sum_{k,l} \mathbf{E}(G_{kk}G_{ll}) = N \sum_{k,l} \mathbf{E}(G_{k,l}V'(M)_{lk}). \quad \square$$

The relation (3-1) can be naturally regarded as a generating function for the second order matrix cumulants of M when written in the equivalent form

$$E((g(z))^2) - E((g(z)))^2 = N E(\text{Tr}(G \cdot V'(M))) - E((g(z)))^2. \quad (3-1)$$

To proceed further we will need to introduce some more notation. First, we will use the following general expression for the potential

$$V(z) = \sum_{j=0}^{\infty} t_j z^j$$

which is understood to have only finitely many nonzero t_j 's. We also have the formal *vertex operator*

$$\frac{d}{dV} = - \sum_{j=0}^{\infty} \frac{1}{z^{j+1}} \frac{d}{dt_j}. \quad (3-2)$$

(A precise meaning for this formal relation will be given in the beginning of the next section.) This can be used to give a compact formal representation of a generating function for matrix moments in terms of the RM partition function (1-1):

$$\frac{d}{dV} \frac{1}{N^2} \log Z_N = \sum_{j=0}^{\infty} \frac{1}{z^{j+1}} E\left(\frac{1}{N} \text{Tr} M^j\right). \quad (3-3)$$

Asymptotic expansions. To make formal relations such as (3-3) meaningful we need to use some fundamental asymptotic facts. The trace of $G(z)$ has a standard integral representation in terms of the RM one-point function (1-3)

$$g(z) = N \int_{-\infty}^{\infty} \frac{\rho_N^{(1)}(\lambda)}{z - \lambda} d\lambda. \quad (3-4)$$

By boundedness and exponential decay of $\rho_N^{(1)}(\lambda)$, $g(z)$ has a valid asymptotic expansion in large z as

$$\int_{-\infty}^{\infty} \frac{\rho_N^{(1)}(\lambda)}{z - \lambda} d\lambda \sim \sum_{j=0}^{\infty} \frac{1}{z^{j+1}} E\left(\frac{1}{N} \text{Tr} M^j\right). \quad (3-5)$$

Thus (3-3) can be precisely understood as saying that for each m and large z ,

$$\begin{aligned} \frac{d}{dV^{(m)}} \frac{1}{N^2} \log Z_N &= \sum_{j=0}^{m-1} \frac{1}{z^{j+1}} \int_{-\infty}^{\infty} \frac{\lambda^j \rho_N^{(1)}(\lambda)}{z - \lambda} d\lambda \\ &= \int_{-\infty}^{\infty} \frac{\rho_N^{(1)}(\lambda)}{z - \lambda} d\lambda + \mathcal{O}(z^{-(m+1)}), \end{aligned} \quad (3-6)$$

where

$$\frac{d}{dV^{(m)}} = - \sum_{j=0}^{m-1} \frac{1}{z^{j+1}} \frac{d}{dt_j}.$$

In a similar sense we have for each m and large z the asymptotic equation

$$\begin{aligned} & \mathbf{E}((g(z))^2) - \mathbf{E}(g(z))^2 \\ &= \sum_{j,k=0}^m \frac{1}{z^{j+k+2}} (\mathbf{E}(\text{Tr } M^j \cdot \text{Tr } M^k) - \mathbf{E}(\text{Tr } M^j) \mathbf{E}(\text{Tr } M^k)) + \mathcal{O}(z^{-(2m+3)}) \\ &= \frac{d}{dV^{(m)}} \frac{d}{dV^{(m)}} \frac{1}{N^2} \log Z_N + \mathcal{O}(z^{-(2m+3)}) \\ &= \frac{d}{dV^{(m)}} \int_{-\infty}^{\infty} \frac{\rho_N^{(1)}(\lambda)}{z-\lambda} d\lambda + \mathcal{O}(z^{-(2m+3)}). \end{aligned} \tag{3-7}$$

In what follows, we will use d/dV instead of $d/dV^{(m)}$ but with the above asymptotic interpretation understood. In the rest of this section we need to establish that there are estimates controlling the errors in the asymptotic expansions (3-6) and (3-7) that remain valid uniformly as $N \rightarrow \infty$. To this end we first note that for (3-6) the error has the form

$$\begin{aligned} & \frac{1}{z^{m+1}} \int_{-\infty}^{\infty} \frac{\lambda^m \rho_N^{(1)}(\lambda)}{z-\lambda} d\lambda \\ &= \frac{1}{z^{m+1}} (f_0^{(m)}(z) + N^{-2} f_1^{(m)}(z) + N^{-4} f_2^{(m)}(z) + \dots) \end{aligned}$$

for $\mathbf{t} \in \mathbb{T}(T, \gamma)$. The right-hand side is an asymptotic expansion valid uniformly (in \mathbb{T}), which follows from the fundamental Theorem 2.1. Similarly for (3-7) the error has the form

$$\begin{aligned} & - \sum_{k=m}^{\infty} \frac{1}{z^{k+1}} \frac{d}{dt_k} \frac{1}{z^{m+1}} (f_0^{(m)}(z) + N^{-2} f_1^{(m)}(z) + N^{-4} f_2^{(m)}(z) + \dots) \\ &= \sum_{k=m}^{\nu} \frac{1}{z^{k+m+2}} (g_0^{(m)}(z) + N^{-2} g_1^{(m)}(z) + N^{-4} g_2^{(m)}(z) + \dots), \end{aligned}$$

in which the sum on the right is finite since V depends on only finitely many distinct t_k . We use here the fact stated in Theorem 2.1 that these asymptotic expansions in N can be differentiated term by term, preserving uniformity.

With the observations of this section we can express the relation (3-1) in terms of integral representations involving the one-point function:

$$\begin{aligned} \frac{d}{dV} \int_{-\infty}^{\infty} \frac{\rho_N^{(1)}(\lambda)}{z-\lambda} d\lambda \\ = N^2 \int_{-\infty}^{\infty} \frac{V'(\lambda)\rho_N^{(1)}(\lambda)}{z-\lambda} d\lambda - N^2 \left(\int_{-\infty}^{\infty} \frac{\rho_N^{(1)}(\lambda)}{z-\lambda} d\lambda \right)^2, \end{aligned} \quad (3-8)$$

to be understood in the sense of an asymptotic expansion in large z whose coefficients moreover have asymptotic expansions in *even* powers of N which are uniform in admissible t . We note that one consequence of this is that the two terms on the right-hand side of (3-8) cancel at leading order so that the difference has leading order $\mathcal{O}(1)$ for large N .

Loop equations. To prepare for the transformation to a recursive loop equation we parse the first integral on the right-hand side of (3-8) as

$$\int_{-\infty}^{\infty} \frac{V'(\lambda)\rho_N^{(1)}(\lambda)}{z-\lambda} d\lambda = \int_{\alpha-\delta}^{\beta+\delta} \frac{V'(\lambda)\rho_N^{(1)}(\lambda)}{z-\lambda} d\lambda + \mathcal{O}(e^{-cN}), \quad (3-9)$$

where $c > 0$ depends on the choice of the positive constant δ . The justification for the exponential error term is part of the proof of the fundamental asymptotic relation presented in [Ercolani and McLaughlin 2003].

From (3-9), we may further transform this term:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{V'(\lambda)\rho_N^{(1)}(\lambda)}{z-\lambda} d\lambda \\ = \frac{1}{2\pi i} \int_{\alpha-\delta}^{\beta+\delta} \left(\oint_{\mathcal{C}} \frac{V'(x)}{x-\lambda} dx \right) \frac{\rho_N^{(1)}(\lambda)}{z-\lambda} d\lambda + \mathcal{O}(e^{-cN}), \\ = \frac{1}{2\pi i} \oint_{\mathcal{C}} \int_{\alpha-\delta}^{\beta+\delta} \left(\frac{V'(x)}{x-\lambda} \right) \frac{\rho_N^{(1)}(\lambda)}{z-\lambda} d\lambda dx + \mathcal{O}(e^{-cN}), \\ = \frac{1}{2\pi i} \oint_{\mathcal{C}} \int_{\alpha-\delta}^{\beta+\delta} V'(x)\rho_N^{(1)}(\lambda) \left(\frac{1}{x-\lambda} - \frac{1}{z-\lambda} \right) \left(\frac{1}{z-x} \right) d\lambda dx + \mathcal{O}(e^{-cN}) \\ = \frac{1}{2\pi i} \oint_{\mathcal{C}} \int_{\alpha-\delta}^{\beta+\delta} V'(x)\rho_N^{(1)}(\lambda) \left(\frac{1}{x-\lambda} \right) \left(\frac{1}{z-x} \right) d\lambda dx + \mathcal{O}(e^{-cN}) \\ = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{V'(x)}{z-x} \int_{\alpha-\delta}^{\beta+\delta} \frac{\rho_N^{(1)}(\lambda)}{x-\lambda} d\lambda dx + \mathcal{O}(e^{-cN}), \end{aligned}$$

where on the first line we have expressed $V'(\lambda)$ as a loop integral à la Cauchy's Theorem, where the contour \mathcal{C} encircles the interval $(\alpha-\delta, \beta+\delta)$, with z outside

the contour of integration, and on the penultimate line one term has vanished by Cauchy's Theorem and analyticity.

Inserting the last expression above into (3-8), we have derived the final form of the loop equation generating function:

$$\begin{aligned} & \frac{1}{2\pi i} \oint_{\mathcal{C}_\epsilon} \frac{V'(x)}{z-x} \int_{\alpha-\delta}^{\beta+\delta} \frac{\rho_N^{(1)}(\lambda)}{x-\lambda} d\lambda dx \\ &= -N^{-2} \frac{d}{dV} \int_{\alpha-\delta}^{\beta+\delta} \frac{\rho_N^{(1)}(\lambda)}{z-\lambda} d\lambda - \left(\int_{\alpha-\delta}^{\beta+\delta} \frac{\rho_N^{(1)}(\lambda)}{z-\lambda} d\lambda \right)^2 + \mathcal{O}(e^{-cN}). \end{aligned} \quad (3-10)$$

Using Theorem 2.3, the term in parentheses on the last line is easily seen to possess an asymptotic expansion in large N , each of whose coefficients possesses a Laurent expansion in large z :

$$\int_{\alpha-\delta}^{\beta+\delta} \frac{\rho_N^{(1)}(\lambda)}{z-\lambda} d\lambda \sim \sum_{g=0}^{\infty} N^{-2g} P_g(z). \quad (3-11)$$

We note that the terms $P_g(z)$ are independent of the parameter δ . Combining (3-6) and (2-4) we see that

$$P_g(z) = \frac{d}{dV} e_g(\mathbf{t}) = \frac{d}{dV^{(v+1)}} e_g(\mathbf{t}) = - \sum_{j=0}^v \frac{1}{z^{j+1}} \frac{de_g(\mathbf{t})}{dt_j}. \quad (3-12)$$

Inserting (3-11) into the loop equation generating function (3-10) yields the hierarchy of loop equations:

$$\begin{aligned} & \frac{1}{2\pi i} \oint_{\mathcal{C}_\epsilon} \frac{V'(x)}{z-x} P_g(x) dx = -\frac{d}{dV} P_{g-1}(z) - \sum_{g'=0}^g P_{g'}(z) P_{g-g'}(z), \\ & \frac{1}{2\pi i} \oint_{\mathcal{C}_\epsilon} \frac{(V'(x) - 2P_0(x))}{z-x} P_g(x) dx = -\frac{d}{dV} P_{g-1}(z) - \sum_{g'=1}^{g-1} P_{g'}(z) P_{g-g'}(z), \\ & \frac{1}{2\pi i} \oint_{\mathcal{C}_\epsilon} \frac{\psi(x)}{x-z} P_g(x) dx = \frac{d}{dV} P_{g-1}(z) + \sum_{g'=1}^{g-1} P_{g'}(z) P_{g-g'}(z), \end{aligned} \quad (3-13)$$

where the transition to the final recursion formula is mediated by the identity (1-9):

$$2\pi i \psi(x) = V'(x) - 2 \int_{-\infty}^{\infty} \frac{\psi(\lambda)}{x-\lambda} d\lambda = V'(x) - 2P_0(x) \quad (3-14)$$

where $\psi(x)$ here is interpreted as the analytic extension of the density for the equilibrium measure off of the slit $[\alpha, \beta]$ as given by (1-8).

With this result in hand it is now possible to consider a recursive derivation of the terms P_g starting with P_0 as given by (3-14). These terms are related to the map enumeration generating functions, $e_g(\mathbf{t})$, through (3-12). In the physics literature there are instances in which loop equations are used to formally derive expressions for some of the e_g . In particular, we refer the reader to [Ambjørn et al. 1993].

A natural application of our derivation of (3-13) would be to the derivation of closed form expressions for $e_g(\mathbf{t})$ which extend our results in [Ercolani, McLaughlin and Pierce 2006] for potentials V depending only on a single nonzero time $t_{2\nu}$. We may also be able to use these equations to say something about the general qualitative and asymptotic behavior of the e_g . Finally, the $P_g(z, \mathbf{t})$ contain information about the large N asymptotic behavior of the matrix moments, such as (1-2), which could be used to explore whether or not the general unitary ensembles are asymptotically free.

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