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Ordinal partizan End Nim

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Introduction

Partizan End Nim is a game played by two players called Left and Right. Initially there are n stacks of boxes in a row, each stack containing at least one box. Players take turns reducing the number of boxes from the stack on their respective side (Left removes from the leftmost stack, while Right removes from the rightmost stack). For example, the position 3 | 5 | 2 (or, denoted more tersely, 352) has three boxes in its leftmost pile, five boxes in the middle pile, and two boxes in the rightmost pile. When the game starts, Left can only remove boxes from the pile of size three, and Right can only remove boxes from the pile of size two. The first player that cannot move loses. This particular position should be a win for the first player (whether that be Left or Right). The first player should remove a whole pile, for the stack of size 5 dominates the remaining stack. Notice that if one player has a legal move from a position, the other player can also legally move, making the game *all small*, as defined in [3, p. 101] and [2, vol. 1, p. 221].

In our version of Partizan End Nim, piles are not limited to finite size. A move in Partizan End Nim requires the player to change the size of the closest pile to a smaller *ordinal* number (possibly 0). For instance, a move from a pile of size ω, the smallest nonfinite ordinal, consists of changing the size of that pile to some finite ordinal height. A player can move a pile of size $\omega+1$ to any natural number or to a pile of size ω.

For an example of a position with ordinal pile sizes, the reader can confirm that Left can win from the three pile position $(\omega+\omega+1) \mid (\omega+\omega) \mid (\omega+1)$ whether she moves first or second. Moving first, Left can remove the leftmost pile since the middle pile is larger than the rightmost pile. The strategy that Left

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uses when playing second depends on Right's move. If Right takes the whole rightmost pile, then Left's winning move is to remove one box from the leftmost pile. On the other hand, if Right changes the size of the rightmost pile to either ω or some natural number, then the winning move for Left is to remove the entire leftmost pile.

The results and proofs presented here parallel and tighten those found in [1] by allowing for ordinal pile sizes. We give an efficient recursive method to compute the outcome of (and winning moves from) any position. The reader who is not fond of ordinal numbers can safely skip the following section and assume all pile sizes are finite.

Ordinal numbers

Ordinal numbers generalize the natural numbers, allowing us to define transfinite numbers. In this paper, we will represent ordinal numbers as sets by giving a standard recursive definition.

For sets X, Y, define $X < Y$ if $X \subset Y$, the natural partial ordering of sets.

DEFINITION 1. The segment of X determined by α, written X_α, is defined by $X_\alpha = \{x \in X \mid x < \alpha\}$.

DEFINITION 2. An ordinal is a well-ordered set X such that $\alpha = X_\alpha$ for all $\alpha \in X$.

From this definition, we are able to reach the following conclusions:

- If X is an ordinal, then for all $\alpha \in X$, α is also an ordinal.
- If X is an ordinal, then for all $\alpha \in X$, $\alpha \subset X$.
- The set of all ordinals is well-ordered.
- For ordinals α and β, $\alpha < \beta \iff \alpha \subset \beta \iff \alpha \in \beta$.

Now, we define the first ordinal, ϕ, to be 0. From this, we can define $1 = \{0\} = \{\phi\}$, $2 = \{0, 1\} = \{\phi, \{\phi\}\}$, and so on. The least transfinite ordinal is defined as $\omega = \{0, 1, 2, 3, 4, \ldots\}$. Since ordinals are well-ordered, the principle of mathematical induction applies to them. (See, for example, [4, 1.7; 3.1])

Next, we will describe some properties of ordinal numbers.

DEFINITION 3. If we have two ordinals, α and β, then the ordinal sum of α and β is defined by

$$\alpha + \beta = \alpha \cup \{\alpha + \beta'\}_{\beta' \in \beta}$$

and the ordinal difference of α and β, with $\alpha \geq \beta$, is defined by

$$\alpha - \beta = \{\alpha' - \beta\}_{\alpha' \geq \beta, \alpha' \in \alpha}.$$

Ordinal addition is not commutative, but it is associative. For example, $1+\omega = \omega$, while $\omega + 1 = \{1, 2, 3, \ldots, \omega\} > \omega$. With these definitions, we are able to make the following observations:

OBSERVATION 4. If $\beta' \in \beta$, then $(\alpha + \beta') \in (\alpha + \beta)$.

OBSERVATION 5. If $\alpha' \in \alpha$, then $(\alpha' - \beta) \in (\alpha - \beta)$.

Now, we will show that our definitions for ordinal addition and subtraction observe the following identities:

LEMMA 6. *Let α and β be ordinals. Then*

$$(\beta+\alpha)-\beta = \alpha \quad \textit{and} \quad \beta + (\alpha - \beta) = \alpha \ \ (\textit{if}\ \alpha \geq \beta).$$

PROOF.

$$\begin{aligned}
(\beta+\alpha)-\beta &= (\beta \cup \{\beta+\alpha'\})-\beta \\
&= \{(\beta+\alpha')-\beta\} \\
&= \{\alpha'\} && \text{(by induction)} \\
&= \alpha; \\
\beta + (\alpha-\beta) &= \beta + \{\alpha'-\beta\}_{\alpha'\geq\beta} \\
&= \beta \cup \{\beta + (\alpha'-\beta)\}_{\alpha'\geq\beta} \\
&= \beta \cup \{\alpha'\}_{\alpha'\geq\beta} && \text{(by induction)} \\
&= \{\alpha'\} \\
&= \alpha.
\end{aligned}$$

□

Note that $(\alpha+\beta)-\beta$ and $(\alpha-\beta)+\beta$ need not equal α. For example, $(1+\omega)-\omega = 0$ and $(\omega-1)+1 = \omega+1$.

Partizan End Nim

In this section, we will define a recursive algorithm that determines the *outcome class* of a game of Partizan End Nim.

DEFINITION 7. An outcome class describes which player has a winning strategy. The four possible outcome classes are $\mathcal{N}$, $\mathcal{P}$, $\mathcal{L}$, and $\mathcal{R}$. A game is in:

- $\mathcal{N}$ if the first player always has a winning strategy.
- $\mathcal{P}$ if the second player always has a winning strategy.
- $\mathcal{L}$ if the Left player always has a winning strategy.
- $\mathcal{R}$ if the Right player always has a winning strategy.

For the remainder of this paper, we'll encode a position by a string of ordinals, $\mathbf{x}$. We can append and/or prepend additional piles to the string by concatenation, as in $\alpha\mathbf{x}\beta$. For instance, if $\mathbf{x} = 3526$ is a position with 4 piles, we might construct a 6 pile game $\alpha\mathbf{x}\beta$, where $\alpha = \omega$ and $\beta = \omega + 3$.

DEFINITION 8. Let $R(\mathbf{x})$ be defined as the minimum ordinal β such that $\mathbf{x}\beta$ is a win for Right moving second, where $\beta \geq 0$. Similarly, define $L(\mathbf{x})$ as the minimum α such that $\alpha\mathbf{x}$ is a win for Left moving second.

OBSERVATION 9. Using these definitions, we are able to determine the outcome class of $\alpha\mathbf{x}\beta$ for ordinals $\alpha, \beta > 0$ as follows:

$$\alpha\mathbf{x}\beta \in \begin{cases} \mathcal{L} & \text{if } \alpha > L(\mathbf{x}\beta) \text{ and } \beta \leq R(\alpha\mathbf{x}), \\ \mathcal{R} & \text{if } \alpha \leq L(\mathbf{x}\beta) \text{ and } \beta > R(\alpha\mathbf{x}), \\ \mathcal{N} & \text{if } \alpha > L(\mathbf{x}\beta) \text{ and } \beta > R(\alpha\mathbf{x}), \\ \mathcal{P} & \text{if } \alpha \leq L(\mathbf{x}\beta) \text{ and } \beta \leq R(\alpha\mathbf{x}). \end{cases}$$

DEFINITION 10. The triple point of $\mathbf{x}$ is $(L(\mathbf{x}), R(\mathbf{x}))$.

We show in the next proposition that the triple point of $\mathbf{x}$ determines the outcome class of any game of the form $\alpha\mathbf{x}\beta$.

PROPOSITION 11. *We can determine the outcome class of any game $\alpha\mathbf{x}\beta$ for ordinals $\alpha, \beta > 0$ using just $L(\mathbf{x})$ and $R(\mathbf{x})$:*

$$\alpha\mathbf{x}\beta \in \begin{cases} \mathcal{N} & \text{if } \alpha \leq L(\mathbf{x}) \text{ and } \beta \leq R(\mathbf{x}), \\ \mathcal{P} & \text{if } \alpha = L(\mathbf{x}) + \gamma \text{ and } \beta = R(\mathbf{x}) + \gamma \text{ for some } \gamma > 0, \\ \mathcal{L} & \text{if } \alpha = L(\mathbf{x}) + \gamma \text{ and } \beta < R(\mathbf{x}) + \gamma \text{ for some } \gamma > 0, \\ \mathcal{R} & \text{if } \alpha < L(\mathbf{x}) + \gamma \text{ and } \beta = R(\mathbf{x}) + \gamma \text{ for some } \gamma > 0. \end{cases}$$

PROOF. First, assume that $\alpha \leq L(\mathbf{x})$ and $\beta \leq R(\mathbf{x})$. If Left removes all of α, Right cannot win since any move is to $\mathbf{x}\beta'$ where $\beta' < \beta$ and β was the least value such that Right wins moving second on $\mathbf{x}\beta$. Symmetrically, Right can also win moving first by removing all of β. Thus, $\alpha\mathbf{x}\beta \in \mathcal{N}$.

Next, assume that $\alpha = L(\mathbf{x}) + \gamma$ and $\beta = R(\mathbf{x}) + \gamma$ for some $\gamma > 0$. Also, assume that Left moves first. If Left changes the size of α to $\alpha = L(\mathbf{x}) + \gamma'$ where $0 < \gamma' < \gamma$, Right simply responds by moving on β to $\beta = R(\mathbf{x}) + \gamma'$. By induction, this position is in $\mathcal{P}$. On the other hand, if Left changes the size of α to α' where $\alpha' \leq L(\mathbf{x})$, Right can win by removing β as shown in the previous case. Thus, Left loses moving first. Symmetrically, Right also loses moving first. So, $\alpha\mathbf{x}\beta \in \mathcal{P}$.

Finally, assume $\alpha = L(\mathbf{x}) + \gamma$ and $\beta < R(\mathbf{x}) + \gamma$ for some $\gamma > 0$. If $\beta \leq R(\mathbf{x})$, Left can win moving first by removing all of α as shown in the first case. If $\beta > R(\mathbf{x})$, Left wins moving first by changing the size of α to $L(\mathbf{x}) + \gamma'$, where γ' is defined by $\beta = R(\mathbf{x}) + \gamma'$, which is in $\mathcal{P}$ by induction. Left can win moving

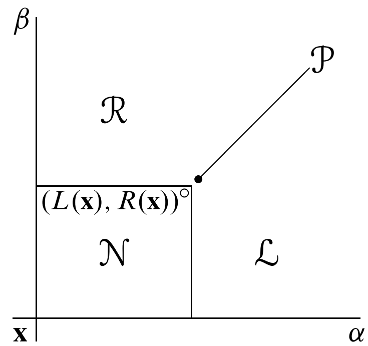

Figure 1. Outcome classes of the game $\alpha\mathbf{x}\beta$ for all ordinals α and β. The unfilled circle represents the triple point, $(L(\mathbf{x}), R(\mathbf{x}))$, which has an outcome class of $\mathcal{N}$. The filled circle is the point $(L(\mathbf{x})+1, R(\mathbf{x})+1)$ and the games on the line originating from the filled circle, which are of the form $(L(\mathbf{x})+\gamma, R(\mathbf{x})+\gamma)$ where $\gamma > 0$, have an outcome class of $\mathcal{P}$.

second from $\alpha\mathbf{x}\beta$ by making the same responses as in the previous case. Thus, $\alpha\mathbf{x}\beta \in \mathcal{L}$. □

OBSERVATION 12. Suppose we have a string of piles called $\mathbf{x}$. If we were to list the outcome classes of $\mathbf{x}\beta$ as β increases from 1 through the ordinals, we would get one of the following results:

- a string (possibly empty) of $\mathcal{N}$'s followed by $\mathcal{R}$'s, or,
- a string of $\mathcal{L}$'s (again possibly empty) followed by a single $\mathcal{P}$, and then $\mathcal{R}$'s.

OBSERVATION 13. If $\alpha\mathbf{x}\beta \in \mathcal{P}$, then $(\alpha+\gamma)\mathbf{x}(\beta+\gamma) \in \mathcal{P}$ for all ordinals $\gamma > 0$.

Notice that by using Proposition 11, the triple point of $\mathbf{x}$ determines the outcome classes of all games of the form $\alpha\mathbf{x}\beta$. Figure 1 represents the outcome classes of the game $\alpha\mathbf{x}\beta$.

Finally, we will give a recursive algorithm to compute $R(\mathbf{x})$ and $L(\mathbf{x})$, which allows us to efficiently analyze any Partizan End Nim position.

PROPOSITION 14. *The functions $R(\mathbf{x})$ and $L(\mathbf{x})$ can be computed recursively using*:

$$R(\alpha\mathbf{x}) = \begin{cases} 0 & \text{if } \alpha \le L(\mathbf{x}), \\ R(\mathbf{x}) + (\alpha - L(\mathbf{x})) & \text{if } \alpha > L(\mathbf{x}), \end{cases}$$

$$L(\mathbf{x}\beta) = \begin{cases} 0 & \text{if } \beta \le R(\mathbf{x}), \\ L(\mathbf{x}) + (\beta - R(\mathbf{x})) & \text{if } \beta > R(\mathbf{x}). \end{cases}$$

For the base case, $R(\mathbf{x}) = L(\mathbf{x}) = 0$ *when x is empty*.

PROOF. As already shown in Proposition 11, Left loses moving first on $\alpha\mathbf{x}$ where $\alpha \le L(\mathbf{x})$, so $R(\alpha\mathbf{x}) = 0$. On the other hand, assume that $\alpha > L(\mathbf{x})$. We know from Lemma 6 that $\alpha = L(\mathbf{x}) + (\alpha - L(\mathbf{x}))$, so let $\gamma = (\alpha - L(\mathbf{x}))$. By Proposition 11, the least value of β that lets Right win moving second on $\alpha\mathbf{x}\beta$ is $\beta = R(\mathbf{x}) + \gamma = R(\mathbf{x}) + (\alpha - L(\mathbf{x}))$. □

An algorithm to compute $R(\mathbf{x})$ and $L(\mathbf{x})$ using the above recurrence can be written to take $\Theta(n2)$ ordinal operations, where n is the number of piles in $\mathbf{x}$, since there are $\frac{1}{2}n(n+1)$ consecutive subsequences of a row of n piles.

Some examples

As an example, we will determine who wins from

$$3 \mid 5 \mid 2 \mid 3 \mid 3 \mid 1 \mid 9$$

when Left moves first and when Right moves first. Fix $\mathbf{x} = 52331$. We wish to compute $L(\mathbf{x})$ and $R(\mathbf{x})$ using Proposition 14. For single-pile positions, $L(\alpha) = R(\alpha) = \alpha$, because $R(\alpha) = R() + (\alpha - L()) = 0 + (\alpha - 0) = \alpha$. For 2-pile positions, we have

$$R(\alpha\beta) = \begin{cases} 0 & \text{if } \alpha \leq \beta, \\ \alpha & \text{if } \alpha > \beta, \end{cases} \qquad L(\alpha\beta) = \begin{cases} 0 & \text{if } \alpha \geq \beta, \\ \beta & \text{if } \alpha < \beta. \end{cases}$$

We can compute $L(52331)$ by first calculating $L(\mathbf{w})$ and $R(\mathbf{w})$ for each shorter substring of piles:

w	L(**w**)	R(**w**)
523	0	2
233	6	2
331	1	6
5233	1	0
2331	0	7
52331	2	12

For instance, $R(523) = R(23) + (5 - L(23)) = 0 + (5 - 3) = 2$.

For the original position $\alpha\mathbf{x}\beta = 3\mathbf{x}9$, we have $3 > L(\mathbf{x}) = 2$ and $9 \leq R(\mathbf{x}) = 12$, and hence, by Observation 9, 3523319 is an $\mathcal{L}$-position.

As a last example, we will tabulate $L(\mathbf{x})$ for three 3-pile positions, the first of which is from the introduction:

x left pile	**x** middle pile	**x** right pile	$L(\mathbf{x})$
$\omega+\omega+1$	$\omega+\omega$	$\omega+1$	0
$\omega+\omega+1$	$\omega+\omega+1$	$\omega+1$	$\omega+1$
$\omega+\omega+1$	$\omega+\omega+2$	$\omega+1$	$\omega+\omega+\omega+1$

A small change to an individual pile size (in this case, the middle pile) can have a large effect on the triple-point.

Open question

While the values of arbitrary Partizan End Nim positions appear quite complicated, we conjecture that the atomic weights are all integers.

References

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