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Integrable systems and 2D gravitation: How a soliton illuminates a black hole

SHABNAM BEHESHTI

1. Introduction

The interlacing of number theory with modern physics has a long and fruitful history. Indeed, the inital seeds were sown by Riemann himself, from paving the way to the Einstein equations with his introduction of the curvature tensor to setting the stage for quantum correction to black hole entropy with his careful study of the zeta function. This note indicates several elegant connections between two-dimensional gravitation, an eigenvalue problem of interest, extended objects (1D bosonic strings), and zeta regularization in 2D quantum gravity; we also note the presence of modular forms when possible and connect our results to the classical three-dimensional theory. It is our aim to find points of tangency with themes from the 2008 MSRI Summer School on Zeta and Modular Physics and motivate the reader for further exploration.

2. JT Gravitation: A simple 2D metric-scalar field theory

Consider the vacuum Einstein equations with vanishing cosmological constant

$$R_{ij} - \frac{R}{2} g_{ij} = 0, \qquad 1 \le i, j \le n \tag{1}$$

where R_{ij} and R denote the Ricci tensor and scalar curvature, respectively, and the solution (M^n, g) is an n-dimensional Riemannian manifold with metric tensor $g = g_{ij}$. A simple calculation shows that for dim $M = n = 2$, Equation (1) is trivially satisfied. Thus, to make a meaningful interpretation

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The title of this communication is both in reference and in honour of Geoffrey Mason's lecture *Vertex operators and arithmetic: how a single photon illuminates number theory*, at the 2004 ICMS Conference on Moonshine Conjectures and Vertex Algebras.

of these field equations, one is compelled to modify the Einstein-Hilbert action $\int_{M^n} R(g)\sqrt{|\det g|}d^n x$ from which they arise [40]. Introducing a dilaton $\tau = \tau(x) = \tau(x_1, x_2)$, a scalar field in the two variables of the manifold, a potential function $V = V(y)$, and a nonzero constant m, the modified action becomes

$$\int_{M^2} \big(R(g)\tau - m^2 V \circ \tau\big) \sqrt{|\det g|}\, d^2 x. \tag{2}$$

Standard variational principles yield a corresponding set of field equations

$$\begin{aligned} R(g) - m^2(V' \circ \tau) &= 0, \\ \nabla_i \nabla_j \tau - \frac{m^2}{2} g_{ij}(V \circ \tau) &= 0. \qquad 1 \le i, j \le 2 \end{aligned} \tag{3}$$

Here, $\nabla_i \nabla_j \tau$ denotes the Hessian of the field τ computed with respect to the metric g_{ij} [9]. The first equation in (3) is referred to as the Einstein equation of the system, with the remainder being called equations of motion for the dilaton τ. In contrast to (1), solutions to the modified two-dimensional model consist of a metric-dilaton pair (g, τ). This toy model has proved useful in understanding several key problems of interest, including

- relating exact solutions of system (3) to nonlinear equations having known special function solutions [5; 6; 41; 44],
- understanding the statistical origin for black hole entropy [20; 30],
- studying the endpoint of gravitational collapse [14; 21],
- examining the thermodynamics of black hole solutions in two and three dimensions [11; 12; 37],
- comparing 2D string gravity with higher dimensional counterparts [28; 45],
- providing a stepping stone for finding a consistent theory of quantum gravity (e.g. computing a one-loop effective action in a 2D model [16; 18]).

To better understand (3), we will assume the potential takes the form $V(y) = 2y$ until otherwise indicated. Independently studied by Jackiw [25] in the context of Liouville theory and Teitelboim [36] in the context of Hamiltonian dynamics, this specific case of the action is known as the *JT action* having *JT field equations*. Notice the Einstein equation of this system is a constant curvature condition on the manifold, namely $R(g) - 2m^2 = 0$. Having reduced the problem considerably with this choice of potential, we shall state a few solutions without proof.

EXAMPLE 1. Let $(x_1, x_2) = (T, r)$. If one makes the simplification $\tau(T, r) = \tau(r)$, then the remaining field equations may be solved to find a static solution to the JT field equations:

$$ds^2_{\text{bh}} = (M - m^2 r^2)\, dT^2 - \frac{1}{M - m^2 r^2} dr^2, \quad \tau_{\text{bh}}(T, r) = mr, \tag{4}$$

for M a constant. Excluding the surface $\tau = 0$, one may show that the Penrose diagram of this metric is identical to a two-dimensional section of the Schwarzschild black hole [1; 11; 27]. We use this to justify using the subscript "bh" in (4) and to refer to $ds^2_{\rm bh}$ as a *black hole metric*.

On the other hand, setting $(x_1, x_2) = (x, t)$ and making the metric ansatz

$$ds^2 \overset{g}{=} \cos^2 \frac{u(x,t)}{2} dx^2 - \sin^2 \frac{u(x,t)}{2} dt^2 \tag{5}$$

for an arbitrary function $u = u(x,t)$, one finds the Einstein equation in (3) is satisfied if and only if

$$\triangle u = m^2 \sin u, \tag{6}$$

i.e., u solves the *elliptic* sine-Gordon equation. This well-studied nonlinear partial differential equation is completely integrable in the sense that it has infinitely many conservation laws, a Lax formulation, a Backlünd Transformation and can be successfully treated with the inverse scattering technique [26; 32; 35; 48]. Consequently, equation (6) falls into a special class of nonlinear equations possessing *soliton* solutions, or localised wave solutions which maintain their shape and velocity upon collisions. Solving system (3) thus reduces to fixing a soliton solution u of (6) in the metric ansatz above and solving the equations of motion for the dilaton $\tau = \tau(x,t)$. We thus use the subscript "sol" and refer to $ds^2_{\rm sol}$ as a *soliton metric*.

EXAMPLE 2. Set u to be the simplest nontrivial solution to (6), namely the *kink soliton* $u(x,t) = 4 \arctan e^{m(x-vt)/a}$, with a, v constants such that $a^2 = 1 + v^2$. Then a solution of (3) is given by

$$ds^2_{\rm sol} = \cos^2 \frac{u}{2} dx^2 - \sin^2 \frac{u}{2} dt^2 \qquad \tau_{\rm sol}(x,t) = a \operatorname{sech} \frac{m}{a}(x - vt). \tag{7}$$

EXAMPLE 3. Choosing a slightly more complicated solution to (6), the oscillating kink-antikink soliton

$$u = u(x,t) = 4 \arctan \frac{v \sinh amx}{a \cos vmt},$$

with a and v as before, one may verify the pair

$$ds^2_{\rm sol} = \cos^2 \frac{u}{2} dx^2 - \sin^2 \frac{u}{2} dt^2, \quad \tau_{\rm sol}(x,t) = \frac{4v^2 am \sin vmt \sinh amx}{a^2 \cos^2 vmt + v^2 \sinh^2 amx} \tag{8}$$

solves system (3). Further details as to the derivation of these two examples may be found in [5; 44].

Since one expects the two dimensional metrics ds^2_{bh} and ds^2_{sol} to be locally equivalent, it is reasonable to pose whether it is possible to find an explicit correspondence between the solution spaces $(ds^2_{\text{bh}}, \tau_{\text{bh}})$ and $(ds^2_{\text{sol}}, \tau_{\text{sol}})$. When $M = v^2$ in (4), an explicit map is known between the black hole metric and the kink soliton metric described in Example 2 [44]. PDE conditions for a general mapping $\Theta(x,t)$ were later found, establishing a correspondence between $(ds^2_{\text{sol}}, \tau(x,t))$ for an arbitrary solution $u(x,t)$ of (6) and a generalised black hole solution $(ds^2_{\text{bh}}, \tau(T,r) = mr)$ of the form

$$ds^2_{\text{bh}} = -\big(|\nabla\tau|^2_{\text{sol}} \circ \Psi\big)/m^2 dT^2 + m^2/\big(|\nabla\tau|^2_{\text{sol}} \circ \Psi\big)\, dr^2.$$

The notation $|\nabla\tau|^2_{\text{sol}}$ denotes the length of the gradient of τ with respect to the soliton metric ds^2_{sol} and $\Psi(T,r) = \Theta(x,t)^{-1}$. To further elevate the status of the dilaton, it is also worth noting that τ plays a crucial role in determining the geometry of the two-dimensional black hole, as the Killing vectors are known once τ is given [5; 20]. Remarkably, the mappings Θ, Ψ constructed in [5; 6] turn out to be isometries. Specifically, they are transformations of the solution spaces of the field equations defined by the Laplace Beltrami operators of the soliton and black hole metrics.

3. Application of special functions to JT theory

3.1. Illuminating an eigenvalue problem. We rephrase the final statment of the last section in a particularly useful way. If $f = f(T,r)$, then a mapping Θ satisfying the PDE system found in [5] satisfies

$$\triangle_{\text{sol}}(f \circ \Theta) = (\triangle_{\text{bh}} f) \circ \Theta. \tag{9}$$

Therefore, $\triangle_{\text{bh}} f = \nu f$ if and only if $\triangle_{\text{sol}} F = \nu F$ with $F = f \circ \Theta$. This gives us a mechanism by which to solve eigenvalue problems in soliton coordinates by examining the vastly simpler equation $\triangle_{\text{bh}} f = \nu f$. Using the separation of variables $f(T,r) = e^{\omega T} h(r)$, one obtains a *differential equation of hypergeometric type* in r

$$\sigma(r)h''(r) + \sigma(r)\tilde{\tau}(r)h'(r) + \tilde{\sigma}(r)h(r) = 0, \tag{10}$$

where σ, and $\tilde{\tau}$ and $\tilde{\sigma}$ are polynomials in r satisfying deg σ, deg $\tilde{\sigma} \leq 2$, deg $\tilde{\tau} \leq 1$. Using the methods in [31; 43], Equation (10) is expressed in canonical form and quantization conditions are derived, from which infinite families of solutions may be written down. In special cases, the final solutions will involve functions such as Jacobi elliptic functions, or Gauss' hypergeometric functions, among others [5; 6; 43].

EXAMPLE 4. Under the mentioned separation of variables, the eigenvalue problem $\triangle_{\mathrm{bh}} f = \nu f$ reduces to

$$B(r)h''(r) - 2m^2 r B(r) h'(r) + (2m^2(B(r) - \nu)h(r) = 0,$$

where $B(r) = M - m^2 r^2$. Setting $\nu = \omega^2$, it is possible to reduce the equation to one of the form $r(1-r)v''(r) + [\gamma + r(\alpha + \beta + 1)]v'(r) - \alpha\beta v(r) = 0$, with $m_0 = 1/(m\sqrt{M})$, $\alpha = 2 + |\omega| m_0$, $\beta = -1 + |\omega| m_0$, $\gamma = 1 + |\omega| m_0$ and $v(r) = \frac{1}{2} h(m^2 m_0 r + 1)$. The equation is now in the standard form of Gauss' hypergeometric equation, having as solutions generalised hypergeometric functions $F(\alpha, \beta, \gamma; r)$:

$$F(\alpha, \beta, \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} z^n \qquad \text{for} |z| < 1, \gamma \neq \mathbb{Z}_{\leq 0}, \tag{11}$$

and where $(a)_n := \Gamma(a+n)/\Gamma(a) = a(a+1)(a+2)\cdots(a+n-1)$ is the Pochhammer symbol [22; 43]. Thus, one solves the eigenvalue problem as $f(T, r) = e^{\omega T} h(r)$, with $h(r) = F(\alpha, \beta, \gamma; mr/(2\sqrt{M}) + 1)$, and α, β, γ defined above. It is important to note that properties of this F (e.g., Saalschütz Theorem, Dougall-Ramanujan identity, etc.), and consequently properties of the solution f are intimately dependent on number theoretic and complex analytic results of the Gamma function [15; 34; 39].

Interestingly, hypergeometric differential equations also lend themselves to the study zero-weight modular forms and vertex operator algebras [38; 47].

3.2. Solitons and black hole entropy. A second way in which one may examine the interplay between the two-dimensional black hole and the soliton gauge, is by computing quantities of physical interest, such as entropy. In explicating a correspondence between ds^2_{bh} and ds^2_{sol}, one finds a relationship between the black hole mass M and several soliton parameters (the constants a, v in (7), for instance). In particular, we find the M is nonnegative in all the cases studied. More, [20; 27] compute the ADM energy ($mM/2G$, with G=Newton's coupling constant), Hawking temperature ($m\sqrt{M}/2\pi$) and associated Bekenstein-Hawking entropy ($2\pi\sqrt{M}/G$) for the black hole solutions ds^2_{bh} given in (4), so each of these quantities may be expressed in terms of the soliton parameters as well. The physical interpretation of these correspondences is still under investigation. Attempts have also been made to recover the asymptotic behaviour of the entropy using N-solitons and partition functions [19]. We shall outline a *fairly speculative* argument by Gegenberg and Kunstatter here with the hope that further discussion can shed light on the matter. Takhtadjan and Faddeev [35] compute the total energy for an N-soliton to be $E = \sum_{j=1}^{N} \left(m^2/\beta^2 + p_j^2\right)^2$, where p_j is the canonical momentum of the jth wave packet and β is a constant.

The rest energy of the state is thus $E_0 = Nm/\beta$. The claim is that "degeneracy of the state comes from the fact that the wave packets of an N-soliton are indistinguishable"; that is, degeneracy is the number of different ways to write N as a sum of non-negative integers. The Hardy–Ramanujan partition function $\mathrm{p}(N)$ counts precisely this value and is given asymptotically by

$$\mathrm{p}(N) \sim \frac{1}{4N\sqrt{3}} e^{K\sqrt{N}},$$

where $K = \pi\sqrt{2/3}$; see [23]. Thus, for large N, the entropy grows as $S \sim \log \mathrm{p}(N) \sim \sqrt{N} \sim \sqrt{E_0}$, up to order one multiplicative constant factors; this coincides with the Bekenstein-Hawking entropy stated above, found in [3; 20]. We remark that in order to make any of the above discussion rigourous, it is first necessary to make an argument which will include solutions of the sine-Gordon equation which do not fit the form of an N-soliton (e.g., breathers, non-soliton solutions). Furthermore, black hole energy has not been proven to be given by the rest energy of the N-soliton solution. We remark that the partition function can be cast as a special case of the Rademacher-Zuckerman formula for the coefficients of a modular form of negative weight -$\frac{1}{2}$ [33]. It is possible that a modular forms perspective will clarify these points.

4. Other two-dimensional considerations: Strings and quantum gravity

Two-dimensional theory is not restricted to the study of the JT field equations, of course. We touch upon two possible directions of exploration by altering the potential function V appearing in the action (2) and consequently, the resulting field equations (3).

4.1. 1D bosonic strings. If we now assume $V(y) = \gamma y^\alpha$, the original model not only encompasses the JT Theory ($\gamma = 2$, $\alpha = 1$), but several other gravitational theories of interest, including string-inspired gravity and spherically symmetric gravity as well. We shall only discuss the first of these two. Let $\alpha = 0$ so that $V(y) = \gamma \geq 0$. Correspondingly, the action appearing in (2) is modified to

$$I[g,\tau] = \int_{M^2} \left(R(g)\tau - m^2\gamma\right)\sqrt{|\det(g)|}\, d^2x, \tag{12}$$

and first field equation becomes $R(g) = 0$. Thus, the metric ansatz in (5) gives rise to the harmonicity condition $\triangle u = 0$, rather than the sine-Gordon equation. In this sense, the integrable systems content of the field equations changes qualitatively. However, under the conformal change of coordinates $\hat{g} = ge^{\varphi}$, one may, in fact, recover the classical Polyakov (bosonic) string action [7; 28; 29]

$$I[\hat{g},\varphi,\beta]=\int_{M^2}\big(R(\hat{g})-4|\nabla\varphi|^2_{\hat{g}}+\beta\big)e^{-2\varphi}\sqrt{|\det\hat{g}|}\,d^2x, \tag{13}$$

for $\beta = m^2\gamma$ and $\tau = e^{-2\varphi}$. The physical and geometric role of the dilaton appears in a new context, as the square of the conformal factor. Although the two actions are related, black hole solutions exist in the string model having vastly different geometry than the static Schwarzschild-type case we have discussed. An example of this follows.

EXAMPLE 5. Consider the target space action given in [28]

$$S(g,\varphi,T)=\int_{M^2}\big(R(g)-4|\nabla\varphi|^2+|\nabla T|^2+V(T)\big)e^{-2\varphi}\sqrt{|\det g|}\,d^2x, \tag{14}$$

where g is a two-dimensional metric, φ and T are scalar fields, known as the dilaton field and the tachyon field, respectively, and V is a polynomial potential satisfying $V(0)=0$. Supposing the tachyon field vanishes and $\varphi(T,r)=\kappa r$ for some constant κ, the field equations reduce to an inhomogeneous second order ODE, which yield the metric-dilaton solution

$$ds^2=-(1-ae^{Qr})\,dT^2+\frac{1}{1-ae^{Qr}}dr^2 \qquad \varphi(T,r)=\frac{Q}{2}r, \tag{15}$$

where $Q^2=-C$ is related to the central charge. The asymptotic and topological behaviour of this solution is clearly not of Schwarzschild type. Investigations of this example and string theories in general are detailed in [2; 28; 45].

In connection to the previous section, we comment that the parition function $p(N)$ has been used to count the microstates of a bosonic string; for further details, see [14; 46] and references therein.

4.2. From classical to quantum gravity: Zeta regularization. In a dimensionally reduced model, it is often possible to exactly compute various quantities of interest, both classically and quantum mechanically. We mention the value of two-dimensional models in the context of quantum gravity and zeta functions. Elizalde and Odintsov consider the action

$$\int_{M^2}\Big(R\frac{1}{\Delta}R+\Lambda\Big)\sqrt{|\det g|}\,d^2x \tag{16}$$

of induced two-dimensional gravity on the background $M^2=\mathbb{R}^1\times\mathbb{S}^1$; $R=R(g)$ is the scalar curvature, $\frac{1}{\Delta}$ is the resolvent operator, and Λ is a constant [16; 18]. Upon consideration of the one-loop gauge-independent effective action, the effective potential V is computed via regularization. Derived in terms of the differential operator Δ (see [17]) and a particular constant β, the zeta function is given by

$$\zeta_{-\Delta+m^2}\left(\frac{s}{2}\right) = -\frac{S}{\pi}\int_0^\infty \sum_{n=-\infty}^{+\infty}\left(k^2 + \frac{2\pi n}{\beta} + m^2\right)^{-s/2} dk. \tag{17}$$

Defining the variables $x = \Lambda/4(2-a)$ and $y = R\sqrt{x}$, with a =constant, the effective potential is computed as

$$V = \sqrt{x}\left(8\pi(2-a)y + \frac{y}{8}(1-\ln x) - \frac{1}{4} + \frac{1}{24y} - F(y)\right), \tag{18}$$

for

$$F(y) = \frac{1}{4\pi}\sum_{k=0}^{\infty}\left(\frac{(16\pi)^{-k}}{k!}y^{-k-\frac{1}{2}}\prod_{j=1}^{k}\left(4-(2j-1)^2\right)\sum_{n=1}^{\infty} n^{-k-\frac{3}{2}}e^{-2\pi n y}\right).$$

From this, a minimum for V is found and the authors conclude the compactification is stable [18]. The result is in marked contast to multidimensional quantum gravity on $\mathbb{R}^d \times \mathbb{S}^1$, which is known to be one-loop unstable [10; 24].

5. Relation to the 3D BTZ black hole

Two-dimensional models are studied with the ultimate goal of understanding higher dimensional theories. Naturally, we would like to connect the dilaton theory to higher dimensions in an explicit fashion. The Einstein Equations, arising from the classical Einstein-Hilbert action from the first section have also been examinined for *three dimensions* via

$$\int_{M^3}(R(g)-2\Lambda)\sqrt{|\det(g)|}d^3x, \tag{19}$$

with $x = (x_1, x_2, x_3)$ and Λ a constant. One solution of particular interest is the black hole metric discovered by Bañados, Teitelboim and Zanelli

$$ds^2_{BTZ} = -N(r)^2 dT^2 + \frac{1}{N(r)^2}dr^2 + r^2\left(N^\phi(r)\,dT + d\phi\right)^2, \tag{20}$$

where $x = (T, r, \phi)$, $N(r) = \Lambda r^2 - M + J^2/(4r^2)$ and $N^\phi(r) = -J/(2r)$; see [4]. The constants M and J correspond to the mass and angular momentum of the black hole, respectively. The field equations afforded by the three-dimensional case have been carefully studied, with the geometry and physics of the BTZ black hole outlined in [1; 3; 4; 11]. We notice an immediate relationship between the BTZ black hole and the JT black hole from Section 1. Keeping the ϕ-coordinate constant and setting $J = 0$, $\Lambda = m^2$, the two-dimensional metric in (4) is recovered as a static slice of (20). This motivates the following discussion: let $x = (x_1, x_2, \phi)$ and impose axial symmetry on the 3D metric g as $ds^2 = h_{ij}(\tilde{x})\, dx_i\, dx_j + \tau(\tilde{x})\, d\phi^2$, for $\tilde{x} = (x_1, x_2)$, $1 \le i, j \le 2$. Then (19) reduces to a two-dimensional action from which the JT field equations arise

$$I[g, \tau] = \int (R(h) - 2m^2)\tau(\tilde{x})\sqrt{|\det(h)|}\, d^2\tilde{x}, \tag{21}$$

where $R(h)$ is the scalar curvature of the two dimensional metric $h = h_{ij}(x_1, x_2)$ and $\tau(\tilde{x}) = \tau(x_1, x_2)$ is the dilaton, as before; compare with (2) for $V(y) = 2y$. In this way, the scalar field τ can be viewed as a radius along the direction of symmetry (the $\phi\phi$ direction) of the the surface defined by the metric h. Clearly then, the soliton content of the three-dimensional case can be considered, as well as pertinent questions on the presence of exact solutions involving special functions and physical quantities of interest. In this context, modular forms of negative weight also appear. The Rademacher-Zuckerman formula asymptotically yields the Cardy entropy formula of conformal field theory and as a special case, the statistical derivation of the Bekenstein-Hawking entropy of the BTZ black hole [8; 13; 14]. Further, quantum correction to entopy can be realised as a deformation of zeta and thus close connections between zeta functions and BTZ black hole thermodynamics have been suggested [42]. It is an interesting question whether the integrability structure in two dimensions sheds any light on the three dimensional case. Such avenues are currently under exploration and will be discussed in a future communication.

6. Conclusion

In the context of classical two-dimensional gravitation, we have only touched upon the possible mergers of pure mathematics with black hole physics and cosmology. For further exploration, consult the references.

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SHABNAM BEHESHTI
DEPARTMENT OF MATHEMATICS
110 FRELINGHUYSEN ROAD
RUTGERS UNIVERSITY
PISCATAWAY, NJ 08854-8019
UNITED STATES
beheshti@math.rutgers.edu