Nonintersecting Brownian motions, integrable systems and orthogonal polynomials

PIERRE VAN MOERBEKE

To Henry, teacher and friend, with admiration and gratitude

ABSTRACT. Consider n nonintersecting Brownian motions on $\mathbb{R}$, which leave from p definite points and are forced to end up at q points at time $t = 1$. When $n \to \infty$, the equilibrium measure for these Brownian particles has its support on p intervals, for $t \sim 0$, and on q intervals, for $t \sim 1$. Hence it is clear that, when t evolves, intervals must merge, must disappear and be created, leading to various *phase transitions* between times $t = 0$ and 1.

Near these moments of phase transitions, there appears an infinite-dimensional diffusion, *a Markov cloud*, in the limit $n \nearrow \infty$, which one expects to depend only on the nature of the singularity associated with this phase change. The transition probabilities for these Markov clouds satisfy nonlinear PDE's, which are obtained from taking limits of the Brownian motion model with finite particles; the finite model is closely related to Hermitian matrix integrals, which themselves satisfy nonlinear PDE's. The latter are obtained from investigating the connection between the Karlin-McGregor formula, moment matrices, the theory of orthogonal polynomials and the associated integrable systems. Various special cases are provided to illustrate these general ideas. This is based on work by Adler and van Moerbeke.

1. Introduction

This lecture in honor of Henry McKean forms a step in the direction of understanding the behavior of nonintersecting Brownian motions on $\mathbb{R}$ (Dyson's Brownian motions), when the number of particles tends to ∞. It explains a novel interface between diffusion theory, integrable systems and the theory of orthogonal polynomials. These subjects have been at the center of Henry McKean's oeuvre. I am delighted to dedicate this paper to Henry, teacher and friend, with admiration for his pioneering work in these fields.

Consider n Brownian particles leaving from points $a_1 < \cdots < a_p$ and forced to end up at $b_1 < \cdots < b_q$ at time $t = 1$. It is clear that, when $n \to \infty$, the equilibrium measure for $t \sim 0$ has its support on p intervals and for $t \sim 1$ on q intervals. It is also clear that, when t evolves, intervals must merge, must disappear and be created, leading to various *phase transitions*, depending on the respective fraction of particles leaving from the points a_i and arriving at the points b_j. Therefore the region $\mathcal{R}$ in the space-time strip (x, t) formed by the support ($\subset \mathbb{R}$) of the equilibrium measure as a function of time $0 \le t \le 1$ will typically present singularities of different types.

Near the moments, where a phase transition takes place, one would expect to find in the limit $n \nearrow \infty$ an infinite-dimensional diffusion, *a Markov cloud*, having some universality properties. Universality here means that the infinite-dimensional diffusion is to depend on the type of singularity only. These Markov clouds are infinite-dimensional diffusions, which 'in principle' could be described by an infinite-dimensional Laplacian with a drift term. We conjecture that each of the Markov clouds obtained in this fashion is related to some *integrable system*, which enables one to derive a nonlinear (finite-dimensional) PDE, satisfied by the joint probabilities. The purpose of this lecture is to show the intimate relationship between these subjects: nonintersecting Brownian motions and integrable systems, via the theory of orthogonal polynomials. Special cases have also shown an intimate connection between the integrable system and the Riemann-Hilbert problem associated with the singularity. These ideas will then be applied to a simple model, where we show that the transition probabilities for the infinite-dimensional Brownian motions near a cusp satisfy a nonlinear PDE. The interrelations between all such equations, "initial" and "final" ($t \to \pm\infty$) conditions, are interesting and challenging open problems. Universality in this context is a largely open field. For references, see later.

2. Biorthogonal polynomials and the 2-component KP hierarchy

Consider the inner product for the weight $\rho(x, y)$ on $\mathbb{R}^2$,

$$\langle f \mid g \rangle := \iint_{\mathbb{R}^2} f(x)g(y)\rho(x, y)dxdy.$$

and an inner product for this weight, augmented with an extra-exponential factor, depending on "time" parameters $t := (t_1, t_2, \dots)$ and $s := (s_1, s_2, \dots)$,

$$\langle f \mid g \rangle_{t,s} := \iint_{\mathbb{R}^2} f(x)g(y)\rho(x, y)e^{\sum_1^\infty (t_i y^i - s_i x^i)}dxdy.$$

Construct monic biorthogonal polynomials $p_m^{(1)}(y)$ and $p_n^{(2)}(x)$ (also depending on the parameters t and s) with regard to this deformed weight,

$$\left\langle p_n^{(2)}(x)e^{-\sum_1^\infty s_i x^i} \mid p_m^{(1)}(y)e^{\sum_1^\infty t_i y^i} \right\rangle$$

$$= \iint_{\mathbb{R}^2} p_n^{(2)}(x) p_m^{(1)}(y)\rho(x, y)e^{\sum_1^\infty (t_i y^i - s_i x^i)} \, dx \, dy$$

$$= \delta_{nm} h_n,$$

and let τ_n be the determinant of the moment matrix

$$\tau_n(t, s) := \det\left(\left\langle x^k e^{-\sum_1^\infty s_i x^i} \mid y^\ell e^{\sum_1^\infty t_i y^i} \right\rangle \right)_{0 \le k, \ell \le n-1}.$$

The following theorem and its corollary, due to Adler and van Moerbeke [1997; 1999b] and inspired by Sato's theory, establishes a link between the functions τ_n and the biorthogonal polynomials:

THEOREM 2.1. *Given these data, the determinant $\tau_n(t, s)$ and the biorthogonal polynomials are related by the following relations, where we have set $[\alpha] := (\alpha, \frac{1}{2}\alpha^2, \frac{1}{3}\alpha^3, \dots)$ for $\alpha \in \mathbb{C}$:*

$$z^n \frac{\tau_n(t - [z^{-1}], s)}{\tau_n(t, s)} = p_n^{(1)}(z),$$

$$z^n \frac{\tau_n(t, s + [z^{-1}])}{\tau_n(t, s)} = p_n^{(2)}(z),$$

$$z^{-n-1} \frac{\tau_{n+1}(t + [z^{-1}], s)}{\tau_n(t, s)} = \iint_{\mathbb{R}^2} \frac{p_n^{(2)}(x)}{z - y} e^{\sum_1^\infty (t_i y^i - s_i x^i)} \rho(x, y) \, dx \, dy,$$

$$z^{-n-1} \frac{\tau_{n+1}(t, s - [z^{-1}])}{\tau_n(t, s)} = \iint_{\mathbb{R}^2} \frac{p_n^{(1)}(y)}{z - x} e^{\sum_1^\infty (t_i y^i - s_i x^i)} \rho(x, y) \, dx \, dy, \quad (2\text{-}1)$$

with the $\tau_n(t, s)$ satisfying bilinear equations, for all integers $n, m \geq 0$ and all $t, t', s, s' \in \mathbb{C}^{\infty}$:

$$\oint_{z=\infty} \tau_{n-1}(t - [z^{-1}], s)\tau_{m+1}(t' + [z^{-1}], s')e^{\sum_1^{\infty}(t_i - t_i')z^i} z^{n-m-2} dz$$

$$= \oint_{z=\infty} \tau_n(t, s - [z^{-1}])\tau_m(t', s' + [z^{-1}])e^{\sum_1^{\infty}(s_i - s_i')z^i} z^{m-n} dz.$$

Two-component KP hierarchy. Define the *Hirota symbol* between functions $f = f(t_1, t_2, \ldots)$ and $g = g(t_1, t_2, \ldots)$ by

$$p\left(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \ldots\right) f \circ g := p\left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \ldots\right) f(t + y)g(t - y)\Big|_{y=0}.$$

The elementary Schur polynomials $\boldsymbol{S}_{\ell}$ are defined by $e^{\sum_1^{\infty} t_i z^i} := \sum_{i \geq 0} \boldsymbol{S}_i(t)z^i$ for $\ell \geq 0$ and $\boldsymbol{S}_{\ell}(t) = 0$ for $\ell < 0$; moreover, set for later use

$$\boldsymbol{S}_{\ell}(\tilde{\partial}_t) := \boldsymbol{S}_{\ell}\left(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, \ldots\right).$$

Finally, recall that the Wronskian $\{f, g\}_x$ of f and g is given by

$$\frac{\partial f}{\partial x}g(x) - \frac{\partial g}{\partial x}f(x).$$

COROLLARY. *From Theorem 2.1, one deduces the equations*

$$\boldsymbol{S}_j\left(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \ldots\right)\tau_{n+1} \circ \tau_{n-1} = -\tau_n^2 \frac{\partial^2}{\partial s_1 \partial t_{j+1}} \log \tau_n,$$

$$\boldsymbol{S}_j\left(\frac{\partial}{\partial s_1}, \frac{1}{2}\frac{\partial}{\partial s_2}, \ldots\right)\tau_{n-1} \circ \tau_{n+1} = -\tau_n^2 \frac{\partial^2}{\partial t_1 \partial s_{j+1}} \log \tau_n,$$

(2-2)

and finally a single partial differential equation for τ_n in terms of Wronskians,

$$\left\{\frac{\partial^2 \log \tau_n}{\partial t_1 \partial s_2}, \frac{\partial^2 \log \tau_n}{\partial t_1 \partial s_1}\right\}_{t_1} + \left\{\frac{\partial^2 \log \tau_n}{\partial s_1 \partial t_2}, \frac{\partial^2 \log \tau_n}{\partial t_1 \partial s_1}\right\}_{s_1} = 0. \quad (2\text{-}3)$$

SKETCH OF PROOF OF THEOREM 2.1 AND ITS COROLLARY. The following double integral can be expanded in two different ways with regard to the parameters $a := (a_1, a_2, \ldots)$:

$$\tau_n(t,s)\tau_{n+1}(t',s')$$

$$\iint_{\mathbb{R}^2} dx\,dy\, p_{n+1}^{(2)}(t',s';x)\,p_n^{(1)}(t,s;y)e^{\sum_1^\infty(t_iy^i-s_i'x^i)}\rho(x,y)\bigg|_{\substack{t\mapsto t-a\\t'\mapsto t+a\\s'=s}}$$

$$= \left(\sum_{j=0}^\infty -2a_{j+1}S_j\left(\frac{\partial}{\partial t_1},\frac{1}{2}\frac{\partial}{\partial t_2},\frac{1}{3}\frac{\partial}{\partial t_3},\dots\right)\tau_{n+2}\circ\tau_n + O(a^2)\right)$$

$$= \left(\sum_{j=0}^\infty 2a_{j+1}\tau_{n+1}^2\frac{\partial^2}{\partial s_1\partial t_{j+1}}\log\tau_{n+1} + O(a^2)\right), \tag{2-4}$$

using the fact that the space $\mathcal{H} := \mathrm{span}\{z^i,\ i\in\mathbb{Z}\}$ can be equipped with two (formal) inner products:

(i) $\langle f,g\rangle = \int_{\mathbb{R}} f(z)g(z)\,dz$,

(ii) a residue pairing about $z=\infty$ between $f = \sum_{i\geq 0} a_i z^i \in \mathcal{H}^+$ and $h = \sum_{j\in\mathbb{Z}} b_j z^{-j-1} \in \mathcal{H}$:

$$\langle f,h\rangle_\infty = \oint_{z=\infty} f(z)h(z)\frac{dz}{2\pi i} = \sum_{i\geq 0} a_i b_i.$$

The two inner products are related by

$$\langle f,g\rangle = \int_{\mathbb{R}} f(z)g(z)\,dz = \left\langle f,\int_{\mathbb{R}}\frac{g(u)}{z-u}du\right\rangle_\infty.$$

Then the two expansions (2-4) are obtained, using the τ-function representation (2-1) of the biorthogonal polynomials, transforming the double integral (2-4) into a contour integral about ∞ and finally computing the residues. Upon equating the two series in (2-4) for arbitrary a_j, one finds the first identity (2-2). Application of a similar shift $s\mapsto s-a$, $s'\mapsto s+a$, $t'=t$ yields the second identity (2-2). Then combining the identities (2-2) for $j=0$ and 1 leads to the PDE (2-3). $\qquad\square$

3. Orthogonal polynomials with regard to several weights and the n-component KP hierarchy

Now considering two sets of weights,

$$\psi_1,\dots,\psi_q \quad\text{and}\quad \varphi_1,\dots,\varphi_p,$$

and deform each weight with its own set of times:

$$\psi_k^{-s}(x) := \psi_k(x)e^{-\sum_1^\infty s_{ki}x^i} \quad\text{and}\quad \varphi_k^t(y) := \varphi_k(y)e^{\sum_1^\infty t_{ki}y^i},$$

the time parameters being

$$s_k = (s_{k1}, s_{k2}, \dots) \text{ for } 1 \le k \le q \quad \text{and} \quad t_k = (t_{k1}, t_{k2}, \dots) \text{ for } 1 \le k \le p.$$

Take a moment matrix consisting of $p \times q$ blocks of sizes $m_i \times n_j$, formed of moments with regard to all the different combinations of ψ_i and φ_j's; of course for the full matrix to be a square matrix, the integers $m_1, m_2, \dots \ge 0$ and $n_1, n_2, \dots \ge 0$ must satisfy $\sum_1^q m_i = \sum_1^p n_i$. Define the determinant τ_{mn} of these moment matrices (the inner product is the same as in Section 2):

$$\tau_{m_1,\dots,m_q;n_1,\dots,n_p}(s_1, \dots, s_q; t_1, \dots, t_p) :=$$

$$\det \begin{pmatrix} \left(x^k \psi_1^{-s_1}(x) \mid y^\ell \varphi_1^{t_1}(y) \right)_{\substack{0 \le k < m_1 \\ 0 \le \ell < n_1}} & \cdots & \left(x^k \psi_1^{-s_1}(x) \mid y^\ell \varphi_p^{t_p}(y) \right)_{\substack{0 \le k < m_1 \\ 0 \le \ell < n_p}} \\ \vdots & & \vdots \\ \left(x^k \psi_q^{-s_q}(x) \mid y^\ell \varphi_1^{t_1}(y) \right)_{\substack{0 \le k < m_q \\ 0 \le \ell < n_1}} & \cdots & \left(x^k \psi_q^{-s_q}(x) \mid y^\ell \varphi_p^{t_p}(y) \right)_{\substack{0 \le k < m_q \\ 0 \le \ell < n_p}} \end{pmatrix}.$$

$$(3\text{-}1)$$

Notice that Section 1 is a special case of this situation, where $p = q = 1$. In this general setup, the analogue of Theorem 2.1 is the following statement, due to [Adler et al. 2006]. (The precise signs $\pm$, which we omit here, can be found in that reference. The symbol e_α stands for $0, \dots, 0, 1, 0, \dots$), with 1 at the α-th place. The meaning of $\tau_{mn}(t_\ell - [z^{-1}])$ is that only the t_ℓ variable gets shifted and no other, i.e., reference to the unshifted variables is omitted.)

I. The expressions

$$z^{n_\ell} \frac{\tau_{mn}(t_\ell - [z^{-1}])}{\tau_{mn}} := Q_{mn}^{(\ell\ell)}(z) = z^{n_\ell} + \cdots,$$

$$z^{n_\alpha - 1} \frac{\tau_{m,n+e_\ell - e_\alpha}(t_\alpha - [z^{-1}])}{\tau_{mn}} = Q_{mn}^{(\ell\alpha)}(z) = c_\alpha z^{n_\alpha - 1} + \cdots \quad \text{for } \alpha \ne \ell$$

are polynomials (involving $\sum_1^p n_\alpha$ coefficients), satisfying $\sum_1^q m_\alpha$ orthogonality conditions

$$\left\langle x^j \psi_\alpha^{-s}(x) \,\Big|\, \sum_{i=1}^p Q_{mn}^{(\ell i)}(y)\varphi_i^t(y) \right\rangle = 0 \quad \text{for} \quad \begin{cases} 1 \le \alpha \le q, \\ 0 \le j \le m_\alpha - 1. \end{cases}$$

II. Similarly, the expressions

$$\pm z^{m_\alpha - 1} \frac{\tau_{m-e_\alpha,n-e_\ell}(s_\alpha + [z^{-1}])}{\tau_{mn}} = P_{nm}^{(\ell\alpha)}(z) \quad \text{of degree} < m_\alpha$$

are polynomials (involving $\sum_1^q m_\alpha$ coefficients), satisfying $\sum_1^p n_\alpha$ orthogonality conditions:

$$\left\langle \sum_{i=1}^q P_{nm}^{(\ell i)}(x)\psi_i^{-s}(x) \,\middle|\, y^j \varphi_\alpha^t(y) \right\rangle = 0 \quad \text{for} \begin{cases} 1 \le \alpha \le p, \ 0 \le j \le n_\alpha - 1 \\ \text{except } \alpha = \ell, \ j = n_\ell - 1, \end{cases}$$

$$\left\langle \sum_{i=1}^q P_{nm}^{(\ell i)}(x)\psi_i^{-s}(x) \,\middle|\, y^{n_\ell - 1}\varphi_\ell^t(y) \right\rangle = 1.$$

III. The Cauchy transforms of the polynomials in II are

$$z^{-n_\ell}\frac{\tau_{mn}(t_\ell + [z^{-1}])}{\tau_{mn}} := \left\langle \sum_{i=1}^q P_{nm}^{(\ell i)}(x)\psi_i^{-s}(x) \,\middle|\, \frac{\varphi_\ell^t(y)}{z-y} \right\rangle,$$

$$\pm z^{-n_\ell - 1}\frac{\tau_{m,n+e_\ell-e_\alpha}(t_\ell + [z^{-1}])}{\tau_{mn}} = \left\langle \sum_{i=1}^q P_{nm}^{(\alpha i)}(x)\psi_i^{-s}(x) \,\middle|\, \frac{\varphi_\ell^t(y)}{z-y} \right\rangle.$$

IV. The Cauchy transforms of the polynomials in I are

$$\pm z^{-m_\alpha - 1}\,\frac{\tau_{m+e_\alpha,n+e_\ell}(s_\alpha - [z^{-1}])}{\tau_{mn}} = \left\langle \frac{\psi_\alpha^{-s}(x)}{z-x} \,\middle|\, \sum_{i=1}^p Q_{mn}^{(\ell i)}(y)\varphi_i^t(y) \right\rangle.$$

The orthogonality conditions for these polynomials lead to the following statement:

PROPOSITION 3.1. *The determinants τ_{mn} defined in* (3-1) *satisfy the $\underline{(p+q)\text{-}KP}$ hierarchy; that is,*

$$\sum_{\beta=1}^p \oint_\infty \tau_{m,n-e_\beta}(t_\beta - [z^{-1}])\tau_{m',n'+e_\beta}(t_\beta' + [z^{-1}])e^{\sum_1^\infty (t_{\beta i}-t_{\beta i}')z^i}\,z^{n_\beta - n_\beta' - 2}\,dz =$$

$$\pm \sum_{\alpha=1}^q \oint_\infty \tau_{m+e_\alpha,n}(s_\alpha - [z^{-1}])\tau_{m'-e_\alpha,n'}(s_\alpha' + [z^{-1}])e^{\sum_1^\infty (s_{\alpha i}-s_{\alpha i}')z^i}\,z^{m_\alpha' - m_\alpha - 2}\,dz,$$

where $\sum m_\alpha' = \sum n_\alpha' + 1$ and $\sum m_\alpha = \sum n_\alpha + 1$.

These polynomials happen to be the so-called multiple orthogonal polynomials of mixed type, introduced in [Daems and Kuijlaars 2007] in the context of nonintersecting Brownian motions; they generalize multiple orthogonal polynomials, introduced in [Aptekarev 1998; Aptekarev et al. 2003; Adler and van Moerbeke 1999a]. This will now be used in the next section.

4. Nonintersecting Brownian motions

If the transition density for standard Brownian motion $x(t)$ in $\mathbb{R}$, leaving from x and arriving at y, is given by

$$p(t, x, y) = \frac{1}{\sqrt{\pi t}} e^{-(x-y)^2/t},$$

then the probability that N nonintersecting Brownian motions $x_1(t), \ldots, x_N(t)$ in $\mathbb{R}$, leaving at $\alpha := (\alpha_1, \ldots, \alpha_N)$ and arriving at $\beta := (\beta_1, \ldots, \beta_N)$, belong to E at time t, is given by the Karlin–McGregor formula [1959]:

$$\int_{E^N} \det\left[p(t, \alpha_i, x_j)\right]_{1 \leq i,j \leq N} \det\left[p(1-t, x_i, \beta_j)\right]_{1 \leq i,j \leq N} \prod_{i=1}^{N} dx_i.$$

Considering the particular case where several points coincide, i.e., where

$$\alpha := a = (\overbrace{a_1, a_1, \ldots, a_1}^{m_1}, \overbrace{a_2, a_2, \ldots, a_2}^{m_2}, \ldots, \overbrace{a_q, a_q, \ldots, a_q}^{m_q}) \in \mathbb{R}^N$$

$$\beta := b = (\overbrace{b_1, b_1, \ldots, b_1}^{n_1}, \overbrace{b_2, b_2, \ldots, b_2}^{n_2}, \ldots, \overbrace{b_p, b_p, \ldots, b_p}^{n_p}) \in \mathbb{R}^N, \tag{4-1}$$

one verifies that the probability below can be expressed as a determinant of a moment matrix of the form (3-1) with $p \times q$ blocks,

$$\mathbb{P}\left(\text{all } x_i(t) \in E \,\middle|\, \begin{array}{l} (x_1(0), \ldots, x_N(0)) = \alpha \\ (x_1(1), \ldots, x_N(1)) = \beta \end{array}\right) \quad (0 < t < 1)$$

$$= \lim_{\substack{(\alpha_1, \ldots, \alpha_N) \to a \\ (\beta_1, \ldots, \beta_N) \to b}}$$

$$\frac{1}{Z_N} \int_{E^N} \det[p(t, \alpha_i, x_j)]_{1 \leq i,j \leq N} \det\left[p(1-t, x_i, \beta_j)\right]_{1 \leq i,j \leq N} \prod_{i=1}^{N} dx_i$$

$$= \frac{N!}{Z_N'} \det\left(\left(\int_{\tilde{E}} dy \, e^{-\frac{y^2}{2}} y^{i+j} e^{(\tilde{a}_\alpha + \tilde{b}_\beta)y}\right)_{\substack{0 \leq i < m_\alpha \\ 0 \leq j < n_\beta}}\right)_{\substack{1 \leq \alpha \leq q \\ 1 \leq \beta \leq p}}, \tag{4-2}$$

where

$$\tilde{E} = E\sqrt{\frac{2}{t(1-t)}}, \quad \tilde{a}_i = \sqrt{\frac{2(1-t)}{t}} a_i, \quad \tilde{b}_i = \sqrt{\frac{2t}{1-t}} b_i.$$

PROOF. It is based on the matrix identity

$$\det\left(A_{ik}\right)_{1 \leq i,k \leq n} \det\left(B_{ik}\right)_{1 \leq i,k \leq n} = \sum_{\sigma \in S_n} \det\left(A_{i,\sigma(j)} \, B_{j,\sigma(j)}\right)_{1 \leq i,j \leq n}. \qquad \square$$

Upon adding extra-time parameters

$$t_\beta = (t_{\beta,1}, t_{\beta,2}, \dots) \quad \text{and} \quad s_\alpha = (s_{\alpha,1}, s_{\alpha,2}, \dots)$$

to

$$\det\left(\left(\int_{\tilde{E}} dy \; e^{-\frac{y^2}{2}} y^{i+j} e^{(\tilde{a}_\alpha + \tilde{b}_\beta)y}\right)_{\substack{0 \le i < m_\alpha \\ 0 \le j < n_\beta}}\right)_{\substack{1 \le \alpha \le q \\ 1 \le \beta \le p}},$$

it follows automatically from Section 3 that the expression

$$\tau_{m_1,\dots,m_q;n_1,\dots,n_p}(t_1,\dots,t_p;s_1,\dots,s_q)$$

$$= \det\left(\left(\int_{\tilde{E}} dy \; e^{-\frac{y^2}{2}} y^{i+j} e^{(\tilde{a}_\alpha + \tilde{b}_\beta)y + \sum_1^\infty (t_{\beta,k} - s_{\alpha,k})y^k}\right)_{\substack{0 \le i < m_\alpha \\ 0 \le j < n_\beta}}\right)_{\substack{1 \le \alpha \le q \\ 1 \le \beta \le p}}$$

satisfies the $p + q$-KP hierarchy, where p denotes the number of starting points and q the number of end points of the Brownian motions; see (4-1). Noninter-secting Brownian motions have been studied in [Karlin and McGregor 1959; Dyson 1962; Grabiner 1999; Johansson 2001; Bleher and Kuijlaars 2004b; 2004a; Daems and Kuijlaars 2007; Tracy and Widom 2004; 2006; Adler and van Moerbeke 2005; 2006].

In the next section, I work out the example where the Brownian motions all depart from 0 and end up at the points $-a$ and a.

5. Nonintersecting Brownian motions leaving from the origin and forced to end up at two points

Consider $n = n_1 + n_2$ nonintersecting Brownian motions on $\mathbb{R}$, all leaving from the origin, with n_1 paths forced to go to $-a$ and n_2 paths forced to go to a, at time $t = 1$. The probability that all the particles belong to the set E at time $0 < t < 1$ can be expressed as a Gaussian Hermitian random matrix "with external potential", specified by the diagonal matrix

$$A := \begin{pmatrix} \alpha & & & & & \\ & \ddots & & & \mathbf{O} & \\ & & \alpha & & & \\ & & & -\alpha & & \\ & \mathbf{O} & & & \ddots & \\ & & & & & -\alpha \end{pmatrix} \begin{matrix} \updownarrow n_1 \\ \\ \updownarrow n_2 \end{matrix} \quad \text{with} \quad \alpha = a\sqrt{\frac{2t}{1-t}},$$

but also as a determinant of a moment matrix, a consequence of Section 4. This

gives (with $n = n_1 + n_2$),

$$\mathbb{P}_0^{\pm a}\left(\text{all } x_i(t) \in E \,\middle|\, \begin{array}{c} \text{all } x_j(0) = 0, \\ n_1 \text{ left paths end up at } -a \text{ at time } t = 1, \\ n_2 \text{ right paths end up at } +a \text{ at time } t = 1 \end{array}\right)$$

$$= \mathbb{P}_n\left(a\sqrt{\frac{2t}{1-t}}; E\sqrt{\frac{2}{t(1-t)}}\right), \quad (5\text{-}1)$$

with $\mathbb{P}_n$ being an integral over the space $\mathcal{H}_n(E')$ of Hermitian matrices with spectrum belonging to the set $E' \subseteq \mathbb{R}$:

$$\mathbb{P}_n(\alpha; E') := \frac{1}{Z_n}\int_{\mathcal{H}_n(E')} dM \; e^{-\operatorname{Tr}(\frac{1}{2}M^2 - AM)}$$

$$= \frac{1}{Z_n}\det\left(\begin{array}{c} \left(\displaystyle\int_{E'} z^{i+j-1}e^{-z^2/2+\alpha z}dz\right)_{\substack{1 \le i \le n_1, \\ 1 \le j \le n_1+n_2}} \\[2em] \left(\displaystyle\int_{E'} z^{i+j-1}e^{-z^2/2-\alpha z}dz\right)_{\substack{1 \le i \le n_2, \\ 1 \le j \le n_1+n_2}} \end{array}\right) \quad (5\text{-}2)$$

THEOREM 5.1 [Adler and van Moerbeke 2007]. *The log of the probability* $\mathbb{P}_n(\alpha; E)$ *satisfies a* fourth-order *PDE in* α *and in the boundary points* $b_1, \ldots, b_{2r}$ *of the set* E, *with* quartic nonlinearity:

$$\det\begin{pmatrix} F^+ & F^- & 0 \\ \mathcal{B}_{-1}F^+ & \mathcal{B}_{-1}F^- & F^-G^+ + F^+G^- \\ \mathcal{B}_{-1}^2 F^+ & \mathcal{B}_{-1}^2 F^- & F^-\mathcal{B}_{-1}G^+ + F^+\mathcal{B}_{-1}G^- \end{pmatrix} = 0, \quad (5\text{-}3)$$

where $\mathcal{B}_k := \sum_{i=1}^{2r} b_i^{k+1}\partial/\partial b_i$ *and*

$$F^+ := 2\mathcal{B}_{-1}\left(\frac{\partial}{\partial\alpha} - \mathcal{B}_{-1}\right)\log\mathbb{P}_n - 4n_1, \qquad F^- = F^+\big|_{\substack{\alpha \to -\alpha \\ n_1 \leftrightarrow n_2}}$$

$$2G^+ := \{H_1^+, F^+\}_{\mathcal{B}_{-1}} - \{H_2^+, F^+\}_{\partial/\partial\alpha}, \qquad G^- = G^+\big|_{\substack{\alpha \to -\alpha \\ n_1 \leftrightarrow n_2}},$$

with

$$H_1^+ := \frac{\partial}{\partial\alpha}\left(\mathcal{B}_0 - \alpha\frac{\partial}{\partial\alpha} - \alpha\mathcal{B}_{-1}\right)\log\mathbb{P}_n + \left(\mathcal{B}_0\mathcal{B}_{-1} + 4\frac{\partial}{\partial\alpha}\right)\log\mathbb{P}_n + 4n_1\left(\alpha + \frac{n_2}{\alpha}\right),$$

$$H_2^+ := \frac{\partial}{\partial\alpha}\left(\mathcal{B}_0 - \alpha\frac{\partial}{\partial\alpha} - \alpha\mathcal{B}_{-1}\right)\log\mathbb{P}_n - (\mathcal{B}_0 - 2\alpha\mathcal{B}_{-1} - 2)\,\mathcal{B}_{-1}\log\mathbb{P}_n.$$

SKETCH OF PROOF. In view of the results in Section 3, we add extra parameters $t_1, t_2, \ldots, s_1, s_2, \ldots$ and β to the integrals in the moment matrix above (5-2). In terms of the Vandermonde determinants $\Delta_k(x) = \prod_{1 \le i < j \le k}(x_i - x_j)$ for the

variables $x_1, \ldots, x_{n_1}$ and $\Delta_n(x, y)$ for all variables $x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2}$, we obtain from the results in Section 4 that (again with $n = n_1 + n_2$) the function

$$
\tau_{n_1 n_2}(t, s, u; \alpha, \beta; E) := \det \begin{pmatrix} (\mu_{ij}^+(t, s, \alpha, \beta, E))_{1 \leq i \leq n_1, \ 1 \leq j \leq n_1 + n_2} \\ (\mu_{ij}^-(t, u, \alpha, \beta, E))_{1 \leq i \leq n_2, \ 1 \leq j \leq n_1 + n_2} \end{pmatrix}
$$

$$
= \frac{1}{n_1! \, n_2!} \int_{E^n} \Delta_n(x, y) \prod_{j=1}^{n_1} e^{\sum_1^\infty t_i x_j^i} \prod_{j=1}^{n_2} e^{\sum_1^\infty t_i y_j^i}
$$

$$
\times \left(\Delta_{n_1}(x) \prod_{j=1}^{n_1} e^{-x_j^2/2 + \alpha x_j + \beta x_j^2} e^{-\sum_1^\infty s_i x_j^i} dx_j \right)
$$

$$
\times \left(\Delta_{n_2}(y) \prod_{j=1}^{n_2} e^{-y_j^2/2 - \alpha y_j - \beta y_j^2} e^{-\sum_1^\infty u_i y_j^i} dy_j \right) \quad (5\text{-}4)
$$

satisfies the 3-component KP equation, since $p + q = 2 + 1 = 3$, since this matrix corresponds to $p = 2$, $q = 1$. The function $\tau_{n_1 n_2}(t, s, u; \alpha, \beta; E)$ also satisfies Virasoro constraints, to be explained below.

(i)   The three-component KP bilinear equations of Proposition 3.1 imply, using a standard residue computation on the bilinear equation (equations of the type (2-2) for $j = 0$ and $j = 1$, except that the three-component KP bilinear equations give rise to τ-functions depending on two integer indices)

$$
\frac{\partial^2 \log \tau_{n_1, n_2}}{\partial t_1 \partial s_1} = - \frac{\tau_{n_1 + 1, n_2} \tau_{n_1 - 1, n_2}}{\tau_{n_1, n_2}^2} \quad (5\text{-}5)
$$

and

$$
\frac{\partial}{\partial t_1} \log \frac{\tau_{n_1 + 1, n_2}}{\tau_{n_1 - 1, n_2}} = \frac{(\partial^2/\partial t_2 \partial s_1) \log \tau_{n_1, n_2}}{(\partial^2/\partial t_1 \partial s_1) \log \tau_{n_1, n_2}} \quad (5\text{-}6)
$$

$$
- \frac{\partial}{\partial s_1} \log \frac{\tau_{n_1 + 1, n_2}}{\tau_{n_1 - 1, n_2}} = \frac{(\partial^2/\partial t_1 \partial s_2) \log \tau_{n_1, n_2}}{(\partial^2/\partial t_1 \partial s_1) \log \tau_{n_1, n_2}}. \quad (5\text{-}7)
$$

(ii)   The Virasoro equations are as follows: The integral $\tau_{n_1 n_2}(t, s, u; \alpha, \beta; E)$, as defined in (5-4), satisfies

$$
\mathcal{B}_m \tau_{n_1, n_2} = \mathbb{V}_m^{n_1, n_2} \tau_{n_1, n_2} \quad \text{for} \quad m \geq -1, \quad (5\text{-}8)
$$

where $\mathcal{B}_m$ and $\mathbb{V}_m$ are differential operators:

$$
\mathcal{B}_m = \sum_1^{2r} b_i^{m+1} \frac{\partial}{\partial b_i}, \quad \text{for } E = \bigcup_1^{2r} [b_{2i-1}, b_{2i}] \subset \mathbb{R}
$$

and (with the convention that t_i is omitted whenever it appears for $i = 0, -1, \ldots$)

$$
\begin{aligned}
\mathbb{V}_m^{n_1 n_2} := {} & \frac{1}{2} \sum_{i+j=m} \left(\frac{\partial^2}{\partial t_i \partial t_j} + \frac{\partial^2}{\partial s_i \partial s_j} + \frac{\partial^2}{\partial u_i \partial u_j} \right) \\
& + \sum_{i \geq 1} \left(i\, t_i \frac{\partial}{\partial t_{i+m}} + i\, s_i \frac{\partial}{\partial s_{i+m}} + i\, u_i \frac{\partial}{\partial u_{i+m}} \right) \\
& + (n_1 + n_2) \left(\frac{\partial}{\partial t_m} + (-m) t_{-m} \right) - n_1 \left(\frac{\partial}{\partial s_m} + (-m) s_{-m} \right) \\
& - n_2 \left(\frac{\partial}{\partial u_m} + (-m) u_{-m} \right) + (n_1^2 + n_1 n_2 + n_2^2) \delta_{m0} \\
& + \alpha (n_1 - n_2) \delta_{m+1,0} + \frac{m(m+1)}{2} (t_{-m} + s_{-m} + u_{-m}) \\
& - \frac{\partial}{\partial t_{m+2}} + \alpha \left(- \frac{\partial}{\partial s_{m+1}} + \frac{\partial}{\partial u_{m+1}} + (m+1)(s_{-m+1} - u_{-m+1}) \right) \\
& + 2\beta \left(\frac{\partial}{\partial u_{m+2}} - \frac{\partial}{\partial s_{m+2}} \right).
\end{aligned}
$$

These Virasoro equations are obtained by setting

$$
\begin{aligned}
x_i &\mapsto x_i + \varepsilon x_i^{m+1}, \\
y_i &\mapsto y_i + \varepsilon y_i^{m+1}
\end{aligned}
$$

in the integral (5-4) and observing that this substitution does not change the value of the integral, provided the boundary is changed infinitesimally as well.

The Virasoro constraints (5-8) above for $m = -1$ and $m = 0$ lead to the following equations for $f = \log \tau_{n_1 n_2}(t, s, u; \alpha, \beta; E)$ along the locus $\mathcal{L}$ of points where $t = s = u = 0$, $\beta = 0$:

$$
\frac{\partial f}{\partial t_1} = -\mathcal{B}_{-1} f + \alpha(n_1 - n_2),
$$

$$
\frac{\partial f}{\partial s_1} = \frac{1}{2} \left(\mathcal{B}_{-1} - \frac{\partial}{\partial \alpha} \right) f + \frac{\alpha}{2}(n_2 - n_1),
$$

$$
2 \frac{\partial^2 f}{\partial t_1 \partial s_1} = \mathcal{B}_{-1} \left(\frac{\partial}{\partial \alpha} - \mathcal{B}_{-1} \right) f - 2n_1,
$$

$$
2 \frac{\partial^2 f}{\partial t_1 \partial s_2} = \left(\alpha \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta} - \mathcal{B}_0 + 1 \right) \mathcal{B}_{-1} f - 2 \frac{\partial f}{\partial \alpha} - 2\alpha(n_1 - n_2),
$$

$$
2 \frac{\partial^2 f}{\partial t_2 \partial s_1} = \frac{\partial}{\partial \alpha} (\mathcal{B}_0 - \alpha \frac{\partial}{\partial \alpha} + \alpha \mathcal{B}_{-1}) f - \mathcal{B}_{-1}(\mathcal{B}_0 - 1) f - 2\alpha(n_1 - n_2). \quad (5\text{-}9)
$$

From the differential equations (5-6)–(5-7) and from the two first two Virasoro equations (5-9) it follows that, along the locus $\mathcal{L}$, and for the indices $n_1 \pm 1, n_2$,

$$\frac{\dfrac{\partial^2}{\partial t_2 \partial s_1} \log \tau_{n_1 n_2}}{\dfrac{\partial^2}{\partial t_1 \partial s_1} \log \tau_{n_1 n_2}} = \frac{\partial}{\partial t_1} \log \frac{\tau_{n_1+1,n_2}}{\tau_{n_1-1,n_2}} = -\mathcal{B}_{-1} \log \frac{\tau_{n_1+1,n_2}}{\tau_{n_1-1,n_2}} + 2\alpha,$$

$$-\frac{\dfrac{\partial^2}{\partial t_1 \partial s_2} \log \tau_{n_1 n_2}}{\dfrac{\partial^2}{\partial t_1 \partial s_1} \log \tau_{n_1 n_2}} = \frac{\partial}{\partial s_1} \log \frac{\tau_{n_1+1,n_2}}{\tau_{n_1-1,n_2}} = \frac{1}{2}\left(\mathcal{B}_{-1} - \frac{\partial}{\partial \alpha}\right) \log \frac{\tau_{n_1+1,n_2}}{\tau_{n_1-1,n_2}} - \alpha.$$

From these two equations, the logarithmic expression on the right can be eliminated, by acting on the first equation with the operator $\frac{1}{2}\left(\mathcal{B}_{-1} - (\partial/\partial\alpha)\right)$ and on the second with $-\mathcal{B}_{-1}$ and subtracting, thus yielding

$$\frac{1}{2}\left(\mathcal{B}_{-1} - \frac{\partial}{\partial\alpha}\right)\left(\frac{\dfrac{\partial^2}{\partial t_2 \partial s_1} \log \tau_{n_1 n_2}}{\dfrac{\partial^2}{\partial t_1 \partial s_1} \log \tau_{n_1 n_2}} - 2\alpha\right) = \mathcal{B}_{-1}\left(\frac{\dfrac{\partial^2}{\partial t_1 \partial s_2} \log \tau_{n_1 n_2}}{\dfrac{\partial^2}{\partial t_1 \partial s_1} \log \tau_{n_1 n_2}} - \alpha\right)$$

or, equivalently,

$$\mathcal{B}_{-1}\left(\frac{\left(\dfrac{\partial^2}{\partial t_2 \partial s_1} - 2\dfrac{\partial^2}{\partial t_1 \partial s_2}\right) \log \tau_{n_1 n_2}}{\dfrac{\partial^2}{\partial t_1 \partial s_1} \log \tau_{n_1 n_2}}\right)$$
$$-\frac{\partial}{\partial\alpha}\left(\frac{\left(\dfrac{\partial^2}{\partial t_2 \partial s_1} - 2\alpha\dfrac{\partial^2}{\partial t_1 \partial s_1}\right) \log \tau_{n_1 n_2}}{\dfrac{\partial^2}{\partial t_1 \partial s_1} \log \tau_{n_1 n_2}}\right) = 0. \quad (5\text{-}10)$$

Using the remaining Virasoro relations (5-9), one obtains along $\mathcal{L}$ the equalities

$$4\frac{\partial^2}{\partial t_1 \partial s_1} \log \tau_{n_1 n_2} = F^+, \quad 2\left(\frac{\partial^2}{\partial t_2 \partial s_1} - 2\alpha\frac{\partial^2}{\partial t_1 \partial s_1}\right) \log \tau_{n_1 n_2} = H_2^+,$$

$$2\left(\frac{\partial^2}{\partial t_2 \partial s_1} - 2\frac{\partial^2}{\partial t_1 \partial s_2}\right) \log \tau_{n_1 n_2} = H_1^+ - 2\mathcal{B}_{-1}\frac{\partial}{\partial\beta} \log \tau_{n_1 n_2}$$

where we have set[1]

$$F^+ := 2\mathcal{B}_{-1}\left(\frac{\partial}{\partial\alpha} - \mathcal{B}_{-1}\right) \log \tau_{n_1 n_2} - 4n_1 = 2\mathcal{B}_{-1}\left(\frac{\partial}{\partial\alpha} - \mathcal{B}_{-1}\right) \log \mathbb{P}_n - 4n_1,$$

---

[1] One checks that $\tau_{n_1 n_2}(t,s,u;\alpha,\beta,\mathbb{R})|_{\mathcal{L}} = (-2)^{n_1 n_2}(2\pi)^{\frac{n_1+n_2}{2}} \prod_0^{n_1-1} j! \prod_0^{n_2-1} j! \, \alpha^{n_1 n_2} e^{\frac{n_1+n_2}{2}\alpha^2}$.

$$H_1^+ := \frac{\partial}{\partial \alpha}\left(\mathcal{B}_0 - \alpha\frac{\partial}{\partial \alpha} - \alpha\mathcal{B}_{-1}\right)\log \tau_{n_1 n_2} + \left(\mathcal{B}_0\mathcal{B}_{-1} + 4\frac{\partial}{\partial \alpha}\right)\log \tau_{n_1 n_2} + 2\alpha(n_1 - n_2)$$

$$= \frac{\partial}{\partial \alpha}\left(\mathcal{B}_0 - \alpha\frac{\partial}{\partial \alpha} - \alpha\mathcal{B}_{-1}\right)\log \mathbb{P}_n + \left(\mathcal{B}_0\mathcal{B}_{-1} + 4\frac{\partial}{\partial \alpha}\right)\log \mathbb{P}_n + 4\alpha n_1 + \frac{4n_1 n_2}{\alpha}$$

$$H_2^+ := \frac{\partial}{\partial \alpha}\left(\mathcal{B}_0 - \alpha\frac{\partial}{\partial \alpha} - \alpha\mathcal{B}_{-1}\right)\log \tau_{n_1 n_2} + (2\alpha\mathcal{B}_{-1} - \mathcal{B}_0 + 2)\mathcal{B}_{-1}\log \tau_{n_1 n_2} + 2\alpha(n_1 + n_2)$$

$$= \frac{\partial}{\partial \alpha}\left(\mathcal{B}_0 - \alpha\frac{\partial}{\partial \alpha} - \alpha\mathcal{B}_{-1}\right)\log \mathbb{P}_n + (2\alpha\mathcal{B}_{-1} - \mathcal{B}_0 + 2)\mathcal{B}_{-1}\log \mathbb{P}_n.$$

Further define

$$F^- = F^+\big|_{\substack{\alpha \to -\alpha \\ n_1 \leftrightarrow n_2}}, \qquad H_i^- = H_i^+\big|_{\substack{\alpha \to -\alpha \\ n_1 \leftrightarrow n_2}}.$$

With this notation, equation 5-10 becomes

$$\left\{\mathcal{B}_{-1}\frac{\partial}{\partial \beta}\log \tau_{n_1 n_2}\Big|_{\mathcal{L}}, F^+\right\}_{\mathcal{B}_{-1}} = \left\{H_1^+, \tfrac{1}{2}F^+\right\}_{\mathcal{B}_{-1}} - \left\{H_2^+, \tfrac{1}{2}F^+\right\}_{\partial/\partial \alpha} =: G^+,$$

yielding automatically a second equation, using the involution $\alpha \mapsto -\alpha$, $\beta \mapsto -\beta$, $n_1 \leftrightarrow n_2$ (which leaves (5-4) unchanged):

$$-\left\{\mathcal{B}_{-1}\frac{\partial}{\partial \beta}\log \tau_{n_1 n_2}\Big|_{\mathcal{L}}, F^-\right\}_{\mathcal{B}_{-1}} = \left\{H_1^-, \tfrac{1}{2}F^-\right\}_{\mathcal{B}_{-1}} - \left\{H_2^-, \tfrac{1}{2}F^-\right\}_{-\partial/\partial \alpha} =: G^-.$$

The last two displays yield a linear system of equations in

$$\mathcal{B}_{-1}\frac{\partial \log \tau_{n_1 n_2}}{\partial \beta}\Big|_{\mathcal{L}} \quad \text{and} \quad \mathcal{B}_{-1}^2\frac{\partial \log \tau_{n_1 n_2}}{\partial \beta}\Big|_{\mathcal{L}}$$

from which

$$\mathcal{B}_{-1}\frac{\partial \log \tau_{n_1 n_2}}{\partial \beta}\Big|_{\mathcal{L}} = \frac{G^- F^+ + G^+ F^-}{-F^-(\mathcal{B}_{-1}F^+) + F^+(\mathcal{B}_{-1}F^-)},$$

$$\mathcal{B}_{-1}^2\frac{\partial \log \tau_{n_1 n_2}}{\partial \beta}\Big|_{\mathcal{L}} = \frac{G^-(\mathcal{B}_{-1}F^+) + G^+(\mathcal{B}_{-1}F^-)}{-F^-(\mathcal{B}_{-1}F^+) + F^+(\mathcal{B}_{-1}F^-)}.$$

Subtracting the second equation from $\mathcal{B}_{-1}$ of the first equation yields the differential equation

$$\left(F^+\mathcal{B}_{-1}G^- + F^-\mathcal{B}_{-1}G^+\right)\left(F^+\mathcal{B}_{-1}F^- - F^-\mathcal{B}_{-1}F^+\right)$$
$$- \left(F^+G^- + F^-G^+\right)\left(F^+\mathcal{B}_{-1}^2F^- - F^-\mathcal{B}_{-1}^2F^+\right) = 0, \quad (5\text{-}11)$$

which can be rewritten as

$$F^+ F^- \det \begin{pmatrix} F^+ & F^- & 0 \\ \mathcal{B}_{-1}F^+ & \mathcal{B}_{-1}F^- & \dfrac{G^+}{F^+} + \dfrac{G^-}{F^-} \\ \mathcal{B}_{-1}^2 F^+ & \mathcal{B}_{-1}^2 F^- & \dfrac{\mathcal{B}_{-1}G^+}{F^+} + \dfrac{\mathcal{B}_{-1}G^-}{F^-} \end{pmatrix} = 0, \qquad (5\text{-}12)$$

establishing (5-3) for $\log \mathbb{P}_n$. $\qquad\qquad\qquad\square$

6. The Pearcey process

As in section 5, consider $n = 2k$ nonintersecting Brownian motions on $\mathbb{R}$ (Dyson's Brownian motions), all starting at the origin, such that the k left paths end up at $-a$ and the k right paths end up at $+a$ at time $t = 1$.

Also as observed in section 5, the transition probability can be expressed in terms of the Gaussian Hermitian random matrix probability $\mathbb{P}_n(\alpha; E)$ with external source, for which the PDE (5-3) was deduced.

Let now the number $n = 2k$ of particles go to infinity, and let the points a and $-a$, properly rescaled, go to $\pm\infty$. This forces the left k particles to $-\infty$ at $t = 1$ and the right k particles to $+\infty$ at $t = 1$. Since the particles all leave from the origin at $t = 0$, it is natural to believe that for small times the equilibrium measure (mean density of particles) is supported by one interval, and for times close to 1, the equilibrium measure is supported by two intervals. With a precise scaling, $t = 1/2$ is critical in the sense that for $t < 1/2$, the equilibrium measure for the particles is supported by one, and for $t > 1/2$, it is supported by two intervals. The Pearcey process $\mathcal{P}(t)$ is now defined [Tracy and Widom 2006] as the motion of an infinite number of nonintersecting Brownian paths, just around time $t = 1/2$ near $x = 0$, with the precise scaling (upon introducing the scaling parameter z):

$$n = 2k = \frac{2}{z^4}, \quad \pm a = \pm\frac{1}{z^2}, \quad x_i \mapsto x_i z, \quad t \mapsto \frac{1}{2} + tz^2, \quad \text{for } z \to 0. \quad (6\text{-}1)$$

The Pearcey process has also arisen in the context of various growth models [Okounkov and Reshitikhin 2005]. Even though the pathwise interpretation of $\mathcal{P}(t)$ is unclear and deserves investigation, it is natural to define the following probability for $t \in \mathbb{R}$, in terms of the probability (5-1),

$$\mathbb{P}(\mathcal{P}(t) \cap E = \varnothing) := \lim_{z \to 0} \mathbb{P}_0^{\pm 1/z^2} \left(\text{all } x_j\left(\tfrac{1}{2} + tz^2\right) \notin zE; \ 1 \leq j \leq n \right)\Big|_{n=2/z^4}.$$

The results of Brézin and Hikami [1996; 1997; 1998b; 1998a] for the Pearcey kernel and Tracy and Widom [2006] for the extended kernels show that this limit exists and equals a Fredholm determinant:

$$\mathbb{P}(\mathcal{P}(t) \cap E = \varnothing) = \det\left(I - K_t \chi_E\right),$$

where $K_t(x, y)$ is the Pearcey kernel, defined as follows:

$$
\begin{aligned}
K_t(x, y) &:= \frac{p(x)q''(y) - p'(x)q'(y) + p''(x)q(y) - tp(x)q(y)}{x - y} \\
&= \int_0^\infty p(x + z)q(y + z)\, dz, \quad\quad\quad (6\text{-}2)
\end{aligned}
$$

where (note that $\omega = e^{i\pi/4}$)

$$p(x) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-u^4/4 - tu^2/2 - iux} \, du,$$

$$q(y) := \frac{1}{2\pi i} \int_X e^{u^4/4 - tu^2/2 + uy} \, du$$

$$= \mathrm{Im}\left(\frac{\omega}{\pi} \int_0^{\infty} du \; e^{-u^4/4 - (it/2)u^2} (e^{\omega uy} - e^{-\omega uy}) \right)$$

satisfy the differential equations

$$p''' - tp' - xp = 0 \quad \text{and} \quad q''' - tq' + yq = 0.$$

The contour X is given by the ingoing rays from $\pm\infty e^{i\pi/4}$ to 0 and the outgoing rays from 0 to $\pm\infty e^{-i\pi/4}$, i.e., X stands for the contour

For compact $E = \bigcup_{i=1}^r [x_{2i-1}, x_{2i}] \subset \mathbb{R}$, define the gradient and the Euler operator with regard to the boundary points of E,

$$\mathcal{B}_{-1} = \sum_1^{2r} \frac{\partial}{\partial x_i}, \quad \mathcal{B}_0 = \sum_1^{2r} x_i \frac{\partial}{\partial x_i}. \tag{6-3}$$

THEOREM 6.1 [Adler and van Moerbeke 2007].

$$Q(t; x_1, \ldots, x_{2r}) := \log \mathbb{P}\Big(\mathcal{P}(t) \cap E = \varnothing \Big) = \log \det (I - K_t \chi_E) \tag{6-4}$$

satisfies a fourth-order, third-degree PDE, which can be written as a single Wronskian:

$$\left\{ \frac{1}{2} \frac{\partial^3 Q}{\partial t^3} + (\mathcal{B}_0 - 2)\mathcal{B}_{-1}^2 Q + \frac{1}{16} \left\{ \mathcal{B}_{-1} \frac{\partial Q}{\partial t}, \mathcal{B}_{-1}^2 Q \right\}_{\mathcal{B}_{-1}}, \mathcal{B}_{-1}^2 \frac{\partial Q}{\partial t} \right\}_{\mathcal{B}_{-1}} = 0. \tag{6-5}$$

REMARK. A similar PDE can be written for the transition probability involving several times; see [Adler and van Moerbeke 2006]. Such equations can be used to compute the asymptotic behavior of the Pearcey process for $t \to -\infty$.

SKETCH OF PROOF. Consider the function $Q_z(s; x_1, \ldots, x_{2r})$, defined in terms of the probabilities $\mathbb{P}_0^{\pm a}$, defined in (5-1) and $\mathbb{P}_n$, defined in (5-2), as follows:

$$Q_z(s; x_1, \ldots, x_{2r}) := \log \mathbb{P}_0^{\pm a}(t; b_1, \ldots, b_{2r}) \big|_{\substack{n=2/z^4, \; a=1/z^2, \\ b_i = x_i z, \; t = \frac{1}{2} + sz^2}}$$

$$= \log \mathbb{P}_n \left(a \sqrt{\frac{2t}{1-t}}; \ b_1 \sqrt{\frac{2}{t(1-t)}}, \ \ldots, \ b_{2r} \sqrt{\frac{2}{t(1-t)}} \right) \Bigg|_{\substack{n=2/z^4, \ a=1/z^2, \\ b_i = x_i z, \ t=\frac{1}{2}+sz^2}}$$

$$= \log \mathbb{P}_{2/z^4} \left(\frac{\sqrt{2}}{z^2} \sqrt{\frac{\frac{1}{2}+sz^2}{\frac{1}{2}-sz^2}}; \ \frac{x_1 z \sqrt{2}}{\sqrt{\frac{1}{4}-s^2 z^4}}, \ \ldots, \ \frac{x_{2r} z \sqrt{2}}{\sqrt{\frac{1}{4}-s^2 z^4}} \right),$$

from which it follows, by inversion, that

$$Q_z \left(\frac{u^2 z^4 - 2}{2z^2 \left(u^2 z^4 + 2 \right)}; \frac{v_1 u z}{u^2 z^4 + 2}, \ldots, \frac{v_{2r} u z}{u^2 z^4 + 2} \right)$$

$$= \log \mathbb{P}_{2/z^4}(u; v_1, \ldots, v_{2r}). \quad (6\text{-}6)$$

This expression satisfies the PDE (5-3), with α and $b_1, \ldots, b_{2r}$ replaced by u and $v_1, \ldots, v_{2r}$. Therefore all the partials of $\log \mathbb{P}$ with regard to these variables u and $v_1, \ldots, v_r$, as appears in the PDE (5-3), can be expressed, by virtue of (6-6), by partials of Q_z with regard to s and $x_1, \ldots, x_{2r}$.

For this, we need to compute the expressions $F^{\pm}, \tilde{\mathcal{B}}_{-1} F^{\pm}, \tilde{\mathcal{B}}^2_{-1} F^{\pm}, G^{\pm}$ and $\tilde{\mathcal{B}}_{-1} G^{\pm}$ appearing in (5-3) (where we use tildes in contrast to the operators defined in (6-3)), in terms of

$$Q_z(s; x_1, \ldots, x_{2r})$$

$$= \log \mathbb{P}_{2/z^4} \left(\frac{\sqrt{2}}{z^2} \sqrt{\frac{\frac{1}{2}+sz^2}{\frac{1}{2}-sz^2}}; \ x_1 \frac{z\sqrt{2}}{\sqrt{\frac{1}{4}-s^2 z^4}}, \ldots, x_{2r} \frac{z\sqrt{2}}{\sqrt{\frac{1}{4}-s^2 z^4}} \right)$$

$$= Q(s; x_1, \ldots, x_{2r}) + O(z), \quad (6\text{-}7)$$

with

$$Q(s; x_1, \ldots, x_{2r}) = \log \det \left(I - K_s \chi_{E^c} \right). \quad (6\text{-}8)$$

Without taking the limit $z \to 0$ on $\mathbb{Q}_z(s; x_1, \ldots, x_{2r})$ yet, one computes, upon setting $\varepsilon := \pm$,

$$F^{\varepsilon} = -\frac{4}{z^4} - \frac{1}{4z^2} \mathcal{B}^2_{-1} Q_z + \frac{\varepsilon}{4z} \mathcal{B}_{-1} \frac{\partial Q_z}{\partial s} + O(z),$$

$$\frac{1}{\sqrt{2}} \tilde{\mathcal{B}}_{-1} F^{\varepsilon} = -\frac{1}{16z^3} \mathcal{B}^3_{-1} Q_z + \frac{\varepsilon}{16z^2} \mathcal{B}^2_{-1} \frac{\partial Q_z}{\partial s} - \frac{\varepsilon s}{8} \mathcal{B}^2_{-1} \frac{\partial Q_z}{\partial s} + O(z),$$

$$\tilde{\mathcal{B}}^2_{-1} F^{\varepsilon} = -\frac{1}{32z^4} \mathcal{B}^4_{-1} Q_z + \frac{\varepsilon}{32z^3} \mathcal{B}^3_{-1} \frac{\partial Q_z}{\partial s} - \frac{\varepsilon s}{16z} \mathcal{B}^3_{-1} \frac{\partial Q_z}{\partial s} + O(1),$$

$$G^\varepsilon = \frac{3\varepsilon}{8z^9}\mathcal{B}^3_{-1}Q_z + \frac{\varepsilon s}{4z^7}\mathcal{B}^3_{-1}Q_z$$

$$-\frac{1}{128z^6}\left(\left(\mathcal{B}_{-1}\frac{\partial Q_z}{\partial s}\right)(\mathcal{B}^3_{-1}Q_z) + 32\mathcal{B}_0\mathcal{B}^2_{-1}Q_z\right.$$

$$\left. -(\mathcal{B}^2_{-1}Q_z + 64s)\mathcal{B}^2_{-1}\frac{\partial Q_z}{\partial s} - 64\mathcal{B}^2_{-1}Q_z + 16\frac{\partial^3 Q_z}{\partial s^3}\right) + O\left(\frac{1}{z^5}\right),$$

$$\frac{1}{\sqrt{2}}\tilde{\mathcal{B}}_{-1}G^\varepsilon = \frac{3\varepsilon}{32z^{10}}\mathcal{B}^4_{-1}Q_z + \frac{\varepsilon s}{16z^8}\mathcal{B}^4_{-1}Q_z$$

$$+\frac{1}{512z^7}\left(-\left(\mathcal{B}_{-1}\frac{\partial Q_z}{\partial s}\right)(\mathcal{B}^4_{-1}Q_z) - 32\mathcal{B}_0\mathcal{B}^3_{-1}Q_z\right.$$

$$+(\mathcal{B}^2_{-1}Q + 64s)\mathcal{B}^3_{-1}\frac{\partial Q_z}{\partial s}$$

$$\left. + 32\mathcal{B}^3_{-1}Q_z - 16\mathcal{B}_{-1}\frac{\partial^3 Q_z}{\partial s^3}\right) + O\left(\frac{1}{z^6}\right).$$

Using these expressions, one easily deduces for small z,

$$0 = \left(F^+\tilde{\mathcal{B}}_{-1}G^- + F^-\tilde{\mathcal{B}}_{-1}G^+\right)\left(F^+\tilde{\mathcal{B}}_{-1}F^- - F^-\tilde{\mathcal{B}}_{-1}F^+\right)$$

$$-\left(F^+G^- + F^-G^+\right)\left(F^+\tilde{\mathcal{B}}^2_{-1}F^- - F^-\tilde{\mathcal{B}}^2_{-1}F^+\right)$$

$$= -\frac{\varepsilon}{2z^{17}}\left(\left\{\mathcal{B}^2_{-1}\frac{\partial Q_z}{\partial s}, \frac{1}{2}\frac{\partial^3 Q_z}{\partial s^3} + (\mathcal{B}_0 - 2)\mathcal{B}^2_{-1}Q_z\right\}_{\mathcal{B}_{-1}}\right.$$

$$\left. + \frac{1}{16}\mathcal{B}_{-1}\frac{\partial Q_z}{\partial s}\left\{\mathcal{B}^3_{-1}Q_z, \mathcal{B}^2_{-1}\frac{\partial Q_z}{\partial s}\right\}_{\mathcal{B}_{-1}}\right) + O\left(\frac{1}{z^{15}}\right)$$

$$= -\frac{\varepsilon}{2z^{17}}\left(\text{the same expression for } Q(s; x_1, \ldots, x_{2r})\right) + O\left(\frac{1}{z^{16}}\right),$$

using (6-8) in the last equality. Taking the limit when $z \to 0$ yields equation 6-5 of Theorem 6.1. $\square$

7. The Airy process

Consider n nonintersecting Brownian motions on $\mathbb{R}$, all leaving from the origin and forced to return to the origin. According to formula (4-2), this probability,

$$\Pi := \mathbb{P}^0_0\left(\text{all } x_i(t) \in E \mid \text{all } x_j(0) = x_j(1) = 0\right),$$

can be expressed in terms of the determinant of a moment matrix and further as an integral over Hermitian matrices, both with rescaled space, for $0 \le t \le 1$. To do this we let $\mathcal{H}_n(E)$ denote the space of $n \times n$ Hermitian matrices with

spectrum in the set $E \subset \mathbb{R}$, and one checks that

$$\Pi = \frac{1}{Z_n} \det\left(\int_{E(\sqrt{2}/\sqrt{t(1-t)})} dy \; y^{i+j} e^{-y^2/2} \right)_{0 \le i,j \le n-1}$$

$$= \frac{1}{Z_n'} \int_{\mathcal{H}_n(E(1/\sqrt{t(1-t)}))} e^{-\operatorname{Tr} M^2} dM.$$

The Airy process $A(\tau)$ describes the nonintersecting Brownian motions above for large n, but viewed from the (right-hand) edge $\sqrt{2nt(1-t)}$ of the set of particles, with time and space properly rescaled, so that the new time scale τ equals 0 when $t = 1/2$. Random matrix theory suggests the following time and space rescaling (edge rescaling):

$$t = \frac{1}{1 + e^{-2\tau/n^{1/3}}}, \quad E = \frac{\sqrt{2n} + \dfrac{(-\infty, x)}{\sqrt{2}n^{1/6}}}{2 \cosh \dfrac{\tau}{n^{1/3}}}.$$

Taking the limit when $n \to \infty$, one finds that the rescaled motion becomes time-independent (stationary),

$$P(A(\tau) \le x)$$

$$:= \lim_{n\to\infty} \mathbb{P}_0^0\left(\text{all } x_i\left(\frac{1}{1 + e^{-2\tau/n^{1/3}}} \right) \in \frac{\sqrt{2n} + \dfrac{(-\infty, x)}{\sqrt{2}n^{1/6}}}{2\cosh(\tau/n^{1/3})} \;\middle|\; \text{all } x_j(0) = x_j(1) = 0 \right)$$

$$= \lim_{n\to\infty} \frac{1}{Z_n} \int_{\mathcal{H}_n\left(\sqrt{2n} + ((-\infty,x)/\sqrt{2}n^{1/6})\right)} e^{-\operatorname{Tr} M^2} dM$$

$$= \lim_{n\to\infty} \operatorname{Prob}\left((\text{all eigenvalues of } M) \le \sqrt{2n} + \frac{x}{\sqrt{2}n^{1/6}} \right)$$

$$= \exp\left(-\int_x^\infty (\alpha - x) g^2(\alpha) d\alpha \right)$$

$$=: F_2(x) = \text{Tracy–Widom distribution,}$$

with $g(\alpha)$ the unique solution of

$$\begin{cases} g'' = \alpha g + 2g^3 \\ g(\alpha) \cong -\dfrac{e^{-(2/3)\alpha^{3/2}}}{2\sqrt{\pi}\alpha^{1/4}} \text{ for } \alpha \nearrow \infty. \end{cases} \quad \textbf{(Painlevé II).} \qquad (7\text{-}1)$$

This is to say the outmost particle in the nonintersecting Brownian motions fluctuates according to the Tracy–Widom distribution [1994] for $n \to \infty$.

Since the Airy process is stationary, the joint distribution for two times $t_1 < t_2$ in $[0, 1]$ is of interest; here one checks that

$$\mathbb{P}_0^0 \left(\text{all } x_i(t_1) \in E_1, \text{all } x_i(t_2) \in E_2 \mid \text{all } x_j(0) = x_j(1) = 0 \right)$$

$$= \mathbb{P}_n \left(\sqrt{\frac{t_1(1-t_2)}{t_2(1-t_1)}}; E_1 \sqrt{\frac{2t_2}{(t_2-t_1)t_1}}, E_2 \sqrt{\frac{2(1-t_1)}{(1-t_2)(t_2-t_1)}} \right), \quad (7\text{-}2)$$

where

$$\mathbb{P}_n(c; E_1', E_2') := \frac{1}{Z_n} \iint_{\mathcal{H}(E_1') \times \mathcal{H}(E_2')} dM_1\, dM_2\, e^{-\frac{1}{2}\text{Tr}(M_1^2 + M_2^2 - 2cM_1 M_2)}$$

$$= c_N' \iint_{E^N} \Delta_N(x)\Delta_N(y) \prod_{k=1}^{N} e^{-\frac{1}{2}(x_k^2 + y_k^2 - 2cx_k y_k)}\, dx_k\, dy_k.$$

According to [Adler and van Moerbeke 1999b], given

$$E = E_1 \times E_2 := \bigcup_{i=1}^{r} [a_{2i-1}, a_{2i}] \times \bigcup_{i=1}^{s} [b_{2i-1}, b_{2i}] \subset \mathbb{R}^2, \qquad (7\text{-}3)$$

$\log \mathbb{P}_n(c; E_1, E_2)$ satisfies a nonlinear third-order partial differential equation (in terms of the Wronskian $\{f, g\}_X = g(Xf) - f(Xg)$, with regard to the first order differential operator X):

$$\left\{ \mathcal{B}_2\mathcal{A}_1 \log \mathbb{P}_n,\ \mathcal{B}_1\mathcal{A}_1 \log \mathbb{P}_n + \frac{nc}{c^2 - 1} \right\}_{\mathcal{A}_1}$$

$$- \left\{ \mathcal{A}_2\mathcal{B}_1 \log \mathbb{P}_n,\ \mathcal{A}_1\mathcal{B}_1 \log \mathbb{P}_n + \frac{nc}{c^2 - 1} \right\}_{\mathcal{B}_1} = 0. \quad (7\text{-}4)$$

in terms of the differential operators, depending on the coupling term c and the boundary of E,

$$\mathcal{A}_1 = \frac{1}{c^2 - 1}\left(\sum_{1}^{r} \frac{\partial}{\partial a_j} + c \sum_{1}^{s} \frac{\partial}{\partial b_j} \right), \quad \mathcal{B}_1 = \frac{1}{1 - c^2}\left(c \sum_{1}^{r} \frac{\partial}{\partial a_j} + \sum_{1}^{s} \frac{\partial}{\partial b_j} \right),$$

$$\mathcal{A}_2 = \sum_{j=1}^{r} a_j \frac{\partial}{\partial a_j} - c \frac{\partial}{\partial c}, \qquad\qquad \mathcal{B}_2 = \sum_{j=1}^{s} b_j \frac{\partial}{\partial b_j} - c \frac{\partial}{\partial c}. \qquad (7\text{-}5)$$

Using the same rescaled space and time variables, as before, introduce new times $\tau_1 < \tau_2$ and points $x,\ y \in \mathbb{R}$, defined as

$$t_i = \frac{1}{1 + e^{-2\tau_i/n^{1/3}}}, \quad E_1 = \frac{\sqrt{2n} + \dfrac{(-\infty, x)}{\sqrt{2}n^{1/6}}}{2\cosh\dfrac{\tau_1}{n^{1/3}}}, \quad E_2 = \frac{\sqrt{2n} + \dfrac{(-\infty, y)}{\sqrt{2}n^{1/6}}}{2\cosh\dfrac{\tau_2}{n^{1/3}}}.$$

One verifies, in view of (7-2), that

$$E_1 \sqrt{\frac{2t_2}{(t_2 - t_1)t_1}} = \frac{\sqrt{2}\left(\sqrt{2n} + \frac{(-\infty, x)}{\sqrt{2}n^{1/6}}\right)}{\sqrt{1 - e^{-2(\tau_2 - \tau_1)/n^{1/3}}}},$$

$$E_2 \sqrt{\frac{2(1 - t_1)}{(1 - t_2)(t_2 - t_1)}} = \frac{\sqrt{2}\left(\sqrt{2n} + \frac{(-\infty, y)}{\sqrt{2}n^{1/6}}\right)}{\sqrt{1 - e^{-2(\tau_2 - \tau_1)/n^{1/3}}}},$$

$$c = \sqrt{\frac{t_1(1 - t_2)}{t_2(1 - t_1)}} = e^{-(\tau_2 - \tau_1)/n^{1/3}}.$$

Defining

$$\mathbb{Q}(\tau_2 - \tau_1; x, y) :=$$

$$\log \mathbb{P}_n\left(e^{-(\tau_2 - \tau_1)/n^{1/3}}; \frac{\left(2\sqrt{n} + \frac{x}{n^{1/6}}\right)}{\sqrt{1 - e^{-2(\tau_2 - \tau_1)/n^{1/3}}}}, \frac{\left(2\sqrt{n} + \frac{y}{n^{1/6}}\right)}{\sqrt{1 - e^{-2(\tau_2 - \tau_1)/n^{1/3}}}}\right),$$

one checks, setting $z = n^{-1/6}$ and using the inverse map, that

$$\log \mathbb{P}_n(c; a, b) = \mathbb{Q}\left(-z^{-2}\log c; az^{-1}\sqrt{1 - c^2} - 2z^{-4}, bz^{-1}\sqrt{1 - c^2} - 2z^{-4}\right).$$

But $\log \mathbb{P}_n(c; E_1, E_2)$ satisfies the PDE (7-4), which induces a PDE for $\mathbb{Q}$; then letting $z \to \infty$, the leading term in this series must be $= 0$. One finds thus the following PDE for the Airy joint probability, namely

$$H(t; x, y) := \log P\left(A(\tau_1) \le y + x, A(\tau_2) \le y - x\right),$$

takes on the following simple form in x, y and t^2, with $t = \tau_2 - \tau_1$, also involving a Wronskian (see [Adler and van Moerbeke 2005])

$$2t \frac{\partial^3 H}{\partial t \partial x \partial y} = \left(\frac{t^2}{2} \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}\right)\left(\frac{\partial^2 H}{\partial x^2} - \frac{\partial^2 H}{\partial y^2}\right) + \left\{\frac{\partial^2 H}{\partial x \partial y}, \frac{\partial^2 H}{\partial y^2}\right\}_y, \quad (7\text{-}6)$$

with initial condition

$$\lim_{t \searrow 0} H(t; x, y) = \log F_2\left(\min(y + x, y - x)\right).$$

The edge $\sup A(t)$ of the cloud is non-Markovian, as is the largest particle in the finite nonintersecting Brownian problem. As $t = \tau_2 - \tau_1 \to \infty$, the edges $\sup A(\tau_1)$ and $\sup A(\tau_2)$ become independent. This poses the question: *How*

much does the process remember from the remote past? The following asymptotics for the covariance of the edge of the cloud, for large $t = \tau_2 - \tau_1$, is deduced from the PDE:

$$E(\sup A(\tau_2) \sup A(\tau_1)) - E(\sup A(\tau_2)) E(\sup A(\tau_1))$$
$$= \frac{1}{t^2} + \frac{2}{t^4} \iint_{\mathbb{R}^2} \Phi(u, v) \, du \, dv + \cdots,$$

where

$$\Phi(u, v) := F_2(u) F_2(v) \left(\frac{1}{4} \left(\int_u^\infty g^2 d\alpha \right)^2 \left(\int_v^\infty g^2 d\alpha \right)^2 \right.$$
$$+ g^2(u) \left(\frac{1}{4} g^2(v) - \frac{1}{2} \left(\int_v^\infty g^2 d\alpha \right)^2 \right)$$
$$\left. + \int_v^\infty d\alpha \left(2(v - \alpha) g^2 + g'^2 - g^4 \right) \int_u^\infty g^2 d\alpha \right).$$

(Here $g = g(\alpha)$ is the function (7-1) and $F_2(u)$ is the Tracy–Widom distribution.)

The Airy process was introduced by Spohn and Prähofer [2002] in the context of polynuclear growth models. It has been further investigated by Johansson [2001; 2003; 2005], by Tracy and Widom [2004] and by Adler and van Moerbeke [2005]; see also [Widom 2004].

References

[Adler and van Moerbeke 1997] M. Adler and P. van Moerbeke, "String-orthogonal polynomials, string equations, and 2-Toda symmetries", *Comm. Pure Appl. Math.* **50**:3 (1997), 241–290. Available at arxiv.org/abs/hep-th/9706182.

[Adler and van Moerbeke 1999a] M. Adler and P. van Moerbeke, "Generalized orthogonal polynomials, discrete KP and Riemann-Hilbert problems", *Comm. Math. Phys.* **207**:3 (1999), 589–620. Available at arxiv.org/abs/nlin.SI/0009002.

[Adler and van Moerbeke 1999b] M. Adler and P. van Moerbeke, "The spectrum of coupled random matrices", *Ann. of Math.* (2) **149**:3 (1999), 921–976.

[Adler and van Moerbeke 2005] M. Adler and P. van Moerbeke, "PDEs for the joint distributions of the Dyson, Airy and sine processes", *Ann. Probab.* **33**:4 (2005), 1326–1361. Available at arxiv.org/abs/math.PR/0302329.

[Adler and van Moerbeke 2006] M. Adler and P. van Moerbeke, "Joint probability for the Pearcey process", preprint, 2006. Available at arxiv.org/abs/math.PR/0612393.

[Adler and van Moerbeke 2007] M. Adler and P. van Moerbeke, "PDEs for the Gaussian ensemble with external source and the Pearcey distribution", *Comm. Pure Appl. Math.* **60**:9 (2007), 1261–1292. Changed page range following MathSciNet; please check.

[Adler et al. 2006] M. Adler, P. van Moerbeke, and P. Vanhaecke, "Moment matrices and multi-component KP, with applications to random matrix theory", preprint, 2006. Available at arxiv.org/abs/math-ph/0612064.

[Aptekarev 1998] A. I. Aptekarev, "Multiple orthogonal polynomials", *J. Comput. Appl. Math.* **99**:1-2 (1998), 423–447.

[Aptekarev et al. 2003] A. I. Aptekarev, A. Branquinho, and W. Van Assche, "Multiple orthogonal polynomials for classical weights", *Trans. Amer. Math. Soc.* **355**:10 (2003), 3887–3914.

[Bleher and Kuijlaars 2004a] P. Bleher and A. B. J. Kuijlaars, "Large n limit of Gaussian random matrices with external source. I", *Comm. Math. Phys.* **252**:1-3 (2004), 43–76.

[Bleher and Kuijlaars 2004b] P. M. Bleher and A. B. J. Kuijlaars, "Random matrices with external source and multiple orthogonal polynomials", *Int. Math. Res. Not.* no. 3 (2004), 109–129. Available at arxiv.org/abs/math-ph/0307055.

[Brézin and Hikami 1996] E. Brézin and S. Hikami, "Correlations of nearby levels induced by a random potential", *Nuclear Phys. B* **479**:3 (1996), 697–706.

[Brézin and Hikami 1997] E. Brézin and S. Hikami, "Extension of level spacing universality", *Phys. Rev. E* **56** (1997), 264–269.

[Brézin and Hikami 1998a] E. Brézin and S. Hikami, "Level spacing of random matrices in an external source", *Phys. Rev. E* (3) **58**:6, part A (1998), 7176–7185.

[Brézin and Hikami 1998b] E. Brézin and S. Hikami, "Universal singularity at the closure of a gap in a random matrix theory", *Phys. Rev. E* (3) **57**:4 (1998), 4140–4149.

[Daems and Kuijlaars 2007] E. Daems and A. B. J. Kuijlaars, "Multiple orthogonal polynomials of mixed type and non-intersecting Brownian motions", *J. Approx. Theory* **146**:1 (2007), 91–114. Available at arxiv.org/abs/math.CA/0511470.

[Dyson 1962] F. J. Dyson, "A Brownian-motion model for the eigenvalues of a random matrix", *J. Mathematical Phys.* **3** (1962), 1191–1198.

[Grabiner 1999] D. J. Grabiner, "Brownian motion in a Weyl chamber, non-colliding particles, and random matrices", *Ann. Inst. H. Poincaré Probab. Statist.* **35**:2 (1999), 177–204.

[Johansson 2001] K. Johansson, "Universality of the local spacing distribution in certain ensembles of Hermitian Wigner matrices", *Comm. Math. Phys.* **215**:3 (2001), 683–705.

[Johansson 2003] K. Johansson, "Discrete polynuclear growth and determinantal processes", *Comm. Math. Phys.* **242**:1-2 (2003), 277–329.

[Johansson 2005] K. Johansson, "The arctic circle boundary and the Airy process", *Ann. Probab.* **33**:1 (2005), 1–30. Available at arxiv.org/abs/math.PR/0306216.

[Karlin and McGregor 1959] S. Karlin and J. McGregor, "Coincidence probabilities", *Pacific J. Math.* **9** (1959), 1141–1164.

[Okounkov and Reshitikhin 2005] A. Okounkov and N. Reshitikhin, "Random skew plane partitions and the Pearcey process", preprint, 2005. Available at arxiv.org/abs/ math.CO/0503508.

[Prähofer and Spohn 2002] M. Prähofer and H. Spohn, "Scale invariance of the PNG droplet and the Airy process", *J. Statist. Phys.* **108**:5-6 (2002), 1071–1106. Dedicated to David Ruelle and Yasha Sinai on the occasion of their 65th birthdays.

[Tracy and Widom 1994] C. A. Tracy and H. Widom, "Level-spacing distributions and the Airy kernel", *Comm. Math. Phys.* **159**:1 (1994), 151–174.

[Tracy and Widom 2004] C. A. Tracy and H. Widom, "Differential equations for Dyson processes", *Comm. Math. Phys.* **252**:1-3 (2004), 7–41. Available at arxiv.org/abs/ math.PR/0309082.

[Tracy and Widom 2006] C. A. Tracy and H. Widom, "The Pearcey process", *Comm. Math. Phys.* **263**:2 (2006), 381–400. Available at arxiv.org/abs/math.PR/0412005.

[Widom 2004] H. Widom, "On asymptotics for the Airy process", *J. Statist. Phys.* **115**:3-4 (2004), 1129–1134. Available at arxiv.org/abs/math.PR/0308157.

PIERRE VAN MOERBEKE
DEPARTMENT OF MATHEMATICS
UNIVERSITÉ CATHOLIQUE DE LOUVAIN
1348 LOUVAIN-LA-NEUVE
BELGIUM
and
BRANDEIS UNIVERSITY
WALTHAM, MA 02454
UNITED STATES
    pierre.vanmoerbeke@uclouvain.be