Topology of Stratified Spaces
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Preface

This volume is based on lectures given at the Workshop on the Topology of Stratified Spaces that was held at the Mathematical Sciences Research Institute in Berkeley, CA, from September 8 to September 12, 2008. Stratified spaces are usually not quite manifolds — they may possess singularities — but they are composed of manifold layers, the strata. Examples of such spaces include algebraic varieties, quotients of manifolds and varieties by group actions, homotopy stratified spaces, topological and piecewise linear pseudomanifolds, and even manifolds, augmented by filtrations that can arise, for example, via embeddings and their singularities. In recent years, there has been extensive interest and success in expanding to stratified spaces the triumphs of algebraic topology in manifold theory, including the vast progress in the mid-twentieth century on signatures, characteristic classes, surgery theories, and the special homological properties of nonsingular analytic and algebraic varieties, such as the Kähler package and Hodge theories. Such extensions from manifold theory to stratified space theory are rarely straightforward — they tend to involve the discovery and study of subtle interactions between local and global behavior — but vast progress has been made, particularly using such topological tools as intersection homology and the related analytic L^2 cohomology. The goal of the workshop was, and of this proceedings is, to provide an overview of this progress as well as of current research results, with a particular emphasis on communication across the boundaries of the different fields of mathematics that encompass stratified space research. Thus there is an emphasis in this volume on expository papers that give introductions to and overviews of topics in the area of stratified spaces.

Four main areas were featured in the MSRI workshop: L^2 cohomology and Hodge theorems, topology of algebraic varieties, signature theory on singular spaces, and mixed Hodge theory and singularities. For the purpose of giving some organization to the volume, we have grouped the papers roughly into these topics, although some papers overlap more than one area.

There are three papers on analysis and topology. The paper by Dai is an introduction to L^2 cohomology, which discusses some of the analytic considerations that arise in the study of L^2 cohomology and L^2 signatures on stratified

spaces as well as relationships between these analytic objects and topological analogues. Building on these basics, the paper of Carron gives an example of how his idea of the exterior derivative on a space having "almost closed range" can be used to calculate the L^2 cohomology in the case of two QALE-type spaces, the Hilbert schemes of two and of three points on $\mathbb{CP}^2$. The paper by Waelder comes from a different angle. It considers "rigid" differential operators on a smooth manifold commuting with an S^1-action, and infinite dimensional analog of the Dirac operator, whose index yields the complex elliptic genus. In this survey, Waelder discusses how rigidity theorems for such operators are related to the problem of defining Chern numbers on singular varieties. The paper of Waelder also fits under the second category: complex algebraic varieties.

The next section, on algebraic varieties, starts with the survey by Kovács and Schwede, which gives an introduction to the study of singularities that are unavoidable in classification problems of smooth algebraic varieties, and especially in the minimal model program, including log-canonical and Du Bois singularities. The paper of Libgober is an overview of the work of the author and his collaborators on extensions of the elliptic genus to singular varieties. It contains a new description of the class of holomorphic functions (quasi-Jacobi forms) that are elliptic genera of complex manifolds (possibly without the Calabi–Yau condition) and includes a survey of recent developments such as the "higher elliptic genera", which contain information on non-simply connected spaces. It also constructs the elliptic genus for singular real algebraic spaces. In their contribution, McCrory and Parusiński make a thorough study of the weight filtration on the Borel–Moore cohomology of real algebraic varieties. The authors give several descriptions of this filtration and show some nice applications of their construction to real algebraic and analytic geometry. Finally, the paper of Maxim gives a new formulation of results about Milnor classes, which generalize Parusiński's Milnor numbers to non-isolated hypersurface singularities. He considers these classes for singular hypersurfaces in complex manifolds and, using his new formulation, gives comparisons to the topologically defined L-classes of Cappell–Shaneson and Goresky–MacPherson.

Three papers discuss aspects of intersection homology and signatures on stratified spaces. The paper by Friedman is an expository survey of perversities in intersection homology, starting with the classical perversities of Goresky and MacPherson and describing various ways in which these have been generalized in the past three decades. It can also be read as a general introduction to intersection homology. Banagl's contribution is an overview of bordism invariant signatures on singular spaces, including the category of Witt space, where the signature on intersection cohomology provides a Witt-bordism invariant signature, and Banagl's new signature on a category of non-Witt spaces under a

new type of bordism. Levikov's paper describes the existence of a Wang-type sequence for intersection homology, which has implications for the non-Witt signatures defined by Banagl.

The section on mixed Hodge structures includes a comprehensive survey by Kerr and Pearlstein of recent developments in the theory of normal functions. It begins with the classical theory of Griffiths, Steenbrink, and Zucker, and includes the work of Green, Griffiths, and Kerr on limits of Abel–Jacobi mappings, the equivalence (due to Brosnan, Fang, Nie, and Pearlstein) of the Hodge conjecture to a question about singularities of certain normal functions; work by Brosnan and Pearlstein on the algebraicity of the zero locus of an admissible normal function; and the construction of a Néron model by Green, Griffiths, and Kerr, and by Brosnan, Pearlstein, and Saito. This survey includes several concrete examples and an extensive bibliography, and it concludes with a discussion of open questions in the field. The papers by Yokura and Schürmann, respectively, describe recent work by the authors, together with Brasselet, which gives a positive answer to the question of MacPherson as to whether there exists a unified theory of characteristic classes for singular varieties that is analogous to the classical Hirzebruch theory. Yokura's paper emphasizes the motivic touch of the story, whereas Schürmann's approach relies on Saito's powerful theory of algebraic mixed Hodge modules.

This volume concludes with an annotated list of problems proposed by participants in the workshop.

Neither the workshop nor this volume could have come about without the help of many people. Firstly, MSRI provided funding for the workshop and its staff helped enormously in making it a success. Secondly, Silvio Levy helped arrange publication of this volume and gave us good advice on how to make it as useful a book as possible. Finally, several of our colleagues have been insightful and efficient referees for the various papers herein. We are grateful to everyone for their assistance. We hope that readers will find this volume useful and inspiring.

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