Applications of elliptic and theta functions to Friedmann–Robertson–Lemaître–Walker cosmology with cosmological constant

JENNIE D'AMBROISE

1. Introduction

Elliptic functions are known to appear in many problems, applied and theoretical. A less known application is in the study of exact solutions to Einstein's gravitational field equations in a Friedmann–Robertson–Lemaître–Walker, or FRLW, cosmology [Abdalla and Correa-Borbonet 2002; Aurich and Steiner 2001; Aurich et al. 2004; Basarab-Horwath et al. 2004; Kharbediya 1976; Kraniotis and Whitehouse 2002]. We will show explicitly how Jacobi and Weierstrass elliptic functions arise in this context, and will additionally show connections with theta functions. In Section 2, we review the definitions of various elliptic functions. In Section 3, we record relations between elliptic functions and theta functions. In Section 4 we introduce the FRLW cosmological model and we then proceed to show how elliptic functions appear as solutions to Einstein's gravitational equations in sections 5 and 6. The author thanks Floyd Williams for helpful discussions.

2. Elliptic functions

An *elliptic integral* is one of the form $\int R\big(x, \sqrt{P(x)}\big)\, dx$, where $P(x)$ is a polynomial in x of degree three or four and R is a rational function of its arguments. Such integrals are called elliptic since an integral of this kind arises in the computation of the arclength of an ellipse. Legendre showed that any elliptic integral can be written in terms of the three fundamental or *normal elliptic integrals*

$$F(x, k) \overset{\text{def.}}{=} \int_0^x \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}},$$

$$E(x,k) \stackrel{\text{def.}}{=} \int_0^x \sqrt{\frac{1-k^2t^2}{1-t^2}}\,dt, \qquad (2.1)$$

$$\Pi(x,\alpha^2,k) \stackrel{\text{def.}}{=} \int_0^x \frac{dt}{(1-\alpha^2t^2)\sqrt{(1-t^2)(1-k^2t^2)}},$$

which are referred to as normal elliptic integrals of the first, second and third kind, respectively. The parameter α is any real number and k is referred to as the *modulus*. For many problems in which real quantities are desired, $0 < k, x < 1$, although this is not required in the above definitions (for $k = 0$ and $k = 1$, the integral can be expressed in terms of elementary functions and is therefore *pseudo-elliptic*).

Elliptic functions are inverse functions of elliptic integrals. They are known to be the simplest of nonelementary functions and have applications in the study of classical equations of motion of various systems in physics including the pendulum. One can easily show that if $f(u)$ denotes the inverse function of an elliptic integral $y(x) = \int R(x, \sqrt{P(x)})\,dx$, then since $y(f(u)) = u$, which implies $y'(f(u)) = 1/f'(u)$, we have

$$f'(u)^2 = \frac{1}{R(f(u), \sqrt{P(f(u))})^2}. \qquad (2.2)$$

The *Jacobi elliptic function* $\text{sn}(u,k)$ is the inverse of $F(x,k)$ defined above, and eleven other Jacobi elliptic functions can be written in terms of $\text{sn}(u,k)$: $\text{cn}(u,k)$ and $\text{dn}(u,k)$ satisfy $\text{sn}^2 u + \text{cn}^2 u = 1$ and $k^2 \text{sn}^2 u + \text{dn}^2 u = 1$, respectively, and

$$\text{ns} \stackrel{\text{def.}}{=} \frac{1}{\text{sn}}, \quad \text{nc} \stackrel{\text{def.}}{=} \frac{1}{\text{cn}}, \quad \text{nd} \stackrel{\text{def.}}{=} \frac{1}{\text{dn}},$$

$$\text{sc} \stackrel{\text{def.}}{=} \frac{\text{sn}}{\text{cn}}, \quad \text{sd} \stackrel{\text{def.}}{=} \frac{\text{sn}}{\text{dn}}, \quad \text{cd} \stackrel{\text{def.}}{=} \frac{\text{cn}}{\text{dn}}, \qquad (2.3)$$

$$\text{cs} \stackrel{\text{def.}}{=} \frac{1}{\text{sc}}, \quad \text{ds} \stackrel{\text{def.}}{=} \frac{1}{\text{sd}}, \quad \text{dc} \stackrel{\text{def.}}{=} \frac{1}{\text{cd}}.$$

By (2.2) and (2.3), one can see that the Jacobi elliptic functions satisfy the differential equation

$$f'(u)^2 + af(u)^2 + bf(u)^4 = c \qquad (2.4)$$

for a, b, c in terms of the modulus k according to the following table.

$f(u)$	a	b	c
$\text{sn}(u,k)$	$1+k^2$	$-k^2$	1
$\text{cn}(u,k)$	$1-2k^2$	k^2	$1-k^2$
$\text{dn}(u,k)$	k^2-2	1	k^2-1
$\text{ns}(u,k)$	$1+k^2$	-1	k^2
$\text{nc}(u,k)$	$1-2k^2$	k^2-1	$-k^2$
$\text{nd}(u,k)$	k^2-2	$1-k^2$	-1
$\text{sc}(u,k)$	k^2-2	k^2-1	1
$\text{sd}(u,k)$	$1-2k^2$	$k^2(1-k^2)$	1
$\text{cd}(u,k)$	$1+k^2$	$-k^2$	1
$\text{cs}(u,k)$	k^2-2	-1	$1-k^2$
$\text{ds}(u,k)$	$1-2k^2$	-1	$k^2(k^2-1)$
$\text{dc}(u,k)$	$1+k^2$	-1	k^2

(The a-value for $f(u) = \text{dn}(u,k)$ seen above corrects an error in [Basarab-Horwath et al. 2004].)

The *Weierstrass elliptic function*

$$\wp(z;\omega_1,\omega_2) = \frac{1}{z^2} + \sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}-\{(0,0)\}} \frac{1}{(z-m\omega_1-n\omega_2)^2} - \frac{1}{(m\omega_1+n\omega_2)^2} \tag{2.5}$$

is a doubly periodic elliptic function of $z \in \mathbb{C}$ with periods $\omega_1,\omega_2 \in \mathbb{C}$ such that $Im(\omega_1/\omega_2) > 0$. The function $\wp$ is the inverse of the elliptic integral

$$\wp^{-1}(x;g_2,g_3) = \int_x^\infty \frac{1}{\sqrt{4t^3-g_2t-g_3}}\,dt \tag{2.6}$$

where $g_2,g_3 \in \mathbb{C}$ are known as *Weierstrass invariants*. Given periods ω_1,ω_2, the invariants are

$$g_2 = 60 \sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}-\{(0,0)\}} \frac{1}{(m\omega_1+n\omega_2)^4},$$
$$g_3 = 140 \sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}-\{(0,0)\}} \frac{1}{(m\omega_1+n\omega_2)^6}. \tag{2.7}$$

Alternately given invariants g_2,g_3, periods ω_1,ω_2 can be constructed if the *discriminant* $\Delta \overset{\text{def.}}{=} g_2^3 - 27g_3^2$ is nonzero — that is, when the *Weierstrass cubic* $4t^3-g_2t-g_3$ does not have repeated roots (see 21·73 of [Whittaker and Watson 1927]). Therefore we refer to $\wp(z;\omega_1,\omega_2)$ as either $\wp(z)$ or $\wp(z;g_2,g_3)$ and consider only cases where the invariants are such that $g_2^3 \neq 27g_3^2$. By (2.2) and

(2.6) one can see that the Weierstrass elliptic function satisfies

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3. \tag{2.8}$$

Note that in Michael Tuite's lecture in this volume, $\omega_{m,n}$ there is equal to $m\omega_1 + n\omega_2$ here with our ω_1, ω_2 specialized to $2\pi i\tau$ and $2\pi i$ in his lecture.

In the special case that the discriminant $\Delta > 0$, the roots of $4t^3 - g_2t - g_3$ are real and distinct, and are conventionally notated by $e_1 > e_2 > e_3$ for $e_1 + e_2 + e_3 = 0$. In this case $4t^3 - g_2t - g_3 = 4(t - e_1)(t - e_2)(t - e_3)$, the Weierstrass invariants are given in terms of the roots by

$$g_2 = -4(e_2e_3 + e_1e_3 + e_1e_2), \qquad g_3 = 4e_1e_2e_3, \tag{2.9}$$

and $\wp$ can be written in terms of the Jacobi elliptic functions by

$$\begin{aligned}
\wp(z) &= e_3 + \gamma^2 \operatorname{ns}^2(\gamma z, k), \\
\wp(z) &= e_2 + \gamma^2 \operatorname{ds}^2(\gamma z, k), \\
\wp(z) &= e_1 + \gamma^2 \operatorname{cs}^2(\gamma z, k)
\end{aligned} \tag{2.10}$$

for $e_1 < z \in \mathbb{R}$, $\gamma^2 \overset{\text{def.}}{=} e_1 - e_3$ and modulus k such that $k^2 = \frac{e_2 - e_3}{e_1 - e_3}$ (similar equations hold if $z \in \mathbb{R}$ and z is in a different range in relation to the real roots e_1, e_2, e_3, and alternate relations hold for nonreal roots when $\Delta < 0$; see chapter II of [Greenhill 1959]).

Note that the Jacobi elliptic functions solve a differential equation which contains only even powers of $f(u)$, and $\wp$ solves an equation with no squared or quartic powers of $\wp$. Weierstrass elliptic functions have the advantage of being easily implemented in the case that the cubic $4t^3 - g_2t - g_3$ is not factored in terms of its roots. Elliptic integrals of the type $\int R(x)/\sqrt{P(x)}dx$, where $P(x)$ is a cubic polynomial and R is a rational function of x, can be written in terms of three fundamental *Weierstrassian normal elliptic integrals* although we will not record the details here (see Appendix of [Byrd and Friedman 1954]). In Section 6 we will see a method which allows one to write the elliptic integral $\int_{x_0}^x 1/\sqrt{F(t)}dt$, for $F(t)$ a quartic polynomial, in terms of the Weierstrassian normal elliptic integral *of the first kind* (2.6) by reducing the quartic to a cubic.

3. Jacobi theta functions

Jacobi theta functions are functions of two arguments, $z \in \mathbb{C}$ a complex number and $\tau \in \mathbb{H}$ in the upper-half plane. Every elliptic function can be written as the ratio of two theta functions. Doing so elucidates the meromorphic nature of elliptic functions and is useful in the numerical evaluation of elliptic functions. One must be cautious with the notation of theta functions, since many different

conventions are used. We will use the notation of [Whittaker and Watson 1927] to define

$$\theta_1(z,\tau) \stackrel{\text{def.}}{=} 2q^{1/4} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)} \sin((2n+1)z),$$

$$\theta_2(z,\tau) \stackrel{\text{def.}}{=} 2q^{1/4} \sum_{n=0}^{\infty} q^{n(n+1)} \cos((2n+1)z),$$

$$\theta_3(z,\tau) \stackrel{\text{def.}}{=} 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos(2nz),$$

$$\theta_4(z,\tau) \stackrel{\text{def.}}{=} 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos(2nz),$$

(3.1)

where $q \stackrel{\text{def.}}{=} e^{\pi i \tau}$ is called the *nome*. We also define the special values $\theta_i \stackrel{\text{def.}}{=} \theta_i(0,\tau)$.

In terms of theta functions, the Jacobi elliptic functions are

$$\text{sn}(u,k) = \frac{\theta_3 \, \theta_1(u/\theta_3^2, \tau)}{\theta_2 \, \theta_4(u/\theta_3^2, \tau)},$$

$$\text{cn}(u,k) = \frac{\theta_4 \, \theta_2(u/\theta_3^2, \tau)}{\theta_2 \, \theta_4(u/\theta_3^2, \tau)},$$

(3.2)

$$\text{dn}(u,k) = \frac{\theta_4 \, \theta_3(u/\theta_3^2, \tau)}{\theta_3 \, \theta_4(u/\theta_3^2, \tau)},$$

where τ is chosen such that $k^2 = \theta_2^4/\theta_3^4$. By [Whittaker and Watson 1927, 22·11], if $0 < k^2 < 1$ there exists a value of τ for which the quotient θ_2^4/θ_3^4 equals k^2.

4. The FRLW cosmological model

The Friedmann–Robertson–Lemaître–Walker cosmological model assumes that our current expanding universe is on large scales homogeneous and isotropic. On a d-dimensional spacetime this assumption translates into a metric of the form

$$ds^2 = -dt^2 + \tilde{a}(t)^2 \left(\frac{dr^2}{1 - k'r^2} + r^2 d\Omega_{d-2}^2 \right), \qquad (4.1)$$

where $\tilde{a}(t)$ is the *cosmic scale factor* and $k' \in \{-1, 0, 1\}$ is the *curvature parameter*.

The Einstein field equations

$$G_{ij} = -\kappa_d T_{ij} + \Lambda g_{ij}$$

then govern the evolution of the universe over time. In these equations, the Einstein tensor

$$G_{ij} \stackrel{\text{def.}}{=} R_{ij} - \tfrac{1}{2} R g_{ij}$$

is computed directly from the metric g_{ij} by calculating the Ricci tensor R_{ij} and the scalar curvature R. Also $\kappa_d = 8\pi G_d$, where G_d is a generalization of Newton's constant to d-dimensional spacetime and $\Lambda > 0$ is the cosmological constant. The form of the energy-momentum tensor T_{ij} depends on what sort of matter content one is assuming, and in this lecture will be that of a perfect-fluid — that is, $T_{ij} = (p+\rho)g_{i0}g_{j0} + pg_{ij}$, where $\rho(t)$ and $p(t)$ are the density and pressure of the fluid, respectively.

For the metric (4.1), Einstein's equations are

$$\frac{(d-1)(d-2)}{2}\left(H^2 + \frac{k'}{\tilde{a}^2}\right) = \kappa_d \rho(t) + \Lambda, \qquad \text{(i)}$$

$$(d-2)\dot{H} + \frac{(d-1)(d-2)}{2}H^2 + \frac{(d-2)(d-3)}{2}\frac{k'}{\tilde{a}^2} = -\kappa_d p(t) + \Lambda, \qquad \text{(ii)}$$

for $H(t) \stackrel{\text{def.}}{=} \dot{\tilde{a}}(t)/\tilde{a}(t)$ and where dot denotes differentiation with respect to t. In this lecture, only equation (i) will be required to relate $\tilde{a}(t)$ to elliptic and theta functions. We rewrite equation (i) in terms of *conformal time* η by defining the *conformal scale factor* $a(\eta) \stackrel{\text{def.}}{=} \tilde{a}(f(\eta))$, where $f(\eta)$ is the inverse function of $\eta(t)$, which satisfies $\dot{\eta}(t) = 1/\tilde{a}(t)$. In terms of $a(\eta)$, (i) becomes

$$a'(\eta)^2 = \tilde{\Lambda}a(\eta)^4 + \tilde{\kappa}_d\rho(f(\eta))a(\eta)^4 - k'a(\eta)^2, \qquad (4.2)$$

where we use notation $\tilde{\Lambda} \stackrel{\text{def.}}{=} 2\Lambda/(d-1)(d-2)$, $\tilde{\kappa}_d \stackrel{\text{def.}}{=} 2\kappa_d/(d-1)(d-2)$ and we take spacetime dimension $d > 2$.

5. FRLW and Jacobi elliptic and theta functions

In general, if $f(u)$ is a solution to $f'(u)^2 + af(u)^2 + bf(u)^4 = c$, then $g(u) = \beta f(\alpha u)$ is a solution to

$$g'(u)^2 + Ag(u)^2 + Bg(u)^4 = \frac{A^2bc}{a^2B} \qquad (5.1)$$

for $\alpha = \sqrt{A/a}$ and $\beta = \sqrt{Ab/(aB)}$, where we may choose either the positive or negative square root for each of α and β. We will construct solutions to (4.2), given that Jacobi elliptic functions solve (2.4), and also proceed to write these solutions in terms of theta functions by the relations in (3.2).

For the special case of density $\rho(t) = D/\tilde{a}(t)^4$ with $D > 0$, (4.2) becomes

$$a'(\eta)^2 + k'a(\eta)^2 - \tilde{\Lambda}a(\eta)^4 = \tilde{\kappa}_d D. \tag{5.2}$$

Therefore in (5.1) we take $A = k'$, $B = -\tilde{\Lambda}$, and a, b, c as in the table in Section 2. To restrict to real solutions we only consider entries in the table for which the ratios a/A and b/B are positive, and we also restrict D to be such that the right side of (5.2) agrees with the right side of (5.1).

For positive curvature $k' = 1$, and for $D = \dfrac{k^2}{\tilde{\kappa}_d\,\tilde{\Lambda}(1+k^2)^2}$ with $0 < k < 1$, we solve

$$a'(\eta)^2 + a(\eta)^2 - \tilde{\Lambda}a(\eta)^4 = \frac{k^2}{\tilde{\Lambda}(1+k^2)^2} \tag{5.3}$$

in conformal time in terms of Jacobi elliptic functions to obtain

$$
\begin{aligned}
a_{\text{sn}}(\eta) &= \frac{k}{\sqrt{\tilde{\Lambda}(1+k^2)}}\ \text{sn}\left(\frac{\eta}{\sqrt{1+k^2}}, k\right) = \frac{\theta_2\theta_3}{\sqrt{\tilde{\Lambda}(1+\theta_2^4\theta_3^4)}}\ \frac{\theta_1\left(\eta/\sqrt{\theta_2^4+\theta_3^4}, \tau\right)}{\theta_4\left(\eta/\sqrt{\theta_2^4+\theta_3^4}, \tau\right)}, \\[2mm]
a_{\text{ns}}(\eta) &= \frac{1}{\sqrt{\tilde{\Lambda}(1+k^2)}}\ \text{ns}\left(\frac{\eta}{\sqrt{1+k^2}}, k\right) = \frac{\theta_2\theta_3}{\sqrt{\tilde{\Lambda}(1+\theta_2^4\theta_3^4)}}\ \frac{\theta_4\left(\eta/\sqrt{\theta_2^4+\theta_3^4}, \tau\right)}{\theta_1\left(\eta/\sqrt{\theta_2^4+\theta_3^4}, \tau\right)}, \\[2mm]
a_{\text{cd}}(\eta) &= \frac{k}{\sqrt{\tilde{\Lambda}(1+k^2)}}\ \text{cd}\left(\frac{\eta}{\sqrt{1+k^2}}, k\right) = \frac{\theta_2\theta_3}{\sqrt{\tilde{\Lambda}(1+\theta_2^4\theta_3^4)}}\ \frac{\theta_2\left(\eta/\sqrt{\theta_2^4+\theta_3^4}, \tau\right)}{\theta_3\left(\eta/\sqrt{\theta_2^4+\theta_3^4}, \tau\right)}, \\[2mm]
a_{\text{dc}}(\eta) &= \frac{1}{\sqrt{\tilde{\Lambda}(1+k^2)}}\ \text{dc}\left(\frac{\eta}{\sqrt{1+k^2}}, k\right) = \frac{\theta_2\theta_3}{\sqrt{\tilde{\Lambda}(1+\theta_2^4\theta_3^4)}}\ \frac{\theta_3\left(\eta/\sqrt{\theta_2^4+\theta_3^4}, \tau\right)}{\theta_2\left(\eta/\sqrt{\theta_2^4+\theta_3^4}, \tau\right)},
\end{aligned}
\tag{5.4}
$$

where τ is chosen such that $k^2 = \theta_2^4/\theta_3^4$.

The first two solutions, $a_{\text{sn}}(\eta)$ and $a_{\text{ns}}(\eta)$, reduce to hyperbolic trigonometric functions in the case of modulus $k = 1$, since $\text{sn}(u, 1) = \tanh(u)$ and $\text{ns}(u, 1) = \coth(u)$. That is, two additional solutions in terms of elementary functions are

$$a_1(\eta) = \frac{1}{\sqrt{2\tilde{\Lambda}}}\ \tanh(\eta/\sqrt{2}), \qquad a_2(\eta) = \frac{1}{\sqrt{2\tilde{\Lambda}}}\ \coth(\eta/\sqrt{2}). \tag{5.5}$$

For these two solutions, one may solve the differential equation $\dot{\eta}(t) = 1/\tilde{a}(t) = 1/a(\eta(t))$ for $\eta(t)$, and therefore obtain the cosmic scale factor $\tilde{a}(t) = a(\eta(t))$ which solves the Einstein field equation (i) in Section 4 for $\rho(t) = D/\tilde{a}(t)^4$ with special value $D = k^2/\tilde{\Lambda}\tilde{\kappa}_d(1+k^2)^2$. Doing so, we obtain

$$
\begin{aligned}
\tilde{a}_1(t) &= a_1(\eta(t)) = \frac{1}{\sqrt{2\tilde{\Lambda}}}\ \tanh\ln\left(e^{\sqrt{\tilde{\Lambda}}t} + \sqrt{e^{2\sqrt{\tilde{\Lambda}}t} - 1}\right), \\[2mm]
\tilde{a}_2(t) &= a_2(\eta(t)), = \frac{1}{\sqrt{2\tilde{\Lambda}}}\ \coth\ln\left(e^{\sqrt{\tilde{\Lambda}}t} + \sqrt{e^{2\sqrt{\tilde{\Lambda}}t} + 1}\right),
\end{aligned}
\tag{5.6}
$$

for $t > 0$.

For negative curvature $k' = -1$, and for

$$D = \frac{1 - k^2}{\widetilde{\Lambda}\,\widetilde{\kappa}_d\,(k^2 - 2)^2}$$

with $0 < k < 1$, equation (5.2) becomes

$$a'(\eta)^2 - a(\eta)^2 - \widetilde{\Lambda}a(\eta)^4 = \frac{1 - k^2}{(k^2 - 2)^2\,\widetilde{\Lambda}}. \tag{5.7}$$

In terms of Jacobi elliptic functions and theta functions, we obtain the two solutions for the scale factor in conformal time,

$$a_{\mathrm{sc}}(\eta) = \frac{\sqrt{1-k^2}}{\sqrt{\widetilde{\Lambda}(2-k^2)}}\,\mathrm{sc}\!\left(\frac{\eta}{\sqrt{2-k^2}}, k\right) = \frac{\sqrt{\theta_3^4 - \theta_2^4}}{\sqrt{\widetilde{\Lambda}(2\theta_3^4 - \theta_2^4)}}\,\frac{\theta_3\,\theta_1\left(\eta/\theta_3^2\sqrt{2\theta_3^4 - \theta_2^4}, \tau\right)}{\theta_4\,\theta_2\left(\eta/\theta_3^2\sqrt{2\theta_3^4 - \theta_2^4}, \tau\right)},$$

$$a_{\mathrm{cs}}(\eta) = \frac{1}{\sqrt{\widetilde{\Lambda}(2-k^2)}}\,\mathrm{cs}\!\left(\frac{\eta}{\sqrt{2-k^2}}, k\right) = \frac{\theta_3\,\theta_4}{\sqrt{\widetilde{\Lambda}(2\theta_3^4 - \theta_2^4)}}\,\frac{\theta_2\left(\eta/\theta_3^2\sqrt{2\theta_3^4 - \theta_2^4}, \tau\right)}{\theta_1\left(\eta/\theta_3^2\sqrt{2\theta_3^4 - \theta_2^4}, \tau\right)},$$

$$\tag{5.8}$$

where τ is such that $k^2 = \theta_2^4/\theta_3^4$.

Note that by the comments following equation (2.10), it is possible to express the solutions obtained in this section in terms of Weierstrass functions. E. Abdalla and L. Correa-Borbonet [2002] have also considered the Einstein equation (i) with $\rho(t) = D/\tilde{a}(t)^4$, and have found connections with Weierstrass functions in cosmic time (as opposed to the conformal time argument given here). In Section 6 of this lecture we will find more general solutions to the conformal time equation (4.2) in terms of $\wp$, for arbitrary curvature k' and D-value. We will also consider the density functions $\rho(t) = D/\tilde{a}(t)^3$ and $\rho(t) = D_1/\tilde{a}(t)^3 + D_2/\tilde{a}(t)^4$ in Section 6.

6. FRLW and Weierstrass elliptic functions

In general, if $g(0) = x_0$ and $g(u)$ satisfies

$$g'(u)^2 = F(g(u)) \tag{6.1}$$

for $F(x) = A_4 x^4 + A_3 x^3 + A_2 x^2 + A_1 x + A_0$ any quartic polynomial with no repeated roots, then the inverse function $y(x)$ of $g(u)$ is the elliptic integral

$$y(x) = \int_{x_0}^x \frac{dt}{\sqrt{F(t)}}. \tag{6.2}$$

For the initial condition $g'(0) = 0$, x_0 is a root of the polynomial $F(x)$ by (6.1). In this case the integral (6.2) can be rewritten as $\int_\xi^\infty dz/\sqrt{P(z)}$, where $\xi = 1/(x - x_0)$ and $P(z)$ is a cubic polynomial. To do this, first expand $F(t)$

into its Taylor series about x_0 and then perform the change of variables $z = 1/(t - x_0)$. Furthermore one may obtain the form $\int_\tau^\infty d\sigma / \sqrt{Q(\sigma)}$ for $\tau = \frac{1}{4}\left(F'(x_0)\xi + \frac{1}{6}F''(x_0)\right)$, where $Q(\sigma) = 4\sigma^3 - g_2\sigma - g_3$. This is done by setting $z = (4\sigma - B_2/3)/B_3$ where $B_2 = F''(x_0)/2$ and $B_3 = F'(x_0)$ are the quadratic and cubic coefficients of $P(z)$ respectively. Note that since $F(x)$ has no repeated roots, x_0 is not a double root, $F'(x_0) \neq 0$ and the variable z is well-defined. Therefore we have obtained

$$y\left(\frac{6F'(x_0)}{24\tau - F''(x_0)} + x_0\right) = \wp^{-1}(\tau).$$

Writing this in terms of x, setting $x = g(u)$ and solving for $g(u)$, one obtains the solution to (6.1):

$$g(u) = x_0 + \frac{F'(x_0)}{4\wp(u; g_2, g_3) - \frac{1}{6}F''(x_0)}, \tag{6.3}$$

where

$$\begin{aligned}
g_2 &= A_0 A_4 - \tfrac{1}{4}A_1 A_3 + \tfrac{1}{12}A_2^2 \quad \text{and} \\
g_3 &= \tfrac{1}{6}A_0 A_2 A_4 + \tfrac{1}{48}A_1 A_2 A_3 - \tfrac{1}{16}A_1^2 A_4 - \tfrac{1}{16}A_0 A_3^2 - \tfrac{1}{216}A_2^3
\end{aligned} \tag{6.4}$$

are referred to as the *invariants of the quartic* $F(x)$. Since $F(x)$ has no repeated roots, the discriminant $\Delta = g_2^3 - 27g_3^2 \neq 0$. Here if $x_0 = 0$, (6.3) becomes

$$g(u) = \frac{A_1}{4\wp(u; g_2, g_3) - \frac{1}{3}A_2}. \tag{6.5}$$

If the initial condition on the first derivative is such that $g'(0) \neq 0$ then x_0 is not a root of $F(x)$ and a more general solution to (6.1) is due to Weierstrass. The proof (which we will not include here) was published by Biermann in 1865 (see [Biermann 1865; Reynolds 1989]). The solution is

$$g(u) = x_0 + \frac{\sqrt{F(x_0)}\wp'(u) + \frac{1}{2}F'(x_0)\left(\wp(u) - \frac{1}{24}F''(x_0)\right) + \frac{1}{24}F(x_0)F'''(x_0)}{2\left(\wp(u) - \frac{1}{24}F''(x_0)\right)^2 - \frac{1}{48}F(x_0)F''''(x_0)} \tag{6.6}$$

where $\wp$ is formed with the invariants of the quartic seen in (6.4) such that $\Delta \neq 0$. Here if $x_0 = 0$, (6.6) becomes

$$g(u) = \frac{\sqrt{A_0}\wp'(u) + \frac{1}{2}A_1\left(\wp(u) - \frac{1}{12}A_2\right) + \frac{1}{4}A_0 A_3}{2\left(\wp(u) - \frac{1}{12}A_2\right)^2 - \frac{1}{2}A_0 A_4}. \tag{6.7}$$

To generate a number of examples, we consider the conformal time Einstein equation (4.2) for density $\rho(t) = D_1/\tilde{a}(t)^3 + D_2/\tilde{a}(t)^4$ with $D_1, D_2 > 0$. In

this case (4.2) becomes

$$a'(\eta)^2 = \tilde{\Lambda} a(\eta)^4 - k' a(\eta)^2 + \tilde{\kappa}_d D_1 a(\eta) + \tilde{\kappa}_d D_2 \qquad (6.8)$$

and we take $A_4 = \tilde{\Lambda}$, $A_3 = 0$, $A_2 = -k'$, $A_1 = \tilde{\kappa}_d D_1$ and $A_0 = \tilde{\kappa}_d D_2$. The most general solution seen here is with initial conditions $a'(0) \neq 0$ so that $a(0) \overset{\text{def.}}{=} a_0$ is not a root of the polynomial

$$F(t) = \tilde{\Lambda} t^4 - k' t^2 + \tilde{\kappa}_d D_1 t + \tilde{\kappa}_d D_2. \qquad (6.9)$$

The solution is given by (6.6) as

$$a(\eta) = a_0 + \frac{\sqrt{F(a_0)} \wp'(\eta) + \frac{1}{2} F'(a_0)\big(\wp(\eta) - \frac{1}{24} F''(a_0)\big) + \frac{1}{24} F(a_0) F'''(a_0)}{2\big(\wp(\eta) - \frac{1}{24} F''(a_0)\big)^2 - \frac{1}{48} F(a_0) F''''(a_0)},$$
$$(6.10)$$

with Weierstrass invariants

$$g_2 = \tilde{\Lambda} \tilde{\kappa}_d D_2 + \tfrac{1}{12}(k')^2 \quad \text{and}$$
$$g_3 = -\tfrac{1}{6} \tilde{\Lambda} \tilde{\kappa}_d D_2 k' - \tfrac{1}{16} \tilde{\Lambda} \tilde{\kappa}_d^2 D_1^2 + \tfrac{1}{216}(k')^3 \qquad (6.11)$$

restricted to be such that $\Delta = g_2^3 - 27 g_3^2 \neq 0$ so that $F(t)$ does not have repeated roots.

Since $D_2, \tilde{\kappa}_d > 0$, by (6.9) zero is not a root of $F(t)$ and therefore for initial condition $a'(0) \neq 0$ and $a_0 = 0$, the solution to (6.8) is given by (6.7) as

$$a(\eta) = \frac{\sqrt{\tilde{\kappa}_d D_2} \wp'(\eta) + \frac{1}{2} \tilde{\kappa}_d D_1 \big(\wp(\eta) + \frac{k'}{12}\big)}{2\big(\wp(\eta) + \frac{k'}{12}\big)^2 - \frac{1}{2} \tilde{\Lambda} \tilde{\kappa}_d D_2} \qquad (6.12)$$

for invariants g_2, g_3 as in (6.11) with $\Delta \neq 0$. One can compare this with the results in papers by Aurich, Steiner and Then, where curvature is taken to be $k' = -1$ [Aurich and Steiner 2001; Aurich et al. 2004].

For $a'(0) = 0$ and a_0 a root of $F(t)$ in (6.9), the solution to (6.8) is given by (6.3),

$$a(\eta) = a_0 + \frac{F'(a_0)}{4\wp(\eta) - \frac{1}{6} F''(a_0)} \qquad (6.13)$$

again with invariants (6.11) such that $\Delta \neq 0$.

For a more concrete example, consider the density function $\rho(t) = D/\tilde{a}(t)^3$ for $D > 0$ so that conformal time equation (4.2) becomes

$$a'(\eta)^2 = \tilde{\Lambda} a(\eta)^4 - k' a(\eta)^2 + \tilde{\kappa}_d D a(\eta). \qquad (6.14)$$

Here zero is a root of the polynomial $F(t)$ with $A_4 = \tilde{\Lambda}$, $A_3 = A_0 = 0$, $A_2 = -k'$ and $A_1 = \tilde{\kappa}_d D$. Therefore with initial conditions $a'(0) = a(0) = 0$, the solution

to (6.14) is given by (6.5) as

$$a(\eta) = \frac{3\widetilde{\kappa}_d D}{12\wp(\eta) + k'} \tag{6.15}$$

with invariants

$$g_2 = \tfrac{1}{12}(k')^2 \quad \text{and} \quad g_3 = -\tfrac{1}{16}\widetilde{\Lambda}\widetilde{\kappa}_d^2 D^2 + \tfrac{1}{216}(k')^3 \tag{6.16}$$

restricted to be such that $\Delta \neq 0$. As noted in the comments following equations (2.10), one can write this solution in terms of Jacobi elliptic functions (by using equations (2.10) if the roots of the reduced cubic $4t^3 - g_2 t - g_3$ are real). To demonstrate this, we choose

$$D = \frac{1}{\widetilde{\kappa}_d}\sqrt{\frac{2}{27\widetilde{\Lambda}}}$$

and $k' = 1$, so that $g_3 = 0$ and $g_2 = \tfrac{1}{12}$. For this positive curvature case, (6.15) becomes

$$a(\eta) = \frac{\sqrt{2}}{\sqrt{3\widetilde{\Lambda}}\big(12\wp(\eta) + 1\big)}, \tag{6.17}$$

and the reduced cubic is $4t^3 - (1/12)t = 4t(t - 1/4\sqrt{3})(t + 1/4\sqrt{3})$. Applying (2.10) with $e_3 = -1/4\sqrt{3}$, $e_2 = 0$, $e_1 = 1/4\sqrt{3}$, (6.17) can be equivalently written in terms of Jacobi elliptic functions as

$$a(\eta) = \frac{\sqrt{2/\widetilde{\Lambda}}}{\sqrt{3} - 3 + 6\,\mathrm{ns}^2\left(\eta/\sqrt{2\sqrt{3}}, 1/\sqrt{2}\right)} = \frac{\sqrt{2/\widetilde{\Lambda}}}{\sqrt{3} + 6\,\mathrm{ds}^2\left(\eta/\sqrt{2\sqrt{3}}, 1/\sqrt{2}\right)}$$

$$= \frac{\sqrt{2/\widetilde{\Lambda}}}{\sqrt{3} + 3 + 6\,\mathrm{cs}^2\left(\eta/\sqrt{2\sqrt{3}}, 1/\sqrt{2}\right)}, \tag{6.18}$$

since $k^2 = \dfrac{e_2 - e_3}{e_1 - e_3} = \dfrac{1}{2}$ and $\gamma^2 = e_1 - e_3 = \dfrac{1}{2\sqrt{3}}$. In terms of theta functions, this is

$$a(\eta) = \frac{\sqrt{2/\widetilde{\Lambda}}\,\theta_3^2\,\theta_1^2\left(\eta/\sqrt{2\sqrt{3}}\,\theta_3^2, \tau\right)}{\left(\sqrt{3} - 3\right)\theta_3^2\,\theta_1^2\left(\eta/\sqrt{2\sqrt{3}}\,\theta_3^2, \tau\right) + 6\,\theta_2^2\,\theta_4^2\left(\eta/\sqrt{2\sqrt{3}}\,\theta_3^2, \tau\right)}$$

$$= \frac{\sqrt{2/\widetilde{\Lambda}}\,\theta_3^4\,\theta_1^2\left(\eta/\sqrt{2\sqrt{3}}\,\theta_3^2, \tau\right)}{\sqrt{3}\,\theta_3^4\,\theta_1^2\left(\eta/\sqrt{2\sqrt{3}}\,\theta_3^2, \tau\right) + 6\,\theta_2^2\,\theta_4^2\,\theta_3^2\left(\eta/\sqrt{2\sqrt{3}}\,\theta_3^2, \tau\right)}$$

$$= \frac{\sqrt{2/\widetilde{\Lambda}}\,\theta_3^2\,\theta_1^2\left(\eta/\sqrt{2\sqrt{3}}\,\theta_3^2, \tau\right)}{\left(\sqrt{3} + 3\right)\theta_3^2\,\theta_1^2\left(\eta/\sqrt{2\sqrt{3}}\,\theta_3^2, \tau\right) + 6\,\theta_4^2\,\theta_2^2\left(\eta/\sqrt{2\sqrt{3}}\,\theta_3^2, \tau\right)} \tag{6.19}$$

where τ is taken such that $\tfrac{1}{2} = \theta_2^4/\theta_3^4$.

As a final example, we return to $\rho(t) = D/\tilde{a}(t)^4$ for $D > 0$ considered in Section 5. That is, we will obtain alternate solutions to equation (5.2). Since $D > 0$, zero is not a root of the polynomial $F(t)$ with $A_4 = \tilde{\Lambda}, A_3 = A_1 = 0, A_2 = -k'$ and $A_0 = \tilde{\kappa}_d D$. Therefore for initial conditions $a'(0) \neq 0$ and $a(0) = 0$, (6.7) gives the solution

$$a(\eta) = \frac{\sqrt{\tilde{\kappa}_d D}\wp'(\eta)}{2(\wp(\eta) + \frac{1}{12}k')^2 - \frac{1}{2}\tilde{\Lambda}\tilde{\kappa}_d D} \qquad (6.20)$$

for invariants

$$g_2 = \tilde{\Lambda}\tilde{\kappa}_d D + \frac{1}{12}(k')^2 \quad \text{and} \quad g_3 = -\frac{1}{6}\tilde{\Lambda}\tilde{\kappa}_d Dk' + \frac{1}{216}(k')^3 \qquad (6.21)$$

restricted to be such that $\Delta \neq 0$. (6.20) is more general than the solutions to (5.2) in Section 5, since here the curvature k' and the constant D are unspecified. To see this solution expressed in terms of Jacobi elliptic functions, take curvature $k' = 1$ and $D = 1/(36\tilde{\Lambda}\tilde{\kappa}_d)$ for $0 < k < 1$ so that $g_3 = 0$ and $g_2 = \frac{1}{9}$. Then the reduced cubic is $4t^3 - \frac{1}{9}t = 4(t - \frac{1}{6})(t + \frac{1}{6})$ so that $e_3 = -\frac{1}{6}, e_2 = 0, e_1 = \frac{1}{6}$ and (6.20) becomes

$$a(\eta) = \frac{(12/\sqrt{\tilde{\Lambda}})\ \wp'(\eta)}{144(\wp(\eta) + \frac{1}{12})^2 - 1}. \qquad (6.22)$$

By (2.8) and (2.10), we write this solution in terms of Jacobi elliptic functions and obtain

$$a(\eta) = \frac{\sqrt{2 - 3\operatorname{sn}^2\left(\frac{\eta}{\sqrt{3}}, \frac{1}{\sqrt{2}}\right) + \operatorname{sn}^4\left(\frac{\eta}{\sqrt{3}}, \frac{1}{\sqrt{2}}\right)}}{\sqrt{6\tilde{\Lambda}}\left(2\operatorname{ns}\left(\frac{\eta}{\sqrt{3}}, \frac{1}{\sqrt{2}}\right) - \operatorname{sn}\left(\frac{\eta}{\sqrt{3}}, \frac{1}{\sqrt{2}}\right)\right)}$$

$$= \frac{1}{2\sqrt{3\tilde{\Lambda}}\operatorname{ds}\left(\frac{\eta}{\sqrt{3}}, \frac{1}{\sqrt{2}}\right)}\sqrt{\frac{2\operatorname{ds}^2\left(\frac{\eta}{\sqrt{3}}, \frac{1}{\sqrt{2}}\right) - 1}{2\operatorname{ds}^2\left(\frac{\eta}{\sqrt{3}}, \frac{1}{\sqrt{2}}\right) + 1}}$$

$$= \frac{\operatorname{cs}\left(\frac{\eta}{\sqrt{3}}, \frac{1}{\sqrt{2}}\right)}{\sqrt{6\tilde{\Lambda}}\sqrt{\left(1 + \operatorname{cs}^2\left(\frac{\eta}{\sqrt{3}}, \frac{1}{\sqrt{2}}\right)\right)\left(1 + 2\operatorname{cs}^2\left(\frac{\eta}{\sqrt{3}}, \frac{1}{\sqrt{2}}\right)\right)}} \qquad (6.23)$$

where each of the positive and negative square roots solve (5.2) for $k' = 1$, $D = 1/(36\tilde{\Lambda}\tilde{\kappa}_d)$ and where $\gamma^2 = e_1 - e_3 = \frac{1}{3}$ and $k^2 = (e_2 - e_3)/(e_1 - e_3) = \frac{1}{2}$. Writing (6.23) in terms of theta functions, and defining $\mu = \eta/\sqrt{3}\theta_3^2$, we

find that

$$a(\eta) = \frac{\theta_3\theta_1(\mu)\sqrt{2\theta_2^4\theta_4^4(\mu) - 3\theta_2^2\theta_3^2\theta_1^2(\mu)\theta_4^2(\mu) + \theta_3^4\theta_1^4(\mu)}}{\sqrt{6\widetilde{\Lambda}}\,\theta_2\theta_4(\mu)\left(2\theta_2^2\theta_4^2(\mu) - \theta_3^2\theta_1^2(\mu)\right)}$$

$$= \frac{\theta_3^2\theta_1(\mu)}{2\sqrt{3\widetilde{\Lambda}}\,\theta_2\theta_4\theta_3(\mu)}\sqrt{\frac{2\theta_2^2\theta_4^2\theta_3^2(\mu) - \theta_3^4\theta_1^2(\mu)}{2\theta_2^2\theta_4^2\theta_3^2(\mu) + \theta_3^4\theta_1^2(\mu)}}$$

$$= \frac{\theta_4\theta_3\theta_2(\mu)\theta_1(\mu)}{\sqrt{6\widetilde{\Lambda}\left(\theta_3^2\theta_1^2(\mu) + \theta_4^2\theta_2^2(\mu)\right)\left(\theta_3^2\theta_1^2(\mu) + 2\theta_4^2\theta_2^2(\mu)\right)}} \qquad (6.24)$$

by the theta function representations for sn, ds and cs respectively. Here the τ that forms the theta functions is suppressed and is taken to satisfy $\frac{1}{2} = \theta_2^4/\theta_3^4$.

7. Summary

There are a number of ways to see that elliptic and theta functions solve the d-dimensional Einstein gravitational field equations in a FRLW cosmology with a cosmological constant. Here we considered a scenario with no scalar field and with density functions $\rho(t) = D_1/\tilde{a}(t)^3 + D_2/\tilde{a}(t)^4$, $\rho(t) = D/\tilde{a}(t)^3$ and $\rho(t) = D/\tilde{a}(t)^4$ scaling in inverse proportion to the scale factor $\tilde{a}(t)$. In these cases the first Einstein equation (i) takes the form $\dot{\tilde{a}}(t)^2 =$ an expression containing negative powers of the cosmic scale factor $\tilde{a}(t)$. At this point, one could have introduced the inverse function $y(x)$ of $\tilde{a}(t)$ to obtain an expression for $y(x)$ as the integral of a power of x divided by the square root of a polynomial in x. That is, $y(x)$ would be an elliptic integral that is not normal; other authors have taken this approach [Abdalla and Correa-Borbonet 2002; Kraniotis and Whitehouse 2002]. Here, we switched to conformal time by a change of variables $a(\eta) \overset{\text{def.}}{=} \tilde{a}(f(\eta))$. This produced an equation of the form $a'(\eta)^2 =$ an expression containing nonnegative powers of the conformal scale factor $a(\eta)$.

After reviewing the definitions and properties of elliptic and theta functions in sections 2 and 3, we introduced the FRLW cosmological model in Section 4. In Section 5 for $\rho(t) = D/\tilde{a}(t)^3$, we obtained a differential equation for $a(\eta)$ containing only even powers of $a(\eta)$ and constructed solutions in terms of Jacobi elliptic functions, restricted to particular values of the constant D, parameterized by modulus $0 < k < 1$. The equivalent theta function representations for these solutions were recorded, and we noted the special cases for which the elliptic solutions reduce to elementary functions and the corresponding solution in cosmic time was also computed. In Section 6, we considered each of $\rho(t) = D_1/\tilde{a}(t)^3 + D_2/\tilde{a}(t)^4$, $\rho(t) = D/\tilde{a}(t)^3$ and $\rho(t) = D/\tilde{a}(t)^4$

with various initial conditions and obtained solutions in terms of Weierstrass functions for general curvature k' and constants D_1, D_2, $D > 0$. By considering these solutions restricted to certain D-values (again, parameterized by modulus $0 < k < 1$), we wrote $a(\eta)$ equivalently in terms of Jacobi elliptic and theta functions.

In current joint work with Floyd Williams [D'Ambroise and Williams $\geq$ 2010], we have seen that elliptic functions also appear in the presence of a scalar field $\phi(t)$, for both the FRLW and Bianchi I d-dimensional cosmological models with a $\Lambda \neq 0$ and with a similar density function scaling in inverse proportion to $\tilde{a}(t)$. There we note that the equations of each of these cosmological models can be rewritten in terms of a generalized Ermakov–Milne–Pinney differential equation [Lidsey 2004; D'Ambroise and Williams 2007], a type which the square root of the second moment of the wave function of the Bose–Einstein condensate (BEC) also satisfies. On the cosmological side of the FRLW-BEC correspondence, imposing an equation of state $\rho_\phi(t) = w p_\phi(t)$ (w constant) on the density $\rho_\phi(t)$ and pressure $p_\phi(t)$ of the scalar field $\phi(t)$ allows one to obtain the differential equation for an elliptic function on the side of the BECs.

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JENNIE D'AMBROISE
DEPARTMENT OF MATHEMATICS AND STATISTICS
LEDERLE GRADUATE RESEARCH TOWER
UNIVERSITY OF MASSACHUSETTS
AMHERST, MA 01003-9305
UNITED STATES
dambroise@math.umass.edu