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On the geometry of combinatorial games: A renormalization approach

ERIC J. FRIEDMAN AND ADAM S. LANDSBERG

ABSTRACT. We describe the application of a physics-inspired renormalization technique to combinatorial games. Although this approach is not rigorous, it allows one to calculate detailed, probabilistic properties of the geometry of the P-positions in a game. The resulting geometric insights provide explanations for a number of numerical and theoretical observations about various games that have appeared in the literature. This methodology also provides a natural framework for several new avenues of research in combinatorial games, including notions of "universality," "sensitivity-to-initial-conditions," and "crystal-like growth," and suggests surprising connections between combinatorial games, nonlinear dynamics, and physics. We demonstrate the utility of this approach for a variety of games — three-row Chomp, 3-D Wythoff's game, Sprague–Grundy values for 2-D Wythoff's game, and Nim and its generalizations — and show how it explains existing results, addresses longstanding questions, and generates new predictions and insights.

1. Introduction

In this paper we introduce a method for analyzing combinatorial games based on renormalization. As a mathematical tool, renormalization has enjoyed great success in virtually all branches of modern physics, from statistical mechanics [Goldenfeld 1992] to particle physics [Rivasseau 2003] to chaos theory [Feigenbaum 1980], where it is used to calculate properties of physical systems or objects that exhibit so-called 'scaling' behavior (i.e., geometric similarity on different spatial scales). In the present context we adapt this methodology to the study of combinatorial games. Here, the main "object" we study is the set of

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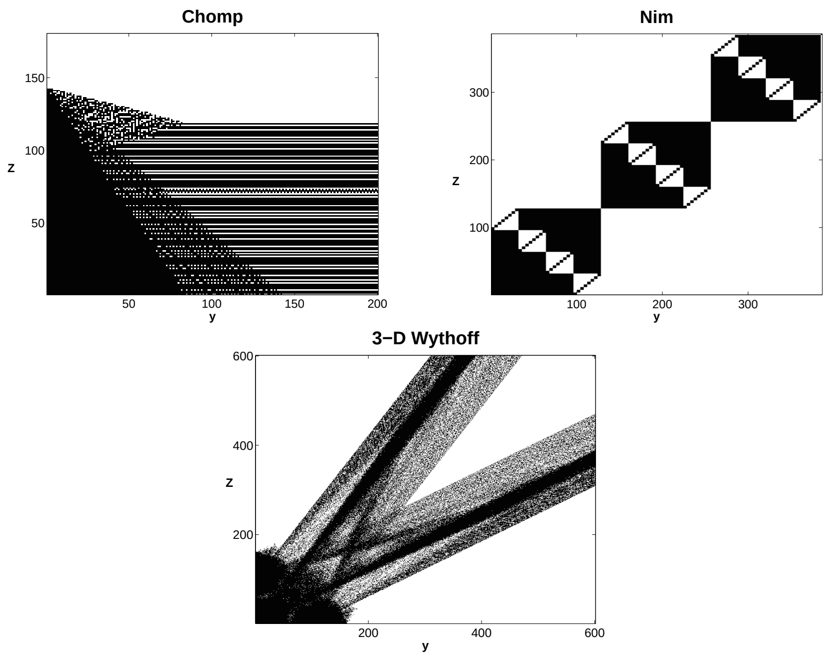

Figure 1. The underlying geometries of combinatorial games. Shown are the IN-sheet structures for Chomp, Nim, and 3-D Wythoff's game.

P-positions[1] of the game, viewed as a geometric entity in the abstract position space of the game (see, e.g., Figure 1, which will be explained later). As we will show, this geometric object exhibits a strong scaling property, and hence can be analyzed via a suitably adapted renormalization technique. Since all critical information about a game is encoded in this geometry, as a methodology renormalization has broad explanatory powers and impressive (numerically verifiable) predictive capabilities, as will be demonstrated through examples.

When we compare renormalization to other traditional analytical techniques for analyzing combinatorial games, such as Sprague–Grundy theory, nimbers, algebraic approaches, and so on [Berlekamp et al. 1979; Bouton 1902; Sprague 1936; Grundy 1939; Conway 1976], several significant distinctions, advantages, and disadvantages emerge:

I: As a mathematical technique, the renormalization procedure described here does not, at present, possess the strict level of rigor needed for formal mathematical proof. In this respect, this renormalization procedure for games 'suffers' the same defect as the renormalization of modern physics: even though renormalization is a highly successful, well-established technique in physics

[1] In games without ties or draws, every game position is either of type N (Next player to move wins) or P (Previous player wins).

that is routinely used to correctly predict physical phenomena with (sometimes unwarranted!) accuracy, there are very few cases where it has been rigorously proven to be correct. In this same spirit, we hope that the reader will find the insights provided by the renormalization analysis of games to be sufficiently compelling so as to warrant its serious consideration as a practical and powerful method of analysis for combinatorial games, despite its non-rigorous status at present.

II: Unlike Sprague–Grundy theory [Bouton 1902; Sprague 1936; Grundy 1939] and its extensions to numbers and nimbers [Berlekamp et al. 1979] which have proven extremely successful for analyzing games that can be decomposed (i.e., expressed as a disjunctive sum of simpler games) such as Dots-and-Boxes [Berlekamp 2000] and Go endgames [Berlekamp 1994], the renormalization approach to games works equally well for decomposable and non-decomposable games. Indeed, many interesting games, such as the early play in chess and Go, have resisted analysis using traditional methods due to their intrinsic non-decomposability. Very little is in fact understood about the optimal strategies in non-decomposable games, even such "elementary" ones as Chomp. We will demonstrate how renormalization can readily handle a non-decomposable game such as Chomp and raise the possibility that such an approach could be extended to more complex games such as go, chess or checkers.

III: One of the interesting features of renormalization is that it results in probabilistic information about the game, despite the fact that we consider only purely deterministic games (i.e., games of no chance). In particular, rather than providing a description of the exact locations (in position space) of the P-positions of a game, renormalization only specifies the probability that a given position is P. However, there are often relatively sharp boundaries associated with these probabilities. So even though renormalization cannot give us the precise (point-by-point) geometry of the game, it will allow us to calculate its broad, overall geometric features, which in fact provides significant information about the game. Indeed, we believe that this inherent "imprecision" in the methodology, rather than being a shortcoming, is in fact what allows renormalization to proceed and what gives it its power. We conjecture that for many combinatorial games there do not exist any simple formulas or polynomial-time algorithms for efficiently computing the exact location of the P-positions, but that probabilistic information is possible. By sacrificing exact geometric information about the game for probabilistic information, significant insights into "hard" combinatorial games can be obtained. This is reminiscent of chaotic dynamical systems and strange attractors in which there does not exist formulas for specific trajectories [Cvitanovic 1989], but global information about the overall structure of attractor does exist. We discuss this further in later sections, but comment here that this view is supported by numerical evidence on the "sensitive dependence on initial conditions" displayed by the game Chomp and discussed in Section 3.2.

IV: The renormalization analysis brings to light several previously unexplored features of combinatorial games, and indeed in certain respects we consider these new lines of inquiry to be one of the highlights of this new approach. In Section 3.1 we introduce the notion of universality, and describe how renormalization provides a natural classification scheme for combinatorial games, wherein games can be grouped into "universality classes" such that all members of a class share key features in common. In Section 3.2 we show how it is possible to discuss the "sensitivity" of a game to certain types of perturbations (i.e., rule changes) within the renormalization framework. And in Section 3.3 we describe how this method reveals unexpected similarities between the geometric structures seen in games and various crystal-growth models and aggregation processes in physics.

V: As a final comment, we emphasize that as a new approach to combinatorial games, renormalization is still very much in its infancy; its limitations, shortcomings, and scope of applicability are not fully understood at present. Hence, in what follows we will simply give a number of worked examples of this method applied to specific games, so that the reader might develop a working feel for how the procedure is actually implemented, and perhaps appreciate its potential utility.

2. Renormalization framework

We begin with a schematic overview of the general renormalization procedure.

The first step is to create a natural geometry for the game. Towards this end, consider the abstract "space" of all positions of a game. Typically, this space can be realized by mapping game positions to a subset of the integer lattice $\mathbb{Z}^d$ for some dimension $d > 1$. The set of all P-positions in this d-dimensional position space, which we call the "P-set", is the key geometric 'object' for study.

To proceed, we next define various sets of $(d-1)$-dimensional hyperplanes ("sheets") that foliate position space. Here, we will let $x \in \mathbb{Z}$ specify the index of a sheet, and $y \in \mathbb{Z}^{d-1}$ the coordinates on the sheet. As we will see, there exist various types of recursion relations and nonlinear operators that relate the different sheets to one another. These sheets will prove instrumental for determining the overall geometric structure of the game's P-set.

There are several basic types of foliating sheets to consider. The first are the P-sheets. These simply mark the location of the game's P-positions within each hyperplane. More precisely, define P_x, the P-sheet at level x, to be a $(d-1)$-

dimensional, semi-infinite matrix consisting of zeros and ones, with ones marking the locations of the P-positions (i.e., $P_x(y) = 1$ if game position $[x, y] \in \mathbb{Z}^d$ is a P-position and 0 otherwise). We will be interested in the geometric patterns (of the ones and zeros) on these sheets, since, taken together, they capture the full geometry of the P-set in d-dimensional position space.

A second type of foliating sheets are the instant-N sheets (IN-sheets for short). They are constructed as follows: Following [Zeilberger 2004], we declare an N-position in the game, $[x, y]$, to be an IN if there exists a legal move from that position to some P-position $[x', y']$ on a lower sheet, $x' < x$. The IN-sheets are simply hyperplanes through position space that mark the locations of the IN's (i.e., defining matrix W_x to be the IN-sheet at level x, set $W_x(y) = 1$ if position $[x, y]$ is an IN and 0 otherwise). As we will see, their key significance lies in the fact that the P-sheets (and, ultimately, the P-set itself) can be computed directly from the IN-sheets via the relation $P_x = \mathcal{M} W_x$, where $\mathcal{M}$ denotes a "supermex" operator (a generalization of the standard Mex operator). Hence, we can think of the IN-sheets as effectively encoding the critical information about the game. Moreover, they will prove useful for visualizing a game's geometric features.

Now, in many examples (e.g., the first three discussed in this paper), it is possible to write down a recursion relation on the IN-sheets:

$$W_x = \mathcal{R} W_{x-1}.$$

As will become clear, this is the key step in the renormalization analysis, since it allows the IN-sheet at level x to be generated directly from its immediate predecessor.[2]

We note, however, that in general it is not always possible to construct a recursive formulation on the IN-sheets themselves, as shown in our 4th example. In such cases, we show how to construct auxiliary sheets $V_x^1, \ldots, V_x^k$ for which a (vector) recursion relation of the form $V_x = \mathcal{R} V_{x-1}$ does exist. (The W_x can then be computed from the vector of sheets V_x.) For ease of presentation we will assume for the remainder of this section that there exists a direct recursion relation for the IN-sheets themselves (making auxiliary sheets unnecessary); however, we will demonstrate the alternate case in an example.

Thus far, the overall scheme is as follows (see Figure 2): We first recursively generate the IN-sheets using the recursion operator $\mathcal{R}$, then use the supermex operator $\mathcal{M}$ to construct the P-sheets. The final key to the renormalization

[2] We remark that for all the games considered here, a judicious choice of position-space coordinates allows one to recursively compute the P-sheet at level x, P_x, from all the preceding sheets, $P_0, P_1, \ldots P_{x-1}$. However, this type of 'infinite'-dimensional recursive formulation is not directly useful for renormalization purposes, since one has to know *all* preceding sheets just to compute the current one; to apply renormalization effectively we require a 'finite'-dimensional recursion relation, like that for the IN-sheets. (Nonetheless, the assumption that P_x can be expressed in terms of all preceding sheets is useful for other parts of the analysis, and we will assume that this is always the case.)

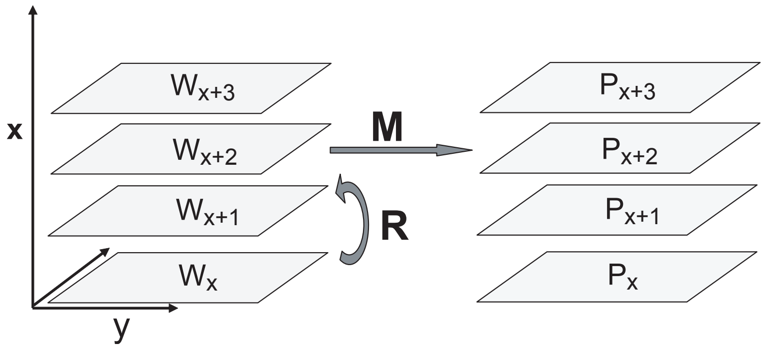

Figure 2. Foliating sheets and associated operator relations.

scheme is the observation that the IN-sheets exhibit a form of 'geometrical invariance'. Loosely speaking, the geometric patterns on sheets at different x levels all look similar to one another (i.e., they exhibit "scaling" — here, shape-preserving growth with increasing x). This allows for a compact description of W_x for large x. In the examples considered we will see that, in some sense, the sheets W_x converge to a specific geometry. However, understanding and defining this 'convergence' relies on two key observations:

The first is that since the geometric structures (i.e. patterns of 0's and 1's) on these IN-sheets 'grow' with increasing x (but maintain their overall shape), we must re-scale (shrink) the sheets with larger x values to see the convergence. We will introduce a rescaling operator $\mathcal{S}$ for this purpose. Second, while the precise structure (point by point) of the IN-sheets does not converge, the overall probabilistic structure (i.e., densities of points) on the IN-sheets does converge. Formally, we must define a probability measure on the sheets for this purpose.

Hence, the asymptotic behavior of the IN-sheets (i.e., their 'invariant geometry') is described by the limiting probability measure $\mathbf{W} = \lim_{x\to\infty}(\mathcal{S}\mathcal{R})^x W_0$. Here, we think of the operator $\mathcal{R}$ as 'growing' an IN-sheet at a given level to the next higher level (since $\mathcal{R}W_x = W_{x+1}$), and we think of the operator $\mathcal{S}$ as inducing a simple geometrical rescaling of the grown sheet back to the original size. Repeated application of these growing and rescaling operators, starting at the initial sheet W_0, yields the desired limiting probability measure $\mathbf{W}$. However, since this limit is independent of the initial sheet W_0 for most interesting cases, we can alternatively express $\mathbf{W}$ as a fixed point of the equation

$$\mathbf{W} = \mathcal{S}\mathcal{R}\mathbf{W}.$$

This "renormalization" equation (with $\mathcal{S}\mathcal{R}$ the "renormalization operator") is our key equation. It states that the invariant measure on a sheet is unchanged if you grow the sheet and then rescale it, thereby expressing the invariance of the

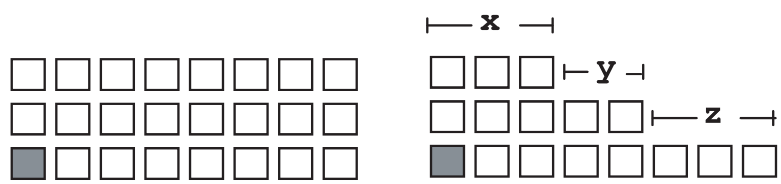

Figure 3. The game of Chomp. Left: the starting configuration of three--row ($M = 3$) Chomp. Right: a sample game configuration after play has begun (describable by coordinates $[x, y, z]$).

geometry on the different IN-sheets. The solution to this equation will provide a complete probabilistic description of the game (including the geometry of its P-positions), and will allow us to understand much about the game. As will be illustrated, in practice the above renormalization equation is most easily solved by deriving a series of related algebraic self-consistency conditions.

We now present several examples. All the games we have chosen share certain features in common. First, they are all impartial games, and do not allow draws or ties. They are also all poset games for which it is possible to define a complete ordering on position space such that the position values strictly decrease during play. Whether these conditions are inherent limitations on the scope of applicability of the renormalization methodology remains to be seen, although we strongly suspect that they are not. For convenience, we have chosen all of our examples to have three-dimensional position spaces, so as to make the visualization of the resulting patterns (and the analytic calculations) more transparent.

2.1. Chomp. We focus first on the game of Chomp, which is, in some sense, among the simplest of the "unsolved" games. Its history is marked by some significant theoretical advances [Gale 1974; Schuh 1952; Zeilberger 2001; Zeilberger 2004; Sun 2002; Byrnes 2003], but it has yet to succumb to a complete analysis in the 30 years since its introduction by Gale [1974] and Schuh [1952]. (Chomp is an example of a non-decomposable game, where traditional methods so far have not proven to be especially effective.)

The rules of Chomp are easily explained. Play begins with an $M \times N$ array of tokens, with the (dark) token in the southwest corner considered "poison" (Figure 3, left). On each turn a player selects a token and removes ("chomps") it along with all tokens to the north and east of it. (Figure 3, right, shows a sample token configuration after two chomps.) Players alternate turns until one player takes the poison token, thereby losing the game.

For simplicity, we consider here the case of three-row ($M = 3$) Chomp, a subject of recent study [Zeilberger 2001; Zeilberger 2004; Sun 2002; Byrnes

2003]. To start, note that the token configuration at any stage of play can be described (using Zeilberger's coordinates) by the position $p = [x, y, z]$, where x, y, z specify the number of columns of height three, two, and one, respectively (Figure 3, right). (Note here that the first coordinate x will eventually serve as our sheet index, and $[y, z]$ the coordinates within a sheet.)

In these coordinates, the game's starting position is $[x, 0, 0]$, while the opening move must be to a position of the form $[x-r, r, 0]$, $[x-s, 0, s]$ or $[x-t, 0, 0]$ (these are the "children" of the starting position). Every position may be classified as either an N-position — if a player starting from that position can guarantee a win under perfect play — or as a P-position otherwise (draws and ties not being possible). The computation of N- and P-positions rests on the standard observation that all children of a P-position must be N-positions, and at least one child of an N-position must be a P-position. For example, position $[0, 0, 1]$ (where only the poison token remains) is a P-position by definition, so $[0, 1, 0]$ must be an N-position since its child is $[0, 0, 1]$. (Note that a *winning move* in the game is always from an N-position to a P-position.)

An intriguing feature of Chomp, as shown by Gale [1974], is that the player who moves first can always win under optimal play (i.e., $[x, 0, 0]$ is an N-position). The proof uses an elegant strategy-stealing argument: Consider the "nibble" move to $[x-1, 1, 0]$. If this is a winning move, then we are done. If it is not a winning move, then the second player must have a winning response, in which case the first player could have chosen this as the opening move instead of the nibble, leading to a win. Observe that this argument provides no information as to what the desired opening move for the first player should be (or even whether it is unique), only that it exists — a longstanding question that the renormalization analysis will address.

In previous numerical studies of the game by Brouwer [2004] and others, several linear scaling relations were noticed. For example, for every x (under $\approx 80,000$) there is a P-position of the form $[x, 0, z]$ where $z = 0.7x \pm 1.75$; other sequences with similar linear scaling behavior were also observed. Zeilberger [2001], Sun [2002] and Byrnes [2003] also find more complex patterns in the P-positions, including periodic orbits and intimations of possible chaotic-like behavior. The existence of these numerically observed scaling behaviors provides the first hint that some type of renormalization approach may be possible, as we now describe.

To begin, we introduce the foliating sheets, indexed by their x values, as in [Zeilberger 2004]. (As noted previously, sheet index x here corresponds to the first coordinate of Chomp's three-dimensional position space $[x, y, z]$.) For any x, recall that the P-sheet P_x is a two-dimensional, semi-infinite matrix that marks the location of all P-positions at the specified x value. The $(y, z)^{\text{th}}$

element of this matrix is denoted $P_x(y,z)$. (We note for future reference the easily proven fact that for every x there exists at most one z, call it $z^*(x)$, such that $[x,0,z^*(x)]$ is a P-position, i.e. $P_x(0,z^*(x)) = 1$.) The IN-sheets are defined as in the previous section: $W_x(y,z) = 1$ if $[x,y,z]$ is an IN, and 0 otherwise[3].

As noted earlier, the IN-sheet W_x contains all the necessary information for computing the corresponding P-sheet P_x, and, moreover, one can calculate W_{x+1} directly from W_x. To see this, we define the following operators:

Identity I: for any sheet A, let $(IA)(y,z) = A(y,z)$.

Left shift $\mathcal{L}$: for any sheet A, let $(\mathcal{L}A)(y,z) = A(y+1,z)$.

Diagonal $\mathcal{D}$: for any x the action of $\mathcal{D}$ on the P-sheet P_x is given by

$$(\mathcal{D}P_x)(z^*(x)-t,t) = 1 \text{ for } 0 \le t \le z^*(x),$$
$$(\mathcal{D}P_x)(y,z) = P_x(y,z) \text{ otherwise.}$$

Supermex $\mathcal{M}$: for any x the action of $\mathcal{M}$ on W_x is defined via the following algorithm:

(1) Set $\mathcal{M}W_x = 0$, $T_x = W_x$, $y = 0$.
(2) Let z_s be the smallest z such that $T_x(y,z) = 0$ and set $(\mathcal{M}W_x)(y,z_s) = 1$, $T_x(y+t, z_s-t) = 1$ for all $0 \le t \le z_s$.
(3) If $z_s = 0$ stop; else let $y \to y+1$ and go to step 2.

A direct computation shows (see [Friedman and Landsberg 2007] for details):

$$P_x = \mathcal{M}W_x,\ W_{x+1} = \mathcal{L}(I+\mathcal{D}\mathcal{M})W_x.$$

Thus, defining $\mathcal{R} \equiv \mathcal{L}(I+\mathcal{D}\mathcal{M})$ yields $W_{x+1} = \mathcal{R}W_x$. (These relations all follow simply from a careful application of the game rules.) This provides the setting for a renormalization analysis. (For future reference, we also mention one additional relation which is sometimes of use: $W_x = \sum_{t=1}^{x} \mathcal{L}^t \mathcal{D} P_{x-t}$, where all sums are binary and interpreted as logical OR's. This relation follows from the observation that the IN positions at level x are generated from the parents[4] of P-positions at all lower levels.)

Numerical solution of the recursion equation $W_{x+1} = \mathcal{R}W_x$ reveals an interesting structure for the IN-sheets (W_x), characterized by several distinct regions (Figure 4a). Most crucially, the IN-sheets at different x levels 'scale' (see, e.g., Figure 4b): their overall geometric structures are identical (in a probabilistic sense) up to a linear scale factor. In particular, the boundary-line slopes and densities of points in the interior regions of the sheets W_x are the same for all

[3] This definition differs in a small but important way from the "instant-winner" sheets introduced in [Zeilberger 2004].

[4] A *parent* of position p is defined as any position from which it is possible to reach p in a single move.

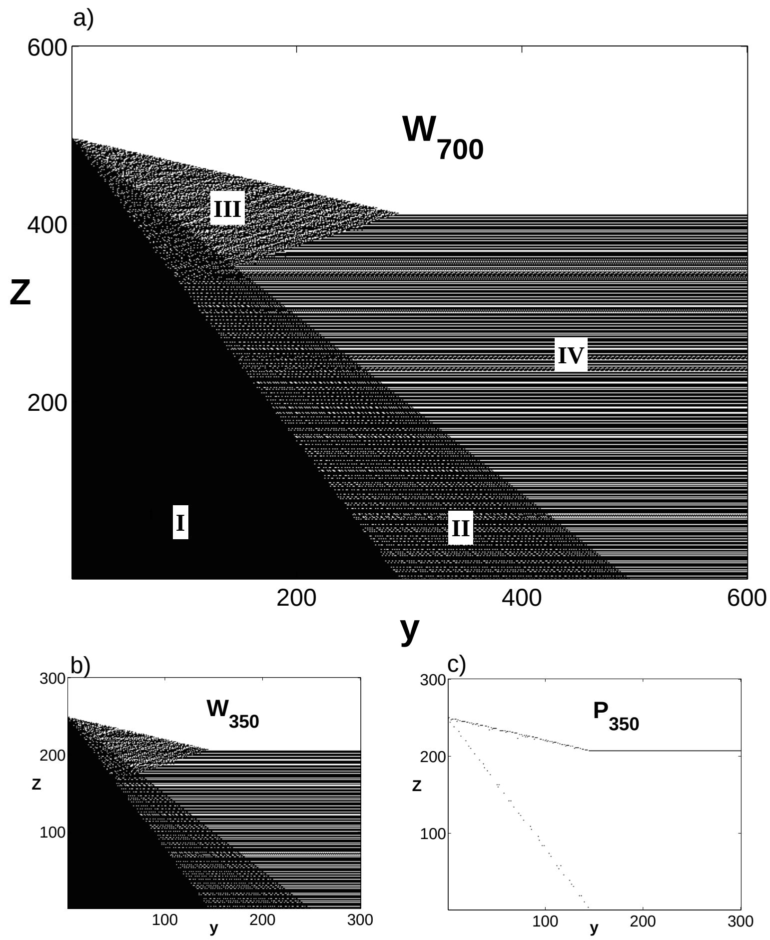

Figure 4. The geometry of Chomp. (a) the IN-sheet geometry W_x for three-row Chomp, shown for $x = 700$. Here, IN locations in the y-z plane (i.e., the 1's in the matrix) are shown in black. (b) The IN-sheet W_x for $x = 350$. Comparison with W_{700} illustrates the geometric invariance of the sheets. (c) The geometry of P_x, shown for $x = 350$. The P-sheets also exhibit geometrical invariance, i.e., the P_x for all x exhibit identical structure (in the probabilistic sense) up to an overall scale factor.

x (though the actual point-by-point locations of the instant-N positions within the sheets will differ).

Thus, the invariant geometry $\mathbf{W}$ of the sheets satisfies the renormalization fixed-point equation $\mathbf{W} = \mathcal{S}\mathcal{R}\mathbf{W}$, with operators $\mathcal{S}$ and $\mathcal{R}$ defined above. We now show how to analyze this equation, and ultimately determine the structure of Chomp's P-set. We point out that a direct, formal assault on the renormalization equation would require that the renormalization operator be carefully defined on the space of probability distributions over IN-sheets, which proves somewhat delicate. In practice, the desired result can be more easily obtained through a somewhat less formal procedure, as we now describe.

We begin by considering the structure of a typical P-sheet P_x (Figure 4c). Numerically, it is found to consist of three (diffuse) lines (heretofore called P-lines) that may be characterized by six fundamental geometric parameters: a lower P-line of slope m_L and density of points λ_L, an upper line of slope m_U and density λ_U, and a flat line extending to infinity. The upper and lower P-lines originate from a point whose height (i.e., z-value) is αx. The flat line (with density one) is only present with probability γ in randomly selected P-sheets. Our goal is to determine analytical values for these six geometric parameters that characterize the P-set. (Recall that the IN-sheet geometry can be directly linked with this P-sheet geometry via $W_x = \sum_{t=1}^{x} \mathcal{L}^t \mathcal{D} P_{x-t}$.) Hence, a determination of the parameters $m_L, \lambda_L, m_U, \lambda_U, \alpha, \gamma$ will provide a complete probabilistic description of the entire geometric structure of the game[5]

To get at this geometry, we will derive a set of algebraic self-consistency equations relating the six geometric parameters. Intuitively, these equations arise from the demand that as an IN-sheet at level x (W_x) 'grows' to W_{x+1} under the action of the recursion operator $\mathcal{R}$, its overall geometry is preserved. The key to actually implementing this analysis is to observe that the P-positions in sheet P_x (i.e., the 1's of the matrix; see Figure 4c) are constrained to lie along certain boundaries in W_x (Figure 4b); the various interior regions of W_x remain "forbidden" to P-positions. Geometric invariance of the sheets demands that these forbidden regions be preserved as an IN-sheet grows under the recursion operator. Each such forbidden region yields a constraint on the allowable geometry of the W_x's, and may be formulated as an algebraic equation relating the hitherto unknown parameters $m_L, \lambda_L, m_U, \lambda_U, \alpha, \gamma$ that define the P-sheets. In all, we find six independent geometric constraints:

$$\frac{\lambda_U}{1+m_U} = 1, \tag{2-1}$$

$$\frac{1}{1+\alpha} - \frac{\lambda_L}{1+m_L} = 1, \tag{2-2}$$

[5]Here, we will not be addressing the interesting issue of the small 'scatter' of points around the P-lines. Numerical simulations show that the range of scatter is in fact extremely narrow (e.g., the distance from a P-position to the idealized P-line is always less than 5 for $x < 1,000$, and does not appear to increase with x). Despite its smallness, the scatter is not at all irrelevant, and indeed, we believe it is largely the scatter that makes a purely deterministic analysis of Chomp hard. Our probabilistic description provides a means of bypassing much of this difficulty while still extracting useful information about the game. We will revisit this notion briefly in Section 4.

$$(\gamma-1)\frac{m_L}{\alpha-m_L}+\frac{1}{1+\alpha}=1, \tag{2-3}$$

$$\lambda_U+\lambda_L=1, \tag{2-4}$$

$$\frac{\alpha\lambda_L}{\alpha-m_L}\left(\frac{m_U-m_L}{(m_U-m_L)\alpha+m_L\gamma}\right)+\frac{1}{1+\alpha}=1, \tag{2-5}$$

$$\frac{\lambda_L}{\alpha-m_L}-\frac{\alpha}{\alpha+1}\left(1-\frac{\lambda_U}{\alpha-m_U}\right)=0. \tag{2-6}$$

These six constraints arise as follows (see Figure 4a): (1) arises from forbidden region III; (2) from region II; (3) from the bottom row of region I; (4) from operator $\mathcal{M}$ in regions I,II,III; (5) from the lower part of region I; and (6) from the upper part of region I. To illustrate we derive constraint (3) here. (For detailed derivations of the others see [Friedman and Landsberg 2007].)

Recall that $W_x=\sum_{t=1}^{x}\mathcal{L}^t\mathcal{D}P_{x-t}$, so that the IN-sheet at level x is 'built up' from a series of earlier P-lines (coming from lower-level sheets) and diagonal lines (associated with operator $\mathcal{D}$). Constraint (3) arises because the lower P-lines and diagonal lines each contribute points (i.e., instant-N's) to the bottom-most row of region I and completely fill it up, thereby rendering it forbidden. Now, the density of the diagonal lines along the bottom row of W_x can be computed from elementary geometry to be $(1+\alpha)^{-1}$. The density of the lower P-lines is $-m_L/(\alpha-m_L)$. However, each lower P-line will only contribute a point to the bottom row (at $z=0$) with probability $(1-\gamma)$, since this equals the probability that the flat P-line doesn't exist (by step 3 in the supermex algorithm). Hence, the actual density of points contributed by the lower P-lines to the bottom row becomes $-(1-\gamma)m_L/(\alpha-m_L)$. Summing this density with that of the diagonals and equating to unity yields constraint (3) (i.e., this is the condition that the bottom row of the IN-sheets always remains forbidden to P-positions even as the sheets grow.)

Taken together, the above six constraints may be solved exactly, yielding precise values for the geometric parameters of the game. These are:

$$\alpha=\tfrac{1}{\sqrt{2}},\ \lambda_L=1-\tfrac{1}{\sqrt{2}},\ \lambda_U=\tfrac{1}{\sqrt{2}},$$
$$m_L=-1-\tfrac{1}{\sqrt{2}},\ m_U=-1+\tfrac{1}{\sqrt{2}},\ \gamma=\sqrt{2}-1.$$

Thus, we have found the renormalization fixed point, and hence have a complete (probabilistic) characterization of the game of Chomp — i.e., the global geometric structure of its P-set.

With this geometric insight, it becomes straightforward to explain virtually all numerical properties of the game previously reported in the literature, including various numerical conjectures by Brouwer [2004], along with a variety of new results. As an illustration of its utility, we show how this geometric result lets us decide (in a probabilistic sense) the optimal first move of the game, which has been a longstanding open question.

To start, recall that the possible opening moves from the starting position $[x, 0, 0]$ are to positions of the form $[x-r, r, 0]$, $[x-s, 0, s]$ or $[x-t, 0, 0]$ (bearing in mind that the desired (winning) opening move will be to a P-position). The last of these, $[x-t, 0, 0]$, can never be a P-position, by Gale's strategy-stealing argument. Next consider $[x-r, r, 0]$, which we will refer to as an "r"-position. From the geometric structure of the P-sheets, a simple calculation shows that the only accessible P-position of this form is for $r(x) \approx \alpha x/(\alpha - m_L)$. Likewise, the only possible P-position of the form $[x-s, 0, s]$ (i.e., an "s"-position) is for $s(x) \approx \alpha x/(\alpha + 1)$. (Note that we use $\approx$ here since these are asymptotic values; for any finite x there are small deviations owing to the slight scatter of P-positions around the P-lines in Figure 4c.) Thus, the P-set geometry has allowed us to identify the asymptotic locations of the only two possible winning opening moves in the game! Moreover, since $r(x) < s(x)$, the "s"-position is a child of the "r"-position, so only one of these two positions can be an actual P-position (for a given x). Hence the winning opening move is unique — a result which was previously only known numerically [Brouwer 2004] for x values up to a certain level. Taking this further, we can also compute the probabilities that this unique winning opening move will be to the "r"-position or to the "s"-position, as follows: For each starting position $[x, 0, 0]$ from $x = 1 \ldots x_{\max}$ there is an associated "r"-position $[x - r(x), r(x), 0]$, which may or may not be a P-position. The total number of actual "r"-type P-positions with an x-value less than or equal to $x_{\max} - r(x_{\max})$ is just $\gamma(x_{\max} - r(x_{\max}))$. So the fraction of "r"-positions which are actually P-positions is $\gamma(x_{\max} - r(x_{\max}))/x_{\max} = \sqrt{2} - 1$. Thus, the winning opening move is to the "r"-position with probability $\sqrt{2} - 1$, and to the "s"-position with probability $2 - \sqrt{2}$.

The ease with which the above results were obtained (once the P-set geometry was determined) illustrates the utility of this geometrically based, renormalization approach as a potentially powerful tool for analyzing games.

2.2. Nim. We next consider three-heap Nim [Bouton 1902]. Note that this simple game is decomposable and "solvable" (in the sense that a simple criterion exists for deciding if a given position is N or P). In this game we let $[x, y, z]$ represent the heights of the three heaps. An allowed move in Nim consists of reducing a single position coordinate by an arbitrary amount. As in Chomp, we will let the various hyperplanes be indexed by x, and the coordinates within each plane by $[y, z]$. (Note that while these coordinates break the natural permutation symmetry of the heaps, the resulting analysis is not affected.)

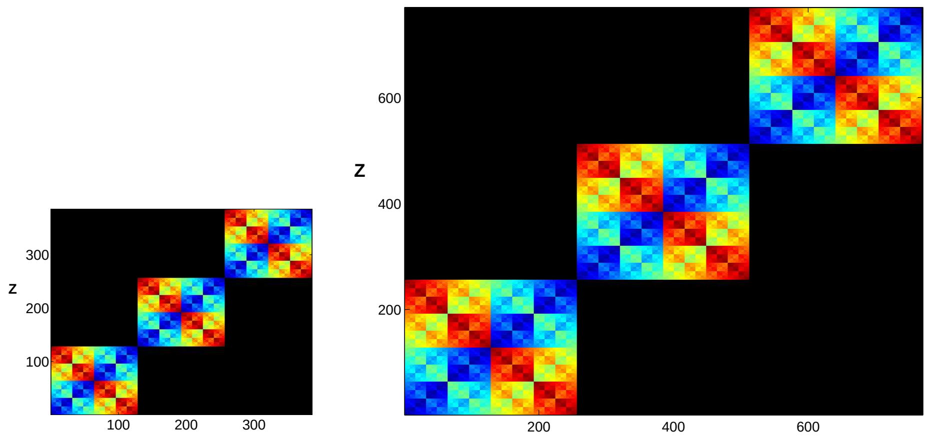

Figure 5. IN-sheet geometry for ordinary Nim: W_{128} (left) and W_{256} (right). (Note here that the instant-N positions have been color-coded based on the order in which they were recursively generated; the black background corresponds to non-instant-N's.

A straightforward calculation based on the game rules shows that the IN-sheets are related to the P-sheets by

$$W_x = \sum_{x'=0}^{x-1} P_{x'}, \qquad (*)$$

where addition denotes the logical OR operation. This relation simply reflects the fact that the IN's at level x are, by definition, determined by the parents of the P-positions at the lower levels.

We define the action of the supermex operator $\mathcal{M}$ on W_x via the following algorithm:

(1) Set $\mathcal{M}W_x = 0$, $T_x = W_x$, $y = 0$.
(2) Let z_s be the smallest z such that $T_x(y, z) = 0$ and set $(\mathcal{M}W_x)(y, z_s) = 1$, $T_x(y + t, z_s) = 1$ for all $0 \le t$.
(3) Let $y \to y + 1$ and go to step 2.

This yields the relation,

$$P_x = \mathcal{M}W_x, \qquad (**)$$

as the reader may verify. Combining $(*)$ and $(**)$ yields the desired recursion formula,

$$W_{x+1} = \mathcal{R}W_x,$$

with $\mathcal{R} = I + \mathcal{M}$ and I the identity operator.

The IN-sheets are readily constructed using this recursion operator. Figure 5 displays the sheet geometry W_x for $x = 128$ and $x = 256$. Again, we observe that the geometry of the game's IN-sheets scale (linearly with x), just as in the preceding case of Chomp. As before, one could construct algebraic self-consistency conditions that exploit this scaling, and thereby develop a geometric characterization of the sheets. This is unnecessary here since the game of Nim is easily solvable, and all sheets can be directly constructed using nimbers instead. So for the moment we will content ourselves with having set up the basic renormalization framework for the game. However, we will revisit this issue when we discuss a modified (nontrivial) version of Nim later in this paper.

Before moving to our next example, we remark briefly on one unique feature of Nim (not seen in the earlier Chomp example). In Chomp, all W_x's (regardless of x) look geometrically similar up to linear rescaling. In Nim, the sheets exhibit linear scaling, but also display a periodicity (in x) in powers of 2. For instance, the sequence $W_{128}, W_{256}, W_{512}, \dots$ exhibits geometric invariance (up to rescaling), as does the sequence $W_{100}, W_{200}, W_{400}, \dots$. However, the basic patterns for these two sequences will differ somewhat (e.g., Figure 1b illustrates the geometry for this second sequence.) The existence of this periodicity means that the original renormalization equation will not have a true fixed point; however, in practice this can be handled by using a slightly modified renormalization equation which exploits this periodicity (loosely speaking, $W_{2x} = (\mathcal{S}\mathcal{R})^x W_x$), but we do not pursue this further. (Nim is the only one of our examples to display this feature.)

2.3. 2-D Wythoff's game and Sprague–Grundy values. In our next example we show how renormalization can be used to compute the Sprague–Grundy values of a game. We illustrate here with Wythoff's game [Wythoff 1907], whose Grundy values have been the subject of a recent study by Nivasch [2004]. Wythoff's game is equivalent to two-heap Nim, where in addition to removing an arbitrary number from either heap a player can also remove the same number from both heaps. Thus if coordinates $[y, z]$ represent the heights of the two heaps then a legal move reduces either of the coordinates by an arbitrary amount, or both by the same amount.

It is well known that the P-positions of Wythoff's game are all of the form $(\lfloor \phi k \rfloor, \lfloor \phi^2 k \rfloor)$ and $(\lfloor \phi^2 k \rfloor, \lfloor \phi k \rfloor)$ for all positive integers $k > 0$, where $\phi = (\sqrt{5} + 1)/2$ is the golden ratio [Wythoff 1907]. Thus, they lie near the lines through the origin with slopes ϕ and ϕ^{-1}. However, the characterization of the Sprague–Grundy values for Wythoff's game is significantly more difficult. Nivasch [2004] has shown that these Grundy values also lie 'close' to these lines; specifically, that a position with Grundy value g is bounded within a distance $O(g)$ of these lines. We show that this result follows directly from a straightforward renormalization analysis — although our proof is not rigorous, whereas Nivasch's is.

The general procedure for computing the Sprague–Grundy values of a game via renormalization is straightforward: Add a single Nim heap to be played in conjunction (i.e., disjunctive sum) with the game of interest, and then do ordinary renormalization on the combined game. In the present case we represent the position space of the combined game by coordinates $[x, y, z]$, where a player can either move in Wythoff's game, $[y, z]$, or play on the Nim pile by reducing x by an arbitrary amount. Then, a standard argument shows that Wythoff position $[y, z]$ has Grundy value x if $[x, y, z]$ is a P-position of the combined game. Thus the sheets P_x correspond to the set of all positions in Wythoff's game with Grundy value x.

As in Nim, we use the IN-sheets and note that[6]

$$W_x = \sum_{x'=0}^{x-1} P_{x'}.$$

One can compute $\mathcal{M}$ from the properties of Wythoff's game:

(1) Set $\mathcal{M}W_x = 0$, $T_x = W_x$, $y = 0$.
(2) Let z_s be the smallest z such that $T_x(y, z) = 0$ and set $(\mathcal{M}W_x)(y, z_s) = 1$, $T_x(y + t, z_s) = 1$, $T_x(y + t, z_s + t) = 1$ for all $0 \le t$.
(3) Let $y \to y + 1$ and go to step 2.

Combining this supermex algorithm with the preceding expression yields the recursion operator

$$\mathcal{R} = I + \mathcal{M}.$$

We now analyze the invariant geometry of the game, and show how it explains Nivasch's result.

A representative IN-sheet and P-sheet are shown in Figure 6. We will focus on the outer regions of these graphs (i.e., the large $[y, z]$ regime), avoiding the more complicated structures near the origin. Here, the IN-sheet consists of two (thick) lines, an upper and lower one, whose slopes we denote by m_U, m_L. The P-sheet also exhibits two related lines (P-lines) of the same slopes as in the IN-sheet (we neglect the fact that closer inspection shows that each P-line is a double line, as this will not affect the calculation). We let λ_U, λ_L denote the density of points (i.e., P-positions) along these lines (per unit horizontal).

The renormalization analysis of the invariant geometry is simplified by the observation that the regions of Figure 6 labeled I, II, III, IV are entirely devoid of P-positions. (More precisely, if the P-sheet and IN-sheet plots are superimposed, the P-positions do not appear in the four labeled regions.) These four regions are thus "forbidden." The fact that they contain no P-positions provides

[6] This holds true for a sum of any game with a single Nim pile, which arises whenever one analyzes the Sprague–Grundy values.

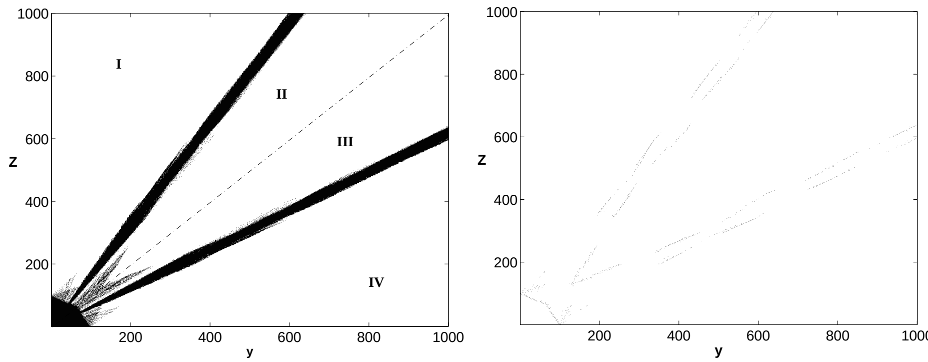

Figure 6. IN-sheet and P-sheet associated with the Sprague–Grundy values of Wythoff's game: W_{100} (left) and P_{100} (right). Note that a 45^o-line has been artificially added to the W_{100} plot so as to demarcate regions II and III.

constraints that allow us to compute analytical values for the four parameters $m_U, m_L, \lambda_U, \lambda_L$ that characterize the invariant geometry of the P-set.

The absence of P-positions in the forbidden regions is due to the fact that these empty regions get completely filled up (during the supermex operation $P_x = \mathcal{M}W_x$) by parents of the P-positions. (Note that these parents cannot themselves be P-positions.) Within any given sheet, the parents of a position $[y, z]$ lie along three lines (one vertical, one horizontal, and one diagonal): $V = \{[y+k, z] \mid k > 0\}$, $H = \{[y, z+k] \mid k > 0\}$, and $D = \{[y+k, z+k] \mid k > 0\}$. Forbidden region (I) gets completely covered by the vertical lines V arising from (parents of) P-positions along the lower and upper P-lines. The density of these (per unit y) is given by $\lambda_L + \lambda_U$ which must equal 1, since they can't overlap and must completely fill the region. Likewise, Forbidden region (IV) is completely filled by horizontal lines arising from the upper and lower P-lines. Since their densities (per unit z) are λ_i / m_i, $i = L, U$, it follows that $\lambda_L/m_L + \lambda_U/m_U = 1$. Forbidden region (II) is filled by the diagonal lines emerging from the upper P-line. Elementary geometry shows the density of these lines to be $\lambda_U/(m_U - 1)$, yielding $\lambda_U/(m_U - 1) = 1$. (We note that horizontal lines from the upper P-line and vertical lines from the lower P-line also contribute to region (II), but since neither of these — either alone or in combination — is sufficient to completely fill the region, and since they are not well correlated with the diagonal line, it must be the case that the diagonal lines alone are sufficient to fill the region.) A similar argument for region (III) shows that the diagonals emerging from the lower P-line completely fill the region, yielding $\lambda_L/(1 - m_L) = 1$.

Solving these four constraints we find that $m_U = \phi = m_L^{-1}$ and $\lambda_U = \phi^{-1}$, $\lambda_L = 1 - \phi^{-1}$, which agree with numerical observations. Thus, we see that the Sprague–Grundy values lie near the rays defined by the game's P-positions, in agreement with Nivasch's result. It is also straightforward to show that the deviation from these lines must be $O(x)$ (where x corresponds to the Grundy value). This follows from the game's recursion relation, $W_{x+1} = (I + \mathcal{M})W_x = W_x + P_x$, which shows that an IN-sheet at level x is built up from a series of lower-level P-sheets (whose total number is x). Since the P-positions in the P-sheets can never overlap with one another as they are being laid down to form the IN-sheet, it follows that the width of the two N-sheet lines must be $O(x)$, also in agreement with [Nivasch 2004].

The geometric picture emerging from our analysis actually suggests a way to compute a crude estimate for the tightness of this bound. This bound is related to the width of the two (thick) lines in the sheet W_x, which we can calculate as follows: Consider a section of horizontal extent S of one of these lines. The area occupied by this section of line is just that of a rectangle with length $S\sqrt{1+m^2}$ and thickness w (where m denotes the slope of the line, either m_U or m_L). The total number of points making up this area is just $\lambda S x$ (since the thick line is built from x P-lines, each one contributing λS points, with $\lambda = \lambda_U$ or λ_L). Since this area is completely filled, it must have density 1. Solving, we find that the line thickness is $w = x/(\phi\sqrt{1+\phi^2})$. Thus, our probabilistic estimate is that, asymptotically speaking, a game position with Grundy value g will roughly lie within a distance of $g/(\phi\sqrt{1+\phi^2})$ of the known P-lines. (Here, asymptotic refers to game positions with suitably large values of $[y, z]$, so that we are far from the complex structure located near the origin of Figure 6.)

In summary, this analysis illustrates how the renormalization method can be used to rather easily (albeit nonrigorously) obtain results that are difficult to obtain by more traditional methods, including Sprague–Grundy results. Moreover, this type of geometric analysis reveals insights that are less apparent by other means. In the present case, for instance, we find a complex structure near the origin of the IN- and P-sheets, which (to our knowledge) has not been recognized before. Separate treatments of the local and asymptotic structures in this game would presumably allow one to derive even tighter analytical bounds than were obtained in [Nivasch 2004].

2.4. Three-dimensional Wythoff's game. As our last example, we consider a 3-D generalization of the ordinary (2-D) Wythoff's game, for which, as far as we know, relatively little is known. Here, we will not carry out the complete renormalization analysis, but will derive the necessary analytical operators and recursion relations (which in this case will require the use of auxiliary sheets in addition to the IN-sheets), and also numerically illustrate the geometric scaling property of these sheets.

The rules of the game are as follows: Letting $[x, y, z]$ denote the three heap sizes, one can remove one or more tokens from a single heap, or the same number from any pair of heaps. Other versions of the game are also possible: For instance, one could replace the rule about removing tokens from any pair of heaps with one allowing removal of an equal number of tokens from all three heaps, or keep all the original rules and supplement them with this additional one. In any case, the derivation of the recursion operators for these alternate versions will be entirely analogous to the game version we will illustrate here.

We note that in this example there does not exist a recursion relation among the IN-sheets. This is related to the set of legal moves in the game. In this case, there are three distinct types of legal moves from a higher sheet at level x to a lower sheet at level x', with $x' < x$. These are the 'straight' move $[x, y, z] \to [x', y, z]$ and the two 'diagonal' moves $[x, y, z] \to [x', y-(x-x'), z]$ and $[x, y, z] \to [x', y, z-(x-x')]$. We will require one auxiliary sheet for each of these moves, V^1, V^2, V^3. The first sheet, associated with the straight move, is constructed as a sum (logical OR's): $V_x^1 = \sum_{x'=0}^{x-1} P_{x'}$. The second sheet is constructed via right-shifted sums (i.e., shifts along the y-axis in $[y, z]$ space) $V_x^2 = \sum_{x'=0}^{x-1} \mathcal{Y}^{x-x'} P_{x'}$ and the third via 'upward'-shifts along the z-axis, $V_x^3 = \sum_{x'=0}^{x-1} \mathcal{Z}^{x-x'} P_{x'}$.[7] We also note that the IN-sheets can be expressed as sums (logical ORs) of the auxiliary sheets, $W_x = V_x^1 + V_x^2 + V_x^3$, and that the supermex operator for this game is the same as that for our previous example, the Sprague–Grundy values for 2-dimensional Wythoff's game. Thus $P_x = \mathcal{M}W_x = \mathcal{M}(V_x^1 + V_x^2 + V_x^3)$.

The key observation is that the auxiliary sheets have been constructed so as to obey a recursion relation:

$$V_{x+1}^1 = V_x^1 + \mathcal{M}(V_x^1 + V_x^2 + V_x^3),$$
$$V_{x+1}^2 = \mathcal{Y}V_x^2 + \mathcal{Y}\mathcal{M}(V_x^1 + V_x^2 + V_x^3),$$
$$V_{x+1}^3 = \mathcal{Z}V_x^2 + \mathcal{Z}\mathcal{M}(V_x^1 + V_x^2 + V_x^3),$$

and hence can be recursively generated from one another. The IN-sheets and the P-sheets can in turn be derived from these.

Plots of the IN-sheets and P-sheets for this game are given in Figure 7. They display complex probabilistic geometrical structures (which, as in our other examples, exhibit scaling behavior). In theory one should be able to compute the fixed points of the renormalization operator for this game, although this is clearly a complicated calculation that we leave for the future.

Lastly we remark that the IN-sheets for the other versions of 3-D Wythoff mentioned above can be constructed straightforwardly, and display similar (but

[7] Note that when right-shifting (resp. up-shifting), we fill in new columns (resp. rows) with 0's.

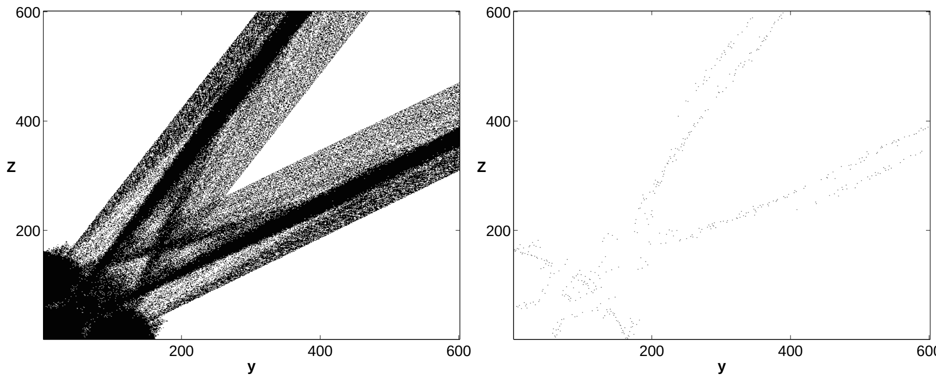

Figure 7. IN-sheet and P-sheet for three-dimensional Wythoff's game: W_{100} (left) and P_{100} (right).

not identical) complex geometrical structure and obey analogous scaling relations.

3. Implications and new directions

Apart from being a practical tool for garnering new insight into specific games, the renormalization methodology opens up several interesting new lines of inquiry into combinatorial games in general, as we now discuss.

3.1. Perturbations, structural stability, and universality in combinatorial games. Having established a renormalization framework, it is natural to inquire about the stability of the associated renormalization fixed point. In other words, is the underlying geometry of a game stable to perturbations? In the present context, we perturb a game by adding one (or more) new points to one of its IN-sheets. We then repeatedly operate on the modified sheet with the recursion operator $\mathcal{R}$, and examine the perturbation's effect on the asymptotic geometry. We can think of such a perturbation to an IN-sheet as creating a variant of the original game with slightly modified rules: In these variant games, one or more of the P-positions of the original game have been arbitrarily "declared" (by the perturbation) to now be N-positions. How does the geometry of these variant games compare to the original?

In the game of three-row Chomp, a numerical analysis shows that for a wide range of perturbations the system quickly returns to the same renormalization fixed point (in the probabilistic sense) as in the original game, i.e., the overall geometric structure seen in Figure 4 re-emerges. Thus, adopting terminology from physics, we would say that these variant games lie in the same "universality class" as ordinary Chomp. In this manner, renormalization provides a natural classification scheme for combinatorial games: games can be grouped

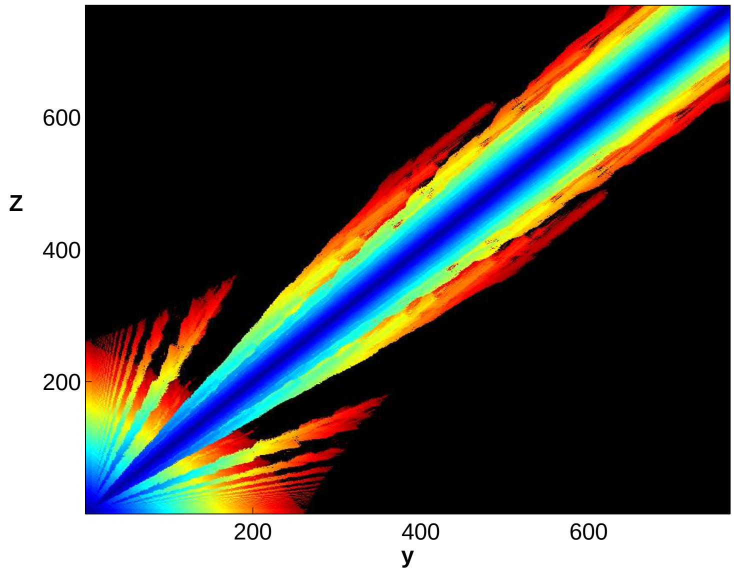

Figure 8. The geometry of generic Nim, illustrated for W_{256}.

into universality classes based on the nature of their associated renormalization fixed point. (We note that, like Chomp, the three-dimensional Wythoff's game discussed in the preceding section also appears to be structurally stable.)

Interestingly, not all games are stable to perturbations (i.e., the perturbation may create a game in a different universality class). For the game of ordinary Nim [Bouton 1902] considered earlier, we find that its IN-sheet geometry is structurally unstable to perturbations (i.e., the renormalization fixed point is unstable), resulting in a radically different geometry. Figure 8 shows the geometry[8] of a typical variant of Nim. We emphasize that this new geometry is stable and reproducible — it is the *typical* geometry that one observes if one makes a random perturbation to ordinary Nim. Hence we think of these variants of Nim as forming their own universality class, with Nim an outlier. In this manner we can see that Nim has a highly delicate (and non-generic) underlying geometric structure.

Why do some games like Chomp and 3-D Wythoff's game possess stable underlying geometries, while a game like Nim does not? We observe that Nim, unlike Chomp, is a solvable[9], decomposable game, and we believe that its inherent instability in the renormalization setting says something deep about the computational complexity of the game. Thus, we are led to this conjecture:

[8]We note that the W_x sheets display a weak periodicity in x in powers of 2, as was the case for ordinary Nim.

[9]i.e., a simple algorithm exists for determining if a given position in Nim is N or P

Conjecture: *Solvable combinatorial games are structurally unstable to perturbations (and hence have unstable sheet geometries), while generic, complex games will be structurally stable.*

— a suggestion which, if true, would relate geometric structure, dynamical stability, and computational complexity! We note an analogous feature from dynamical systems theory — that of *integrability* in Hamiltonian systems. For integrable systems, the solution to the problem can be reduced to quadratures and is characterized by simple behaviors: fixed points, periodic and quasiperiodic orbits. However, integrable systems are highly susceptible to perturbations, and adding a random perturbation will typically render the system non-integrable, destroying (some of) its simple structures and leading to much more complex dynamics, as described by the Poincaré–Birkhoff theorem[Arnold and Avez 1968]. Indeed, most many-degree-of-freedom Hamiltonian systems are non-integrable; integrable ones are exceptional. In this same way, our intuition here is that games which are solvable are rather non-generic — i.e., solvability is a delicate, rare feature that will break under most perturbations.

3.2. Sensitivity to initial conditions. One of the hallmarks of the modern understanding of dynamical systems and chaos theory is the concept of "sensitivity to initial conditions." Colloquially, this is the idea that a butterfly flapping its wings in New York can alter the weather in Chicago a few days later. More formally, it implies that one cannot predict the long term behavior of a dynamical system due to the rapid growth of small uncertainties. (See [Devaney 1986] for an elementary introduction.)

In this section, we will show that games can exhibit a related behavior. To do this, we will view the game's asymptotic distribution (i.e., the IN-sheet or P-sheet geometry) as a type of attractor.

We start, as in the preceding section, by perturbing an IN-sheet W_x in a game, and then iterate (with $\mathcal{R}$). Here, we explicitly assume the game to be structurally stable. We then examine how the precise locations of P-positions in sheets $P_{x'}$, $x' \geq x$ are affected by the initial perturbation. (Recall that by the structurally stability assumption, the same probabilistic structure will emerge in the perturbed and unperturbed cases, but the actual point-by-point locations of the P-positions in the P-set will differ.)

We illustrate this idea with the game of Chomp. Consider Figure 9. The blue data demonstrates that Chomp's attractor appears to exhibit a form of sensitivity to initial conditions. It was generated by changing a single IN on sheet W_{100} and then plotting (as a function of iteration number) the fractional discrepancy between the locations of the P-positions for the perturbed and unperturbed initial conditions (restricting here to P-positions on the lower and upper P-lines in P_x).

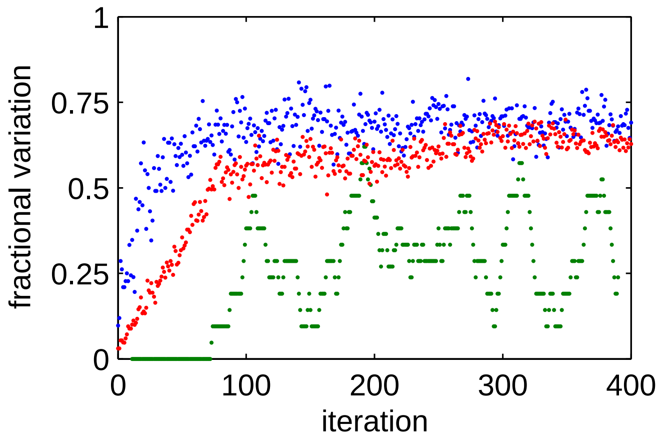

Figure 9. Dependence on initial conditions. The figure shows the fraction of P-positions affected by a small initial perturbation to an IN-sheet, as a function of iteration number.

Remarkably, after only 25 iterations, over half the losing positions have shifted their locations, while still remaining on the attractor. (The red data is similarly computed for an initial perturbation to P_{400}, while the green data shows a rolling average of the corresponding effect for P-positions lying on the flat line of P_x.) Note that despite the strong sensitivity on initial conditions, it is somewhat surprising that the growth of a perturbation appears to be roughly linear, rather than exponential. The resolution of this remains an open problem (as does the formal definition and analysis of Lyapunov exponents in this setting).

3.2.1. Renormalization and correlations. This sensitivity to initial conditions provides some justification for the renormalization procedure. We note that the main (unproven) assumption used in the renormalization analysis is that the various P-lines (at different x levels sufficiently far apart) were essentially uncorrelated with one another. (This was used implicitly, for instance, in the derivation of a few of the algebraic constraints given in Sections 2.1 and 2.3.) In the limit of large x we believe that this is justified because these lines are determined by sheets with large differences in x values, and since the system displays sensitive dependence on initial conditions, the precise point-by-point details of distant sheets should be uncorrelated in the limit.[10] Thus, we see that the renormalization analysis and assumptions about correlations are self-consistent.

[10] We remark that this lack of correlation is not a universal trait of all renormalization analyses. For example, in one of the most famous uses of renormalization, the phase transition in Ising Models (and many other phase transitions), in the 'frozen' case, correlation lengths in fact become infinite. [Ising 1925; Cipra 1987].

3.3. Accretion, crystal growth, and tightness of bounds. We observe here that the "growth" (with increasing x) of the geometric structures W_x (e.g., Figure 1) for games such as Wythoff's game, Nim and Chomp is suggestive of certain crystal growth and aggregation processes in physics [Gouyet 1995; Bar-Yam 1997]. This semblance arises because the recursion operators governing the game evolution (in particular, the supermex operator $\mathcal{M}$) typically act by attaching new points to the boundaries of the current (IN-sheet) structures. Although the details vary, this type of attachment-to-boundaries process is a common feature of many physical growth models (e.g., crystal growth, diffusion-limited aggregation, directional solidification, etc.). Viewed this way, the procedure offers a means of transforming the study of a combinatorial game into that of a shape-preserving growth process - and with it the hope that some of the tools which physicists have developed for analyzing such growth models may be brought to bear on combinatorial games. Most promising in this context would be a PDE description of the evolving boundaries in the game geometry, or a non-markovian diffusion formulation.

4. Open questions

Clearly this work raises many open questions and research problems. We provide a list of some of the key ones below:

1. **Making renormalization rigorous:** Despite its apparent practical capabilities as a tool for analyzing combinatorial games, it would be extremely valuable to make this renormalization approach mathematically rigorous. A first step would be to prove that, for stable games, the fixed point of the renormalization procedure is globally attracting, i.e., all initial conditions converge to the fixed point, or as is more likely, almost all converge. The local version of this stability problem is far more tractable, as it reduces to the computation of the spectrum of the linearization of the renormalization operator at the fixed point and standard techniques should suffice.
2. **3-D Wythoff's Game and Generic ('perturbed') Nim:** Solve analytically for the invariant geometry in these games, which have interesting and complex IN-sheets.
3. **Four-row Chomp:** Solve for the renormalization fixed point in four-row Chomp. Two approaches seem promising and both could be combined with automated procedures described in Item 5 below.

   (i) Consider three-dimensional sheets and apply the analogous renormalization procedure as used for three-row Chomp. (This can be done in a straightforward manner.)

(ii) Given a four-row position (w, x, y, z), fix w and then apply this analysis of Chomp to the subgame with the last three coordinates. Note that in this case, the renormalization equations become inhomogeneous, of the form $\mathbf{W} = \mathcal{R}\mathbf{W} + \mathcal{B}$, where $\mathcal{B}$ comes from the solutions with smaller w.

4. **Sprague–Grundy values for (2-D) Wythoff's game** Analyze the complex structure of the sheets found near the origin (i.e., for small position values).
5. **Automated Renormalization:** Design an algorithm for analytically computing the renormalization fixed point, in the spirit of Zeilberger's automated analysis of Chomp. (An example of an automated renormalization procedure, in a very different setting is given in [Friedman and Landsberg 2001].)
6. **NP-Hardness of Combinatorial Games:** Prove that some game which can be solved by renormalization techniques is NP-hard.
7. **Hardness of Perturbed Games:** Provide a class of solvable games such that 'most' perturbations lead to games that are 'difficult' to solve.
8. **Partisan Games:** Apply renormalization techniques to partisan games.
9. **Accretion and Partial Differential Equations:** Apply modern tools from accretion theory to a combinatorial game. In particular, find a PDE approximation to the renormalization operator to compute the fixed points.
10. **Lyapunov Exponents:** Formally define and calculate Lyapunov exponents to describe the sensitivity to initial conditions of an interesting game.
11. Our analysis of Chomp appears to suggest that one can answer the following two new and fundamental questions in complexity theory in the affirmative.

(i) **Probabilistic Solutions of Hard Problems ("betting on NP"):** Our results suggest that the computation of P-positions in 3-rowed Chomp is not polynomial. (Note that it is not clear whether the problem is in NP or even NP-hard.) Thus we do not expect to find simple formulas or fast algorithms. Nonetheless, this analysis implies that we can compute probabilistic estimates. This raises the question of whether such estimates are possible in complexity theory and raises the following challenging (but fundamental) problem: For an NP-complete (or NP-hard) problem, find a polynomial time algorithm which can accurately estimate the probability that a word is in the language. This would not allow one to solve NP-hard problems (which is not possible if $P \neq NP$), but would allow one to "bet effectively" on such problems. Clearly one needs a more precise formulation to allow one to sensibly evaluate the notion of 'probability'. One possibility is a computational formulation of the notion of 'calibration' from Bayesian analysis [Dawid 1982].

(ii) **Stochastic NP-Hard Problems:** A dual to the previous problem is to consider a set of NP-complete (or NP-hard) languages, generated stochastically and ask whether there exists a polynomial time algorithm which, given a word, can estimate the fraction of languages that it is a member of. For example, one could take a traveling salesman problem on a computer network and assume that each link exists with probability $p \in [0, 1]$. Then one could ask for the probability that a given graph has an expected tour less than some fixed length.

12. **Difficult Combinatorial Games:** Clearly the proof of this new approach is in the pudding. What other combinatorial games can be analyzed using these methods?

5. On the application of renormalization to games

First, we want to emphasize that (at least) some games of no chance have interesting and revealing underlying geometric structures. This suggests that simply computing the geometric structure in a game's position space could, in and of itself, lead to new and potentially powerful insights into a game (even in the absence of a full-blown renormalization analysis). For example, as we saw, the plot of Sprague–Grundy values for 2-D Wythoff's game reveals an interesting structure near the origin.

Second, we wish to reiterate that the renormalization approach to games is still very much in its infancy, with much unexplored terrain — its scope of applicability and limitations are not fully understood. Its primary limitation at present is that, like many renormalization procedures, making it fully rigorous is likely to prove challenging, and most renormalization results do not constitute formal mathematical proofs. Nonetheless, at a minimum, one can view the renormalization results for a game as representing strong conjectures, and then seek independent formal proofs of these conjectures. An alternative is that one can ignore rigor and simply compute — as is done in modern physics — to help understand the complex structure of non-decomposable combinatorial games. Given our lack of knowledge about the solutions of such games, we suggest that this last approach might be extremely valuable.

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ERIC J. FRIEDMAN
SCHOOL OF ORIE
CORNELL UNIVERSITY
ITHACA, NY 14853
UNITED STATES
ejf27@cornell.edu

ADAM S. LANDSBERG
JOINT SCIENCE DEPARTMENT
CLAREMONT MCKENNA, PITZER AND SCRIPPS COLLEGES
CLAREMONT, CA 91711
UNITED STATES
landsberg@jsd.claremont.edu