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ABSTRACT. We survey recent developments in the theory of impartial combinatorial games in misere play, focusing on how Sprague–Grundy theory of normal-play impartial games generalizes to misere play via the *indistinguishability quotient construction* [P2]. This paper is based on a lecture given on 21 June 2005 at the Combinatorial Game Theory Workshop at the Banff International Research Station. It has been extended to include a survey of results on misere games, a list of open problems involving them, and a summary of *MisereSolver* [AS2005], the excellent Java-language program for misere indistinguishability quotient construction recently developed by Aaron Siegel. Many wild misere games that have long appeared intractable may now lie within the grasp of assiduous losers and their faithful computer assistants, particularly those researchers and computers equipped with *MisereSolver*.

1. Introduction

> We've spent a lot of time teaching you how to win games by being the last to move. But suppose you are baby-sitting little Jimmy and want, at least occasionally, to make sure you *lose*? This means that instead of playing the normal play rule in which whoever can't move is the *loser*, you've switched to **misere play** rule when he's the *winner*. Will this make much difference? Not *always*. . .

That's the first paragraph from the thirteenth chapter ("Survival in the Lost World") of Berlekamp, Conway, and Guy's encyclopedic work on combinatorial game theory, *Winning Ways for your Mathematical Plays* [WW].

And why "not *always*?" The misere analysis of an impartial combinatorial game often proves to be far more difficult than it is in normal play. To take a typical example, the normal play analysis of **Dawson's Chess** [D] was published as early as 1956 by Guy and Smith [GS], but even today, a complete misere analysis hasn't been found (see Section 10.1). Guy tells the story [Guy91]:

> [Dawson's chess] is played on a $3 \times n$ board with white pawns on the first rank and black pawns on the third. It was posed as a *losing* game (last-player-losing, now called **misere**) so that capturing was obligatory. Fortunately, (because we *still* don't know how to play misere Dawson's Chess) I assumed, as a number of writers of that time and since have done, that the misere analysis required only a trivial adjustment of the normal (last-player-winning) analysis. This arises because Bouton, in his original analysis of Nim [B1902], had observed that only such a trivial adjustment was necessary to cover both normal and misere play...
>
> But even for *impartial* games, in which the same options are available to both players, regardless of whose turn it is to move, Grundy & Smith [GrS1956] showed that the general situation in misere play soon gets very complicated, and Conway [ONAG], (p. 140) confirmed that the situation can only be simplified to the microscopically small extent noticed by Grundy & Smith.
>
> At first sight Dawson's Chess doesn't look like an impartial game, but if you know how pawns move at Chess, it's easy to verify that it's equivalent to the game played with rows of skittles in which, when it's your turn, you knock down any skittle, together with its immediate neighbors, if any.

So misere play can be difficult. But is it a hopeless situation? It has often seemed so. Returning to chapter 13 in [WW], one encounters the *genus theory of impartial misere disjunctive sums*, extended significantly from its original presentation in chapter 7 ("How to Lose When You Must") of Conway's *On Numbers and Games* [ONAG]. But excluding the *tame games* that play like Nim in misere play, there's a remarkable paucity of example games that the genus theory completely resolves. For example, the section "Misere Kayles" from the 1982 first edition of [WW] promises

> Although several tame games arise in Kayles (see Chapter 4), wild game's abounding and we'll need all our [genus-theoretic] resources to tackle it...

However, it turns out Kayles isn't "tackled" at all — after an extensive table of genus values to heap size 20, one finds the slightly embarrassing question

> Is there a larger single-row P-position?

It was left to the amateur William L. Sibert [SC] to settle misere Kayles using completely different methods. One finds a description of his solution at end of the updated Chapter 13 in the second edition of [WW], and also in [SC]. In 2003, [WW] summarized the situation as follows (p. 451):

> Sibert's remarkable *tour de force* raises once again the question: are misere analyses really so difficult? A referee of a draft of the Sibert–Conway paper wrote "the actual solution will have no bearing on other problems," while another wrote "the ideas are likely to be applicable to some other games. . . "

1.1. Misere play — the natural impartial game convention? When nonmathematicians play impartial games, they tend to choose the misere play convention[1]. This was already recognized by Bouton in his classic paper "Nim, A Game with a Complete Mathematical Theory," [B1902]:

> The game may be modified by agreeing that the player who takes the last counter from the table *loses*. This modification of the three pile [Nim] game seems to be more widely known than that first described, but its theory is not quite so simple. . .

But why do people prefer the misere play convention? The answer may lie in Fraenkel's observation that impartial games lack *boardfeel*, and simple *Schadenfreude*[2]:

> For many MathGames, such as Nim, a player without prior knowledge of the strategy has no inkling whether any given position is "strong" or "weak" for a player. Even two positions before ultimate defeat, the player sustaining it may be in the dark about the outcome, which will stump him. The player has no boardfeel. . . [Fraenkel, p. 3].

If both players are "in the dark," perhaps it's only natural that the last player compelled to make a move in such a pointless game should be deemed the *loser*. Only when a mathematician gets involved are things ever-so-subtly shifted toward the normal play convention, instead — but this is only because there is a simple and beautiful theory of normal-play impartial games, called Sprague–Grundy theory. Secretly computing nim-values, mathematicians win normal-play impartial games time and time again. Papers on normal play impartial games outnumber misere play ones by a factor of perhaps fifty, or even more[3].

[1] "Indeed, if anything, misere Nim is more commonly played than normal Nim. . . " [ONAG], p. 136.

[2] The joy we take in another's misfortune.

[3] Based on an informal count of papers in the [Fraenkel] CGT bibliography.

In the last twelve months it has become clear how to generalize such Sprague–Grundy nim-value computations to misere play via *indistinguishability quotient construction* [P2]. As a result, many misere game problems that have long appeared intractable, or have been passed over in silence as too difficult, have now been solved. Still others, such as a Dawson's Chess, appear to remain out of reach and await new ideas. The remainder of this paper surveys this largely unexplored territory.

2. Two wild games

We begin with two impartial games: **Pascal's Beans** — introduced here for the first time — and **Guiles** (the octal game **0.15**). Each has a relatively simple normal-play solution, but is *wild*[4] in misere play. Wild games are characterized by having misere play that differs in an essential way[5] from the play of misere Nim. They often prove notoriously difficult to analyze completely. Nevertheless, we'll give complete misere analyses for both Pascal's Beans and Guiles by using the key idea of the *misere indistinguishability quotient*, which was first introduced in [P2], and which we take up in earnest in Section 5.

3. Pascal's Beans

Pascal's Beans is a two-player impartial combinatorial game. It's played with heaps of beans placed on Pascal's triangle, which is depicted in Figure 1. A legal move in the game is to slide a single bean either up a single row and to the left one position, or alternatively up a single row and to the right one position in the triangle. For example, in Figure 1, a bean resting on the cell marked 20 could be moved to either cell labelled 10.

The actual numbers in Pascal's triangle are not relevant in the play of the game, except for the 1's that mark the border positions of the board. In play of Pascal's Beans, a bean is considered out of play when it first reaches a border position of the triangle. The game ends when all beans have reached the border.

3.1. Normal play. In *normal play* of Pascal's Beans, the last player to make a legal move is declared the *winner* of the game. Figure 2 shows the pattern of *nim values* that arises in the analysis of the game. Using the figure, it's possible to quickly determine the best-play outcome of an arbitrary starting position in Pascal's Beans using *Sprague–Grundy theory* and the *nim addition* operation $\oplus$. Provided one knows the $\mathbb{Z}_2 \times \mathbb{Z}_2$ addition table in Figure 3, all is well — the

[4] See Chapter 13 ("Survival in the Lost World") in [WW] and Section 7 in this paper for more information on wild misere games.

[5] To be made precise in Section 7.

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 \\
 & & & \vdots & & & & \vdots & & & & \vdots & & &
\end{array}
$$

Figure 1. The Pascal's Beans board.

P-positions (second-player winning positions) are precisely those that have nim value zero (that is, $*0$), and every other position is an *N-position* (or next-player win), of nim value $*1$, $*2$, or $*3$.

$$
\begin{array}{ccccccccccccccccc}
 & & & & & & & & *0 & & & & & & & & \\
 & & & & & & & *0 & & *0 & & & & & & & \\
 & & & & & & *0 & & *1 & & *0 & & & & & & \\
 & & & & & *0 & & *2 & & *2 & & *0 & & & & & \\
 & & & & *0 & & *1 & & *0 & & *1 & & *0 & & & & \\
 & & & *0 & & *2 & & *2 & & *2 & & *2 & & *0 & & & \\
 & & *0 & & *1 & & *0 & & \mathbf{*0} & & *0 & & *1 & & *0 & & \\
 & *0 & & *2 & & *2 & & \mathbf{*1} & & \mathbf{*1} & & *2 & & *2 & & *0 & \\
*0 & & *1 & & *0 & & \mathbf{*0} & & \mathbf{*0} & & \mathbf{*0} & & *0 & & *1 & & *0 \\
 & & & & \vdots & & & & \vdots & & & & \vdots & & & &
\end{array}
$$

Figure 2. The pattern of single-bean nim-values in normal play of Pascal's Beans. Each interior value is the *minimal excludant* (or *mex*) of the two nim values immediately above it. The boldface entries form the first three rows of an infinite subtriangle whose rows alternate between $*0$ and $*1$.

$\oplus$	$*0$	$*1$	$*2$	$*3$
$*0$	$*0$	$*1$	$*2$	$*3$
$*1$	$*1$	$*0$	$*3$	$*2$
$*2$	$*2$	$*3$	$*0$	$*1$
$*3$	$*3$	$*2$	$*1$	$*0$

Figure 3. Addition for normal play of Pascal's Beans.

3.2. Misere play. In *misere play* of Pascal's Beans, the last player to make a move is declared the *loser* of the game. Is it possible to give an analysis of misere Pascal's Beans that resembles the normal play analysis? The answer is yes — but the positions of the triangle can no longer be identified with nim heaps $*k$, and the rule for the misere addition is no longer given by nim addition. Instead, both the values to be identified with particular positions of the triangle and the desired misere addition are given by a particular twelve-element commutative monoid $\mathcal{M}$, the *misere indistinguishability quotient*[6] of Pascal's Beans. The monoid $\mathcal{M}$ has an identity 1 and is presentable using three generators and relations:

$$\mathcal{M} = \langle\, a, b, c \mid a^2 = 1,\ c^2 = 1,\ b^3 = b^2 c \,\rangle.$$

Assiduous readers might enjoy verifying that the identity $b^4 = b^2$ follows from these relations, and that a general word of the form $a^i b^j c^k$ $(i, j, k \geq 0)$ will always reduce to one of the twelve *canonical words*

$$\mathcal{M} = \{1,\ a,\ b,\ ab,\ b^2,\ ab^2,\ c,\ ac,\ bc,\ b^2c,\ abc,\ ab^2c\}.$$

Amongst the twelve canonical words, three represent P-position types

$$\mathcal{P} = \{a,\ b^2, ac\},$$

and the remaining nine represent N-position types:

$$\mathcal{N} = \{1,\ b,\ ab,\ ab^2,\ c,\ bc,\ b^2c,\ abc,\ ab^2c\}.$$

Figure 4 shows the identification of triangle positions with elements of $\mathcal{M}$.

$$\begin{array}{ccccccccccccccccc}
 & & & & & & & & 1 & & & & & & & & \\
 & & & & & & & 1 & & 1 & & & & & & & \\
 & & & & & & 1 & & \boxed{a} & & 1 & & & & & & \\
 & & & & & 1 & & b & & b & & 1 & & & & & \\
 & & & & 1 & & a & & \boxed{b^2} & & a & & 1 & & & & \\
 & & & 1 & & b & & c & & c & & b & & 1 & & & \\
 & & 1 & & a & & b^2 & & \boldsymbol{b^2} & & b^2 & & a & & 1 & & \\
 & 1 & & b & & c & & \boldsymbol{ab^2} & & \boldsymbol{ab^2} & & c & & b & & 1 & \\
1 & & a & & b^2 & & \boldsymbol{b^2} & & \boldsymbol{b^2} & & \boldsymbol{b^2} & & b^2 & & a & & 1 \\
 & & & & \vdots & & & & \vdots & & & & \vdots & & & &
\end{array}$$

Figure 4. Identifications for single-bean positions in misere play of Pascal's Beans. The values are elements of the misere indistinguishability quotient $\mathcal{M}$ of Pascal's Beans. The boldface entries form the first three rows of an infinite subtriangle whose rows alternate between the values b^2 and ab^2.

[6] See Section 5.

Although we've used multiplicative notation to represent the addition operation in the monoid $\mathcal{M}$, we use it to analyze general misere-play Pascal's Beans positions just as we used the nim values of Figure 2 and nim addition in normal play. For example, suppose a Pascal's Beans position involves just two beans — one placed along the central axis of the triangle at each of the two boxed positions in Figure 4. Combining the corresponding entries a and b^2 as monoid elements, we obtain the element ab^2, which we've already asserted is an N-position. What is the winning misere-play move? From the lower bean, at the position marked b^2, the only available moves are both to a cell marked b. This move is of the form

$$ab^2 \to ab,$$

that is, the result is another misere N-position type (here ab). So this option is not a winning misere move. But the cell marked a has an available move is to the border. The resulting winning move is of the form

$$ab^2 \to b^2,$$

that is, the result is b^2, a P-position type.

4. Guiles

Guiles can be played with heaps of beans. The possible moves are to remove a heap of 1 or 2 beans completely, or to take two beans from a sufficiently large heap and partition what is left into two smaller, nonempty heaps. This is the octal game **0.15**.

4.1. Normal play. The nim values of the octal game **Guiles** fall into a period 10 pattern. See Figure 5.

	1	2	3	4	5	6	7	8	9	10
0+	1	1	0	1	1	2	2	1	2	2
10+	1	1	0	1	1	2	2	1	2	2
20+	1	1	0	1	1	2	2	1	2	2
30+	1	1	0	1	1	···				

Figure 5. Nim values for normal play **0.15**.

4.2. Misere play. Using his recently-developed Java-language computer program *MisereSolver*, Aaron Siegel [PS] found that the misere indistinguishability quotient $\mathcal{Q}$ of misere Guiles is a (commutative) monoid of order 42. It has the presentation

$$\begin{aligned}\mathcal{Q} = \langle\ a,\ b,\ c,\ d,\ e,\ f,\ g,\ h,\ i \mid \\ & a^2=1,\ b^4=b^2,\ bc=ab^3,\ c^2=b^2,\ b^2d=d,\\ & cd=ad,\ d^3=ad^2,\ b^2e=b^3,\ de=bd,\ be^2=ace,\\ & ce^2=abe,\ e^4=e^2,\ bf=b^3,\ df=d,\ ef=ace,\\ & cf^2=cf,\ f^3=f^2,\ b^2g=b^3,\ cg=ab^3,\ dg=bd,\\ & eg=be,\ fg=b^3,\ g^2=bg,\ bh=bg,\ ch=ab^3,\\ & dh=bd,\ eh=bg,\ fh=b^3,\ gh=bg,\ h^2=b^2,\\ & bi=bg,\ ci=ab^3,\ di=bd,\ ei=be,\ fi=b^3,\\ & gi=bg,\ hi=b^2,\ i^2=b^2\ \rangle.\end{aligned}$$

In Figure 6 we show the single-heap misere equivalences for Guiles. It is a remarkable fact that this sequence is also periodic of length ten — it's just that the (aperiodic) *preperiod* is longer (length 66), and a person needs to know the monoid $\mathcal{Q}$! The P-positions of Guiles are precisely those positions equivalent to one of the words

$$P = \{\ a,\ b^2,\ bd,\ d^2,\ ae,\ ae^2,\ ae^3,\ af,\ af^2,\ ag,\ ah,\ ai\ \}.$$

	1	2	3	4	5	6	7	8	9	10
0+	a	a	1	a	a	b	b	a	b	b
10+	a	a	1	c	c	b	b	d	b	e
20+	c	c	f	c	c	b	g	d	h	i
30+	ab^2	abg	f	abg	abe	b^3	h	d	h	h
40+	ab^2	abe	f^2	abg	abg	b^3	h	d	h	h
50+	ab^2	abg	f^2	abg	abg	b^3	b^3	d	b^3	b^3
60+	ab^2	abg	f^2	abg	abg	b^3	b^3	d	b^3	b^3
70+	ab^2	ab^2	f^2	ab^2	ab^2	b^3	b^3	d	b^3	b^3
80+	ab^2	ab^2	f^2	ab^2	ab^2	b^3	b^3	d	b^3	b^3
90+	ab^2	ab^2	f^2	ab^2	ab^2	b^3	b^3	d	b^3	b^3
100+										

Figure 6. Misere equivalences for Guiles.

Knowledge of the monoid presentation $\mathcal{Q}$, its partition into N- and P-position types, and the single-heap equivalences in Figure 6 suffices to quickly determine the outcome of an arbitrary misere Guiles position. For example, suppose a position contains four heaps of sizes 4, 58, 68, and 78. Looking up monoid values in Figure 6, we obtain the product

$$\begin{aligned} a \cdot d \cdot d \cdot d &= ad^3 \\ &= a \cdot ad^2 \quad \text{(relation } d^3 = ad^2\text{)} \\ &= d^2 \quad \text{(relation } a^2 = 1\text{)} \end{aligned}$$

We conclude that $4 + 58 + 68 + 78$ is a misere Guiles P-position.

5. The indistinguishability quotient construction

What do these two solutions have in common? They were both obtained via a computer program called *MisereSolver*, by Aaron Siegel. Underpinning *MisereSolver* is the notion of the *indistinguishability quotient construction*. Here, we'll sketch the main ideas of the indistinguishability quotient construction only. They are developed in detail in [P2].

Suppose $\mathscr{A}$ is a set of (normal, or alternatively, misere) impartial game positions that is closed under the operations of game addition and taking options (that is, making moves). Unless we say otherwise, we'll always be taking $\mathscr{A}$ to be the set of all positions that arise in the play of a specific game Γ, which we fix in advance. For example, one might take

$$\begin{aligned} \Gamma &= \text{Normal-play Nim,} \\ \mathscr{A} &= \text{All positions that arise in normal-play Nim,} \end{aligned}$$

or

$$\begin{aligned} \Gamma &= \text{Misere-play Guiles,} \\ \mathscr{A} &= \text{All positions that arise in misere-play Guiles.} \end{aligned}$$

Two games $G, H \in \mathscr{A}$ are then said to be *indistinguishable*, and we write the relation $G \;\rho\; H$, if for every game $X \in \mathscr{A}$, the sums $G + X$ and $H + X$ have the same outcome (that is, are both N-positions, or are both P-positions). Note in particular that if G and H are indistinguishable, then they have the same outcome (choose X to be the *endgame* — that is, the terminal position, with no options).

The indistinguishability relation ρ is easily seen to be an equivalence relation on $\mathscr{A}$, but in fact more is true — it's a *congruence* on $\mathscr{A}$ [P2]. This follows because indistinguishability is *compatible* with addition; that is, for every set of three games $G, H, X \in \mathscr{A}$:

$$G \;\rho\; H \implies (G + X) \;\rho\; (H + X). \tag{5-1}$$

Now let's make the definition

$$\rho G = \{ H \in \mathcal{A} \mid G \; \rho \; H \}.$$

We'll call ρG the *congruence class of* $\mathcal{A}$ *modulo* ρ *containing* G. Because ρ is a congruence, there is a well-defined addition operation

$$\rho G + \rho H = \rho(G + H)$$

on the set $\mathcal{A}/\rho$ of all congruence classes ρG of $\mathcal{A}$ modulo ρ

$$\mathfrak{Q} = \mathfrak{Q}(\Gamma) = \mathcal{A}/\rho = \{ \rho G \mid G \in \mathcal{A}. \} \tag{5-2}$$

The monoid $\mathfrak{Q}$ is called the *indistinguishability quotient* of Γ. It captures the essential information of "how to add" in the play of game Γ, and is the central figure of our drama.

The natural mapping

$$\Phi : G \mapsto \rho G$$

from $\mathcal{A}$ to $\mathcal{A}/\rho$ is called a *pretending function* (see [P2]). Figures 4 and 6 illustrate the (as it happens, provably periodic [P2]) pretending functions of Pascal's Beans and Guiles, respectively. We shall gradually come to see that the recovery of $\mathfrak{Q}$ and Φ from Γ is the essence of impartial combinatorial game analysis in both normal and misere play.

When Γ is chosen as a normal-play impartial game, the elements of $\mathfrak{Q}$ work out to be in 1-1 correspondence with the *nim-heap values* (or G*-values*) that occur in the play of the game Γ. For if G and H are normal-play impartial games with $G = *g$ and $H = *h$, one easily shows that G and H are indistinguishable if and only if $g = h$. Additionally, in normal play, every position G satisfies the equation

$$G + G = 0.$$

As a result, the addition in a normal-play indistinguishability quotient is an abelian group in which every element is its own additive inverse. The addition operation in the quotient $\mathfrak{Q}$ is *nim addition*. Every normal play indistinguishability quotient is therefore isomorphic to a (possibly infinite) direct product

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots,$$

and a position is a P-position precisely if it belongs the congruence class of the identity (that is, $*0$) of this group. In this sense "nothing new" is learned about normal play impartial games via the indistinguishability quotient construction — instead, we've simply recast Sprague–Grundy theory in new language. The fun begins when the construction is applied in *misere play*, instead.

6. Misere indistinguishability quotients

In misere play, the indistinguishability quotient $\mathcal{Q}$ turns out to be a commutative monoid whose structure intimately depends upon the particular game Γ that is chosen for analysis. We need to cover some background material first.

6.1. Preliminaries. Consider the following three concepts in impartial games:

(i) The notion of the *endgame* (or *terminal position*), that is, a game that has no options at all.
(ii) The notion of a *P-position*, that is, a game that is a second-player win in best play of the game.
(iii) The notion of the *sum of two identical games*, that is, $G + G$.

In normal play, these three notions are *indistinguishable* — wherever a person sees (1) in a sum S, he could freely substitute (2) or (3) (or vice-versa, or any combination of such substitutions) without changing the outcome of S.

The three notions do not coincide in misere play. Let's see what happens instead.

The misere endgame. In misere play, the endgame is an N-position, not a P-position: even though there is no move available from the endgame, a player still wants it to be his *turn to move* when facing the endgame in misere play, because that means his opponent just *lost*, on his previous move.

Misere outcome calculation. After the special case of the endgame is taken care of, the recursive rule for outcome calculation in misere play is exactly as it is in normal play: a non-endgame position G is a P-position if and only if all its options are N-positions. Misere games cannot be identified with nim heaps, in general, however — instead, a typical misere game looks like a complicated, usually unsimplifiable tree of options [ONAG], [GrS1956].

Misere P-positions. Since the endgame is not a misere P-position, the simplest misere P-position is the *nim-heap of size one*, that is, the game played using one bean on a table, where the game is to take that bean. To avoid confusion both with what happens in normal play, and with the algebra of the misere indistinguishability quotient to be introduced in the sequel, let's introduce some special symbols for the three simplest misere games:

$\mathbb{0}$ = The misere *endgame*, that is, a position with no moves at all.

$\mathbb{1}$ = The misere *nim heap of size one*, that is, a position with one move (to $\mathbb{0}$).

$\mathbb{2}$ = The misere *nim heap of size two*, that is, the game $\{\mathbb{0}, \mathbb{1}\}$.

Two games that we've intentionally left off this list are $\{\mathbb{1}\}$ and $\mathbb{1} + \mathbb{1}$. Assiduous readers should verify they are both indistinguishable from $\mathbb{0}$.

Misere sums involving P-positions. Suppose that G is an arbitrary misere P-position. Consider the misere sum

$$S = \mathbb{1} + G. \tag{6-1}$$

Who wins S? It's an N-position — a winning first-player move is to simply take the nim heap of size one, leaving the opponent to move first in the P-position G. In terms of outcomes, equation (6-1) looks like

$$N = P + P. \tag{6-2}$$

Equation (6-2) does not remind us of normal play very much — instead, we always have $P + P = P$ in normal play. On the other hand, it's not true that sum of two misere P-positions is *always* a misere N-position — in fact, when two typical misere P-positions G and H are added together with *neither* equal to $\mathbb{1}$, it *usually* happens that their sum is a P-position, also. But that's not *always* the case — it's also possible that two misere impartial P-positions, neither of which is $\mathbb{1}$, can nevertheless result in an N-position when added together. Without knowing the details of the misere P-position involved, little more can be said in general about the outcome when it's added to another game.

Misere sums of the form $G + G$. In normal play, a sum $G + G$ of two identical games is always indistinguishable from the endgame. In misere play, it's true that both $\mathbb{0}+\mathbb{0}$ and $\mathbb{1}+\mathbb{1}$ are indistinguishable from $\mathbb{0}$, but beyond those two sums, positions of the form $G + G$ are rarely indistinguishable from $\mathbb{0}$. It frequently happens that a position G in the play of a game Γ has no $H \in \mathscr{A}$ such that $G + H$ is indistinguishable from $\mathbb{0}$. This lack of natural inverse elements makes the structure of a typical misere indistinguishability quotient a *commutative monoid* rather than an *abelian group*.

The game $\mathbb{2} + \mathbb{2}$. The sum

$$\mathbb{2} + \mathbb{2}$$

is an important one in the theory of impartial misere games. It's a P-position in misere play: for if you move first by taking 1 bean from one summand, I'll take two from the other, forcing you to take the last bean. Similarly, if you choose to take 2 beans, I'll take 1 from the other. So whereas in normal play one has the equation

$$(*2 + *2) \;\; \rho \;\; *0,$$

it's certainly **not** the case in misere play that

$$(\mathbb{2} + \mathbb{2}) \;\; \rho \;\; \mathbb{0},$$

since the two sides of that proposed indistinguishability relation don't even have the same outcome. But perhaps

$$\mathbb{2} + \mathbb{2} \overset{?}{\rho} \mathbb{1} \tag{6-3}$$

is valid? The indistinguishability relation (6-3) looks plausible at first glance — at least the positions on both sides are P-positions. To decide whether it's possible to distinguish between $\mathbb{2}+\mathbb{2}$ and $\mathbb{1}$, we might try adding various fixed games X to both, and see if we ever get differing outcomes:

Misere game X	Misere outcome of $\mathbb{2}+\mathbb{2}+X$	Misere outcome of $\mathbb{1}+X$
$\mathbb{0}$	P	P
$\mathbb{1}$	N	N
$\mathbb{2}$	N	N
$\mathbb{1}+\mathbb{2}$	N	N
$\mathbb{2}+\mathbb{2}$	P	N

The two positions look like they *might* be indistinguishable, until we reach the final row of the table. It reveals that $(\mathbb{2}+\mathbb{2})$ distinguishes between $(\mathbb{2}+\mathbb{2})$ and $\mathbb{1}$. So equation (6-3) fails. Since a set of misere game positions $\mathscr{A}$ that includes $\mathbb{2}$ and is closed under addition and taking options must contain all of the games $\mathbb{1}$, $\mathbb{2}$, and $\mathbb{2}+\mathbb{2}$, we've shown that a game that isn't She-Loves-Me-She-Loves-Me-Not *always* has at least *two* distinguishable P-position types. In normal play, there's just one P-position type up to indistinguishability — the game $*0$.

6.2. Indistinguishability versus canonical forms. In normal play, Sprague–Grundy theory describes how to determine the outcome of a sum $G+H$ of two games G and H by computing *canonical* (or *simplest*) forms for each summand — these turn out to be *nim-heap equivalents* $*k$. In both normal and misere play, canonical forms are obtained by pruning reversible moves from game trees (see [GrS1956], [ONAG] and [WW]).

In [ONAG], Conway succinctly gives the rules for misere game tree simplification to canonical form:

> When H occurs in some sum we should naturally like to replace it by [a] simpler game G. Of course, we will normally be given only H, and have to find the simpler game G for ourselves. How do we do this? Here are two observations which make this fairly easy:
>
> (i) G must be obtained by deleting certain options of H.

(ii) G itself must be an option of any of the deleted options of H, and so G must be itself be a *second option* of H, if we can delete any option at all.

On the other hand, if we obey (1) and (2), the deletion is permissible, except that we can only delete *all* the options of H (making $G = 0$ [the endgame]) if one of the them is a second-player win.

Unlike in normal play, the canonical form of a misere game is not a nim heap in general. In fact, many misere game trees hardly simplify at all under the misere simplification rules. Figure 7, which duplicates information in [ONAG] (its Figure 32), shows the 22 misere game trees born by day 4.

$\mathbb{0} = \{\}$	$\mathbb{2}_{++} = \{\mathbb{2}_+\}$	$\mathbb{2}_+3\mathbb{0} = \{\mathbb{2}_+, 3, \mathbb{0}\}$
$\mathbb{1} = \{\mathbb{0}\}$	$\mathbb{2}_+\mathbb{0} = \{\mathbb{2}_+, \mathbb{0}\}$	$\mathbb{2}_+3\mathbb{1} = \{\mathbb{2}_+, 3, \mathbb{1}\}$
$\mathbb{2} = \{\mathbb{0}, \mathbb{1}\}$	$\mathbb{2}_+\mathbb{1} = \{\mathbb{2}_+, \mathbb{1}\}$	$\mathbb{2}_+3\mathbb{2} = \{\mathbb{2}_+, 3, \mathbb{2}\}$
$3 = \{\mathbb{0}, \mathbb{1}, \mathbb{2}\}$	$\mathbb{2}_+\mathbb{2} = \{\mathbb{2}_+, \mathbb{2}\}$	$\mathbb{2}_+3\mathbb{2}\mathbb{0} = \{\mathbb{2}_+, 3, \mathbb{2}, \mathbb{0}\}$
$4 = \{\mathbb{0}, \mathbb{1}, \mathbb{2}, 3\}$	$\mathbb{2}_+\mathbb{2}\mathbb{0} = \{\mathbb{2}_+, \mathbb{2}, \mathbb{0}\}$	$\mathbb{2}_+3\mathbb{2}\mathbb{1} = \{\mathbb{2}_+, 3, \mathbb{2}, \mathbb{1}\}$
$\mathbb{2}_+ = \{\mathbb{2}\}$	$\mathbb{2}_+\mathbb{2}\mathbb{1} = \{\mathbb{2}_+, \mathbb{2}, \mathbb{1}\}$	$\mathbb{2}_+3\mathbb{2}\mathbb{1}\mathbb{0} = \{\mathbb{2}_+, 3, \mathbb{2}, \mathbb{1}, \mathbb{0}\}$
$3_+ = \{3\}$	$\mathbb{2}_+\mathbb{2}\mathbb{1}\mathbb{0} = \{\mathbb{2}_+, \mathbb{2}, \mathbb{1}, \mathbb{0}\}$	
$\mathbb{2} + \mathbb{2} = \{3, \mathbb{2}\}$	$\mathbb{2}_+3 = \{\mathbb{2}_+, 3\}$	

Figure 7. Canonical forms for misere games born by day 4.

Whereas only one normal-play nim-heap is born at each birthday n, over 4 million nonisomorphic misere canonical forms are born by day five. The number continues to grow very rapidly, roughly like a tower of exponentials of height n ([ONAG]). This very large number of mutually distinguishable trees has often made misere analysis look like a hopeless activity.

Indistinguishability identifies games with different misere canonical forms. The key to the success of the indistinguishability quotient construction is that it is a *construction localized to the play of a particular game* Γ. It therefore has the possibility of identifying misere games with different canonical forms. While it's true that for misere games G, H with different canonical forms that there must be a game X such that $G + X$ and $H + X$ have different outcomes, such an X *might possibly never occur* in play of the fixed game Γ that we've chosen to analyze. Indistinguishability quotients are often *finite*, even for games Γ that involve an infinity of different canonical forms amongst their position sums.

7. What is a wild misere game?

Roughly speaking, a misere impartial game Γ is said to be *tame* when a complete analysis of it can be given by identifying each of its positions with some position that arises in the misere play of Nim. Tameness is therefore an attribute of a *set* of positions, rather than a *particular* position. Games Γ that are not tame are said to be *wild*. Unlike tame games, wild games cannot be completely analyzed by viewing them as disguised versions of misere Nim.

7.1. Tame games. Conway's *genus theory* was first described in chapter 12 of [ONAG]. It describes a method for calculating whether all the positions of particular misere game Γ are tame, and how to give a complete analysis of Γ, if so. For completeness, we've summarized the genus theory in the Appendix (page 81).

For misere games Γ that genus theory identifies as tame, a complete analysis can be given without reference to the indistinguishability quotient construction. Various efforts to extend genus theory to wider classes of games have been made. Example settings where progress has been made are the main subject of papers by of Ferguson [F2], [F3] and Allemang [A1], [A2], [A3].

Indistinguishability quotients for tame games. In this section, we reformulate the genus theory of tame games in terms of the indistinguishability quotient language.

Suppose S is some finite set of misere combinatorial games. We'll use the notation $\text{cl}(S)$ (the *closure* of S) to stand for the smallest set of games that includes every element of S and is closed under addition and taking options. Putting $\mathscr{A} = \text{cl}(S)$ and defining the indistinguishability quotient

$$\mathscr{Q} = A/\rho,$$

the natural question arises, what is the monoid $\mathscr{Q}$? Figure 8 shows answers for $S = \{\mathbb{1}\}$ and $S = \{\mathbb{2}\}$.

S	Presentation for monoid $\mathscr{Q}$	Order	Symbol	Name
$\{\mathbb{1}\}$	$\langle\, a \mid a^2 = 1 \,\rangle$	2	$\mathscr{T}_1$	First tame quotient
$\{\mathbb{2}\}$	$\langle\, a,\ b \mid a^2 = 1,\ b^3 = b \,\rangle$	6	$\mathscr{T}_2$	Second tame quotient

Figure 8. The first and second tame quotients.

$\mathscr{T}_1$ is called the *first tame quotient*. It represents the misere play of *She-Loves-Me, She-Loves-Me-Not*. In $\mathscr{T}_1$, misere P-positions are represented by the monoid (in fact, group) element a, and N-positions by 1.

$\mathcal{T}_2$, the *second tame quotient*, has the presentation

$$\langle\, a, b \mid a^2 = 1,\ b^3 = b\,\rangle.$$

It is a six-element monoid with two P-position types $\{a, b^2\}$. The prototypical game Γ with misere indistinguishability quotient $\mathcal{T}_2$ is the game of Nim, played with heaps of 1 and 2 only. See Figures 9 and 10.

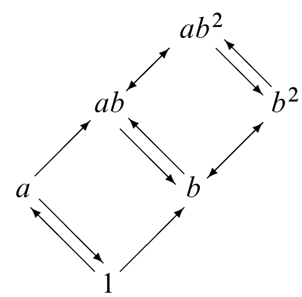

Figure 9. *The misere impartial game theorist's coat of arms,* or the Cayley graph of $\mathcal{T}_2$. Arrows have been drawn to show the action of the generators a (the doubled rungs of the ladder) and b (the southwest-to-northeast-oriented arrows) on $\mathcal{T}_2$. See also Figure 10.

The general tame quotient. For $n \geq 2$, the n-th *tame quotient* is the monoid $\mathcal{T}_n$ with $2^n + 2$ elements and the presentation

$$\mathcal{T}_n = \langle\, a, \underbrace{b, c, d, e, f, g, \dots}_{n-1 \text{ generators}} \mid a^2 = 1, b^3 = b,\ c^3 = c,\ d^3 = d,\ e^3 = e,\ \dots,\ b^2 = c^2 = d^2 = e^2 = \cdots \rangle.$$

$\mathcal{T}_n$ is a disjoint union of its two maximal subgroups $\mathcal{T}_n = U \cup V$. The set

$$U = \{1, a\}$$

is isomorphic to $\mathbb{Z}_2$. The remaining 2^n elements of $\mathcal{T}_n$ form the set

$$V = \{\ a^{a_i} b^{b_i} c^{c_i} d^{d_i} e^{e_i} \cdots \ \mid\ a_i = 0 \text{ or } 1,\ b_i = 1 \text{ or } 2,\ \text{Each of } c_i, d_i, e_i, \dots = 0 \text{ or } 1\ \}.$$

and have an addition isomorphic to

$$\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{n \text{ copies}}.$$

Position type	Misere indistinguishability quotient element	Outcome	Genus
Even #𝟙's only	1	N	0^{120}
Odd #𝟙's only	a	P	1^{031}
Odd #𝟚's and Even #𝟙's	b	N	2^{20}
Odd #𝟚's and Odd #𝟙's	ab	N	3^{31}
Even #𝟚's (≥ 2) and Even #𝟙's	b^2	P	0^{02}
Even #𝟚's (≥ 2) and Odd #𝟙's	ab^2	N	1^{13}

Figure 10. When misere Nim is played with heaps of size 1 and 2 only, the resulting misere indistinguishability quotient is the tame six-element monoid $\mathcal{T}_2$. For more on genus symbols and tameness, see Section 7. See also Figure 9.

The elements a and b^2 are the only P-position types in $\mathcal{T}_n$.

8. More wild quotients

8.1. The commutative monoid $\mathcal{R}_8$. The smallest *wild* misere indistinguishability quotient $\mathcal{R}_8$ has eight elements, and is unique up to isomorphism [S1] amongst misere quotients with eight elements. Its monoid presentation is

$$\mathcal{R}_8 = \langle\, a,\ b,\ c\ \mid a^2 = 1,\ b^3 = b,\ bc = ab,\ c^2 = b^2\,\rangle.$$

The P-positions are $\{a, b^2\}$.

0.75. An example game with misere quotient $\mathcal{R}_8$ is the octal game **0.75**. The first complete analysis of **0.75** was given by Allemang using his *generalized genus theory* [A1]. Alternative formulations of the **0.75** solution are also discussed at length in the appendix of [P] and in [A2]. See Figure 11, left.

	1	2
0+	1	a
2+	b	a
4+	b	c
6+	b	c
8+	b	ab^2
10+	b	ab^2
12+	b	ab^2
14+	...	

	1	2	3	4	5	6	7	8
0+	a	1	a	b	1	a	1	ab
8+	a	c	a	b	1	ac	1	ab
16+	a	c	a	b	1	ac	1	ab

Figure 11. The pretending function for misere play of **0.75** (left) and **0.34**.

8.2. Flanigan's games. Jim Flanigan found solutions to the wild octal games **0.34** and **0.71**; a description of them can be found in the "Extras" of chapter 13 in [WW]. It's interesting to write down the corresponding misere quotients.

0.34. The misere indistinguishability quotient of **0.34** has order 12. There are three P-position types. The pretending function has period 8 (see Figure 11, right).

$$\mathcal{Q}_{\mathbf{0.34}} = \langle\, a, b, c \mid a^2 = 1,\ b^4 = b^2,\ b^2 c = b^3,\ c^2 = 1 \,\rangle, \qquad P = \{a, b^2, ac\}$$

0.71. The game **0.71** has a misere quotient of order 36 with the presentation

$$\mathcal{Q}_{\mathbf{0.71}} = \langle\, a, b, c, d \mid a^2 = 1,\ b^4 = b^2,\ b^2 c = c,\ c^4 = ac^3,\ c^3 d = c^3,\ d^2 = 1 \,\rangle.$$

The P-positions are $\{a, b^2, bc, c^2, ac^3, ad, b^3 d, cd, bc^2 d\}$. The pretending function appears in Figure 12.

	1	2	3	4	5	6
0+	a	b	a	1	c	1
6+	a	d	a	1	c	1
12+	a	d	a	1	c	1
18+	...					

Figure 12. The pretending function for misere play of **0.71**.

8.3. Other quotients. Hundreds more such solutions have been found amongst the octal games. The forthcoming paper [PS] includes a census of such results.

9. Computing presentations and MisereSolver

How are such solutions computed? Aaron Siegel's recently developed Java program *MisereSolver* [AS2005] will do it for you! Some details on the algorithms used in *MisereSolver* are included in [PS]. Here, we simply give a flavor of the some ideas underpinning it and how the software is used.

9.1. Misere periodicity. At the center of Sprague–Grundy theory is the equation $G + G = 0$, which always holds for an arbitrary normal play combinatorial game G. One consequence of $G + G = 0$ is the equation

$$G + G + G = G,$$

in which all we've done is add G to both sides. In general, in normal play,

$$(k + 2) \cdot G = k \cdot G.$$

holds for every $k \geq 0$.

In *misere play*, the relation

$$(G + G) \ \rho \ \mathbb{0}$$

happens to be true for $G = \mathbb{0}$ and $G = \mathbb{1}$, but beyond that, it is only seldom true for occasional rule sets Γ and positions G. On the other hand,

$$(G + G + G) \ \rho \ G$$

is *very often* true in misere play, and it is *always* true, for all G, if Γ is a tame game. And in *wild* games Γ for which the latter equation fails, often a weaker equation such as

$$(G + G + G + G) \ \rho \ (G + G),$$

is still valid, regardless of G.

These considerations suggest that a useful place to look for misere quotients is inside commutative monoids having some (unknown) number of generators x each satisfying a relation of the form

$$x^{k+2} = x^k$$

for each generator x and some value of $k \geq 0$.

9.2. Partial quotients for heap games. A *heap game* is an impartial game Γ whose rules can be expressed in terms of play on separated, noninteracting heaps of beans. In constructing misere quotients for heap games, it's useful to introduce the *n-th partial quotient*, which is just the indistinguishability quotient of Γ when all heaps are required to have n or fewer beans.

9.3. MisereSolver output of partial quotients. Here is an (abbreviated) log of *MisereSolver* output of partial quotients for **0.123**, an octal game that is studied in great detail in [P2]. In this output, monomial exponents have been juxtaposed with the generator names (so that b^2c, for example, appears as b2c). The program stops when it discovers the entire quotient — the partial quotients stabilize in a monoid of order 20, whose single-heap pretending function Φ is periodic of length 5.

```
C:\work>java -jar misere.jar 0.123
=== Normal Play Analysis of 0.123 ===
Max   : G(3) = 2
Period: 5 (5)
=== Misere Play Analysis of 0.123 ===
-- Presentation for 0.123 changed at heap 1 --
Size 2: TAME
P = {a}
Phi = 1 a 1
-- Presentation for 0.123 changed at heap 3 --
Size 6: TAME
P = {a,b2}
Phi = 1 a 1 b b a b2 1
-- Presentation for 0.123 changed at heap 8 --
Size 12: {a,b,c | a2=1,b4=b2,b2c=b3,c2=1}
P = {a,b2,ac}
Phi = 1 a 1 b b a b2 1 c
-- Presentation for 0.123 changed at heap 9 --
Size 20: {a,b,c,d | a2=1,b4=b2,b2c=b3,c2=1,b2d=d,cd=bd,d3=ad2}
P = {a,b2,ac,bd,d2}
Phi = 1 a 1 b b a d2 1 c d a d2 1 c d a d2 1 c d a d2 1
=== Misere Play Analysis Complete for 0.123 ===
Size 20: {a,b,c,d | a2=1,b4=b2,b2c=b3,c2=1,b2d=d,cd=bd,d3=ad2}
P = {a,b2,ac,bd,d2}
Phi = 1 a 1 b b a d2 1 c d a d2 1 c d a d2 1 c d a d2 1
Standard Form : 0.123
Normal Period : 5
Normal Ppd    : 5
Normal Max G  : G(3) = 2
Misere Period : 5
Misere Ppd    : 5
Quotient Order: 20
Heaps Computed: 22
Last Tame Heap: 7
```

9.4. Partial quotients and pretending functions. Let's look more closely at the *MisereSolver* partial quotient output in order to illustrate some of the subtlety of misere quotient presentation calculation.

In Figure 13, we've shown three pretending functions for **0.123**. The first is just the normal play pretending function (that is, the nim-sequence) of the game, to heap six. The second table shows the corresponding misere pretending function for the partial quotient to heap size 6, and the final table shows the initial portion of the pretending function for the entire game (taken over arbitrarily large heaps).

With these three tables in mind, consider the following question:

When is $4+4$ indistinguishable from 6 in **0.123**?

Normal **0.123**

n	1	2	3	4	5	6	7	8	9	10
$G(n)$	$*1$	$*0$	$*2$	$*2$	$*1$	$*0$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

Misere **0.123** to heap 6: $\langle a, b \mid a^2 = 1,\ b^3 = b\rangle$, order 6

n	1	2	3	4	5	6
$\Phi(n)$	a	1	b	b	a	b^2

Complete misere **0.123** quotient, order 20

$$\langle a,b,c,d \mid a^2=1,\ b^4=b^2,\ b^2c=b^3,\ c^2=1,\ b^2d=d,\ cd=bd,\ d^3=ad^2\rangle$$

n	1	2	3	4	5	6	7	8	9	10
$\Phi(n)$	a	1	b	b	a	d^2	1	c	d	$\cdots$

Figure 13. Iterative calculation of misere partial quotients differs in a fundamental way from normal play nim-sequence calculation because sums at larger heap sizes (for example, $8+9$) may distinguish between positions that previously were indistinguishable at earlier partial quotients (e.g., $4+4$ and 6, to heap size six).

Let's answer the question. In normal play (the top table), $4+4$ is indistinguishable from 6 because

$$G(4+4) = G(4) + G(4) = *2 + *2 = *0 = G(6).$$

And in the middle table, $4+4$ is also indistinguishable from 6, since both sums evaluate to b^2. But in the final table,

$$\Phi(4+4) = \Phi(4) + \Phi(4) = b \cdot b = b^2 \neq d^2 = \Phi(6),$$

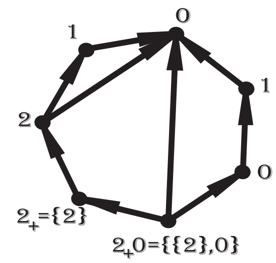

Figure 14. Misere coin-sliding on a directed heptagon with two additional edges. An arbitrary number of coins are placed at the vertices, and two players take turns sliding a single coin along a single directed edge. Play ends when the final coin reaches the topmost (sink) node (labelled $\mathbb{0}$). Whoever makes the last move loses the game. The associated indistinguishability quotient is a commutative monoid of order 14 with presentation $\langle\ a,\ b,\ c\ \mid\ a^2 = 1,\ b^3 = b,\ b^2c = c,\ c^3 = ac^2\ \rangle$ and P-positions $\{\ a,\ b^2,\ bc,\ c^2\}$. See Section 9.5 and Figure 15.

that is, $4+4$ can be distinguished from 6 in play of **0.123** *when no restriction is placed on the heap sizes*. In fact, one verifies that the sum $8+9$, a position of type cd, distinguishes between $4+4$ and 6 in **0.123**.

The fact that the values of partial misere pretending functions may change in this way, as larger heap sizes are encountered, makes it highly desirable to carry out the calculations via computer programs that know how to account for it.

9.5. Quotients from canonical forms. In addition to computing quotients directly from the Guy–Smith code of octal games [GS], *MisereSolver* also can take as input the a canonical form of a misere game G. It then computes the indistinguishability quotient of its closure cl(G). This permits more general games than simply heap games to be analyzed.

A coin-sliding game. For example, suppose we take $G = \{2_+, \mathbb{0}\}$, a game listed in Figure 7. In the output script below, *MisereSolver* calculates that the indistinguishability quotient of cl(G) is a monoid of order 14 with four P-position types:

```
-- Presentation for 2+0 changed at heap 1 --
Size 2: TAME
P = {a}
Phi = 1 a
-- Presentation for 2+0 changed at heap 2 --
Size 6: TAME
P = {a,b2}
Phi = 1 a b b2
-- Presentation for 2+0 changed at heap 4 --
Size 14: {a,b,c | a2=1,b3=b,b2c=c,c3=ac2}
```

```
P = {a,b2,bc,c2}
Phi = 1 a b c2 c
```

Figure 9.5 shows a coin-sliding game that can be played perfectly using this information. Figure 15 shows how the canonical forms at each vertex correspond to elements of the misere quotient.

Canonical form	$\mathbb{0}$	$\mathbb{1}$	$\mathbb{2}$	$\mathbb{2}_+$	$\{\mathbb{2}_+, \mathbb{0}\}$
Quotient element	1	a	b	c^2	c

Figure 15. Assignment of single-coin positions in the heptagon game to misere quotients elements.

10. Outlook

At the time of this writing (December 2005), the indistinguishability quotient construction is only one year old. Several aspects of the theory are ripe for further development, and the misere versions of many impartial games with complete normal play solutions remain to be investigated. We have space only to describe a few of the many interesting topics for further investigation.

10.1. Infinite quotients. Misere quotients are not always finite. Today, it frequently happens that *MisereSolver* will "hang" at a particular heap size as it discovers more and more distinguishable position types. Is it possible to improve upon this behavior and discover algorithms that can handle infinite misere quotients?

Dawson's chess. One important game that seems to have an infinite misere quotient is Dawson's Chess. In the equivalent form **0.07**, (called Dawson's Kayles), Aaron Siegel [PS] found that the order of its misere partial quotients $\mathcal{Q}$ grows as indicated in Figure 16:

Heap size	24	26	29	30	31	32	33	34
$\lvert Q \rvert$	24	144	176	360	520	552	638	∞(?)

Figure 16. Is **0.07** infinite at heap 34?

Since Redei's Theorem (see [P2] for discussion and additional references) asserts that a finitely generated commutative monoid is always finitely presentable, the object being sought in Figure 16 (the misere quotient presentation to heap size 34) certainly exists, although it most likely has a complicated structure of P- and N-positions. New ideas are needed here.

Infinite, but not at bounded heap sizes. Other games seemingly exhibit infinite behavior, but appear to have finite order (rather than simply finitely presentable) partial quotients at all heap sizes. One example is **.54**, which shows considerable structure in the partial misere quotients output by *MisereSolver*. Progress on this game would resolve difficulties with an incorrect solution of this game that appears in the otherwise excellent paper [A3]. Siegel calls this behavior *algebraic periodicity*.

10.2. Classification problem. The *misere quotient classification problem* asks for an enumeration of the possible nonisomorphic misere quotients at each order $2k$, and a better understanding of the category of commutative monoids that arise as misere quotients[7]. Preliminary computations by Aaron Siegel suggest that the number of nonisomorphic misere quotients grows as follows:

Order	2	4	6	8	10	12
# quotients	1	0	1	1	1?	6?

Figure 17. Conjectured number of nonisomorphic misere quotients at small orders.

Evidently misere quotients are far from general commutative semigroups — by comparison, the number of nonisomorphic commutative semigroups at orders 4, 6, and 8 are already 58, 2143, and 221805, respectively [Gril, p. 2].

10.3. Relation between normal and misere play quotients. If a misere quotient is finite, does each of its elements x necessarily satisfy a relation of the form $x^{k+2} = x^k$, for some $k \geq 0$? The question is closely related to the structure of maximal subgroups inside misere finite quotients. Is every maximal subgroup of the form $(\mathbb{Z}_2)^m$, for some m?

At the June 2005 Banff conference on combinatorial games, the author conjectured that an octal game, if misere periodic, had a periodic normal play nim sequence with the two periods (normal and misere) equal. Then Aaron Siegel pointed out that **0.241**, with normal period two, has misere period 10. Must the normal period length *divide* the misere one, if both are periodic?

10.4. Quaternary bounties. Again at the Banff conference, the author distributed the list of wild misere quaternary games in Figure 18.

The author offered a bounty of $25 dollars/game to the first person to exhibit the misere indistinguishability quotient and pretending function of the games in the list. Aaron Siegel swept up 17 of the bounties [PS], but **.3102**, **.3122**, **.3123**, and **.3312** are still open.

[7] It can be shown that a finite misere quotient has even order [PS].

(**.0122**, 1^{20}, 12)	(**.0123**, 1^{20}, 12)	(**.1023**, 2^{1420}, 11)	(**.1032**, 2^{1420}, 12)
(**.1033**, 1^{20}, 11)	(**.1231**, 2^{1420}, 8)	(**.1232**, 2^{1420}, 9)	(**.1233**, 2^{1420}, 9)
(**.1321**, 2^{1420}, 9)	(**.1323**, 2^{1420}, 10)	(**.1331**, 1^{20}, 8)	(**.2012**, 1^{20}, 5)
(**.2112**, 1^{20}, 5)	(**.3101**, 1^{20}, 4)	(**.3102**, 0^{20}, 5)	(**.3103**, 1^{20}, 4)
(**.3112**, 2^{1420}, 7)	(**.3122**, 2^{1420}, 4)	(**.3123**, 1^{31}, 6)	(**.3131**, 2^{1420}, 6)
(**.3312**, 2^{1420}, 5)			

Figure 18. The twenty-one wild four-digit quaternary games (with first wild genus value and corresponding heap size).

10.5. Misere sprouts endgames. Misere Sprouts (see [WW], 2nd edition, Vol III) is perhaps the only misere combinatorial game that is played competitively in an organized forum, the *World Game of Sprouts Association*. It would be interesting to assemble a database of misere sprout endgames and compute the indistinguishability quotient of their misere addition.

10.6. The misere mex mystery. In normal play game computations for heap games, the *mex rule* allows the computation of the heap $n+1$ nim-heap equivalent from the equivalents at heaps of size n and smaller. The *misere mex mystery* asks for the analogue of the normal play mex rule, in misere play. It is evidently closely related to the partial quotient computations performed by *MisereSolver*.

10.7. Commutative algebra. A beginning at application of theoretical results on commutative monoids to misere quotients was begun in [P2]. What more can be said?

Appendix: Genus theory

We summarize Conway's *genus theory*, first described in [ONAG, chapter 12] and used extensively in *Winning Ways*. It describes a method for calculating whether all the positions of particular game Γ are tame, and how to give a complete analysis of Γ, if so. The genus theory assigns to each position G a particular symbol

$$\text{genus}(G) = \mathsf{G}^*(G) = g^{g_0 g_1 g_2 \cdots}. \qquad \text{(A-1)}$$

where the g and the g_i's are always nonnegative integers. We'll define this genus value precisely and illustrate how to calculate genus values for some example games G, below.

To look at this in more detail, we need some preliminary definitions before giving definition of genus values.

A.8. Grundy numbers. Let $*k$ represent the nim heap of size k. The *Grundy number* (or *nim value*) of an impartial game position G is the unique number k such that $G + *k$ is a second-player win. Because Grundy numbers may be defined relative to normal or misere play, we distinguish between the *normal play Grundy number* $\mathsf{G}^+(G)$ and its counterpart $\mathsf{G}^-(G)$, the *misere Grundy number*.

In normal play, Grundy numbers can be calculated using the rules $\mathsf{G}^+(0) = 0$, and otherwise, $\mathsf{G}^+(G)$ is the least number (from $0,1,2,\ldots$) that is *not* the Grundy number of an option of G (the so-called *minimal excludant*, or *mex*).

When normal play is in effect, every game with Grundy number $\mathsf{G}^+(G) = k$ can be thought of as the nim heap $*k$. No information about best play of the game is lost by assuming that G is in fact precisely the nim heap of size k. Moreover, in normal play, the Grundy number of a sum is just the nim-sum of the Grundy numbers of the summands.

The misere Grundy number is also simple to define [p. 140, bottom][ONAG]:

> $\mathsf{G}^-(0) = 1$. Otherwise, $\mathsf{G}^-(G)$ is the least number (from $0,1,2,\ldots$) which is not the G^--value of any option of G. Notice that this is just like the ordinary "mex" rule for computing G^+, except that we have $\mathsf{G}^-(0) = 1$, and $\mathsf{G}^+(0) = 0$.

Misere P-positions are precisely those whose first genus exponent is 0.

A.9. Indistinguishability vs misere Grundy numbers. When misere play is in effect, Grundy numbers can still be defined — as we've already said — but many *distinguishable* games are assigned the *same* Grundy number, and the outcome of a sum is *not* determined by Grundy numbers of the summands. These unfortunate facts lead directly to the apparent great complexity of many misere analyses.

Here is the definition of the genus, directly from [ONAG], now at the bottom of page 141:

> In the analysis of many games, we need even more information than is provided by either of these values [G^+ and G^-], and so we shall define a more complicated symbol that we call the G^*-value, [or *genus*], $\mathsf{G}^*(G)$. This is the symbol
>
> $$g^{g_0 g_1 g_2 \cdots}$$
>
> where
>
> $$\begin{aligned} g &= \mathsf{G}^+(G) \\ g_0 &= \mathsf{G}^-(G) \\ g_1 &= \mathsf{G}^-(G + 2) \\ g_2 &= \mathsf{G}^-(G + 2 + 2) \\ \ldots &= \ldots \end{aligned}$$

> where in general g_n is the G$^-$-value of the sum of G with n other games all equal to [the nim-heap of size] 2.

At first sight, the genus symbol looks to be an potentially infinitely long symbol in its "exponent." In practice, it can be shown that the g_i's always fall into an eventual period two pattern. By convention, a genus symbol is written down with a finite exponent with the understanding that its final two values repeat indefinitely.

The only genus values that arise in misere Nim are the *tame genera*

$$\underbrace{0^{120}, 1^{031}}_{\substack{\text{Genera of normal play } *0 \text{ (resp, } *1\text{) Nim} \\ \text{positions involving nim heaps of size 1 only}}}$$

and

$$\underbrace{0^{02}, 1^{13}, 2^{20}, 3^{31}, 4^{46}, \ldots, n^{n(n\oplus 2)}, \ldots}_{\substack{\text{Genera of } *n \text{ normal-play Nim positions} \\ \text{involving at least one nim heap of size} \geq 2.}}$$

Figure 19. Correspondence between normal play nim positions and tame genera.

The value of genus theory lies in the following result [ONAG, Theorem 73]:

> **Theorem**: If all the positions of some game Γ have tame genera, the genus of a sum $G + H$ can be computed by replacing the summands by Nim-positions of the same genus values, and taking the genus value of the resulting sum.

In order to apply the theorem to analyze a tame game Γ, a person needs to know several things:

(i) How to compute genus symbols for positions G of a game Γ;
(ii) That every position of the game Γ does have a tame genus;
(iii) The correspondence between the tame genera and Nim positions.

We've already given the correspondence between normal-play Nim positions and their misere genus values, in Figure (19). We'll defer the most complicated part — how to compute genera, and verify that they're all tame — to the next section.

The addition rule for tame genera is not complicated. The first two symbols have the $\mathbb{Z}_2$ addition

$$\begin{aligned} 0^{120}+0^{120} &= 0^{120} \\ 0^{120}+1^{031} &= 1^{031} \\ 1^{031}+1^{031} &= 0^{120} \end{aligned}$$

Two positions with genus symbols of the form $n^{n(n \oplus 2)}$ add just like Nim heaps of $*n$. For example,

$$2^{20}+3^{31}=1^{13}.$$

The symbol 0^{120} adds like an identity, for example:

$$4^{46}+0^{120}=4^{46}.$$

When 1^{031} is added to a $n^{n(n \oplus 2)}$, it acts like 1^{13}:

$$4^{46}+1^{031}=5^{57}.$$

It has to emphasized that these rules work *only if all positions in play of Γ are known to have tame genus values*. If, on the other hand, even a single position in a game Γ does *not* have a tame genus, the game is wild and *nothing can be said in general about the addition of tame genera*.

A.10. Genus calculation in octal game 0.123. Let's press on with genus theory, illustrating it in an example game, and keeping in mind the end of Chapter 13 in [WW]:

> *The misere theory of impartial games is the last and most complicated theory in this book. Congratulations if you've followed us so far...*

Genus computations, and the nature of the conclusions that can be drawn from them, are what makes Chapter 13 in *Winning Ways* complicated. In this section we illustrate genus computations by using them to initiate the analysis of a particular wild octal game (**0.123**). Because the game **0.123** is wild, genus theory will *not* lead to a complete analysis of it. A complete analysis can nevertheless be obtained via the indistinguishability quotient construction; for details, see [P2].

The octal game **0.123** can be played with counters arranged in heaps. Two players take turns removing one, two or three counters from a heap, subject to the following additional conditions:

(i) Three counters may be removed from any heap;
(ii) Two counters may be removed from a heap, but only if it has more than two counters; and

+	1	2	3	4	5
0+	1	0	2	2	1
5+	0	0	2	1	1
10+	0	0	2	1	1
15+	⋯				

Figure 20. Normal play nim values of **0.123**.

(iii) One counter may be removed only if it is the only counter in that heap.

Normal play of 0.123. The nim sequence of **0.123**[8] is periodic of length 5, beginning at heap 5. See Figure 20.

Misere play genus computations for 0.123. We exhibit single-heap genus values of **0.123** in Figure 21. It's possible to prove that this sequence is also periodic of length 5. However, a periodic genus sequence is not the same thing as a complete misere analysis. Let's see what happens instead.

+	1	2	3	4	5
0+	1^{031}	0^{120}	2^{20}	2^{20}	1^{031}
5+	0^{02}	0^{120}	2^{1420}	1^{20}	1^{031}
10+	0^{02}	0^{120}	2^{1420}	1^{20}	1^{031}
15+	⋯				

Figure 21. G*-values of **0.123**.

There are some tame genus symbols in Figure 21. They are

$$\begin{aligned} 0 &= 0^{1202020\cdots} = 0^{120} \\ 1 &= 1^{0313131\cdots} = 1^{031} \\ 2 &= 2^{2020202\cdots} = 2^{20} \end{aligned}$$

Despite the presence of these tame genera, the game is still wild — the first wild genus value, 2^{1420}, occurs at heap 8. Conway's Theorem 73 on tame games therefore does *not* apply, since it requires *all* positions to have tame genera in order for the game to be treated as misere Nim. We can say nothing about how genera add — even the tame genera — without examining the game more closely.

Here's what we can (and cannot) do with Figure 21.

[8] See *Winning Ways*, Chapter 4, p. 97, "Other Take-Away Games;" also Table 7(b), p. 104.

+	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8	h_9
h_1	0^{120}	1^{031}	3^{31}	3^{31}	0^{120}	1^{13}	1^{031}	3^{0531}	0^{31}
h_2		0^{120}	2^{20}	2^{20}	1^{031}	0^{02}	0^{120}	2^{1420}	1^{20}
h_3			0^{02}	0^{02}	3^{31}	2^{20}	2^{20}	0^{420}	3^{02}
h_4				0^{02}	3^{31}	2^{20}	2^{20}	0^{420}	3^{02}
h_5					0^{120}	1^{13}	1^{031}	3^{0531}	0^{31}
h_6						0^{02}	0^{02}	2^{20}	1^{13}
h_7							0^{120}	2^{1420}	1^{20}
h_8								0^{120}	3^{02}
h_9									0^{02}

Figure 22. Some genus values of games $h_i + h_j$ in **0.123**.

Single heaps. We *can* determine the outcome class of *single-heap* **0.123** positions. The first superscript in a heap's genus symbol is 0 if and only if that heap size is a P-position. The single heap P-positions of **0.123** therefore occur at heap sizes

$$1, 5, 6, 10, 11, 15, 16, 20, 21, \ldots$$

For example, the genus of the heap of size 7 has its first superscript = 1. It is therefore an N-position. The winning move is $7 \to 5$.

Multiple heaps. Using Figure 21, we *cannot* immediately determine the outcome class of **0.123** positions involving *multiple heaps*. However, the figure does provide a basis for investigating multiheap positions. For example, Figure 22 is a table that shows the genera of two-heap positions up to heap size nine.

A.11. Genus calculation algorithm. Here's how the genus of a particular sum $G = h_8 + h_5$ was computed from the earlier single-heap values in Figure 21. First, we rewrote genus(G) in terms of its options:

$$\text{genus}(G) = \text{genus}(h_8 + h_5) = \text{genus}(\{h_6 + h_5, h_5 + h_5, h_8 + h_3, h_8 + h_2\})$$

The genus of a nonempty game $G = \{A, B, \ldots\}$ can be calculated from the genus of its options $A, B, \ldots$ using the *mex-with-carrying algorithm* ($\diamond$ symbols represent positions with no carry):

$$\begin{aligned}
\text{carry}(\gamma) &= \diamond^{\diamond 05313} \\
\text{carry}(\gamma \oplus 1) &= \diamond^{\diamond 14202} \\
\text{genus}(h_6 + h_5) &= 1^{131313\ldots} \\
\text{genus}(h_5 + h_5) &= 0^{120202\ldots} \\
\text{genus}(h_8 + h_3) &= 0^{420202\ldots} \\
\text{genus}(h_8 + h_2) &= \underline{2^{142020\ldots}}
\end{aligned}$$

$$\text{genus}(G) \quad = \quad 3^{053131\dots}$$

The result $\text{genus}(G) = 3^{053131\cdots} = 3^{0531}$ was computed columnwise, working from left to right. First, the "base" and "first superscript" results

$$\mathsf{G}^{+}(G) = \text{mex}(\{1, 0, 0, 2\}) = 3$$

and

$$\mathsf{G}^{-}(G) = \text{mex}(\{1, 1, 4, 1\}) = 0$$

were computed from the corresponding four positions in each option of G, with no carries present. The "carry out" is then $\gamma = 0$. The second superscript result

$$\mathsf{G}^{-}(G + *2) = \text{mex}(\{3, 2, 2, 4, \mathbf{0}, \mathbf{1}\}) = 5$$

involved a similar computation, but with two *carry values*

$$\{\gamma, \gamma \oplus 1\} = \{0, 1\}.$$

thrown into the mex calculation (they're shown in bold). See the more complete description of this algorithm in the section titled *"But What if They're Wild?" asks the Bad Child* ([WW], page 410). It's also illustrated in [ONAG, p. 143].

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