Singular solutions for geodesic flows
of Vlasov moments

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For Henry McKean, on the occasion of his 75th birthday

ABSTRACT. The Vlasov equation for the collisionless evolution of the single-particle probability distribution function (PDF) is a well-known example of coadjoint motion. Remarkably, the property of coadjoint motion survives the process of taking moments. That is, the evolution of the moments of the Vlasov PDF is also a form of coadjoint motion. We find that *geodesic* coadjoint motion of the Vlasov moments with respect to powers of the single-particle momentum admits singular (weak) solutions concentrated on embedded subspaces of physical space. The motion and interactions of these embedded subspaces are governed by canonical Hamiltonian equations for their geodesic evolution.

1. Introduction

The Vlasov equation. The evolution of N identical particles in phase space with coordinates (q_i, p_i) $i = 1, 2, \ldots, N$, may be described by an evolution equation for their joint probability distribution function. Integrating over all but one of the particle phase-space coordinates yields an evolution equation for the single-particle probability distribution function (PDF). This is the Vlasov equation.

The solutions of the Vlasov equation reflect its heritage in particle dynamics, which may be reclaimed by writing its many-particle PDF as a product of delta functions in phase space. Any number of these delta functions may be integrated out until all that remains is the dynamics of a single particle in the collective field of the others. In plasma physics, this collective field generates the total electromagnetic properties and the self-consistent equations obeyed by the single

particle PDF are the Vlasov–Maxwell equations. In the electrostatic approximation, these become the Vlasov–Poisson equations, which govern the statistical distributions of particle systems ranging from integrated circuits (MOSFETS, metal-oxide semiconductor field-effect transistors), to charged-particle beams, to the distribution of galaxies in the Universe.

A class of singular solutions of the VP equations called the "cold plasma" solutions have a particularly beautiful experimental realization in the Malmberg–Penning trap. In this experiment, the time average of the vertical motion closely parallels the Euler fluid equations. In fact, the cold plasma singular Vlasov–Poisson solution turns out to obey the equations of point-vortex dynamics in an incompressible ideal flow. This coincidence allows the discrete arrays of "vortex crystals" envisioned by J. J. Thomson for fluid vortices to be realized experimentally as solutions of the Vlasov–Poisson equations. For a survey of these experimental cold-plasma results see [Dubin and O'Neil 1990].

Vlasov moments. The Euler fluid equations arise by imposing a closure relation on the first three momentum moments, or p-moments of the Vlasov PDF $f(p, q, t)$. The zeroth p-moment is the spatial density of particles. The first p-moment is the mean momentum and its ratio with the zeroth p-moment is the Eulerian fluid velocity. Introducing an expression for the fluid pressure in terms of the density and velocity closes the system of p-moment equations, which otherwise would possess a countably infinite number of dependent variables.

The operation of taking p-moments preserves the geometric nature of Vlasov's equation. It's closure after the first p-moment results in Euler's useful and beautiful theory of ideal fluids. As its primary geometric characteristic, Euler's fluid theory represents fluid flow as Hamiltonian geodesic motion on the space of smooth invertible maps acting on the flow domain and possessing smooth inverses. These smooth maps (called diffeomorphisms) act on the fluid reference configuration so as to move the fluid particles around in their container. And their smooth inverses recall the initial reference configuration (or label) for the fluid particle currently occupying any given position in space. Thus, the motion of all the fluid particles in a container is represented as a time-dependent curve in the infinite-dimensional group of diffeomorphisms. Moreover, this curve describing the sequential actions of the diffeomorphisms on the fluid domain is a special optimal curve that distills the fluid motion into a single statement. Namely, "A fluid moves to get out of its own way as efficiently as possible." Put more mathematically, fluid flow occurs along a curve in the diffeomorphism group which is a geodesic with respect to the metric on its tangent space supplied by its kinetic energy.

Given the beauty and utility of the solution behavior for Euler's equation for the first p-moment, one is intrigued to know more about the dynamics of

the other moments of Vlasov's equation. Of course, the dynamics of the p-moments of the Vlasov–Poisson equation is one of the mainstream subjects of plasma physics and space physics.

Summary. This paper formulates the problem of Vlasov p-moments governed by *quadratic* Hamiltonians. This dynamics is a certain type of geodesic motion on the symplectomorphisms, rather than the diffeomorphisms. The symplecto-morphisms are smooth invertible maps acting on the phase space and possessing smooth inverses. We shall consider the singular solutions of the geodesic dynamics of the Vlasov p-moments. Remarkably, these equations turn out to be related to integrable systems governing shallow water wave theory. In fact, when the Vlasov p-moment equations for geodesic motion on the symplecto-morphisms are closed at the level of the first p-moment, their singular solutions are found to recover the peaked soliton of the integrable Camassa–Holm equation for shallow water waves [Camassa and Holm 1993].

Thus, geodesic symplectic dynamics of the Vlasov p-moments is found to possess singular solutions whose closure at the fluid level recovers the peakon solutions of shallow water theory. Being solitons, the peakons superpose and undergo elastic collisions in fully nonlinear interactions. The singular solutions for Vlasov p-moments presented here also superpose and interact nonlinearly as coherent structures.

The plan of the paper follows:

Section 2 defines the Vlasov p-moment equations and formulates them as Hamiltonian system using the Kupershmidt–Manin Lie–Poisson bracket. This formulation identifies the p-moment equations as coadjoint motion under the action of a Lie algebra $\mathfrak{g}$ on its dual Lie algebra $\mathfrak{g}^*$, in any number of spatial dimensions.

Section 3 derives variational formulations of the p-moment dynamics in both their Lagrangian and Hamiltonian forms.

Section 4 formulates the problem of geodesic motion on the symplectomor-phisms in terms of the Vlasov p-moments and identifies the singular solutions of this problem, whose support is concentrated on delta functions in position space. In a special case, the truncation of geodesic symplectic motion to geodesic dif-feomorphic motion for the first p-moment recovers the singular solutions of the Camassa–Holm equation.

Section 5 discusses how the singular p-moment solutions for geodesic sym-plectic motion are related to the cold plasma solutions. By symmetry under exchange of canonical momentum p and position q, the Vlasov q-moments are also found to admit singular (weak) solutions.

2. Vlasov moment dynamics

The Vlasov equation may be expressed as

$$\frac{\partial f}{\partial t} = \left[f, \frac{\delta h}{\delta f} \right] = \frac{\partial f}{\partial p} \frac{\partial}{\partial q} \frac{\delta h}{\delta f} - \frac{\partial f}{\partial q} \frac{\partial}{\partial p} \frac{\delta h}{\delta f} =: - \operatorname{ad}^*_{\delta h/\delta f} f. \qquad (2\text{-}1)$$

Here the canonical Poisson bracket $[\,\cdot\,,\,\cdot\,]$ is defined for smooth functions on phase space with coordinates (q, p) and $f(q, p, t)$ is the evolving Vlasov single-particle distribution function. The variational derivative $\delta h/\delta f$ is the single particle Hamiltonian.

A functional $g[f]$ of the Vlasov distribution f evolves according to

$$\frac{dg}{dt} = \iint \frac{\delta g}{\delta f} \frac{\partial f}{\partial t} \, dq \, dp = \iint \frac{\delta g}{\delta f} \left[f, \frac{\delta h}{\delta f} \right] dq \, dp$$

$$= - \iint f \left[\frac{\delta g}{\delta f}, \frac{\delta h}{\delta f} \right] dq \, dp =: - \left\langle\!\!\left\langle f, \left[\frac{\delta g}{\delta f}, \frac{\delta h}{\delta f} \right] \right\rangle\!\!\right\rangle =: \{ g, h \}$$

In this calculation boundary terms are neglected upon integrating by parts and the notation $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle$ is introduced for the L^2 pairing in phase space. The quantity $\{ g, h \}$ defined in terms of this pairing is the Lie–Poisson Vlasov (LPV) bracket [Morrison 1980]. This Hamiltonian evolution equation may also be expressed as

$$\frac{dg}{dt} = \{ g, h \} = \left\langle\!\!\left\langle f, \operatorname{ad}_{\delta h/\delta f} \frac{\delta g}{\delta f} \right\rangle\!\!\right\rangle = - \left\langle\!\!\left\langle \operatorname{ad}^*_{\delta h/\delta f} f, \frac{\delta g}{\delta f} \right\rangle\!\!\right\rangle$$

which defines the Lie-algebraic operations ad and ad^* in this case in terms of the L^2 pairing on phase space $\langle\!\langle\,\cdot\,,\,\cdot\,\rangle\!\rangle \colon \mathfrak{s}^* \times \mathfrak{s} \mapsto \mathbb{R}$. Thus, the notation $\operatorname{ad}^*_{\delta h/\delta f} f$ in (2-1) expresses *coadjoint action* of $\delta h/\delta f \in \mathfrak{s}$ on $f \in \mathfrak{s}^*$, where $\mathfrak{s}$ is the Lie algebra of single particle Hamiltonian vector fields and $\mathfrak{s}^*$ is its dual under L^2 pairing in phase space. This is the sense in which the Vlasov equation represents coadjoint motion on the symplectomorphisms.

2.1. Dynamics of Vlasov q, p-moments. The phase space q, p-moments of the Vlasov distribution function are defined by

$$g_{\hat{m}m} = \iint f(q, p) \, q^{\hat{m}} p^m \, dq \, dp.$$

The q, p-moments $g_{\hat{m}m}$ are often used in treating the collisionless dynamics of plasmas and particle beams [Dragt et al. 1990]. This is usually done by considering low order truncations of the potentially infinite sum over phase space moments,

$$g = \sum_{\hat{m},m=0}^{\infty} a_{\hat{m}m} g_{\hat{m},m}, \qquad h = \sum_{\hat{n},n=0}^{\infty} b_{\hat{n}n} g_{\hat{n},n},$$

with constants $a_{\hat{m}m}$ and $b_{\hat{n}n}$, with $\hat{m}, m, \hat{n}, n = 0, 1, \ldots$. If h is the Hamiltonian, the sum over q, p-moments g evolves under the Vlasov dynamics according to the Poisson bracket relation

$$\frac{dg}{dt} = \{\, g, h \,\} = \sum_{\hat{m},m,\hat{n},n=0}^{\infty} a_{\hat{m}m} b_{\hat{n}n} (\hat{m}m - \hat{n}n) g_{\hat{m}+\hat{n}-1,m+n-1}.$$

This Poisson bracket may be identified with the smooth Hamiltonian vector fields on p and q, by invoking the standard Lie-algebra antihomomorphism

$$X_H = \{\, \cdot \, , H \,\},$$

for any function $H(p, q)$, then noticing that the q, p-moments are linear functionals of the canonical variables. The symplectic invariants associated with Hamiltonian flows of the q, p-moments were discovered and classified in [Holm et al. 1990].

2.2. Dynamics of Vlasov p-moments. The momentum moments, or "p-moments," of the Vlasov function are defined as

$$A_m(q,t) = \int p^m \, f(q, p, t) \, dp, \qquad m = 0, 1, \ldots.$$

That is, the p-moments are q-dependent integrals over p of the product of powers p^m, $m = 0, 1, \ldots$, times the Vlasov solution $f(q, p, t)$. We shall consider functionals of these p-moments defined by

$$g = \sum_{m=0}^{\infty} \iint \alpha_m(q) \, p^m \, f \, dq \, dp = \sum_{m=0}^{\infty} \int \alpha_m(q) \, A_m(q) \, dq =: \sum_{m=0}^{\infty} \langle A_m, \alpha_m \rangle,$$

$$h = \sum_{n=0}^{\infty} \iint \beta_n(q) \, p^n \, f \, dq \, dp = \sum_{n=0}^{\infty} \int \beta_n(q) \, A_n(q) \, dq =: \sum_{n=0}^{\infty} \langle A_n, \beta_n \rangle,$$

where $\langle \, \cdot \, , \cdot \, \rangle$ is the L^2 pairing on position space.

The functions α_m and β_n with $m, n = 0, 1, \ldots$ are assumed to be suitably smooth and integrable against the Vlasov p-moments. To assure these properties, one may relate the p-moments to the previous sums of Vlasov q, p-moments by choosing

$$\alpha_m(q) = \sum_{\hat{m}=0}^{\infty} a_{\hat{m}m} q^{\hat{m}}, \qquad \beta_n(q) = \sum_{\hat{n}=0}^{\infty} b_{\hat{n}n} q^{\hat{n}}.$$

For these choices of $\alpha_m(q)$ and $\beta_n(q)$, the sums of p-moments will recover the full set of Vlasov (q, p)-moments. Thus, as long as the q, p-moments of the distribution $f(q, p)$ continue to exist under the Vlasov evolution, one may assume that the dual variables $\alpha_m(q)$ and $\beta_n(q)$ are smooth functions whose

Taylor series expands the p-moments in the q, p-moments. These functions are dual to the p-moments $A_m(q)$ with $m = 0, 1, \ldots$ under the L^2 pairing $\langle \cdot, \cdot \rangle$ in the spatial variable q. In what follows we will assume *homogeneous* boundary conditions. This means, for example, that we will ignore boundary terms arising from integrations by parts.

2.3. Poisson bracket for Vlasov p-moments. The Poisson bracket among the p-moments is obtained from the LPV bracket through explicit calculation:

$$
\begin{aligned}
\{ g, h \} &= - \sum_{m,n=0}^{\infty} \iint f\big[\alpha_m(q)\, p^m, \, \beta_n(q)\, p^n\big] \, dq\, dp \\
&= - \sum_{m,n=0}^{\infty} \iint \big[m\alpha_m\beta_n' - n\beta_n\alpha_m'\big] f \, p^{m+n-1} \, dq\, dp \\
&= - \sum_{m,n=0}^{\infty} \int A_{m+n-1}(q)\big[m\alpha_m\beta_n' - n\beta_n\alpha_m'\big] \, dq \\
&=: \sum_{m,n=0}^{\infty} \big\langle A_{m+n-1}, \, \mathrm{ad}_{\beta_n}\alpha_m \big\rangle \\
&= - \sum_{m,n=0}^{\infty} \int \big[n\beta_n A_{m+n-1}' + (m+n)A_{m+n-1}\beta_n'\big]\alpha_m \, dq \\
&=: - \sum_{m,n=0}^{\infty} \big\langle \mathrm{ad}^*_{\beta_n} A_{m+n-1}, \, \alpha_m \big\rangle,
\end{aligned}
$$

where we have integrated by parts and the symbols ad and ad* stand for the adjoint and coadjoint actions. This is done by again invoking the Lie-algebra antihomomorphism with the smooth Hamiltonian vector fields, since the smooth functions $\alpha_m(q)$ and $\beta_n(q)$ are assumed to possess convergent Taylor series.

Upon recalling the dual relations

$$
\alpha_m = \frac{\delta g}{\delta A_m} \quad \text{and} \quad \beta_n = \frac{\delta h}{\delta A_n}
$$

the LPV bracket in terms of the p-moments may be expressed as

$$
\begin{aligned}
\{ g, h \}(\{A\}) \\
= - \sum_{m,n=0}^{\infty} \int \frac{\delta g}{\delta A_m}\Big[n\frac{\delta h}{\delta A_n}\frac{\partial}{\partial q}A_{m+n-1} + (m+n)A_{m+n-1}\frac{\partial}{\partial q}\frac{\delta h}{\delta A_n}\Big] dq \\
=: - \sum_{m,n=0}^{\infty} \Big\langle A_{m+n-1}, \, \Big[\!\!\Big[\frac{\delta g}{\delta A_m}, \frac{\delta h}{\delta A_n}\Big]\!\!\Big] \Big\rangle.
\end{aligned}
$$

This is the Kupershmidt–Manin Lie–Poisson (KMLP) bracket [Kupershmidt and Manin 1978], which is defined for functions on the dual of the Lie algebra with bracket

$$\llbracket \, \alpha_m, \, \beta_n \, \rrbracket = m\alpha_m \partial_q \beta_n - n\beta_n \partial_q \alpha_m.$$

This Lie algebra bracket inherits the Jacobi identity from its definition in terms of the canonical Hamiltonian vector fields. Thus, we have shown:

THEOREM 2.1 [Gibbons 1981]. *The operation of taking p-moments of Vlasov solutions is a Poisson map. It takes the LPV bracket describing the evolution of $f(q, p)$ into the KMLP bracket, describing the evolution of the p-moments $A_n(x)$.*

REMARK 2.2. A result related to theorem 2.1 for the Benney hierarchy [Benney 1966] was also noted by Lebedev and Manin [Lebedev and Manin 1979].

The evolution of a particular p-moment $A_m(q, t)$ is obtained from the KMLP bracket by

$$\frac{\partial A_m}{\partial t} = \{ \, A_m, \, h \, \} = -\sum_{n=0}^{\infty} \left(n \frac{\delta h}{\delta A_n} \frac{\partial}{\partial q} A_{m+n-1} + (m+n)A_{m+n-1} \frac{\partial}{\partial q} \frac{\delta h}{\delta A_n} \right).$$

The KMLP bracket among the p-moments is given by

$$\{ \, A_m, \, A_n \, \} = -n \frac{\partial}{\partial q} A_{m+n-1} - m A_{m+n-1} \frac{\partial}{\partial q},$$

expressed as a differential operator acting to the right. This operation is skew-symmetric under the L^2 pairing and the general KMLP bracket can then be written as (see [Gibbons 1981])

$$\{ \, g, h \, \}(\{A\}) = \sum_{m,n=0}^{\infty} \int \frac{\delta g}{\delta A_m} \{ \, A_m, \, A_n \, \} \frac{\delta h}{\delta A_n} \, dq,$$

so that

$$\frac{\partial A_m}{\partial t} = \sum_{n=0}^{\infty} \{ \, A_m, \, A_n \, \} \frac{\delta h}{\delta A_n}.$$

2.4. Multidimensional treatment. We now show that the KMLP bracket and the equations of motion may be written in three dimensions in multi-index notation. By writing $\boldsymbol{p}^{2n+1} = p^{2n}\,\boldsymbol{p}$, and checking that

$$p^{2n} = \sum_{i+j+k=n} \frac{n!}{i!\,j!\,k!}\, p_1^{2i}\, p_2^{2j}\, p_3^{2k},$$

it is easy to see that the multidimensional treatment can be performed in terms of the quantities

$$p^\sigma =: p_1^{\sigma_1} p_2^{\sigma_2} p_3^{\sigma_3},$$

where $\sigma \in \mathbb{N}^3$. Let A_σ be defined as

$$A_\sigma (\boldsymbol{q}, t) =: \int p^\sigma f(\boldsymbol{q}, \boldsymbol{p}, t) \, dp$$

and consider functionals of the form

$$g = \sum_\sigma \iint \alpha_\sigma(\boldsymbol{q}) \, p^\sigma f(\boldsymbol{q}, \boldsymbol{p}, t) \, dq \, dp =: \sum_{\sigma \in \mathbb{N}^3} \langle A_\sigma, \alpha_\sigma \rangle,$$

$$h = \sum_\rho \iint \beta_\rho(\boldsymbol{q}) \, p^\rho f(\boldsymbol{q}, \boldsymbol{p}, t) \, dq \, dp =: \sum_{\rho \in \mathbb{N}^3} \langle A_\rho, \beta_\rho \rangle.$$

With the notation

$$1_j := (0, ..., 1..., 0) \quad (1 \text{ in } j\text{-th position}),$$

so that $(1_j)_i = \delta_{ji}$. the ordinary LPV bracket leads to

$$\{g, h\} = -\sum_{\sigma, \rho} \iint f\left[\alpha_\sigma(\boldsymbol{q}) \, p^\sigma, \beta_\rho(\boldsymbol{q}) \, p^\rho\right] dq \, dp$$

$$= -\sum_{\sigma, \rho} \sum_j \iint f\left(\alpha_\sigma p^\rho \frac{\partial p^\sigma}{\partial p_j} \frac{\partial \beta_\rho}{\partial q_j} - \beta_\rho p^\sigma \frac{\partial p^\rho}{\partial p_j} \frac{\partial \alpha_\sigma}{\partial q_j}\right) dq \, dp$$

$$= -\sum_{\sigma, \rho} \sum_j \iint f\left(\sigma_j \alpha_\sigma p^\rho p^{\sigma - 1_j} \frac{\partial \beta_\rho}{\partial q_j} - \rho_j \beta_\rho p^\sigma p^{\rho - 1_j} \frac{\partial \alpha_\sigma}{\partial q_j}\right) dq \, dp$$

$$= -\sum_{\sigma, \rho} \sum_j \int A_{\sigma + \rho - 1_j} \left(\sigma_j \alpha_\sigma \frac{\partial \beta_\rho}{\partial q_j} - \rho_j \beta_\rho \frac{\partial \alpha_\sigma}{\partial q_j}\right) dq$$

$$=: \sum_{\sigma, \rho} \sum_j \left\langle A_{\sigma + \rho - 1_j}, \left(\text{ad}_{\beta_\rho}\right)_j \alpha_\sigma \right\rangle$$

$$= -\sum_{\sigma, \rho} \sum_j \int \left(\rho_j \beta_\rho \frac{\partial}{\partial q_j} A_{\sigma + \rho - 1_j} + (\sigma_j + \rho_j) A_{\sigma + \rho - 1_j} \frac{\partial \beta_\rho}{\partial q_j}\right) \alpha_\sigma \, dq$$

$$=: -\sum_{\sigma, \rho} \sum_j \left\langle \left(\text{ad}_{\beta_\rho}^*\right)_j A_{\sigma + \rho - 1_j}, \alpha_\sigma \right\rangle,$$

where the sum extends to all $\sigma, \rho \in \mathbb{N}^3$.

The LPV bracket in terms of the p-moments may then be written as

$$\frac{\partial A_\sigma}{\partial t} = -\sum_{\rho \in \mathbb{N}^3} \sum_j \left(\mathrm{ad}^*_{\delta h/\delta A_\rho}\right)_j A_{\sigma+\rho+1_j}$$

where the Lie bracket is now expressed as

$$\left[\!\left[\frac{\delta g}{\delta A_\sigma}, \frac{\delta h}{\delta A_\rho}\right]\!\right]_j = \sigma_j \alpha_\sigma \frac{\partial}{\partial q_j} \frac{\delta h}{\delta A_\rho} - \rho_j \beta_\rho \frac{\partial}{\partial q_j} \frac{\delta g}{\delta A_\sigma}.$$

Moreover the evolution of a particular p-moment A_σ is obtained by

$$\frac{\partial A_\sigma}{\partial t} = \{A_\sigma, h\}$$
$$= -\sum_\rho \sum_j \left(\rho_j \frac{\delta h}{\delta A_\rho} \frac{\partial}{\partial q_j} A_{\sigma+\rho-1_j} + (\sigma_j + \rho_j) A_{\sigma+\rho-1_j} \frac{\partial}{\partial q_j} \frac{\delta h}{\delta A_\rho}\right)$$

and the KMLP bracket among the multidimensional p-moments is given in by

$$\{A_\sigma, A_\rho\} = -\sum_j \left(\sigma_j \frac{\partial}{\partial q_j} A_{\sigma+\rho-1_j} + \rho_j A_{\sigma+\rho-1_j} \frac{\partial}{\partial q_j}\right).$$

Inserting the previous operator in this multidimensional KMLP bracket yields

$$\{g, h\}(\{A\}) = \sum_{\sigma, \rho} \int \frac{\delta g}{\delta A_\sigma} \{A_\sigma, A_\rho\} \frac{\delta h}{\delta A_\rho} \, dq,$$

and the corresponding evolution equation becomes

$$\frac{\partial A_\sigma}{\partial t} = \sum_\rho \{A_\sigma, A_\rho\} \frac{\delta h}{\delta A_\rho}.$$

Thus, in multi-index notation, the form of the Hamiltonian evolution under the KMLP bracket is essentially unchanged in going to higher dimensions.

2.5. Applications of the KMLP bracket. The KMLP bracket was derived in the context of Benney long waves, whose Hamiltonian is

$$H_2 = \tfrac{1}{2}(A_2 + A_0^2).$$

This leads to the moment equations

$$\frac{\partial A_n}{\partial t} + \frac{\partial A_{n+1}}{\partial q} + n A_{n-1} \frac{\partial A_0}{\partial q} = 0$$

derived by Benney [1966] as a description of long waves on a shallow perfect fluid, with a free surface at $y = h(q, t)$. In his interpretation, the A_n were vertical

moments of the horizontal component of the velocity $p(q, y, t)$:

$$A_n = \int_{y=0}^{h} p(q, y, t)^n \, dy.$$

The corresponding system of evolution equations for $p(q, y, t)$ and $h(q, t)$ is related by the hodograph transformation, $y = \int_{-\infty}^{p} f(q, p', t) \, dp'$, to the Vlasov equation

$$\frac{\partial f}{\partial t} + p \frac{\partial f}{\partial q} - \frac{\partial A_0}{\partial q} \frac{\partial f}{\partial p} = 0.$$

The most important fact about the Benney hierarchy is that it is completely integrable. This fact emerges from the following observation. Upon defining a function $\lambda(q, p, t)$ by the principal value integral,

$$\lambda(q, p, t) = p + P \int_{-\infty}^{\infty} \frac{f(q, p', t)}{p - p'} \, dp',$$

it is straightforward to verify [Lebedev and Manin 1979] that

$$\frac{\partial \lambda}{\partial t} + p \frac{\partial \lambda}{\partial q} - \frac{\partial A_0}{\partial q} \frac{\partial \lambda}{\partial p} = 0;$$

so that f and λ are advected along the same characteristics.

In higher dimensions, particularly $n = 3$, we may take the direct sum of the KMLP bracket, together with the Poisson bracket for an electromagnetic field (in the Coulomb gauge) where the electric field $\boldsymbol{E}$ and magnetic vector potential $\boldsymbol{A}$ are canonically conjugate; then the Hamiltonian

$$H_{MV} = \iint \left(\frac{1}{2m} |\boldsymbol{p} - e\boldsymbol{A}|^2 \right) f(\boldsymbol{p}, \boldsymbol{q}) \, d^n \boldsymbol{p} \, d^n \boldsymbol{q}$$
$$+ \int \left(\frac{1}{2} |\boldsymbol{E}|^2 + \frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} (A_{i,j} - A_{j,i})^2 \right) d^n \boldsymbol{q}$$

yields the Maxwell–Vlasov (MV) equations for systems of interacting charged particles. For a discussion of the MV equations from a geometric viewpoint in the same spirit as the present approach, see [Cendra et al. 1998]. For discussions of the Lie-algebraic approach to the control and steering of charged particle beams, see [Dragt et al. 1990].

3. Variational principles and Hamilton–Poincaré formulation

In this section we show how the p-moment dynamics can be derived from Hamilton's principle both in the Hamilton–Poincaré and Euler–Poincaré forms. These variational principles are defined , respectively, on the dual Lie algebra $\mathfrak{g}^*$ containing the moments, and on the Lie algebra $\mathfrak{g}$ itself. For further details

about these dual variational formulations, see [Cendra et al. 2003] and [Holm et al. 1998]. Summation over repeated indices is intended in this section.

3.1. Hamilton–Poincaré hierarchy. We begin with the Hamilton–Poincaré principle for the p-moments written as

$$\delta \int_{t_i}^{t_j} dt \left(\langle A_n, \beta_n \rangle - H\left(\{A\}\right) \right) = 0$$

(where $\beta_n \in \mathfrak{g}$). We shall prove that this leads to the same dynamics as found in the context of the KMLP bracket. To this purpose, we must define the n-th p-moment in terms of the Vlasov distribution function. We check that

$$0 = \delta \int_{t_i}^{t_j} dt \left(\langle A_n, \beta_n \rangle - H\left(\{A\}\right) \right) = \int_{t_i}^{t_j} dt \left(\delta \langle\!\langle f, p^n \beta_n \rangle\!\rangle - \left\langle\!\!\left\langle \delta f, \frac{\delta H}{\delta f} \right\rangle\!\!\right\rangle \right)$$

$$= \int_{t_i}^{t_j} dt \left(\left\langle\!\!\left\langle \delta f, \left(p^n \beta_n - \frac{\delta H}{\delta f} \right) \right\rangle\!\!\right\rangle + \langle\!\langle f, \delta \left(p^n \beta_n \right) \rangle\!\rangle \right).$$

Now recall that any $g = \delta G/\delta f$ belonging to the Lie algebra $\mathfrak{s}$ of the symplectomorphisms (which also contains the distribution function itself) may be expressed as

$$g = \frac{\delta G}{\delta f} = p^m \frac{\delta G}{\delta A_m} = p^m \xi_m \,,$$

by the chain rule. Consequently, one finds the pairing relationship

$$\left\langle\!\!\left\langle \delta f, \left(p^n \beta_n - \frac{\delta H}{\delta f} \right) \right\rangle\!\!\right\rangle = \left\langle \delta A_n, \left(\beta_n - \frac{\delta H}{\delta A_n} \right) \right\rangle.$$

Next, recall from the general theory that variations on a Lie group induce variations on its Lie algebra of the form

$$\delta w = \dot{u} + [g, u]$$

where $u, w \in \mathfrak{s}$ and u vanishes at the endpoints. Writing $u = p^m \eta_m$ then yields

$$\int_{t_i}^{t_j} dt \, \langle\!\langle f, \delta \left(p^n \beta_n \right) \rangle\!\rangle = \int_{t_i}^{t_j} dt \, \langle\!\langle f, \left(\dot{u} + \left[p^n \beta_n, u \right] \right) \rangle\!\rangle$$

$$= - \int_{t_i}^{t_j} dt \left(\langle \dot{A}_m, \eta_m \rangle - \langle A_{n+m-1}, [\![\beta_n, \eta_m]\!] \rangle \right)$$

$$= - \int_{t_i}^{t_j} dt \, \left\langle \left(\dot{A}_m + \mathrm{ad}_{\beta_n}^* A_{m+n-1} \right), \eta_m \right\rangle.$$

Consequently, the Hamilton–Poincaré principle may be written entirely in terms of the moments as

$$\delta S = \int_{t_i}^{t_j} dt \left\{ \left\langle \delta A_n, \left(\beta_n - \frac{\delta H}{\delta A_n} \right) \right\rangle - \left\langle (\dot{A}_m + \mathrm{ad}^*_{\beta_n} A_{m+n-1}), \eta_m \right\rangle \right\} = 0.$$

This expression produces the inverse Legendre transform

$$\beta_n = \frac{\delta H}{\delta A_n}$$

(holding for hyperregular Hamiltonians). It also yields the equations of motion

$$\frac{\partial A_m}{\partial t} = -\mathrm{ad}^*_{\beta_n} A_{m+n-1},$$

which are valid for arbitrary variations δA_m and variations $\delta \beta_m$ of the form

$$\delta \beta_m = \dot{\eta}_m + \mathrm{ad}_{\beta_n} \eta_{m-n+1},$$

where the variations η_m satisfy vanishing endpoint conditions,

$$\eta_m|_{t=t_i} = \eta_m|_{t=t_j} = 0.$$

Thus, the Hamilton–Poincaré variational principle recovers the hierarchy of the evolution equations derived in the previous section using the KMLP bracket.

3.2. Euler–Poincaré hierarchy. The corresponding Lagrangian formulation of the Hamilton's principle now yields

$$\begin{aligned}
\delta \int_{t_i}^{t_j} L\left(\{\beta\}\right) dt &= \int_{t_i}^{t_j} \left\langle \frac{\delta L}{\delta \beta_m}, \delta \beta_m \right\rangle dt \\
&= \int_{t_i}^{t_j} \left\langle \frac{\delta L}{\delta \beta_m}, \left(\dot{\eta}_m + \mathrm{ad}_{\beta_n} \eta_{m-n+1} \right) \right\rangle dt \\
&= -\int_{t_i}^{t_j} \left(\left\langle \frac{\partial}{\partial t} \frac{\delta L}{\delta \beta_m}, \eta_m \right\rangle + \left\langle \mathrm{ad}^*_{\beta_n} \frac{\delta L}{\delta \beta_m}, \eta_{m-n+1} \right\rangle \right) dt \\
&= -\int_{t_i}^{t_j} \left(\left\langle \frac{\partial}{\partial t} \frac{\delta L}{\delta \beta_m}, \eta_m \right\rangle + \left\langle \mathrm{ad}^*_{\beta_n} \frac{\delta L}{\delta \beta_{m+n-1}}, \eta_m \right\rangle \right) dt \\
&= -\int_{t_i}^{t_j} \left\langle \left(\frac{\partial}{\partial t} \frac{\delta L}{\delta \beta_m} + \mathrm{ad}^*_{\beta_n} \frac{\delta L}{\delta \beta_{m+n-1}} \right), \eta_m \right\rangle dt,
\end{aligned}$$

upon using the expression previously found for the variations $\delta \beta_m$ and relabeling indices appropriately. The Euler–Poincaré equations may then be written as

$$\frac{\partial}{\partial t} \frac{\delta L}{\delta \beta_m} + \mathrm{ad}^*_{\beta_n} \frac{\delta L}{\delta \beta_{m+n-1}} = 0$$

with the same constraints on the variations as in the previous paragraph. Applying the Legendre transformation

$$A_m = \frac{\delta L}{\delta \alpha_m}$$

yields the Euler–Poincaré equations (for hyperregular Lagrangians). This again leads to the same hierarchy of equations derived earlier using the KMLP bracket.

To summarize, the calculations in this section have proved this result:

THEOREM 3.1. *With the above notation and hypotheses of hyperregularity the following statements are equivalent*:

(i) (***The Euler–Poincaré variational principle***.) *The curves $\beta_n(t)$ are critical points of the action*

$$\delta \int_{t_i}^{t_j} L\left(\{\beta\}\right) dt = 0$$

for variations of the form

$$\delta \beta_m = \dot{\eta}_m + \mathrm{ad}_{\beta_n} \eta_{m-n+1},$$

in which η_m vanishes at the endpoints

$$\eta_m|_{t=t_i} = \eta_m|_{t=t_j} = 0$$

and the variations δA_n are arbitrary.

(ii) (***The Lie–Poisson variational principle***.) *The curves $(\beta_n, A_n)(t)$ are critical points of the action*

$$\delta \int_{t_i}^{t_j} \left(\langle A_n, \beta_n \rangle - H\left(\{A\}\right)\right) dt = 0$$

for variations of the form

$$\delta \beta_m = \dot{\eta}_m + \mathrm{ad}_{\beta_n} \eta_{m-n+1},$$

where η_m satisfies endpoint conditions

$$\eta_m|_{t=t_i} = \eta_m|_{t=t_j} = 0$$

and the variations δA_n are arbitrary.

(iii) *The **Euler–Poincaré equations** hold*:

$$\frac{\partial}{\partial t} \frac{\delta L}{\delta \beta_m} + \mathrm{ad}_{\beta_n}^* \frac{\delta L}{\delta \beta_{m+n-1}} = 0.$$

(iv) *The **Lie–Poisson equations** hold*:

$$\dot{A}_m = -\mathrm{ad}_{\delta H/\delta A_n}^* A_{m+n-1}.$$

For further details on the proof of this theorem we direct the reader to [Cendra et al. 2003]. An analogous result is also valid in the multidimensional case with slight modifications.

4. Quadratic Hamiltonians

4.1. Geodesic motion. We shall consider the problem of geodesic motion on the space of p-moments. For this, we define the Hamiltonian as the norm on the p-moment given by the following metric and inner product,

$$h = \tfrac{1}{2}\|A\|^2 = \tfrac{1}{2}\sum_{n,s=0}^{\infty}\iint A_n(q)G_{ns}(q,q')A_s(q')\,dq\,dq' \qquad (4\text{-}1)$$

The metric $G_{ns}(q,q')$ is chosen to be positive definite, so it defines a norm for $\{A\}\in\mathfrak{g}^*$. The corresponding geodesic equation with respect to this norm is found as in the previous section to be

$$\frac{\partial A_m}{\partial t} = \{A_m,\,h\} = -\sum_{n=0}^{\infty}\left(n\beta_n\frac{\partial}{\partial q}A_{m+n-1} + (m+n)A_{m+n-1}\frac{\partial}{\partial q}\beta_n\right), \quad (4\text{-}2)$$

with dual variables $\beta_n\in\mathfrak{g}$ defined by

$$\beta_n = \frac{\delta h}{\delta A_n} = \sum_{s=0}^{\infty}\int G_{ns}(q,q')A_s(q')\,dq' = \sum_{s=0}^{\infty}G_{ns}*A_s. \qquad (4\text{-}3)$$

Thus, evolution under (4-2) may be rewritten formally as (infinitesimal) coadjoint motion on $\mathfrak{g}^*$

$$\frac{\partial A_m}{\partial t} = \{A_m,\,h\} =: -\sum_{n=0}^{\infty}\mathrm{ad}^*_{\beta_n}A_{m+n-1}. \qquad (4\text{-}4)$$

The explicit identification of coAdjoint motion by the full group action on the dual Lie algebra is left for a future study. This system comprises an infinite system of nonlinear, nonlocal, coupled evolutionary equations for the p-moments. In this system, evolution of the m-th moment is governed by the potentially infinite sum of contributions of the velocities β_n associated with n-th moment sweeping the $(m+n-1)$-th moment by coadjoint action. Moreover, by equation (4-3), each of the β_n potentially depends nonlocally on all of the moments.

Equations (4-1) and (4-3) may be written in three dimensions in multi-index notation, as follows: the Hamiltonian is given by

$$h = \tfrac{1}{2}\|A\|^2 = \tfrac{1}{2}\sum_{\mu,\nu}\iint A_\mu\left(\boldsymbol{q},t\right)G_{\mu\nu}\left(\boldsymbol{q},\boldsymbol{q}'\right)A_\nu\left(\boldsymbol{q}',t\right)d\boldsymbol{q}\,d\boldsymbol{q}'$$

so the dual variable is written as

$$\beta_\rho = \frac{\delta h}{\delta A_\rho} = \sum_\nu \iint G_{\rho\nu}(q, q') A_\nu(q', t) \, dq \, dq' = \sum_\nu G_{\rho\nu} * A_\nu.$$

4.2. Singular geodesic solutions. Remarkably, in any number of spatial dimensions, the geodesic equation (4-2) possesses exact solutions which are *singular*; that is, they are supported on delta functions in q-space.

THEOREM 4.1 (SINGULAR SOLUTION ANSATZ FOR GEODESIC FLOWS OF VLASOV p-MOMENTS). *Equation* (4-2) *admits singular solutions of the form*

$$A_\sigma(q, t) = \sum_{j=1}^N \int P_j^\sigma(q, t, a_j) \, \delta(q - Q_j(t, a_j)) \, da_j, \qquad (4\text{-}5)$$

in which the integrals over coordinates a_j are performed over N embedded subspaces of the q-space and the parameters (Q_j, P_j) satisfy canonical Hamiltonian equations in which the Hamiltonian is the norm h in (4-1) evaluated on the singular solution Ansatz (4-5).

In one dimension, the coordinates a_j are absent and the singular solutions in (4-5) reduce to

$$A_s(q, t) = \sum_{j=1}^N P_j^s(q, t) \, \delta(q - Q_j(t)). \qquad (4\text{-}6)$$

In order to show this is a solution in one dimension, one checks that these singular solutions satisfy a system of partial differential equations in Hamiltonian form, whose Hamiltonian couples all the moments

$$H_N = \frac{1}{2} \sum_{n,s=0}^\infty \sum_{j,k=1}^N P_j^s(Q_j(t), t) P_k^n(Q_k(t), t) \, G_{ns}(Q_j(t), Q_k(t)).$$

One forms the pairing of the coadjoint equation

$$\dot{A}_m = -\sum_{n,s} \mathrm{ad}^*_{G_{ns} * A_s} A_{m+n-1}$$

with a sequence of smooth functions $\{\varphi_m(q)\}$, so that

$$\langle \dot{A}_m, \varphi_m \rangle = \sum_{n,s} \langle A_{m+n-1}, \mathrm{ad}_{G_{ns} * A_s} \varphi_m \rangle$$

One expands each term and denotes $\widetilde{P}_j(t) := P_j(Q_j, t)$:

$$\langle \dot{A}_m, \varphi_m \rangle = \sum_j \int dq\, \varphi_m(q)\, \frac{\partial}{\partial t}\big(P_j^m(q,t)\delta(q-Q_j)\big)$$

$$= \sum_j \int dq\, \varphi_m(q)\left(\delta(q-Q_j)\frac{\partial P_j^m}{\partial t} - P_j^m \dot{Q}_j \delta'(q-Q_j)\right)$$

$$= \sum_j \left(\frac{d\,\widetilde{P}_j^m}{dt}\varphi_m(Q_j) + \widetilde{P}_j^m \dot{Q}_j \varphi_m'(Q_j)\right)$$

Similarly, expanding

$$\langle A_{m+n-1}, \mathrm{ad}_{G_{ns}*A_s}\varphi_m \rangle$$

$$= \sum_{j,k} \int dq\, \widetilde{P}_k^s\, P_j^{m+n-1}\delta(q-Q_j)\left(n\varphi_m' G_{ns}(q, Q_k) - m\varphi_m \frac{\partial G_{ns}(q, Q_k)}{\partial q}\right)$$

$$= \sum_{j,k} \widetilde{P}_k^s\, \widetilde{P}_j^{m+n-1}\left(n\, \varphi_m'(Q_j)G_{ns}(Q_j, Q_k) - m\, \varphi_m(Q_j)\frac{\partial G_{ns}(Q_j, Q_k)}{\partial Q_j}\right)$$

leads to

$$\widetilde{P}_j^m \frac{dQ_j}{dt} = \sum_{n,s}\sum_k n\, \widetilde{P}_k^s \widetilde{P}_j^{m+n-1} G_{ns}(Q_j, Q_k),$$

$$\frac{d\,\widetilde{P}_j^m}{dt} = -m \sum_{n,s}\sum_k \widetilde{P}_k^s \widetilde{P}_j^{m+n-1}\frac{\partial G_{ns}(Q_j, Q_k)}{\partial Q_j},$$

so we finally obtain equations for Q_j and $\widetilde{P}_j$ in canonical form,

$$\frac{dQ_j}{dt} = \frac{\partial H_N}{\partial \widetilde{P}_j}, \qquad \frac{d\widetilde{P}_j}{dt} = -\frac{\partial H_N}{\partial Q_j}.$$

Remark about higher dimensions. The singular solutions (4-5) with the integrals over coordinates a_j exist in higher dimensions. The higher dimensional singular solutions satisfy a system of canonical Hamiltonian integral-partial differential equations, instead of ordinary differential equations.

5. Discussion

5.1. Remarks about EPSymp and connections with EPDiff. Importantly, geodesic motion for the p-moments is equivalent to geodesic motion for the Euler–Poincaré equations on the symplectomorphisms (EPSymp) given by the

Hamiltonian

$$H[f] = \frac{1}{2} \iint f(q, p, t) \, \mathscr{G}(q, p, q', p') \, f(q', p', t) \, dq \, dp \, dq' dp' \qquad (5\text{-}1)$$

The equivalence with EPSymp emerges when the function $\mathscr{G}$ is written as

$$\mathscr{G}(q, q', p, p') = \sum_{n,m} p^n G_{nm}(q, q') \, p'^m.$$

Thus, whenever the metric $\mathscr{G}$ for EPSymp has a Taylor series, its solutions may be expressed in terms of the geodesic motion for the p-moments.

Moreover the distribution function corresponding to the singular solutions for the moments is a particular case of the *cold-plasma approximation*, given by

$$f(q, p, t) = \sum_j \rho_j(q, t) \, \delta(p - P_j(q, t)),$$

where in our case a summation is introduced and ρ is written as a Lagrangian particle-like density: $\rho_j(q, t) = \delta(q - Q_j(t))$.

To check this is a solution for the geodesic motion of the generating function, one repeats exactly the same procedure as for the moments, in order to find the Hamiltonian equations

$$\frac{dQ_j}{dt} = \frac{\partial}{\partial \widetilde{P}_j} \frac{\delta H}{\delta f}(Q_j, \widetilde{P}_j), \qquad \frac{d \widetilde{P}_j}{dt} = \frac{\partial}{\partial Q_j} \frac{\delta H}{\delta f}(Q_j, \widetilde{P}_j)$$

where $\widetilde{P}_j = P_j \circ Q_j$ denotes the composition of the two functions P_j and Q_j. This recovers single particle motion for density ρ_j defined on a delta function.

As we shall show, these singular solutions of EPSymp are also solutions of the Euler–Poincaré equations on the diffeomorphisms (EPDiff), provided one truncates to consider only first order moments [Holm and Marsden 2005]. With this truncation, the singular solutions in the case of single-particle dynamics reduce in one dimension to the pulson solutions for EPDiff [Camassa and Holm 1993].

5.2. Exchanging variables in EPSymp. One can show that exchanging the variables $q \leftrightarrow p$ in the single particle PDF leads to another nontrivial singular solution of EPSymp, which is different from those found previously. To see this, let f be given by

$$f(q, p, t) = \sum_j \delta(q - Q_j(p, t)) \, \delta(p - P_j(t)).$$

At this stage nothing has changed with respect to the previous solution since the generating function is symmetric with respect to q and p. However, inserting

this expression in the definition of the m-th moment yields

$$A_m(q,t) = \sum_j P_j^m \, \delta(q - Q_j(P_j,t)),$$

which is quite different from the solutions found previously. One again obtains a canonical Hamiltonian structure for P_j and Q_j.

This second expression is an alternative parametrisation of the cold-plasma reduction above and it may be useful in situations where the composition $Q_j \circ P_j$ is more convenient than $P_j \circ Q_j$.

5.3. Remarks about truncations. The problem presented by the coadjoint motion equation (4-4) for geodesic evolution of p-moments under EPDiff needs further simplification. One simplification would be to modify the (doubly) infinite set of equations in (4-4) by truncating the Poisson bracket to a finite set. These moment dynamics may be truncated at any stage by modifying the Lie-algebra in the KMLP bracket to vanish for weights $m + n - 1$ greater than a chosen cut-off value.

5.4. Examples of simplifying truncations and specializations. For example, if we truncate the sums to $m, n = 0, 1, 2$ only, then equation (4-4) produces the coupled system of partial differential equations

$$\frac{\partial A_0}{\partial t} = -\mathrm{ad}^*_{\beta_1} A_0 - \mathrm{ad}^*_{\beta_2} A_1,$$

$$\frac{\partial A_1}{\partial t} = -\mathrm{ad}^*_{\beta_0} A_0 - \mathrm{ad}^*_{\beta_1} A_1 - \mathrm{ad}^*_{\beta_2} A_2,$$

$$\frac{\partial A_2}{\partial t} = -\mathrm{ad}^*_{\beta_0} A_1 - \mathrm{ad}^*_{\beta_1} A_2.$$

Expanding now the expression of the coadjoint operation

$$\mathrm{ad}^*_{\beta_h} A_{k+h-1} = (k+h) \, A_{k+h-1} \partial_q \beta_h + h \beta_h \partial_q A_{k+h-1}$$

and relabeling

$$\mathrm{ad}^*_{\beta_h} A_k = (k+1) \, A_k \partial_q \beta_h + h \beta_h \partial_q A_k$$

one calculates

$$\frac{\partial A_0}{\partial t} = -\partial_q (A_0 \beta_1) - 2 A_1 \partial_q \beta_2 - 2 \beta_2 \partial_q A_1,$$

$$\frac{\partial A_1}{\partial t} = -A_0 \partial_q \beta_0 - 2 A_1 \partial_q \beta_1 - \beta_1 \partial_q A_1 - 3 A_2 \partial_q \beta_2 - 2 \beta_2 \partial_q A_2,$$

$$\frac{\partial A_2}{\partial t} = -2 A_1 \partial_q \beta_0 - 3 A_2 \partial_q \beta_1 - \beta_1 \partial_q A_2.$$

We specialize to the case that each velocity depends only on its corresponding moment, so that $\beta_s = G * A_s$, $s = 0, 1, 2$. If we further specialize by setting A_0 and A_2 initially to zero, then these three equations reduce to the single equation

$$\frac{\partial A_1}{\partial t} = -\beta_1 \, \partial_q A_1 - 2A_1 \, \partial_q \beta_1.$$

Finally, if we assume that G in the convolution $\beta_1 = G * A_1$ is the Green's function for the operator relation

$$A_1 = (1 - \alpha^2 \partial_q^2)\beta_1$$

for a constant lengthscale α, then the evolution equation for A_1 reduces to the integrable Camassa–Holm (CH) equation [1993] in the absence of linear dispersion. This is the one-dimensional EPDiff equation, which has singular (peakon) solutions. Thus, after these various specializations of the EPDiff p-moment equations, one finds the integrable CH peakon equation as a specialization of the coadjoint moment dynamics of equation (4-4).

That such a strong restriction of the p-moment system leads to such an interesting special case bodes well for future investigations of the EPSymp p-moment equations. Further specializations and truncations of these equations will be explored elsewhere. Before closing, we mention one or two other open questions about the solution behavior of the p-moments of EPSymp.

6. Open questions for future work

Several open questions remain for future work. The first is whether the singular solutions found here will emerge spontaneously in EPSymp dynamics from a smooth initial Vlasov PDF. This spontaneous emergence of the singular solutions does occur for EPDiff. Namely, one sees the singular solutions of EPDiff emerging from *any* confined initial distribution of the dual variable. (The dual variable is fluid velocity in the case of EPDiff). In fact, integrability of EPDiff in one dimension by the inverse scattering transform shows that *only* the singular solutions (peakons) are allowed to emerge from any confined initial distribution in that case [Camassa and Holm 1993]. In higher dimensions, numerical simulations of EPDiff show that again only the singular solutions emerge from confined initial distributions. In contrast, the point vortex solutions of Euler's fluid equations (which are isomorphic to the cold plasma singular solutions of the Vlasov Poisson equation) while comprising an invariant manifold of singular solutions, do not spontaneously emerge from smooth initial conditions in Euler fluid dynamics. Nonetheless, something quite analogous to the singular solutions is seen experimentally for cold plasma in a Malmberg–Penning trap [Dubin and O'Neil 1990]. Therefore, one may ask which outcome will prevail

for the singular solutions of EPSymp. Will they emerge from a confined smooth initial distribution, or will they only exist as an invariant manifold for special initial conditions? Of course, the interactions of these singular solutions for various metrics and the properties of their collective dynamics is a question for future work.

Geometric questions also remain to be addressed. In geometric fluid dynamics, Arnold and Khesin [1998] formulate the problem of symplectohydrodynamics, the symplectic counterpart of ordinary ideal hydrodynamics on the special diffeomorphisms SDiff. In this regard, the work of Eliashberg and Ratiu [1991] showed that dynamics on the symplectic group radically differs from ordinary hydrodynamics, mainly because the diameter of $\text{Symp}(M)$ is infinite, whenever M is a compact exact symplectic manifold with a boundary. Of course, the presence of boundaries is important in fluid dynamics. However, generalizing a result by Shnirelman [1985], Arnold and Khesin point out that the diameter of $\text{SDiff}(M)$ is finite for any compact simply connected Riemannian manifold M of dimension greater than two.

In the case under discussion here, the situation again differs from that envisioned by Eliashberg and Ratiu. The EPSymp Hamiltonian (5-1) determines geodesic motion on $\text{Symp}(T^*\mathbb{R}^3)$, which may be regarded as the restriction of the $\text{Diff}(T^*\mathbb{R}^3)$ group, so that the Liouville volume is preserved. The main difference in our case is that $M = T^*\mathbb{R}^3$ is not compact, so one of the conditions for the Eliashberg–Ratiu result does not hold. Thus, one may ask, what are the geometric properties of Symp acting on a symplectic manifold which is not compact? What remarkable differences if any remain to be found between Symp and SDiff in such a situation? Another intriguing possibility is that some relation of the work here may be found with the work of Bloch et al. on integrable geodesic flows on the symplectic group [Bloch et al. 2005]. A final question of interest is whether the present work might be linked with the Lie algebra structure of the BBGKY hierarchy [Marsden et al. 1984].

Yet another interesting case occurs when the particles undergoing Vlasov dynamics are confined in a certain region of position space. In this situation, again the phase space is not compact, since the momentum may be unlimited. The dynamics on a bounded spatial domain descends from that on the unbounded cotangent bundle upon taking the p-moments of the Hamiltonian vector field. Thus, in this topological sense *p-moments and q-moments are not equivalent*. In the present work, this distinction has been ignored by assuming either homogeneous or periodic boundary conditions.

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