Integrable models of waves in shallow water

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Dedicated to Henry McKean, on the occasion of his 75th birthday

ABSTRACT. Integrable partial differential equations have been studied because of their remarkable mathematical structure ever since they were discovered in the 1960s. Some of these equations were originally derived to describe approximately the evolution of water waves as they propagate in shallow water. This paper examines how well these integrable models describe actual waves in shallow water.

1. Introduction

Zabusky and Kruskal [1965] introduced the concept of a *soliton* — a spatially localized solution of a nonlinear partial differential equation with the property that this solution always regains its initial shape and velocity after interacting with another localized disturbance. They were led to the concept of a soliton by their careful computational study of solutions of the Korteweg–de Vries (or KdV) equation,

$$\partial_t u + u\partial_x u + \partial_x^3 u = 0. \tag{1}$$

(See [Zabusky 2005] for his summary of this history.) After that breakthrough, they and their colleagues found that the KdV equation has many remarkable properties, including the property discovered by Gardner, Greene, Kruskal and Miura [Gardner et al. 1967]: *the KdV equation can be solved exactly, as an initial-value problem, starting with arbitrary initial data in a suitable space*. This discovery was revolutionary, and it drew the interest of many people. We note especially the work of Zakharov and Faddeev [1971], who showed that the KdV equation is a nontrivial example of an infinite-dimensional Hamiltonian system that is *completely integrable*. This means that under a canonical

change of variables, the original problem can be written in terms of action-angle variables, in which the action variables are constants of the motion, while the angle variables evolve according to nearly trivial ordinary differential equations (ODEs). Zakharov and Faddeev showed that GGKM's method to solve the KdV equation amounts to transforming from $u(x, t)$ into action-angle variables at the initial time, integrating the ODEs forward in time, and then transforming back to $u(x, t)$ at any later time. In this way the Korteweg-de Vries equation, (1), became the prototype of a completely integrable partial differential equation, the study of which makes up one facet of this conference.

Zabusky and Kruskal had derived (1) as an approximate model of longitudinal vibrations of a one-dimensional crystal lattice. Later they learned that Korteweg and de Vries [1895] had already derived the same equation as an approximate model of the evolution of long waves of moderate amplitude, propagating in one direction in shallow water of uniform depth. So the KdV equation, (1), is of interest to at least two communities of scientists:

- mathematicians, who are primarily interested in its extraordinary mathematical structure; and
- coastal engineers and oceanographers, who use it to make engineering and environmental decisions related to physical processes in shallow water.

The KdV equation is one of several equations that are known to be completely integrable and that also describe approximately waves in shallow water. Other well known examples include the KP equation, a two-dimensional generalization of the KdV equation due to Kadomtsev and Petviashvili [1970]:

$$\partial_x\left(\partial_t u + u\partial_x u + \alpha \cdot \partial_x^3\right) + \partial_y^2 u = 0; \tag{2}$$

an equation first studied by Boussinesq [1871]:

$$\partial_t^2 u = c^2 \partial_x^2 u + \partial_x^2(u^2) + \beta \cdot \partial_x^4 u; \tag{3}$$

and the Camassa–Holm equation [1993],

$$\partial_t m + c\partial_x u + u\partial_x m + 2m\partial_x u + \gamma \cdot \partial_x^3 u = 0, \tag{4}$$

where

$$m = u - \delta^2 \partial_x^2 u.$$

Here are four comments about these equations.

- Among all known integrable equations, this subset is particularly relevant for this conference, because of the many important contributions of Henry McKean and his coauthors to the development of the theory for (1), (3) and (4) with periodic boundary conditions. See [Constantin and McKean 1999; McKean

1977; 1978; 1981a; 1981b; McKean and van Moerbeke 1975; McKean and Trubowitz 1976; 1978].

- The coefficients in (1) are not important, because they can be scaled into $\{u, x, t\}$. But in (2), (3), (4), both the physical meaning and the mathematical structure of each equation changes, depending on the signs of $\{\alpha, \beta, \gamma\}$ respectively. Judgmental nicknames like "good Boussinesq" and "bad Boussinesq" indicate the importance of these signs.

- Equation (1) is a special case of (2), after setting $\partial_y u \equiv 0$ and neglecting a constant of integration. Equation (3) is also a special case of (2), after setting $\partial_t u = -c^2 \partial_x u$, rescaling $u \to 2u$, and then interpreting y in (2) as the time-like variable.

- The KP equation is degenerate: for example, it does not have a unique solution. If $u(x, y, 0) \equiv 0$ at $t = 0$, then both $u = 0$ and $u = t$ solve (2) and satisfy this initial condition. In this paper, we remove this degeneracy by requiring that any solution of (2) also satisfy

$$\int_{-\infty}^{\infty} u(x, y, t) \, dx = 0. \tag{5}$$

Every completely integrable equation possesses extraordinary mathematical structure. Each of the equations listed above is completely integrable, and also describes (approximately) waves in shallow water. This paper addresses the question: *Does this extra mathematical structure provide useful information about the behavior of actual, physical waves in shallow water?*

The outline of the rest of this paper is as follows. Section 2 reviews the derivation of (1)–(4) as approximate models for waves in shallow water. Sections 3, 4, 5 all discuss applications of these approximate models to practical problems involving ocean waves. Section 3 discusses an application of (1) on the whole line (or of (2) on the whole plane): the tsunami of December 26, 2004. Section 4 focusses on spatially periodic, travelling wave solutions of (2), and their relation to periodic travelling waves in shallow water. Section 5 relates doubly periodic waves to the phenomenon of rip currents. Finally Section 6 discusses more complicated, quasiperiodic solutions of (2).

2. Derivation of integrable models from the problem of inviscid water waves

The mathematical theory of water waves goes back at least to Stokes [1847], who first wrote down the equations for the motion of an incompressible, inviscid fluid, subject to a constant (vertical) gravitational force, where the fluid is bounded below by an impermeable bottom and above by a free surface. In the
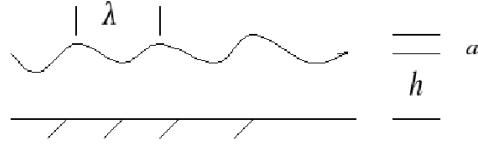
Figure 1. The integrable models of waves in shallow water in (1)–(4) all depend on an ordering of the scales shown here. A fourth scale, the typical length scale out of the page, is not shown.

discussion that follows, we assume that the bottom of the fluid is strictly horizontal (at $z = -h$). In addition to gravity, we also include the effect of surface tension, because including it provides the extra freedom needed to change the signs of $\{\alpha, \beta, \gamma\}$ in (2), (3) and (4).

Without viscosity, we may consider purely irrotational motions. Then the fluid velocity can be written in terms of a velocity potential,

$$\vec{u} = \nabla\phi,$$

and the velocity potential satisfies

$$\nabla^2\phi = 0 \qquad\qquad \text{for } -h < z < \zeta(x, y, t),$$

$$\partial_z\phi = 0 \qquad\qquad \text{on } z = -h,$$

$$\partial_t\zeta + \partial_x\zeta \cdot \partial_x\phi + \partial_y\zeta \cdot \partial_y\phi = \partial_z\phi \qquad\qquad \text{on } z = \zeta(x, y, t),$$

$$\partial_t\phi + \tfrac{1}{2}|\nabla\phi|^2 + g\zeta = T\frac{\partial_x^2\zeta(1+(\partial_y\zeta)^2)+\partial_y^2\zeta(1+(\partial_x\zeta)^2)-2\partial_x\partial_y\zeta\cdot\partial_x\zeta\partial_y\zeta}{(1+(\partial_x\zeta)^2+(\partial_y\zeta)^2)^{3/2}}$$
$$\text{on } z = \zeta(x, y, t), \qquad (6)$$

where g is the acceleration due to gravity, T represents surface tension, and $z = \zeta(x, y, t)$ is the instantaneous location of the free surface.

These equations, known for more than 150 years, are still too difficult to solve in any general sense. Even well-posedness (for short times) was established only recently; see [Wu 1999; Lannes 2005; Coutand and Shkoller 2005]. Progress has been made by focusing on specific limits, in which the equations simplify. The limit of interest here can be stated in terms of length scales, which must be arranged in a certain order. Three of the four relevant lengths are shown in Figure 1. The derivation of either KdV or KP from (6) is based on four assumptions:

(a) long waves (or shallow water), $h \ll \lambda$;
(b) small amplitude; $a \ll h$;
(c) the waves move primarily in one direction;

(d) All these small effects are comparable in size; for KdV, this means

$$\varepsilon = \frac{a}{h} = O\left(\left(\frac{h}{\lambda}\right)^2\right). \qquad (7)$$

If (c) is exactly true, the derivation leads to the KdV equation, (1) If it is approximately true, the derivation can lead to the KP equation, (2).

As we discuss below, assumptions (a)–(d) are also implicit in (3) and (4). One imposes these assumptions on both the velocity potential, $\phi(x, y, z, t)$, and the location of the free surface, $\zeta(x, y, t)$, in (6). (See [Ablowitz and Segur 1981, §4.1] or [Johnson 1997, §3.2] for details.) At leading order ($\varepsilon = 0$ in a formal expansion), the waves in question are infinitely long, infinitesimally small, and the motion at the free surface is exactly one-dimensional. The result is the one-dimensional wave equation:

$$\partial_t^2 \zeta = c^2 \partial_x^2 \zeta, \quad \text{with } c^2 = gh. \qquad (8)$$

Inserting D'Alembert's solution of (8) back into the expansion for $\zeta(x, y, t; \varepsilon)$ yields

$$\zeta(x, y, t; \varepsilon) = \varepsilon h\big(F(x - ct; y, \varepsilon t) + G(x + ct; y, \varepsilon t)\big) + O(\varepsilon^2), \qquad (9)$$

where F and G are determined from the initial data. At leading order, F and G are required only to be bounded, and to be smooth enough for the terms in (6) to make sense.

There are two ways to proceed to the next order. The simpler but more restricted method, used by Johnson [1997], is to ignore one of the two waves in (9). Then we may follow (for example) the F-wave by changing to a coordinate system that moves with that wave, at speed $\sqrt{gh}$. To do so, set

$$\xi = \frac{\sqrt{\varepsilon}}{h}\left(x - t\sqrt{gh}\right). \qquad (10a)$$

At leading order, according to (9), F does not change in this coordinate system, so we may proceed to the next order, $O(\varepsilon^2)$. Now small effects that were ignored at leading order — namely, that the wave amplitude is small but not infinitesimal, that the wavelength is long but not infinitely long, and that slow transverse variations are allowed — can be observed. These small effects can build up over a long distance, to produce a significant cumulative change in F. To capture this slow evolution of F, introduce a slow time scale,

$$\tau = \varepsilon t \sqrt{\frac{\varepsilon g}{h}}, \qquad (10b)$$

and find that F satisfies approximately the KdV equation,

$$2\partial_\tau F + 3F\partial_\xi F + \left(\frac{1}{3} - \frac{T}{gh^2}\right)\partial_\xi^3 F = 0, \tag{11}$$

if the surface waves are strictly one-dimensional (that is, if $\partial_y F \equiv 0$). Or, if the surface patterns are weakly two-dimensional, we can obtain instead the KP equation,

$$\partial_\xi\left(2\partial_\tau F + 3F\partial_\xi F + \left(\frac{1}{3} - \frac{T}{gh^2}\right)\partial_\xi^3 F\right) + \partial_\eta^2 F = 0. \tag{12}$$

After rescaling the variables in (11) or (12) to absorb constants, (11) becomes (1), and (12) becomes (2). The dimensionless parameter, T/gh^2, which appears in both (11) and (12), is the inverse of the Bond number — its value determines the relative strength of gravity and surface tension. The magnitudes and signs of all coefficients in (11) can be scaled out of the problem, but this is not possible in (12) with any real-valued scaling. The sign of $\frac{1}{3} - T/gh^2$ determines the sign of α in (2), of β in (3) and of γ in (4).

In words, (9) says that one wave (F) propagates to the right, while another wave (G) propagates to the left, both with speed $\sqrt{gh}$. Neither wave changes shape as it propagates, during the short time when (9) is valid without correction. On a longer time-scale, the KdV equation (11) describes how F changes slowly, due to weak nonlinearity ($F\partial_\xi F$) and weak dispersion ($\partial_\xi^3 F$). Or, the KP equation (12) shows how F changes because of these two weak effects and also because of weak two-dimensionality ($\partial_\eta^2 F$).

The KdV and KP equations have been derived in many physical contexts, and they always have the same physical meaning: on a short time-scale, the leading-order equation is the one-dimensional, linear wave equation; on a longer time-scale, each of the two free waves that make up the solution of the 1-D wave equation satisfies its own KdV (or KP) equation, so each of the two waves changes slowly because of the cumulative effect of weak nonlinearity, weak dispersion and (for KP) weak two-dimensionality.

The Boussinesq equation, (3), describes approximately the evolution of water waves under the same assumptions as KdV. It is the basis for several numerical codes to model wave propagation in shallow water; see, for instance, [Wei et al. 1995; Bona et al. 2002; 2004; Madsen et al. 2002]. Equation (3) appears to be more general than (1) because (3) allows waves to propagate in two directions, as (1) does not. But Bona et al. [2002] note that the usual derivation of (3) from (6) includes an assumption that the waves are propagating primarily in one direction, so (1) and (3) are formally equivalent.

Conceptually, the Camassa–Holm equation, (4), is based on the same set of assumptions as (1) and (3). But Johnson [2002] shows that the usual derivations

of (4) are logically inconsistent. He then gives a self-consistent derivation of (4) from (6), but the price he pays is that the solution of (4) does not approximate the shape of the free surface — an additional step is needed. The shape of the free surface is an easy quantity to measure experimentally, so this extra cost is not trivial.

The next three sections of this paper compare predictions of (1) and (2) with experimental observations of waves in shallow water.

3. Application: the tsunami of 2004

A dramatic example of a long ocean wave of small amplitude was the tsunami that occurred in the Indian Ocean on December 26, 2004. The tsunami caused terrible destruction in many coastal regions around the Indian Ocean, killing more than 200,000 people. Even so, we show next that until the tsunami neared the shoreline, it was well approximated by the theory leading to (8) and (9). We also show that the nonlinear integrable models, (1)–(4), were *not* relevant for the 2004 tsunami.

The tsunami was generated by a strong, undersea earthquake off the coast of Sumatra. Figure 2 shows a map of the northern Indian Ocean, and the initial shape of the tsunami. (This figure is the first image in an informative animation of the tsunami's propagation, done by Kenji Satake of Japan. To see his entire animation, go to http://staff.aist.go.jp/kenji.satake/animation.html. A comparable simulation by S. Ward can be found at http://www.es.ucsc.edu/~ward.) The fault line of the earthquake, clearly evident in Figure 2, lies on or near the boundary of two tectonic plates, one that carries India and one that holds Burma (Myanmar). Most of the seismic activity in this region occurs because the India plate is slowly sliding beneath the Burma plate.

The original earthquake (near Sumatra) triggered a series of other quakes, which occurred along this fault line, all within about 10 minutes. The north-south distance along this line of quakes was about 900 km. As Figure 2 shows, the effect of the quakes was to *raise* the ocean floor to the west of this (curved) line, and to *lower* it to the east of the line. The lateral extent of this change in the sea floor was about 100 km, on each side of the fault line. The vertical displacement was 1–2 meters or less. The change in level of the sea floor occurred quickly enough that the water above it simply rose (or fell) with the sea floor. These conditions provided the initial conditions for the tsunami. (The estimates quoted here were given by S. Ward. None of the conclusions drawn in this section changes qualitatively if any of these estimates is changed by a factor of 2.)

The ocean depth in the Bay of Bengal (the part of the Indian Ocean west and north of the fault line in Figure 2) is about 3 km. The region east of the fault

line, called the Andaman Sea, is shallower: it average depth is about 1 km.

These estimates provide enough information to consider the theory summarized in Section 2. In the Bay of Bengal, the requirements for KdV theory at leading order are:

• *small amplitude*:
$$\frac{a}{h} = \frac{1}{3000} = 3.3 \cdot 10^{-4} \ll 1;$$

• *long waves*:
$$\left(\frac{h}{\lambda}\right)^2 = \left(\frac{3}{100}\right)^2 = 9 \cdot 10^{-4} \ll 1;$$

• *nearly 1-D surface patterns*:
$$\frac{\lambda}{L} = \frac{100}{900} \ll 1;$$

• *comparable scales*:
$$\varepsilon = \frac{a}{h} = O\left(\left(\frac{h}{\lambda}\right)^2\right).$$

These estimates show that the theory for long waves of small amplitude should work well for the tsunami that propagated westward, across the Bay of Bengal. The reader can verify that this conclusion also holds for the eastward-propagating tsunami, in the Andaman Sea. Therefore, the tsunami in the Bay of Bengal propagated with a speed ($\sqrt{gh}$) of about 620 km/hr, while the speed in the Andaman Sea was about 360 km/hr.

(The analysis that follows also assumes constant ocean depth. This is the weakest assumption in the analysis, but it is easily corrected; see [Segur 2007].) We may take the initial shape of the wave to be that shown in Figure 2, with no vertical motion initially. If we neglect variations along the fault line, then according to (8), a wave with this shape and half of its amplitude propagated to the west, and an identical wave propagated to the east. Neither wave changed its shape as it propagated, so the wave propagating towards India and Sri Lanka consisted of a wave of elevation followed by a wave of depression. The wave propagating towards Thailand was the opposite: a wave of depression, followed by a wave of elevation. These conclusions are consistent with reports from survivors in those two regions.

This description applies on a short time-scale. The KdV (or KP) equation applies on the next time-scale, approximately ε^{-1} longer. Equivalently, the distance required for KdV dynamics to affect the wave forms is approximately ε^{-1} longer than a typical length scale in the problem. Using h as a typical length, this suggests that we need about 3000×3 km $= 9000$ km of propagation distance to see KdV dynamics in the westward propagating wave. But the distance across
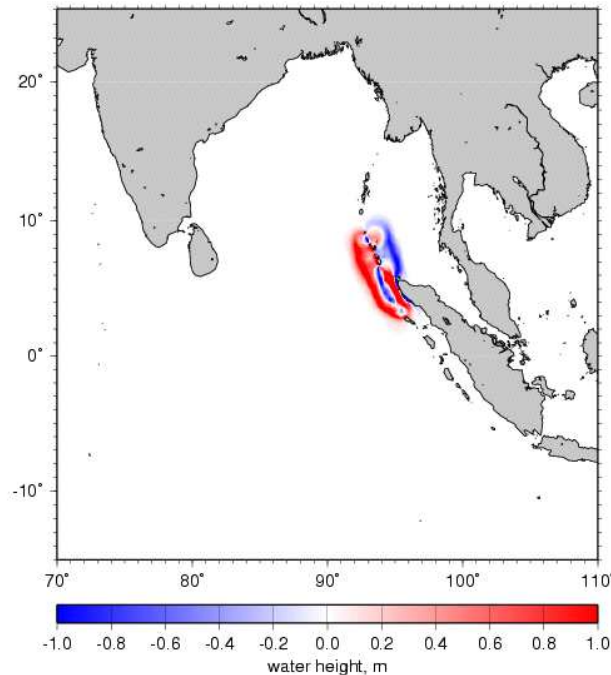
Figure 2. Map of the northern Indian Ocean, showing the shape and intensity of the initial tsunami, 10 minutes after the beginning of the first earthquake. This image is the first frame from a simulation by K. Satake; it shows an elevated water surface west of the island chain and a depressed water surface east of that chain. (For the color coding see this article online at http://www.msri.org/publications/books; Satake's animation can be found at http://staff.aist.go.jp/kenji.satake/animation.html.)

the Bay of Bengal is nowhere larger than about 1500 km, much too short for KdV dynamics to have built up. The numbers for the eastward propagating wave are different, but the conclusion is the same: the distance across the Andaman Sea is too short to see significant KdV dynamics. *Thus, the 2004 tsunami did not propagate far enough for either the KdV or KP equation to apply*.

This conclusion, that propagation distances for tsunamis are too short for soliton dynamics to have an important effect, applies to many tsunamis, but not all. Lakshmanan and Rajasekar [2003] point out that the 1960 Chilean earthquake, the largest earthquake ever recorded (magnitude 9.6 on a Richter scale), produced a tsunami that propagated across the Pacific Ocean. It reached Hawaii after 15 hours, Japan after 22 hours, and it caused massive destruction in both places. This tsunami propagated over a long enough distance that KdV dynamics probably were relevant. For more information about this earthquake

and its tsunami, see the reference just cited, [Scott 1999], or http://neic.usgs.gov/
neis/eq_depot/world/1960_05_22_tsunami.html. More recently, a seismic event
off the Kuril Islands in 2006 sent a wave across the Pacific that damaged the
harbor in Crescent City, CA. KdV (or KP) dynamics probably were relevant for
this wave, which took more than 8 hours to cross the Pacific. For more informa-
tion about this tsunami, see http://www.usc.edu/dept/tsunamis/2005/tsunamis/
Kuril_2006/.

Back to the tsunami of 2004. The discussion presented here might leave
the reader wondering how a wave of such small amplitude, satisfying a linear
equation, could be responsible for so much damage and so much loss of life.
The answer is that the "long waves, small amplitude" model applies only away
from shore — near shore the wave changes its nature entirely. To see this change
in tsunami dynamics, imagine sitting in a boat in the middle of the Indian Ocean
when a tsunami like that in 2004 passes by. The tsunami is 1 m high, 100 km
long, and it travels at 620 km/hr. It would take about 10 minutes to pass the
boat, so in the course of 10 minutes, the boat would rise 1 meter and then fall
1 meter. Hence in the open ocean, it is difficult for a sensor at the free surface
even to detect a passing tsunami.

Near shore, everything changes. The local speed of propagation is still $\sqrt{gh}$,
but h decreases near shore, so the wave slows down. More precisely, the front
of the wave slows down — the back of the wave, still 100 km out at sea, is not
slowing down. The result is that the wave must compress horizontally, as the
back of the wave catches up with the front. But water is nearly incompressible,
so if the wave compresses horizontally, then it must grow vertically as it ap-
proaches shore. The result is that a very long wave that was barely noticeable in
the open ocean becomes shorter (horizontally), larger (vertically), and far more
destructive near shore.

See articles in *Science*, **308** (2005), 1125–1146 or in [Kundu 2007] for more
discussion of the 2004 tsunami.

4. Application: periodic ocean waves

Among ocean waves, tsunamis are anomalous. The vast majority of ocean
waves are approximately periodic, and they are generated by winds and storms
[Munk et al. 1962]. The water surface is two-dimensional, so the KP equation,
(2), is a natural place to seek solutions that might describe approximately peri-
odic waves in shallow water. Gravity dominates surface tension except for very
short waves, so we may set $\alpha = 1$ in (2).

The simplest periodic solution of (2) is a (one-dimensional) plane wave, of
the form

$$u = 12k^2 m^2 \operatorname{cn}^2 \{kx + ly + wt + \phi_0; m\} + u_0, \tag{13}$$

where cn$\{\theta; m\}$ is a Jacobian elliptic function with elliptic modulus m. If we impose (5), then the solution in (13) has four free parameters; for example $\{k, l, \phi_0, m\}$. If $l = 0$, then (13) solves the KdV equation; this solution was first discovered by Korteweg and de Vries [1895], who named it a *cnoidal wave*. Wiegel [1960] brought cnoidal waves to the attention of coastal engineers, who now use them regularly for engineering calculations. (See Chapter 2 of the *Shore Protection Manual* [SPM 1984] for this viewpoint.)

Figure 3 shows ocean waves photographed by Anna Segur near the beach in Lima, Peru. These plane, periodic waves have broad, flat troughs and narrow, sharp crests — typical of cnoidal waves with elliptic modulus near 1, and also typical of plane, periodic waves in shallow water of nearly uniform depth. It is unusual to see such a clean example of a cnoidal wave train, but that might be because few beaches are as flat as the beach in Lima.

Cnoidal waves are appealing because of their simplicity, but they are degenerate in the sense that the water surface is two-dimensional, while cnoidal waves vary only in the direction of propagation. One might wish for a wave pattern that is nontrivially periodic in two spatial directions, and that travels as a wave of permanent form in water of uniform depth.

Figure 4 shows a photograph of a wave pattern photographed by Terry Toedtemeier off the coast of Oregon. This photo can be interpreted in two different ways, each with some validity. The first interpretation is that Figure 4 shows two plane solitary waves, interacting obliquely in shallow water of nearly uniform depth. A basic rule of soliton theory is that the interaction of two solitons results in a phase shift. A phase shift is evident in Figure 4: each wave crest is shifted beyond the interaction region from where it would have been without the interaction. The KP equation admits a 2-soliton solution that looks very much like the wave pattern in Figure 4. Equivalently, one can identify this wave pattern with a 2-soliton solution of the Boussinesq equation, (3).
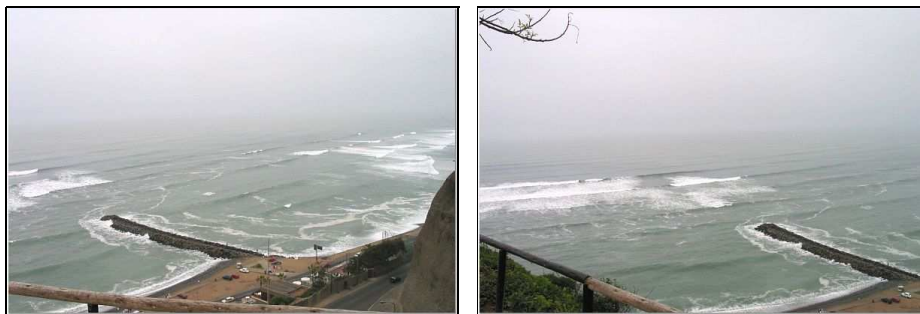

Figure 3. Periodic plane waves in shallow water, off the coast of Lima, Peru. (Photographs courtesy of A. Segur)
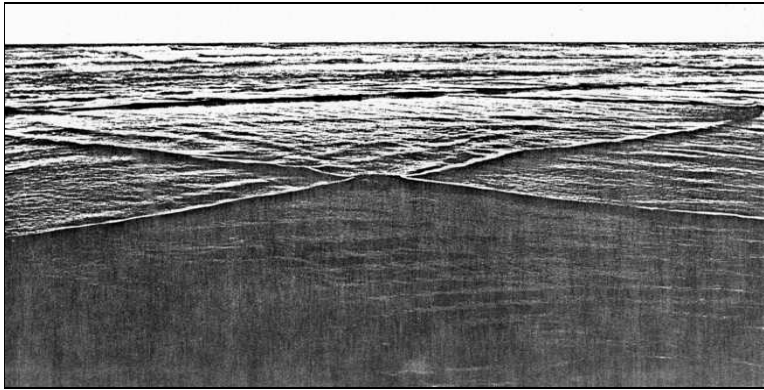
Figure 4. Oblique interaction of two shallow water waves, off the coast of Oregon. (Photograph courtesy of T. Toedtemeier)

The other interpretation is that Figure 4 shows the oblique, nonlinear interaction of two (plane) cnoidal wave trains. Each wave train exhibits the flat-trough, sharp-crest pattern seen in Figure 3, but in Figure 4 successive wave crests in the same train are so far apart that each crest acts nearly like a solitary wave. Even so, if one looks carefully at Figure 4, one can see the next crest after the prominent crest in each wave train. With this interpretation, Figure 4 shows a wave pattern that propagates with permanent form, and is nontrivially periodic in two horizontal directions. Each wave crest in one wavetrain undergoes a phase shift every time it interacts with a wave crest from the other wavetrain. The result is a two-dimensional, periodic wave pattern, in which the basic "tile" of the pattern is a hexagon: two parallel sides of the hexagon are crests from one wavetrain, two sides are crests from the other wavetrain, and the last two sides are two, short interaction regions. (Only one interaction region is evident in Figure 4.)

The KP equation admits an 8-parameter family of real-valued solutions like this — each KP solution is a travelling wave of permanent form, nontrivially periodic in two spatial directions; see [Segur and Finkel 1985] for details. In terms of Riemann surface theory, every cnoidal wave solution of KdV or KP corresponds to a Riemann surface of genus 1; each of the KP solutions considered here corresponds to a Riemann surface of genus 2. These two-dimensional, doubly periodic wave patterns are the simplest periodic or quasiperiodic solutions of the KP equation beyond cnoidal waves.

In a series of experiments, Joe Hammack and Norm Scheffner created waves in shallow water with spatial periodicity in two directions, in order to test the KP model of such waves [Hammack et al. 1989; 1995]. Figure 5 shows overhead photographs of three of their propagating wave patterns. Each wave pattern in

these photos is generated by a complicated set of paddles at one end of a long
wave tank. The photos are oriented so that the waves propagate downward in
each pair of photos. The pattern in the top photos is symmetrical, and it propa-
gates directly away from the paddles (so straight down in Figure 5). The other
two patterns are asymmetric; these patterns propagate with nearly permanent
form but not directly away from the paddles — there is also a uniform drift to the
left or right for each wave pattern. The corresponding KP solution predicts this
direction of propagation, along with the detailed shape of the two-dimensional,
doubly periodic wave pattern. Hammack et al. [1995] showed experimentally
that for each wave pattern they generated, the appropriate KP solution of genus
2 predicts the detailed shape of that pattern with remarkable accuracy. See their
paper for these comparisons.

In addition to these photographs, they also made videos of the experiments.
There was no convenient way to present those videos in 1989, but now they are
archived at the MSRI website. Go to http://phoebe.msri.org:8080/vicksburg/
vicksburg.mov to see the first video. The experiments were conducted in a
large (30 m x 56 m) wave tank at the US Army Corps of Engineers Waterways
Experiment Station, in Vicksburg, MS. A segmented wavemaker, consisting of
60 piston-type paddles that spanned the tank width, is shown in the first scene.
In this scene the paddles all move together, and they generate a train of plane,
periodic (*i.e.*, cnoidal) waves that propagate to the other end of the tank, where
they are absorbed.

In the second scene, the camera looks down on the tank from above; the
paddles are visible along the end of the tank at the upper right. The paddles were
programmed to create approximately a KP solution of genus 2, with a specific
set of choices of the free parameters. Hence the wave pattern coming off the
paddles is periodic in two spatial directions. The experiment shows that the
entire two-dimensional pattern propagates as a wave of nearly permanent form.
As in Figures 4 and 5, the basic tile of the periodic pattern is a hexagon, but
the long, straight, dominant crests seen in the video are the interaction regions,
which are quite short in Figure 4. The relatively narrow, zigzag region that
connects adjacent cells contains the other four edges of the hexagonal tile. Wave
amplitudes in the zigzag region are smaller than those of the long, dominant
crests, and the KP solution shows that horizontal velocities are smaller in this
region as well.

The relative length of the long, straight, dominant crests within a hexagonal
tile is a parameter that one can chose by choosing properly the free parameters
of the KP solution. This freedom of choice is demonstrated in the third scene
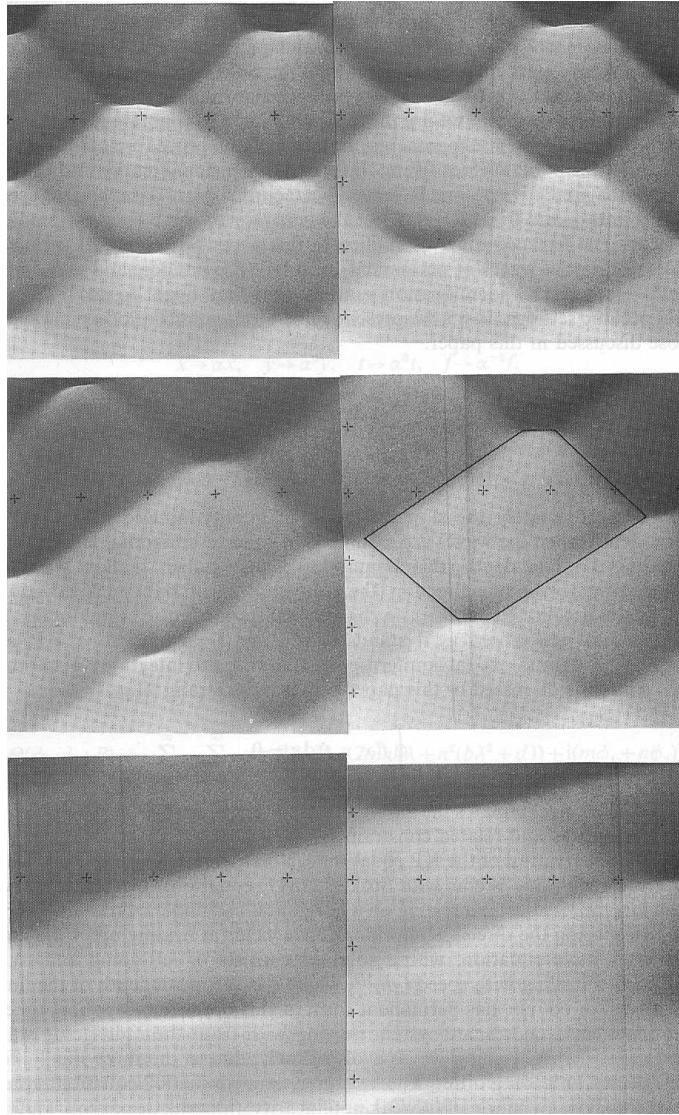in the video. In this experiment, all of the parameters of the KP solution are the

Figure 5. Mosaics of two overhead photos, showing three surface patterns of travelling waves of nearly permanent form, with periodicity in two spatial directions, in shallow water of uniform depth. The basic tile of each pattern is a hexagon; one hexagon is drawn in the middle photos. Each wave pattern is generated by a set of paddles, above the top of each pair of photos. The wave pattern propagates away from the paddles, so downward in these photos. The waves are illuminated by a light that shines towards the paddles, so a bright region identifies the front of a wave crest, a dark region lies behind the crest, and a sharp transition from bright to dark represents a steep wave crest. (Figure taken from [Hammack et al. 1995].)

same as those in the previous experiment, except that the length of the dominant crest is shorter.

Almost all real-valued genus 2 solutions of the KP equation define wave patterns like this: travelling waves of permanent form that are periodic in two spatial directions. The existence of these KP solutions does not guarantee that such solutions are stable within the KP equation, or that the corresponding water waves are stable. The video shows that these water waves look stable. See Section 6 for more discussion of stability.

Wiegel [1960] pointed out the practical, engineering value of KdV (or KP) solutions of genus 1, and the experiments shown here demonstrate the practical value of KP solutions of genus 2 — both sets of KP solutions describe accurately waves of nearly permanent form in shallow water of uniform depth. See Section 6 for a discussion of KP solutions of higher genus.

We end this discussion of spatially periodic waves of permanent form by noting the work of Craig and Nicholls [2000; 2002]. Motivated in part by the experimental results shown here, these authors proved directly that the equations of inviscid water waves, (6), admit travelling wave solutions that are spatially periodic in two horizontal directions, like those shown here. The parameter range of their family of solutions is not identical with the parameter range of KP solutions of genus 2, but the two overlap. (The KP equation approximates water waves only in shallow water, but Craig and Nicholls find solutions in water of any depth. In the other direction, they have not yet found asymmetric solutions, like those shown in the middle and bottom photos of Figure 5.) One value of an approximate model, like KP, is that it provides hypotheses about what might be true in the unapproximated problem. The success of Craig and Nicholls demonstrates how effective that strategy has been in this particular problem.

5. Application: rip currents

The material in this section can be considered an application of an application. The KP equation predicts the existence of spatially periodic wave patterns of permanent form, approximately like those shown in Figure 5, in shallow water of uniform depth. As these waves propagate into a region near shore where the water depth decreases (to zero at the shoreline), the KP equation no longer applies. But the waves themselves persist, and their behavior near shore can have important practical consequences, as we discuss next.

A rip current is a narrow jet that forms in shallow water near shore under certain circumstances. It carries water away from shore, through the "surf zone" (the region of breaking waves), out to deeper water. A typical rip current flows directly away from the shoreline, and it remains a strong, narrow jet through

Figure 6. Rip currents on two beaches on the Pacific coast. Left: Rosarita Beach in Baja California, Mexico. Starting at the lower left, the dark region shows vegetation, the white strip above it is sandy beach, the shoreline runs from upper left to lower right, the white water beyond that is the surf zone, with deeper water at the upper right. Three separated rip currents are shown. Right: Sand City, California. The land is to the lower left, the Pacific Ocean is to the upper right, with a white surf zone in between. Here the entire coastline is filled with an approximately periodic array of rip currents.

the surf zone. Beyond the surf zone, the current "blossoms" into a wider flow and loses its strength. Figure 6, left, shows three rip currents, while the right-hand part shows an approximately periodic array of rip currents all along the coastline. Rip currents can be dangerous because even a good swimmer cannot overcome the high flow rate in a strong rip current. As a result, every year rip currents carry swimmers out to deep water, where they drown.

What causes rip currents? They form in the presence of breaking waves, but breaking waves alone do not guarantee their formation. Some rip currents are stationary, while others migrate slowly along the beach. Some persist for a few hours, while others last longer.

There is a standard explanation for how rip currents form, which can be found at http://www.ripcurrents.noaa.gov/science.shtml and elsewhere. According to this explanation, rip currents require a long sandbar, parallel to the beach and just beyond the surf zone. Incoming waves break in the surf zone, and then a return flow carries that water back out to sea. Where can the return flow go? The easiest place for the return flow to carry water past the sandbar is where the height of the sandbar has a local minimum. So the return flow goes through this (initially small) pass. In doing so, its flow scours out more sand at the local minimum, which makes the height even lower there. Then more water can go through, and a feedback loop carves a larger hole in the sandbar. As it carves this hole, the return flow strengthens until it forms into a narrow jet — a rip current.

The rip current is located at the hole in the sandbar, and the width of the current is the width of the hole.

This is a sensible explanation, and it is probably correct sometimes. But the left half of Figure 6 shows three rip currents, with some spacing between them. What determines their spacing? The right half shows a long array of rip currents, with an approximately constant spacing between adjacent rips. What determines their spacing? Not all rip currents appear in these approximately periodic arrays, but they often do. And the standard explanation, summarized above, provides no insight into why rips appear in these regular arrays, and no way to predict the spacing between adjacent rips. Separately, this explanation seems to imply that a rip current cannot migrate slowly along the beach, even though some do.

An alternative mechanism to create rip currents was proposed by Hammack et al. [1991]. It requires no sand bars. Recall the doubly periodic wave patterns from their movie, which travel with nearly permanent form (see http:// phoebe.msri.org:8080/vicksburg/vicksburg.mov). What would happen to such a spatially periodic wave pattern, as it traveled up a sloping beach? It is easy to imagine that the large, dominant wave crests would break first, while the smaller crests in the narrow zigzag region would break later, or not at all. After the waves break, a return flow must carry that water back out to deep water. Where can the return flow go? The return flow is likely to go where the incoming flow is the weakest. Where is that?

[Once waves break, the KP equation no longer applies. The next two paragraphs, therefore, define a conjecture, with little mathematical justification at this time.] The derivation of either (1) of (2) from (6) shows that the horizontal velocity of the water is proportional to the wave height, so the strongest horizontal flow occurs where the waves are highest. For the wave patterns shown in the video, therefore, the long, dominant wave crests are also regions of large forward velocity, where a return flow would be resisted by a strong incoming flow. In the narrow zigzag regions, wave heights are smaller, horizontal flows are also smaller, and here the return flow would meet less resistance.

If the return flow travelled through the narrow zigzag region of the incoming flow field, then the return flow would acquire a spatial structure determined by the structure of the incoming waves. Specifically, suppose the incoming wave pattern were one of the hexagonal patterns seen in the previous video. Then:

- The return flow would appear in narrow jets (*i.e.*, rip currents) because incoming flow contains narrow zigzag regions, where the incoming velocities are smaller.
- These narrow jets would be periodically arranged along the beach because the incoming wave patterns are periodic in the direction along the beach.

- The spacing between adjacent rips would be determined by the spacing between adjacent zigzag regions in the incoming waves.
- The width of the jets would be related to the width of the zigzag region in the incoming wave pattern.

Hammack et al. [1991] tested this conjecture with another set of experiments, after some modifications of their wave tank. For the experiments on rip currents, the set of paddles was moved from the end of the tank to the side, so the waves now propagate across the tank instead of along it. In addition, they installed a sloping beach in the tank. The result was that the waves propagated across a region of uniform depth, and then up a uniformly sloping beach. The beach was made of concrete, so there were no sandbars.

The experiments can be viewed in another video, which is available at http://phoebe.msri.org:8080/ripcurrent84/ripcurrent84.mov. As the video opens, one sees dry beach along the bottom of the screen, with quiescent water higher up the screen. A strip of dye (food coloring) has been poured along the water's edge, to mark where the return flow goes. As the camera scans the entire tank, one sees the array of paddles on the far side of the tank. These paddles then create a set of hexagonally shaped, periodic waves of nearly permanent form, which propagate away from the paddles and towards the beach. As the hexagonal wave patterns climb the beach, their crests begin to break. Then a few seconds later, the dye begins to move from the shoreline, away from the beach, in a narrow jet. The jet is clearly in the zigzag region of the incoming flow, because we see the dye zigzagging out to deep water in the video. After the experiment has run for a while, the dye shows the spatial pattern of the rip currents: a periodic array of jets that remain narrow through the surf zone, and then spread out and stop beyond the surf zone. Both the spacing between jets and the width of an individual jet are determined by the spatial structure of the incoming wave pattern. Sand bars are irrelevant for these rip currents.

(In addition to the array of narrow rip currents marked by dye at the end of the experiment, one also sees a weaker, roughly circular glob of dye between each pair of rip currents. This secondary flow of dye might be due to undertow, which exists because the bottom boundary layer of the incoming waves is another place where the incoming flow is weaker. But this is a separate conjecture, with little experimental support at this time.)

6. Quasiperiodic solutions of the KP equation

Krichever [1977a] showed that the KP equation, (2), admits a large family of quasiperiodic solutions of the form

$$u(x, y, t) = 12\partial_x^2 \ln \theta, \tag{14}$$

where θ is a Riemann theta-function, associated with a (compact, connected) Riemann surface of some finite genus. Such a formulation was already known for KdV: (13) with $u_0 = 0$ can be written in this form, with genus 1. The Riemann surface is necessarily hyperelliptic (so only square-root branch points are allowed) for KdV, but Krichever [1976; 1977b] proved that any Riemann surface would do for KP.

As described in detail in [Dubrovin 1981; Belokolos et al. 1994], a Riemann theta function with g phases is defined by a g-fold Fourier series; the coefficients in this series are defined by a $g \times g$ Riemann matrix. If one starts with a Riemann surface of genus g, then by a standard procedure one generates a $g \times g$ Riemann matrix associated with that surface. Krichever showed that (14), with the theta function obtained in this way, solves the KP equation. Then S. P. Novikov conjectured that the connection between the KP equation and Riemann surfaces is even stronger, and that (14) solves KP *only* if its theta function is associated with some Riemann surface. In other words, out of all possible theta functions, of any finite genus, the KP equation can identify those associated with some compact Riemann surface. The conjecture seems remarkable, but Shiota [1986] proved it, after earlier work by Mulase [1984] and by Arbarello and de Concini [1984]. Thus the KP equation, (2), can be studied from at least three perspectives:

- as a completely integrable partial differential equation;
- as an approximate model of waves in shallow water; and
- because of its deep connection with the theory of Riemann surfaces.

An objective of this paper is to relate the extra mathematical structure of the KP equation to the behavior of physical waves in shallow water, if possible. The theory for the KP equation is much less developed than the corresponding theory for the KdV equation, especially for the quasiperiodic KP solutions given by (14). This final section summarizes our current knowledge of four aspects of these solutions: (a) the qualitative nature of quasiperiodic KP solutions; (b) effective methods to construct KP solutions of a given genus; (c) solving the initial-value problem; and (d) stability of quasiperiodic KP solutions.

(a) the qualitative nature of quasiperiodic KP solutions. Here is a summary of what is known about bounded, real-valued, KP and KdV solutions of various genera. (See [Dubrovin 1981] for details.)

$g = 1$: Any KP solution of genus 1 is a cnoidal wave — a plane wave that travels with permanent form, given by (13). The cnoidal wave solutions of KdV are special cases of this, with $l = 0$ in (13).

$g = 2$: A KP solution of genus 2 has two phase variables: $\{k_j x + l_j y + w_j t + \phi_j\}$ for $j = 1, 2$. There are two possibilities.

- If $k_1 l_2 = k_2 l_1$, then all lines of constant phase are parallel, so the spatial pattern of the wave is always one-dimensional. Any such solution of KP can be transformed ("rotated") into a solution of KdV, also of genus 2. These KdV solutions are necessarily time-dependent, in any Galilean coordinate system.

- Otherwise $k_1 l_2 \neq k_2 l_1$, and the solution is spatially periodic — the basic tile of the pattern is a hexagon, as discussed in Section 4. The wave patterns shown in Figure 5 approximate KP solutions of genus 2. Each such solution travels as a wave of permanent form in an appropriately translating coordinate system. Almost all KP solutions of genus 2 have $k_1 l_2 \neq k_2 l_1$, so they are travelling waves of permanent form.

$g \geq 3$: Almost all KP solutions of genus 3 or higher are time-dependent, in every Galilean coordinate system. Hence these KP solutions can describe physical processes that are time-dependent, including energy transfer among modes. Because of this nontrivial time-dependence, snapshots at a particular time, like those in Figure 5, are inadequate to view these solutions.

$g \geq 3$: For $g \geq 3$, the only KP solutions that are waves of permanent form are those that also solve the Boussinesq equation, (3). For $g = 3$, the KP equation admits a 12-parameter family of bounded, real-valued, quasiperiodic solutions, each with three independent phases. The Boussinesq solutions with $g = 3$ comprise an 11-parameter subfamily, so almost all KP solutions with $g = 3$ are intrinsically time-dependent. Even so, this 11-parameter sub-family is much larger than the 8-parameter family of KP solutions of genus 2. Which of these solutions are stable, and in what sense, are open questions.

$g \to \infty$: The development of "finite-gap" solutions of the KdV equation by Novikov [1974], Lax [1975], McKean and van Moerbeke [1975] and others can be considered a nonlinear generalization of a finite Fourier series (which contains only a finite number of terms). Each such finite-gap solution of KdV is based on a hyperelliptic Riemann surface of finite genus. The genus determines the number of open gaps in the spectrum of Hill's equation; it corresponds to the number of terms in a finite Fourier series. McKean and Trubowitz [1976] made this correspondence legitimate, by developing a theory of hyperelliptic curves with infinitely many branch points. In this context, one can discuss convergence of a sequence of finite-gap solutions of KdV, as the number of gaps (and the genus of the Riemann surface) increases without bound.

$g \to \infty$: When we switch from the KdV to the KP equation, we also switch from hyperelliptic to general Riemann surfaces, and things become more complicated. The recent book by Feldman et al. [2003] (see *Bull. Amer. Math. Soc.* **42** (2004), 79–87 for McKean's review) explores Riemann surfaces of infinite

genus, generalizing [McKean and Trubowitz 1976]. When KP solutions of finite genus are understood well enough to consider questions of convergence, one can hope that this recent work will provide a suitable framework in which to address such questions.

$g \geq 3$: Back to finite genus. Time-dependent or not, KP solutions with $g \geq 3$ are typically not periodic in space, but only quasiperiodic.

The fact that KP solutions of higher genus are typically only quasiperiodic in space or time is worth discussing. Physically, it is common to see water waves that are approximately periodic, but truly periodic water waves seem to be rare. In this sense, a mathematical model that naturally produces quasiperiodic solutions is an advantage. In terms of scientific computations, people often use periodic boundary conditions, not because the physical problem is periodic but for computational simplicity. How to build numerical codes that compute efficiently in a space of almost periodic functions seems to be an open problem. In terms of mathematical theory, Dubrovin et al. [1976] developed a theory to construct solutions of the KdV equation that need be only almost periodic. Their task was simpler than for the corresponding problem for KP, because the KdV equation only allows hyperelliptic Riemann surfaces, which are better understood than general Riemann surfaces.

(b) effective methods to construct KP solutions of a given (finite) genus. The method of inverse scattering, to solve the initial-value problem for a completely integrable evolution equation, typically has three parts: map the initial data into scattering data, evolve the scattering data forward in time, and then map back. For solutions of the form (14), one can identify the "scattering data" with the Riemann surface plus a divisor on that surface. Hence, one part of the method of inverse scattering is to produce an explicit Riemann theta function from these scattering data. This procedure is carried out for KP and several other integrable problems in [Belokolos et al. 1994].

Earlier, Bobenko and Bordag [1989] started with different "scattering data", and demonstrated that their method is effective by producing KP solutions. In principle their method can generate solutions of any genus, and they exhibit a solution of genus 4, among others.

Any method that uses the underlying Riemann surface as scattering data faces inherent difficulties related to our inadequate knowledge of Riemann surface theory. A Riemann surface can be defined by an algebraic curve: a polynomial relation of finite degree between two complex variables, $P(w, z) = 0$. But this relation might have singularities, where $\partial P(w, z)/\partial w = 0$ and $\partial P(w, z)/\partial z = 0$ simultaneously. One does not obtain a Riemann surface until all such singularities are resolved. Separately, a given Riemann surface can have more than one

such representation, and it can be difficult to tell whether two such polynomial relations represent the same surface.

Consequently it has been necessary to build computational machinery, to make the abstract theory of Riemann surfaces concrete and effective. See [Deconinck and van Hoeij 2001; Deconinck et al. 2004] for some of this machinery. At this time, computing the ingredients in a Riemann theta function from a representation of its algebraic curve is still not straightforward.

Dubrovin [1981] proposed another approach, which is effective for genus 1, 2 or 3, and only for them. He observed that for these low genera, *any* Riemann matrix that is "irreducible" can be associated with some Riemann surface. Hence one can skip the Riemann surface altogether, and work directly with the Riemann matrix. The papers [Segur and Finkel 1985; Dubrovin et al. 1997] were both based on this approach. These authors demonstrated the effectiveness of their method not only with example solutions, but also with publicly available computer codes that allow an interested reader to compute and to visualize real-valued KP solutions of genus 1, 2 or 3. The limitation of their method is that it fails for any genus larger than 3.

(c) solving the initial-value problem. Let us focus on two published methods to solve the KP equation as an initial-value problem, starting with either periodic or quasiperiodic initial data: by [Krichever 1989; Deconinck and Segur 1998]. Both methods rely on (14) to describe KP solutions of some finite genus.

Krichever requires that the initial data be periodic in x and in y, with fixed periods in each direction (i.e., in a fixed rectangle). Any KP solution that evolves from these initial data then retains that periodicity. He establishes the formal existence of a sequence of KP solutions, each of finite genus and in the form (14), which provide better and better approximations to the given initial data at $t = 0$. An important accomplishment in this work is his approximation theorem, which shows that these finite-genus solutions are dense in a suitable space of KP solutions with the given periods in x and in y.

Our approach in [Deconinck and Segur 1998] differed from that of Krichever in several respects. We considered initial data that are quasiperiodic in space, rather than requiring strict periodicity in x and in y. Even at genus 2, requiring that waves be periodic in x and in y is overly restrictive. Mathematically, the family of real-valued KP solutions of genus 2 that are periodic in x and in y has 5 free parameters, while the full family of real-valued KP solutions of genus 2 has 8. Physically, all three patterns of water waves photographed in Figure 5 are spatially periodic, but only the top pattern is periodic in x and in y.

We paid for the extra flexibility of allowing initial data that are quasiperiodic in space, by requiring that their initial data have the form (14), with some finite number (g) of phases. Then we gave a constructive procedure to determine

g (the number of phases and the genus of the Riemann surface), the Riemann surface itself and the divisor on that surface. Unfortunately, we have no approximation theorem, so we cannot prove that the KP solutions obtained in this way are dense in any suitable space of KP solutions.

The opinion of this writer is that more work is needed to produce a constructive method to solve the initial-value problem for the KP equation with quasiperiodic initial data.

(d) stability of quasiperiodic KP solutions. If one views the KP equation as a mathematical model of a physical system, like waves in shallow water, then the stability of its solutions is an essential piece of information about the model. The video at http://phoebe.msri.org:8080/vicksburg/vicksburg.mov shows water waves that are well approximated by KP solutions of genus 2, and that appear to be stable as they propagate. But one cannot prove stability experimentally — the video only shows that if there is an instability, then its growth rate must be slow enough that it does not appear within the test section of the tank.

At this time, almost nothing is known about the stability of quasiperiodic solutions of the KP equation. The problem is even more difficult than usual because standard numerical methods to test for stability/instability are based on codes with periodic boundary conditions, and these are not suitable for KP solutions that are only quasiperiodic. The problem seems to be completely open at this time.

Acknowledgements

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