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Elliptic genera, real algebraic varieties and quasi-Jacobi forms

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ABSTRACT. We survey the push-forward formula for elliptic class and various applications obtained in the papers by L. Borisov and the author. We then discuss the ring of quasi-Jacobi forms which allows to characterize the functions which are the elliptic genera of almost complex manifolds and extension of Ochanine elliptic genus to certain singular real algebraic varieties.

Introduction

Interest in the elliptic genus of complex manifolds stems from its appearance in a wide variety of geometric and topological problems. The elliptic genus is an invariant of the complex cobordism class modulo torsion, and hence depends only on the Chern numbers of the manifold. On the other hand, the elliptic genus is a holomorphic function defined on $\mathbb{C} \times \mathbb{H}$, where $\mathbb{H}$ is the upper half plane. In one heuristic approach, the elliptic genus is an index of an operator on the loop space (see [53]) and as such it has counterparts defined for C^∞, oriented or Spin manifolds: these were in fact studied before the complex case [43].

The elliptic genus comes up in the study of the geometry and topology of loop spaces and, more specifically, of the chiral de Rham complex [41]; in the study of invariants of singular algebraic varieties [8] — in particular orbifolds; and, more recently, in the study of Gopakumar–Vafa and Nekrasov conjectures [38; 25]. It is closely related to the fast developing subject of elliptic cohomology [46]. There are various versions of the elliptic genus, including the equivariant, higher elliptic genus obtained by twisting by cohomology classes of the fundamental group, the elliptic genus of pairs and the orbifold elliptic genus. There is inter-

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esting connection with singularities of weighted homogeneous polynomials — the so-called Landau–Ginzburg models.

We shall review several recent developments on the elliptic genus; we refer the reader [7] for details on earlier results. Then we shall focus on certain aspects of the elliptic genus: its extension to real singular varieties and its modularity property (or the lack of it). Extensions of the elliptic genus to real singular varieties were suggested by B. Totaro [48]; our approach is based on the push-forward formula for the elliptic class used in [8] to extend elliptic genus from smooth to certain singular complex projective varieties.

In Section 1A we discuss this push-forward formula, which appears as the main technical tool in many applications mentioned later. The rest of Section 1 discusses the relation with other invariants and series of applications based on the material in [9; 8; 6]. It includes a discussion of a relation between elliptic genus and E-function, applications to the McKay correspondence, elliptic genera of non-simply connected manifolds (higher elliptic genera) and generalizations of a formula of R. Dijkgraaf, D. Moore, E. Verlinde, and H. Verlinde. (Other applications in the equivariant context are discussed in R. Waelder's paper in this volume.) The proof of independence of resolutions of the elliptic genus (according to our definition) for certain real algebraic varieties is given later, in Section 3B.

Section 2 deals with modularity properties of the elliptic genus. In the Calabi–Yau cases (of pairs, orbifolds, etc.) the elliptic genus is a weak Jacobi form; see definition below. Also it is important to have a description of the elliptic genus in non-Calabi–Yau situations not just as a function on $\mathbb{C}\times\mathbb{H}$ but as an element of a finite-dimensional algebra of functions. It turns out that in the absence of a Calabi–Yau condition the elliptic genus belong to a very interesting algebra of functions on $\mathbb{C}\times\mathbb{H}$, which we call the algebra of quasi-Jacobi forms, and which is only slightly bigger than the algebra of weak Jacobi forms. This algebra of quasi-Jacobi forms is a counterpart of quasimodular forms [31] and is related to the elliptic genus in the same way as quasimodular forms are related to the Witten genus [55]. The algebra of quasi-Jacobi forms is generated by certain two-variable Eisenstein series, masterfully reviewed by A. Weil in [51], and has many properties parallel to the properties in quasimodular case. A detailed description of the properties of quasi-Jacobi forms appears to be absent in the literature, so we discuss the algebra of such forms in Section 2 — see its introduction. We conclude Section 2 with a discussion of differential operators Rankin–Cohen brackets on the space of Jacobi forms.

Finally in Section 3 we construct an extension of the Ochanine genus to real algebraic varieties with certain class of singularities. This extends results of Totaro [48].

For the readers' convenience we give ample references to prior work on elliptic genus, where more detailed information can be obtained. Section 2, dealing with quasi-Jacobi forms, can be read independently of the rest of the paper.

1. Elliptic genus

1A. Elliptic genus of singular varieties and push-forward formulas. Let X be a projective manifold. We shall use the Chow groups $A_*(X)$ with complex coefficients (see [23]). Let F the ring of functions on $\mathbb{C} \times \mathbb{H}$ where $\mathbb{H}$ is the upper half-plane. The elliptic class of X is an element in $A_*(X) \otimes_{\mathbb{C}} F$ given by

$$\mathcal{E}LL(X) = \prod_i x_i \frac{\theta\left(\frac{x_i}{2\pi \mathrm{i}} - z, \tau\right)}{\theta\left(\frac{x_i}{2\pi \mathrm{i}}, \tau\right)}[X], \tag{1-1}$$

where

$$\theta(z,\tau) = q^{\frac{1}{8}}(2\sin\pi z)\prod_{l=1}^{l=\infty}(1-q^l)\prod_{l=1}^{l=\infty}(1-q^l\mathrm{e}^{2\pi \mathrm{i}z})(1-q^l\mathrm{e}^{-2\pi \mathrm{i}z}) \tag{1-2}$$

is the Jacobi theta function considered as an element in F with $q = e^{2\pi i\tau}$ [12], the x_i are the Chern roots of the tangent bundle of X, and $[X]$ is the fundamental class of X. The component $Ell(X)$ in $A_0(X) = F$ is the elliptic genus of X.

The components of (1-1) in each degree, evaluated on a class in $A^*(X)$, are linear combinations of symmetric functions in c_i: that is, the Chern classes of X. In particular, $Ell(X)$ depends only on the class of X in the ring $\Omega^U \otimes \mathbb{Q}$ of unitary cobordisms.

The homomorphism $\Omega^U \otimes \mathbb{Q} \to F$ taking X to $Ell(X)$ can be described without reference to theta functions. Let M_{3,A_1} be the class of complex analytic spaces "having only A_1-singularities in codimension three", that is, having only singularities of the following type: the singular set $\operatorname{Sing} X$ of $X \in M_{3,A_1}$ is a *manifold* such that $\dim_{\mathbb{C}} \operatorname{Sing} X = \dim X - 3$ and for an embedding $X \to Y$ where Y is a manifold and a transversal H to $\operatorname{Sing} X$ in Y, the pair

$$(H \cap X, H \cap \operatorname{Sing} X)$$

is analytically equivalent to the pair $(\mathbb{C}^4, H_0)$, where H_0 is given by $x^2 + y^2 + z^2 + w^2 = 0$. Each $X \in M_{3,A_1}$ admits two small resolutions $\tilde{X}_1 \to X$ and $\tilde{X}_2 \to X$ in which the exceptional set is a fibration over $\operatorname{Sing} X$ with the fiber $\mathbb{P}^1$. One says that the manifolds underlying the resolutions are obtained from each other by a classical flop.

THEOREM 1.1. (*cf*. [47]) *The kernel of the homomorphism* $Ell : \Omega^U \otimes \mathbb{Q} \to F$ *taking an almost complex manifold* X *to its elliptic genus* $Ell(X)$ *is the ideal generated by the classes of differences* $\tilde{X}_1 - \tilde{X}_2$ *of two small resolutions of a variety in* M_{3,A_1}.

More generally one can fix a class of singular spaces and a type of resolutions and consider the quotient of $\Omega^U \otimes \mathbb{Q}$ by the ideal generated by differences of manifolds underlying resolutions of the same analytic space. The quotient map by this ideal $\Omega^U \otimes \mathbb{Q} \to R$ provides a genus and hence a collection of Chern numbers (linear combination of Chern monomials $c_{i_1} \dots \cdot c_{i_k}[X]$, with $\sum i_s = \dim X$), which can be made explicit via Hirzebruch's procedure with a generating series [28]. These are the Chern numbers which can be defined for the chosen class of singular varieties and chosen class of resolutions. The ideal in Theorem 1.1, it turns out, corresponds to a much larger classes of singular spaces and resolutions. This method of defining Chern classes of singular varieties is an extension of the philosophy underlying a question of Goresky and McPherson [26]: Which Chern numbers can be defined via resolutions independently of the resolution?

DEFINITION 1.2. An analytic space X is called $\mathbb{Q}$-Gorenstein if the divisor D of a meromorphic form $df_1 \wedge \cdots \wedge df_{\dim X}$ is such that for some $n \in \mathbb{Z}$ the divisor nD in locally principal (i.e., K_X is $\mathbb{Q}$-Cartier). In particular, for any codimension-one component E of the exceptional divisor of a map $\pi : \tilde{X} \to X$, the multiplicity $a_E = \mathrm{mult}_E \pi^*(K_X)$ is well defined and a singularity is called log-terminal if there is a resolution π such that $K_{\tilde{X}} = \pi^*(K_X) + \sum a_E E$ and $a_E > -1$. A resolution is called crepant if $a_E = 0$.

THEOREM 1.3. ([8]) *The kernel of the elliptic genus $\Omega^U \otimes \mathbb{Q} \to F$ is generated by the differences of $\tilde{X}_1 - \tilde{X}_2$ of manifolds underlying crepant resolutions of the singular spaces with $\mathbb{Q}$-Gorenstein singularities admitting crepant resolutions*.

The proof of Theorem 1.3 is based on an extension of the elliptic genus $Ell(X)$ of manifolds to the elliptic genus of pairs $Ell(X, D)$, where D is a divisor on X having normal crossings as the only singularities. This is similar to the situation in the study of motivic E-functions of quasiprojective varieties [2; 40]. In fact, other problems such as the the study of McKay correspondence [9] suggest a motivation for looking at triples (X, D, G), where X is a normal variety, G is a finite group acting on X and to introduce the elliptic class $\mathcal{E}LL(X, D, G)$ (see again [9]). More precisely, let $D = \sum a_i D_i$ be a $\mathbb{Q}$-divisor, with the D_i irreducible and $a_i \in \mathbb{Q}$. The pair (X, D) is called Kawamata log-terminal (klt) [35] if $K_X + D$ is $\mathbb{Q}$-Cartier and there is a birational morphism $f : Y \to X$, where Y is smooth and is the union of the proper preimages of components of D, and the components of the exceptional set $E = \bigcup_{\mathcal{J}} E_j$ form a normal crossing divisor such that $K_Y = f^*(K_X + \sum a_i D_i) + \sum \alpha_j E_j$, where $\alpha_j > -1$. (Here K_X, K_Y are the canonical classes of X and Y.) The triple (X, D, G), where X is a nonsingular variety, D is a divisor and G is a finite group of biholomorphic automorphisms is called G-normal [2; 9] if the components of

D form a normal crossings divisor and the isotropy group of any point acts trivially on the components of D containing this point.

DEFINITION 1.4 [9, Definition 3.2]. Let (X, E) be a Kawamata log terminal G-normal pair (in particular, X is smooth and D is a normal crossing divisor) and let $E = -\sum_{k\in\mathcal{K}} \delta_k E_k$. The *orbifold elliptic class* of (X, E, G) is the class in $A_*(X, \mathbb{Q})$ given by

$$\mathcal{E}LL_{\text{orb}}(X, E, G; z, \tau) :=$$
$$\frac{1}{|G|} \sum_{\substack{g,h \\ gh=hg}} \sum_{X^{g,h}} [X^{g,h}] \prod_{\substack{\lambda(g)=0 \\ \lambda(h)=0}} x_\lambda \prod_{\lambda} \frac{\theta\left(\frac{x_\lambda}{2\pi \mathrm{i}} + \lambda(g) - \tau\lambda(h) - z\right)}{\theta\left(\frac{x_\lambda}{2\pi \mathrm{i}} + \lambda(g) - \tau\lambda(h)\right)} \mathrm{e}^{2\pi \mathrm{i}\lambda(h)z}$$
$$\times \prod_k \frac{\theta\left(\frac{e_k}{2\pi \mathrm{i}} + \varepsilon_k(g) - \varepsilon_k(h)\tau - (\delta_k+1)z\right)}{\theta\left(\frac{e_k}{2\pi \mathrm{i}} + \varepsilon_k(g) - \varepsilon_k(h)\tau - z\right)} \frac{\theta(-z)}{\theta(-(\delta_k + 1)z)} \mathrm{e}^{2\pi \mathrm{i}\delta_k \varepsilon_k(h)z}. \tag{1-3}$$

where $X^{g,h}$ denotes an irreducible component of the fixed set of the commuting elements g and h and $[X^{g,h}]$ denotes the image of the fundamental class in $A_*(X)$. The restriction of TX to $X^{g,h}$ splits into linearized bundles according to the ($[0, 1)$-valued) characters λ of $\langle g, h\rangle$, which are sometimes denoted by λ_W, where W is a component of the fixed-point set. Moreover, $e_k = c_1(E_k)$ and ε_k is the character of $\mathcal{O}(E_k)$ restricted to $X^{g,h}$ if E_k contains $X^{g,h}$, and is zero otherwise.

One would like to define the elliptic genus of a Kawamata log-terminal pair (X_0, D_0) as (1-3) calculated for a G-equivariant resolution $(X, E) \to (X_0, D_0)$. Independence of (1-3) of resolution and the proof of (1.3) both depend on the following push-forward formula:

THEOREM 1.5. *Let (X, E) be a Kawamata log-terminal G-normal pair and let Z be a smooth G-equivariant locus in X which is normal crossing to* Supp E. *Let $f : \hat{X} \to X$ denote the blowup of X along Z. Define*

$$\hat{E} = -\sum_k \delta_k \hat{E}_k - \delta \operatorname{Exc} f,$$

where $\hat{E}_k$ is the proper transform of E_k and δ is determined from $K_{\hat{X}} + \hat{E} = f^*(K_X + E)$. Then $(\hat{X}, \hat{E})$ is a Kawamata log-terminal G-normal pair and

$$f_*\mathcal{E}LL_{\text{orb}}(\hat{X}, \hat{E}, G; z, \tau) = \mathcal{E}LL_{\text{orb}}(X, E, G; z, \tau). \tag{1-4}$$

Independence from the resolution is a consequence of the weak factorization theorem [1] and of Theorem 1.5; Theorem 1.3 follows since both $\mathcal{E}LL(X_1)$ and $\mathcal{E}LL(X_2)$ coincide with the elliptic genus of the pair $(\tilde{X}, \tilde{D})$, where $\tilde{X}$ is a resolution of X dominating both X_1 and X_2 (here $D = K_{\tilde{X}/X}$; see [8, Proposition 3.5] and also [52]. For a discussion of the orbifold elliptic genus on orbifolds more general than just global quotients see [20].

1B. Relation to other invariants. V. Batyrev defined in [2], for a G-normal triple (X, D, G), an E-function $E_{\text{orb}}(X, D, G)$ depending on the Hodge theoretical invariants. (There is also a motivic version; see [2; 40].) Firstly for a quasiprojective algebraic variety W one sets, as in [2, Definition 2.10],

$$E(W, u, v) = (-1)^i \sum_{p,q} \dim Gr_F^p Gr_W^{p+q}(H_c^i(W, \mathbb{C}))u^p v^q, \tag{1-5}$$

where F and W are the Hodge and weight filtrations of Deligne's mixed Hodge structure [16; 17]. In particular, $E(W, 1, 1)$ is the topological Euler characteristic of W (with compact support). If W is compact one then obtains Hirzebruch's χ_y-genus [28]:

$$\chi_y(W) = \sum_{i,j} (-1)^q \dim H^q(\Omega_W^p) y^p, \tag{1-6}$$

for $v = -1$, $u = y$, and hence the arithmetic genus, signature and so on are special values of (1-5). Secondly, for a G-normal pair as in Definition 1.4 one stratifies $D = \bigcup_{k \in K} D_k$ by strata $D_J^\circ = \bigcap_{j \in J} D_j - \bigcup_{k \in K-J} D_k$, for $J \subset K$ (the intersection being set to X if $J = \varnothing$), and defines

$$E(X, D, G, u, v) = \sum_{\substack{\{g\} \\ W \subset X^g}} (uv)^{\sum \varepsilon_{D_i}(g)(\delta_i+1)} \sum_{J \subset \mathcal{K}^g} \prod_{j \in J} \frac{uv-1}{(uv)^{\delta_j+1}-1} E(W \cap D_J^\circ / C(g, J)), \tag{1-7}$$

where $C(g, J)$ is the subgroup of the centralizer of g leaving $\bigcap_{j \in J} D_j$ invariant.

One shows that for a Kawamata log-terminal pair (X_0, D_0) the E-function $E(X, D, G)$ of a resolution does not depend on the latter but only on X_0, D_0 and G. Hence (1-7) yields an invariant of Kawamata log-terminal G-pairs. The relation with $Ell(X, D, G)$ is the following (Proposition 3.14 of [9]):

$$\lim_{\tau \to i\infty} Ell(X, D, G, z, \tau) = y^{-\frac{1}{2}\dim X} E(X, D, G, y, 1), \tag{1-8}$$

where $y = \exp(2\pi i z)$. In particular, in the nonequivariant smooth case the elliptic genus for $q \to 0$ specializes into the Hirzebruch χ_y genus (1-6).

On the other hand, in the nonsingular case, Hirzebruch [29; 30] and Witten [53] defined elliptic genera of complex manifolds which are given by modular forms for the subgroup $\Gamma_0(n)$ on level n in $\mathrm{SL}_2(\mathbb{Z})$, provided the canonical class of the manifold in question is divisible by n.

These genera are of course combinations of Chern numbers, but for $n = 2$ one obtains a combination of Pontryagin classes; i.e., an invariant that depends only on the underlying smooth structure, rather than the (almost) complex structure.

This genus was first introduced by S. Ochanine; see [43] and Section 3. These level-n elliptic genera coincide, up to a dimensional factor, with the specialization $z = (\alpha\tau + b)/n$, for appropriate $\alpha, \beta \in \mathbb{Z}$ specifying particular Hirzebruch level n elliptic genus; see Proposition 3.4 of [6].

1C. Application: The McKay correspondence for the elliptic genus. The classical McKay correspondence is a relation between the representations of the binary dihedral groups $G \subset \mathrm{SU}(2)$ (which are classified according to the root systems of type A_n, D_n, E_6, E_7, E_8) and the irreducible components of the exceptional set of the minimal resolution of $\mathbb{C}^2/G$. In particular, the number of conjugacy classes in G is the same as the number of irreducible components of the minimal resolution. The latter is a special case of the relation between the Euler characteristic $e(\widetilde{X/G})$ of a crepant resolution of the quotient X/G of a complex manifold X by an action of a finite group G and the data of the action on X:

$$e(\widetilde{X/G}) = \sum_{\substack{g,h \\ gh=hg}} e(X^{g,h}). \tag{1-9}$$

A refinement of this relation for Hodge numbers and motives is given in [2; 19; 40]. When X is projective one has a refinement in which the Euler characteristic of the manifold in (1-9) is replaced by the elliptic genus of Kawamata log-terminal pairs. More generally, one has the following push-forward formula:

THEOREM 1.6. *Let $(X; D_X)$ be a Kawamata log-terminal pair which is invariant under an effective action of a finite group G on X. Let $\psi\colon X \to X/G$ be the quotient morphism. Let $(X/G; D_{X/G})$ be the quotient pair in the sense that $D_{X/G}$ is the unique divisor on X/G such that $\psi^*(K_{X/G} + D_{X/G}) = K_X + D_X$* (see Definition 2.7 in [9]). *Then*

$$\psi_* \mathcal{E}LL_{\mathrm{orb}}(X, D_X, G; z, \tau) = \mathcal{E}LL(X/G, D_{X/G}; z, \tau).$$

In particular, for the components of degree zero one obtains

$$Ell_{\mathrm{orb}}(X, D_X, G, z, \tau) = Ell(X/G, D_{X/G}, z, \tau). \tag{1-10}$$

When X is nonsingular and X/G admits a crepant resolution $\widetilde{X/G} \to X/G$, for $q = 0$ one obtains $\chi_y(\widetilde{X/G}) = {\chi_y}^{\mathrm{orb}}(X, G)$ and hence for $y = 1$ one recovers (1-9).

1D. Application: Higher elliptic genera and K-equivalences. Another application of the push-forward formula in Theorem 1.5 is the invariance of higher elliptic genera under K-equivalences. A question posed in [44], and answered in [3], concerns the higher arithmetic genus $\chi_\alpha(X)$ of a complex manifold X

corresponding to a cohomology class $\alpha \in H^*(\pi_1(X), \mathbb{Q})$ and defined as

$$\int_X Td_X \cup f^*(\alpha), \tag{1-11}$$

where $f : X \to B(\pi_1(X))$ is the classifying map from X to the classifying space of the fundamental group of X. It asks whether the higher arithmetic genus $\chi_\alpha(X)$ is a birational invariant. This question is motivated by Novikov's conjecture: the higher signatures (i.e., the invariant defined for topological manifold X by (1-11) with the Todd class replaced by the L-class) are homotopy invariant [15]. The higher χ_y-genus defined by (1-11) with the Todd class replaced by Hirzebruch's χ_y class [28] comes into the correction terms describing the nonmultiplicativity of χ_y in topologically locally trivial fibrations $\pi : E \to B$ of projective manifolds with nontrivial action of $\pi_1(B)$ on the cohomology of the fibers of π. See [11] for details.

Recall that two manifolds X_1, X_2 are called K-equivalent if there is a smooth manifold $\tilde{X}$ and a diagram

$$\begin{array}{ccc} & \widetilde{X} & \\ {}^{\phi_1}\swarrow & & \searrow^{\phi_2} \\ X_1 & & X_2 \end{array} \tag{1-12}$$

in which ϕ_1 and ϕ_2 are birational morphisms and $\phi_1^*(K_{X_1})$ and $\phi_2^*(K_{X_2})$ are linearly equivalent.

The push-forward formula (1.5) leads to:

THEOREM 1.7. *For any $\alpha \in H^*(B\pi, \mathbb{Q})$ the higher elliptic genus*

$$(\mathcal{E}LL(X) \cup f^*(\alpha), [X])$$

is an invariant of K-equivalence. Moreover, if (X, D, G) and $(\hat{X}, \hat{D}, G)$ are G-normal and Kawamata log-terminal and if $\phi : (\hat{X}, \hat{D}) \to (X, D)$ is G-equivariant such that

$$\phi^*(K_X + D) = K_{\hat{X}} + \hat{D}, \tag{1-13}$$

then

$$Ell_\alpha(\hat{X}, \hat{D}, G) = Ell_\alpha(X, D, G).$$

In particular the higher elliptic genera (*and hence the higher signatures and $\hat{A}$-genus*) *are invariant for crepant morphisms. The specialization into the Todd class is birationally invariant* (*i.e., the invariance condition* (1-12) *is not needed in the Todd case*).

Another consequence is the possibility of defining higher elliptic genera for singular varieties with Kawamata log-terminal singularities and for G-normal pairs (X, D); see [10].

1E. The DMVV formula. The elliptic genus comes into a beautiful product formula for the generating series for the orbifold elliptic genus associated with the action of the symmetric group S_n on products $X \times \cdots \times X$, for which the first case appears in [18], together with a string-theoretical explanation. A general product formula for orbifold elliptic genus of triples is given in [9].

THEOREM 1.8. *Let (X, D) be a Kawamata log-terminal pair. For every $n \geq 0$ consider the quotient of $(X, D)^n$ by the symmetric group S_n, which we will denote by $(X^n/S_n, D^{(n)}/S_n)$. Here we denote by $D^{(n)}$ the sum of pullbacks of D under n canonical projections to X. Then we have*

$$\sum_{n\geq 0} p^n Ell(X^n/S_n, D^{(n)}/S_n; z, \tau) = \prod_{i=1}^{\infty} \prod_{l,m} \frac{1}{(1 - p^i y^l q^m)^{c(mi,l)}}, \tag{1-14}$$

where the elliptic genus of (X, D) is

$$\sum_{m\geq 0} \sum_{l} c(m, l) y^l q^m$$

and $y = \mathrm{e}^{2\pi \mathrm{i} z}$, $q = \mathrm{e}^{2\pi \mathrm{i}\tau}$.

It is amazing that such a simple-minded construction as the left-hand side of (1-14) leads to the Borcherds lift [4] of Jacobi forms.

1F. Other applications of the elliptic genus. In this section we point out other instances in which the elliptic genus plays a significant role.

The chiral de Rham complex. In [41], the authors construct for a complex manifold X a (bi)-graded sheaf Ω_X^{ch} of vertex operator algebras (with degrees called fermionic charge and conformal weight) with the differential d_{DR}^{ch} having fermionic degree 1 and quasiisomorphic to the de Rham complex of X. An alternative construction using the formal loop space was given in [32]. Each component of fixed conformal weight has a filtration so that graded components are

$$\bigotimes_{n\geq 1} (\Lambda_{-yq^{n-1}} T_X^* \otimes \Lambda_{-y^{-1}q^n} T_X \otimes S_{q^n} T_X^* \otimes S_{q^n} T_X) \tag{1-15}$$

In particular, it follows that

$$Ell(X, q, y) = y^{-\frac{1}{2}\dim X} \chi(\Omega_X^{\mathrm{ch}}) = y^{-\frac{1}{2}\dim X} \mathrm{Supertrace}_{H^*(\Omega_X^{\mathrm{ch}})} y^{J[0]} q^{L[0]}, \tag{1-16}$$

where $J[m]$, $L[n]$ are the operators which are part of the vertex algebra structure. The chiral complex for orbifolds was constructed in [22] and the extension of (1-16) to orbifolds (with discrete torsion) is discussed in [39].

Mirror symmetry. The physics definition of mirror symmetry in terms of conformal field theory suggests that for the elliptic genus, defined as an invariant of a conformal field theory (by an expression similar to the last term in (1-16) — see [54]) one should have for X and its mirror partner $\hat{X}$ the relation

$$Ell(X) = (-1)^{\dim X} Ell(\hat{X}). \tag{1-17}$$

This is indeed the case [6, Remark 6.9] for mirror symmetric hypersurfaces in toric varieties in the sense of Batyrev.

Elliptic genus of Landau–Ginzburg models. The physics literature (see [34], for example) also associates to a weighted homogeneous polynomial a conformal field theory (the Landau–Ginzburg model) and in particular the elliptic genus. Moreover it is expected that the orbifoldized Landau–Ginzburg model will coincide with the conformal field theory of the hypersurface corresponding to this weighted homogeneous polynomial. In particular, one expects a certain identity expressing equality of the orbifoldized elliptic genus corresponding to the weighted homogeneous polynomial (or a more general Landau Ginzburg model) and the elliptic genus of the corresponding hypersurface. In [42] the authors construct a vertex operator algebra related by a correspondence of this type to the cohomology of the chiral de Rham complex of the hypersurface in $\mathbb{P}^n$, and obtain in particular the expression for the elliptic genus of a hypersurface as an orbifoldization. In [27] the authors obtain an expression for the one-variable Hirzebruch's genus as an orbifoldization.

Concluding remarks. There are several other interesting issues which should be mentioned in a discussion of the elliptic genus. It plays an important role in work of J. Li, K. F. Liu and J. Zhou [38] in connection with the Gopakumar–Vafa conjecture (see also [25]). The elliptic genus was defined for proper schemes with 1-perfect obstruction theory [21]. In fact one has well defined cobordism classes in Ω^U associated to such objects [14]. In the case of surfaces with normal singularities, one can extend the definition of elliptic genus beyond log-terminal singularities [50]. The elliptic genus is central in the study of elliptic cohomology [46]. Much of the discussion above can be extended to the equivariant context [49]; a survey of this is given in Waelder's paper in this volume.

2. Quasi-Jacobi forms

The Eisenstein series

$$e_k(\tau) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^k}, \quad \tau \in \mathbb{H},$$

fails to be modular for $k = 2$, but the algebra generated by the functions $e_k(\tau)$, $k \geq 2$, called the algebra of quasimodular forms on $\mathrm{SL}_2(\mathbb{Z})$, has many interesting properties [57]. For example, there is a correspondence between quasimodular forms and real analytic functions on $\mathbb{H}$ which have the same $\mathrm{SL}_2(\mathbb{Z})$ transformation properties as modular forms. Moreover, the algebra of quasimodular forms has a structure of $\mathcal{D}$-module and supports an extension of Rankin–Cohen operations on modular forms.

In this section we show that there is an algebra of functions on $\mathbb{C} \times \mathbb{H}$ closely related to the algebra of Jacobi forms of index zero with similar properties. This algebra is generated by the Eisenstein series $\sum (z+\omega)^{-n}$, the sum being over elements ω of a lattice $W \subset \mathbb{C}$. It has a description in terms of real analytic functions satisfying a functional equation of Jacobi forms and having other properties of quasimodular forms mentioned in the last paragraph. It turns out that the space of functions on $\mathbb{C} \times \mathbb{H}$ generated by elliptic genera of arbitrary (possibly not Calabi–Yau) complex manifolds belong to this algebra of quasi-Jacobi forms.

DEFINITION 2.1. A *weak* (resp. meromorphic) Jacobi form of index $t \in \frac{1}{2}\mathbb{Z}$ and weight k for a finite index subgroup of the Jacobi group $\Gamma_1^J = \mathrm{SL}_2(\mathbb{Z}) \propto \mathbb{Z}^2$ is a holomorphic (resp. meromorphic) function χ on $\mathbb{H} \times \mathbb{C}$ having expansion $\sum c_{n,r} q^n \zeta^r$ in $q = \exp(2\pi\sqrt{-1}\tau)$ with $\operatorname{Im}\tau$ sufficiently large and satisfying the functional equations

$$\chi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k e^{2\pi i t c z^2/(c\tau+d)} \chi(\tau, z),$$
$$\chi(\tau, z+\lambda\tau+\mu) = (-1)^{2t(\lambda+\mu)} e^{-2\pi i t(\lambda^2\tau+2\lambda z)} \chi(\tau, z)$$

for all elements $\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, 0\right]$ and $\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, (a, b)\right]$ in Γ. The algebra of Jacobi forms is the bigraded algebra $J = \bigoplus J_{t,k}$. and the algebra of Jacobi forms of index zero is the subalgebra $J_0 = \bigoplus_k J_{0,k} \subset J$.

For appropriate l a Jacobi form can be expanded in (Fourier) series in $q^{1/l}$, with l depending on Γ. We shall need below the real analytic functions

$$\lambda(z,\tau) = \frac{z-\bar{z}}{\tau-\bar{\tau}} \quad \text{and} \quad \mu(\tau) = \frac{1}{\tau-\bar{\tau}}. \tag{2-1}$$

They have the transformation properties

$$\lambda\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)\lambda(z,\tau) - 2icz, \tag{2-2}$$
$$\lambda(z+m\tau+n, \tau) = \lambda(z,\tau) + m,$$
$$\mu\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2\mu(\tau) - 2ic(c\tau+d). \tag{2-3}$$

DEFINITION 2.2. An *Almost meromorphic Jacobi form* of weight k, index zero and depth (s,t) is a (real) meromorphic function in $\mathbb{C}\{q^{1/l},z\}[z^{-1},\lambda,\mu]$, with λ,μ given by (2-1), which

(a) satisfies the functional equations (2.1) of Jacobi forms of weight k and index zero, and
(b) which has degree at most s in λ and at most t in μ.

DEFINITION 2.3. *A quasi-Jacobi form* is a constant term of an almost meromorphic Jacobi form of index zero considered as a polynomial in the functions λ,μ; in other words, a meromorphic function f_0 on $\mathbb{H}\times\mathbb{C}$ such that there exist meromorphic functions $f_{i,j}$ such that each $f_0+\sum f_{i,j}\lambda^i\mu^j$ is an almost meromorphic Jacobi form.

From the algebraic independence of λ,μ over the field of meromorphic functions in q,z one deduces:

PROPOSITION 2.4. *F is a quasi-Jacobi of depth (s,t) if and only if*

$$(c\tau+d)^{-k}f\Big(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\Big)=\sum_{\substack{i\le s\\ j\le t}}S_{i,j}(f)(\tau,z)\Big(\frac{cz}{c\tau+d}\Big)^i\Big(\frac{c}{c\tau+d}\Big)^j,$$
$$f(\tau,z+a\tau+b)=\sum_{i\le s}T_i(f)(\tau,z)a^i.$$

We turn to some basic examples of quasi-Jacobi forms.

DEFINITION 2.5 [51]. Consider the sequence of functions on $\mathbb{H}\times\mathbb{C}$ given by

$$E_n(z,\tau)=\sum_{(a,b)\in\mathbb{Z}^2}\frac{1}{(z+a\tau+b)^n}$$

(These series were used in [24] under the name twisted Eisenstein series.)

The series $E_n(z,\tau)$ converges absolutely for $n\ge 3$ and for $n=1,2$ defined via “Eisenstein summation” as

$$\sum\nolimits_e(\cdot)=\lim_{A\to\infty}\sum_{a=-A}^{a=A}\lim_{B\to\infty}\sum_{b=-B}^{b=B}(\cdot),$$

though we shall omit the subscript e. The series $E_2(z,\tau)$ is related to the Weierstrass function as follows:

$$\wp(z,\tau)=\frac{1}{z^2}+\sum_{\substack{(a,b)\in\mathbb{Z}\\ (a,b)\ne 0}}\frac{1}{(z+a\tau+b)^2}-\frac{1}{(a\tau+b)^2}$$
$$=E_2(z,\tau)-\lim_{z\to 0}\Big(E_2(z,\tau)-\frac{1}{z^2}\Big).$$

Moreover,

$$e_n = \lim_{z\to 0}\Big(E_n(z,\tau) - \frac{1}{z^n}\Big) = \sum_{\substack{(a,b)\in\mathbb{Z}\\(a,b)\neq 0}} \frac{1}{(a\tau+b)^n}$$

is the Eisenstein series, in the notation of [51]. The algebra of functions of $\mathbb{H}$ generated by the Eisenstein series $e_n(\tau)$ for $n \geq 2$ is the algebra of quasimodular forms for $\mathrm{SL}_2(\mathbb{Z})$ [55; 57].

Now we describe the algebra of quasi-Jacobi forms for the Jacobi group Γ_1^J.

PROPOSITION 2.6. *The functions E_n are weak meromorphic Jacobi forms of index zero and weight n for $n \geq 3$. E_1 is a quasi-Jacobi form of index* 0 *weight* 1 *and depth* $(1,0)$. *$E_2 - e_2$ is a weak Jacobi form of index zero and weight* 2 *and E_2 is a quasi-Jacobi form of weight* 2, *index zero and depth* $(0,1)$.

PROOF. The first part follows from the absolute convergence of the series (2.5) for $n \geq 3$. We have the transformation formulas

$$E_1\Big(a\tau + bc\tau + d, \frac{z}{c\tau+d}\Big) = (c\tau+d)E_1(\tau,z) + \frac{\pi i c}{2}z, \tag{2-4}$$

$$E_1(\tau, z+m\tau+n) = E_1(\tau,z) - 2\pi i m, \tag{2-5}$$

$$E_2\Big(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\Big) = (c\tau+d)^2 E_2(\tau,z) - \tfrac{1}{2}\pi i c(c\tau+d), \tag{2-6}$$

$$E_2(\tau, z+a\tau+b) = E_2(\tau,z). \tag{2-7}$$

Equalities (2-4) and (2-6) follow from $E_1(\tau,z) = 1/z - \sum e_{2k}(\tau)z^{2k-1}$ and $E_2(\tau,z) = 1/z^2 + \sum_k (2k-1)e_{2k}z^{2k-2}$, respectively; see [51, Chapter 3, (10)]. Equality (2-7) is immediate form the definition of Eisenstein summation, while (2-5) follows from [51]. □

REMARK 2.7. The Eisenstein series $e_k(\tau)$, for $k \geq 4$, belong to the algebra of quasi-Jacobi forms. Indeed, from [51, Chapter IV, (7), (35)] one has

$$E_4 = (E_2 - e_2)^2 - 5e_4; \quad E_3^2 = (E_2 - e_2)^2 - 15e_4(E_2 - e_2) - 35e_4.$$

PROPOSITION 2.8. *The algebra of Jacobi forms* (*for* Γ_1^J) *of index zero and weight $t \geq 2$ is generated by* $E_2 - e_2, E_3, E_4$.

A short way to show this is to notice that the ring of such Jacobi forms is isomorphic to the ring of cobordisms of SU-manifolds modulo flops (Section 1A) via an isomorphism sending a complex manifold X of dimension d to $Ell(X)\cdot(\theta'(0)/\theta(z))^d$. This ring of cobordisms in turn is isomorphic to $\mathbb{C}[x_1, x_2, x_3]$, where x_1 is the cobordism class of a K3 surface and x_2, x_3 are the cobordism classes of certain four- and six-manifolds [48]. The graded algebra $\mathbb{C}[E_2 - e_2, E_3, E_4]$ is isomorphic to the same ring of polynomials (Examples 2.14) and the claim follows.

PROPOSITION 2.9. *The algebra of quasi-Jacobi forms is the algebra of functions on $\mathbb{H}\times\mathbb{C}$ generated by the functions $E_n(z,\tau)$ and $e_2(\tau)$.*

PROOF. The coefficient of λ^s for an almost meromorphic Jacobi form $F(\tau,z)=\sum_{i\le s} f_i\lambda^i$ of depth $(s,0)$ is a holomorphic Jacobi form of index zero and weight $k-s$; thus, by the previous proposition, it is a polynomial in E_2-e_2, $E_3,\ldots$. Moreover $f_0-E_1^s f_s$ is a quasi-Jacobi form of index zero and weight at most $s-1$. Hence, by induction, the ring of quasi-Jacobi forms of index zero and depth $(*,0)$ can be identified with $\mathbb{C}[E_1, E_2-e_2, E_3,\ldots]$. Similarly, the coefficient μ^t of an almost meromorphic Jacobi form $F=\sum_{j\le t}\bigl(\sum f_{i,j}\lambda^i\bigr)\mu^j$ is an almost meromorphic Jacobi form of depth $(s,0)$, and $F-\bigl(\sum_i f\bigr)i,s\lambda^i)E_2^t$ has depth (s',t') with $t'<t$. The claim follows. □

Here is an alternative description of the algebra of quasi-Jacobi forms:

PROPOSITION 2.10. *The algebra of functions generated by the coefficients of the Taylor expansion in x of the function*:

$$\frac{\theta(x+z)\theta'(0)}{\theta(x)\theta(z)}-\Bigl(\frac{1}{x}+\frac{1}{z}\Bigr)=\sum_{i\ge 1}F_i x^i$$

is the algebra of quasi-Jacobi forms for $\mathrm{SL}_2(\mathbb{Z})$.

PROOF. From [45] we have the transformation formulas

$$\theta\Bigl(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\Bigr)=\zeta(c\tau+d)^{1/2}e^{\pi icz^2/(c\tau+d)}\theta(\tau,z),$$
$$\theta'\Bigl(\frac{a\tau+b}{c\tau+d},0\Bigr)=\zeta(c\tau+d)^{3/2}\theta'(\tau,0),$$
$$\theta(\tau,z+m\tau+n)=(-1)^{m+n}e^{-2\pi imz-\pi im^2\tau}\theta(\tau,z).$$

They imply that the function

$$\Phi(x,z,\tau)=\frac{x\theta(x+z)\theta'(0)}{\theta(x)\theta(z)} \tag{2-8}$$

satisfies the functional equations

$$\begin{gathered}\Phi(\frac{a\tau+b}{c\tau+d},\frac{x}{c\tau+d},\frac{z}{c\tau+d})=e^{2\pi iczx/(c\tau+d)}\Phi(x,z,\tau),\\ \Phi(x,z+m\tau+n,\tau)=e^{2\pi imx}\Phi(x,z,\tau).\end{gathered} \tag{2-9}$$

In particular, in the expansion

$$\frac{d^2\log\Phi}{dx^2}=\sum H_i x^i, \tag{2-10}$$

the left-hand side is invariant under the transformations in (2-9) and the coefficient H_i is a Jacobi form of weight i and index zero for any i. Moreover the coefficients F_i in $\Phi(x,z,\tau) = 1 + \sum F_i(z,\tau)x^i$ are polynomials in F_1 and the H_i. What remains to show is that the E_i determine F_1 and the H_i, for $i \geq 1$, and vice versa.

Recall that E_i has index zero (is invariant with respect to shifts) and weight i. We shall use the expressions

$$\Phi(x,z,\tau) = \frac{x+z}{z} \exp\biggl(\sum_{k>0} \frac{2}{k!}\bigl(x^k + z^k - (x+z)^k\bigr) G_k(\tau)\biggr), \tag{2-11}$$

where

$$G_k(\tau) = -\frac{B_k}{2k} + \sum_{l=1}^{\infty} \sum_{d\,|\,l} (d^{k-1}) q^l; \tag{2-12}$$

see [56]. On the other hand, from [51, III.7 (10)] we have

$$E_n(z,\tau) = \frac{1}{z^n} + (-1)^n \sum_{2m \geq n}^{\infty} \left(\binom{2m-1}{n-1}\right) e_{2m} z^{2m-n}, \tag{2-13}$$

where

$$e_{2m} = {\sum}' \Bigl(\frac{1}{m\tau+n}\Bigr)^{2m} = \frac{2(2\pi\sqrt{-1})^k}{(k-1)!} G_k \quad \text{for } k = 2m; \tag{2-14}$$

see [51, III.7] and [55, p. 220]. We have

$$\frac{d^2 \log \Phi(x,z,\tau)}{dx^2} = \sum_{i\geq 1} \frac{(-1)^i i x^{i-1}}{z^{i+1}} + \sum_{i \geq 2} \frac{2}{(i-2)!}\bigl(x^{i-2} - (x+z)^{i-2}\bigr) G_i(\tau). \tag{2-15}$$

Now, using (2-14) and identities with binomial coefficients, we obtain for the coefficient of x^{l-2} for $l \geq 2$ in the Laurent expansion the value

$$\begin{aligned} &\frac{(-1)^{l-1}(l-1)}{z^l} - \sum_{i\geq 2, i>l} \frac{2}{(i-2)!} \binom{i-2}{l-2} z^{i-l} G_i(\tau) \\ &= \frac{(-1)^{l-1}(l-1)}{z^{l-1}} - \sum_{\substack{i\geq 2\\ i>l}} \frac{1}{(2\pi\sqrt{-1})^i} (i-1) \binom{i-2}{l-2} z^{i-l} e_i \\ &= \frac{(-1)^{l-1}(l-1)}{z^l} - (l-1) \frac{1}{(2\pi\sqrt{-1})^l} \sum_{\substack{i\geq 2\\ i>l}} \binom{i-1}{l-1} e_i \biggl(\frac{z}{2\pi\sqrt{-1}}\biggr)^{i-l} \end{aligned} \tag{2-16}$$

This yields

$$H_{l-2}(2\pi\sqrt{-1}z,\tau) = (-1)^{l-1}\frac{(l-1)}{(2\pi\sqrt{-1})^l}(E_l - e_l),$$

and the claim follows since formula (15) in [56] yields

$$F_1(z,\tau) = \frac{1}{z} - 2\sum_{r\geq 0} G_{r+1}\frac{z^r}{r!} = \frac{1}{z} - \frac{1}{(2\pi\sqrt{-1})}\sum_{r\geq 0} e_r\left(\frac{z}{2\pi\sqrt{-1}}\right)^r, \quad (2\text{-}17)$$

that is,

$$F_1(2\pi i\sqrt{-1}z,\tau) = \frac{1}{2\pi\sqrt{-1}}E_1(z,\tau) \qquad \square$$

REMARK 2.11. The algebra of quasi-Jacobi forms $\mathbb{C}[e_2, E_1, E_2, \dots]$ is closed under differentiation with respect to τ and ∂_z. Indeed, one has

$$2\pi i\frac{\partial E_1}{\partial\tau} = E_3 - E_1E_2, \qquad \frac{\partial E_1}{\partial z} = -E_2,$$
$$2\pi i\frac{\partial E_2}{\partial\tau} = 3E_4 - 2E_1E_3 - E_2^2, \quad \frac{\partial E_2}{\partial z} = -2E_3,$$

and hence $\mathbb{C}[\dots, E_i, \dots]$ is a $\mathcal{D}$-module, where $\mathcal{D}$ is the ring of differential operators generated by $\partial/\partial\tau$ and $\partial/\partial z$ over the ring of holomorphic Jacobi group invariant functions on $H\times\mathbb{C}$. As is clear from the discussion, the ring of Eisenstein series $\mathbb{C}[\dots, E_i, \dots]$ has a natural identification with the ring of real valued almost meromorphic Jacobi forms $\mathbb{C}[E_1^*, E_2^*, E_3, \dots]$ on $\mathbb{H}\times\mathbb{C}$ having index zero, where

$$E_1^* = E_1 + 2\pi i\frac{\operatorname{Im} x}{\operatorname{Im}\tau}, \quad E_2^* = E_2 + \frac{1}{\operatorname{Im}\tau}. \quad (2\text{-}18)$$

THEOREM 2.12. *The algebra of quasi-Jacobi forms of depth* $(k,0)$, $k\geq 0$, *is isomorphic to the algebra of complex unitary cobordisms modulo flops.*

In another direction, the depth of quasi-Jacobi forms allows one to "measure" the deviation of the elliptic genus of a non-Calabi–Yau manifold from being a Jacobi form.

THEOREM 2.13. *Elliptic genera of manifolds of dimension at most* d *span the subspace of forms of depth* $(d,0)$ *in the algebra of quasi-Jacobi forms. If a complex manifold satisfies* $c_1^k = 0$ *and* $c_1^{k-1}\neq 0$, [1] *its elliptic genus is a quasi-Jacobi form of depth* $(s,0)$, *where* $s\leq k-1$.

[1] More generally, k is the smallest among indices i with $c_1^i\in\operatorname{Ann}(c_2,\dots,c_{\dim M})$; an example of such a manifold is an n-manifold having a $(n-k)$-dimensional Calabi–Yau factor.

PROOF. It follows from the proof of Proposition (2.10) that

$$\frac{d^2 \log \Phi}{dx^2} = \sum_{i \geq 2} (-1)^{i-1} \frac{i-1}{(2\pi\sqrt{-1})^i} (E_i - e_i) x^{i-2},$$

which yields

$$\Phi = e^{E_1 x} \prod_i e^{(1/i)(-1)^{i-1}(i-1)/(2\pi\sqrt{-1})^i (E_i - e_i) x^i}. \tag{2-19}$$

The Hirzebruch characteristic series is

$$\Phi\Big(\frac{x}{2\pi i}\Big)\frac{\theta(z)}{\theta'(0)};$$

compare (1-1). Hence, if $c(TX) = \Pi(1 + x_k)$, then

$$\begin{aligned} Ell(X) &= \left(\frac{\theta(z)}{\theta'(0)}\right)^{\dim X} \prod_{i,k} e^{E_1 x_k} e^{(1/i)(-1)^{i-1}(i-1)(E_i - e_i) x_k^i}[X] \\ &= \left(\frac{\theta(z)}{\theta'(0)}\right)^{\dim X} e^{c_1(X) E_1} \prod_{i,k} e^{(1/i)(-1)^{i-1}(i-1)(E_i - e_i) x_k^i}[X], \end{aligned} \tag{2-20}$$

where $[X]$ is the fundamental class of X. In other words, if $c_1 = 0$, the elliptic class is a polynomial in $E_i - e_i$ with $i \geq 2$, and hence the elliptic genus is a Jacobi form [36]. Moreover of $c_1^k = 0$ the degree of this polynomial is at most k in E_1, and the claim follows. □

EXAMPLE 2.14. Expression (2-20) can be used to get formulas for the elliptic genus of specific examples in terms of Eisenstein series E_n. For example, for a surface in $\mathbb{P}^3$ having degree d one has

$$\big(E_1^2(\tfrac{1}{2}d^2 - 4d + 8)d + (E_2 - e_2)(\tfrac{1}{2}d^2 - 2)d\big)\left(\frac{\theta(z)}{\theta'(0)}\right)^2$$

In particular for $d = 1$ one obtains

$$\big(\tfrac{9}{2}E_1^2 - \tfrac{3}{2}(E_2 - e_2)\big)\left(\frac{\theta(z)}{\theta'(0)}\right)^2.$$

One can compare this with the double series that is a special case of the general formula for the elliptic genus of toric varieties in [6]. This leads to a two-variable version of the identity discussed in [6, Remark 5.9]. In fact following [5] one can define the subalgebra of "toric quasi-Jacobi forms" of the algebra of quasi-Jacobi forms, extending the toric quasimodular forms considered in [5]. This issue will be addressed elsewhere.

Next we consider one more similarity between meromorphic Jacobi forms and modular forms: there is a natural noncommutative deformation of the ordinary product of Jacobi forms similar to the deformation of the product modular forms constructed using Rankin–Cohen brackets [57]. In fact we have the following Jacobi counterpart of Rankin–Cohen brackets:

PROPOSITION 2.15. *Let f and g be Jacobi forms of index zero and weights k and l, respectively. Then*

$$[f,g]=k\Big(\partial_\tau f-\frac{1}{2\pi i}E_1\partial_z f\Big)g-l\Big(\partial_\tau g-\frac{1}{2\pi i}E_1(z,\tau)\partial_z g\Big)f$$

is a Jacobi form of weight $k+l+2$. More generally, let

$$D=\partial_\tau-\frac{1}{2\pi i}E_1\partial_z.$$

Then the Cohen–Kuznetsov series (see [57])

$$\tilde{f}_D(z,\tau,X)=\sum_{n=0}^{\infty}\frac{D^n f(z,\tau)X^n}{n!(k)_n},$$

where $(k)_n=k(k+1)\cdots(k+n-1)$ is the Pochhammer symbol, satisfies

$$\tilde{f}_D\Big(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d},\frac{z}{c\tau+d},\frac{X}{(c\tau+d)^2}\Big)\\ =(c\tau+d)^k\exp\Big(\frac{c}{c\tau+d}\frac{X}{2\pi i}\Big)f_D(\tau,z,X),$$

$$\tilde{f}_D(\tau,z+a\tau+b,X)=\tilde{f}_D(\tau,z,X).$$

In particular, the coefficient $[f,g]_n/(k)_n(l)_n$ of X^n in $\tilde{f}_D(\tau,z,-X)\tilde{g}_D(\tau,z,X)$ is a Jacobi form of weight $k+l+2n$. It is given explicitly in terms of $D^i f$ and $D^j g$ by the same formulas as the classical RC brackets.

PROOF. The main point is that the operator $\partial_\tau-\frac{1}{2\pi i}E_1\partial_z$ has the same deviation from transforming a Jacobi form into another as ∂_τ has on modular forms. Indeed:

$$\begin{aligned}(\partial_\tau-\frac{1}{2\pi i}E_1\partial_z)f\Big(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\Big)\\ =\Big(kc(c\tau+d)^{k+1}f(\tau,z)+zc(c\tau+d)^{k+1}\partial_z f(\tau,z)+(c\tau+d)^{k+2}\partial_\tau f(\tau,z)\\ -\frac{1}{2\pi i}\big((c\tau+d)E_1(\tau,z)+2\pi icz\big)(c\tau+d)^{k+1}\partial_z f\Big)\\ =(c\tau+d)^{k+2}\Big(f(\tau,z)-\frac{1}{2\pi i}E_1 f_z\Big)+kc(c\tau+d)^{k+1}f(\tau,z).\end{aligned}$$

Moreover,

$$\Big(\partial_\tau - \frac{1}{2\pi i}E_1\partial_z\Big)f(\tau, z+a\tau+b) = f_\tau + af_z - \frac{1}{2\pi i}(E_1 - 2\pi i a)f_z$$
$$= \Big(\partial_\tau - \frac{1}{2\pi i}E_1\partial_z\Big)f(\tau, z).$$

The rest of the proof runs as in [57]. □

REMARK 2.16. The brackets introduced in Proposition 2.15 are different from the Rankin–Cohen bracket introduced in [13].

3. Real singular varieties

The Ochanine genus of an oriented differentiable manifold X can be defined using the following series with coefficients in $\mathbb{Q}[\![q]\!]$ as the Hirzebruch characteristic power series (see [37] and references there):

$$Q(x) = \frac{x/2}{\sinh(x/2)}\prod_{n=1}^{\infty}\left(\frac{(1-q^n)^2}{(1-q^n e^x)(1-q^n e^{-x})}\right)^{(-1)^n} \tag{3-1}$$

As was mentioned in Section 1B, this genus is a specialization of the two-variable elliptic genus (at $z=\frac{1}{2}$). Evaluation of the Ochanine genus of a manifold using (3-1) and viewing the result as function of τ on the upper half-plane (where $q=e^{2\pi i\tau}$) yields a modular form on $\Gamma_0(2)\subset SL_2(\mathbb{Z})$; see [37].

In this section we discuss elliptic genera for real algebraic varieties. It particular we address Totaro's proposal [48] that "it should be possible to define Ochanine genus for a large class of compact oriented real analytic spaces." In this direction we have:

THEOREM 3.1 [48]. *The quotient of* MSO *by the ideal generated by oriented real flops and complex flops* (*that is*, *the ideal generated by* $X'-X$, *where* X' *and* X *are related by a real or complex flop*) *is*

$$\mathbb{Z}[\delta, 2\gamma, 2\gamma^2, 2\gamma^4],$$

with $\mathbb{CP}^2$ *corresponding to* δ *and* $\mathbb{CP}^4$ *to* $2\gamma+\delta^2$. *This quotient ring is the the image of* MSO_* *under the Ochanine genus.*

In particular the Ochanine genus of a small resolution is independent of its choice for singular spaces having singularities only along nonsingular strata and having in normal directions only singularities which are cones in $\mathbb{R}^4$ or $\mathbb{C}^4$.

Our goal is to find a wider class of singular real algebraic varieties for which the Ochanine genus of a resolution is independent of the choice of the latter.

3A. Real singularities. For the remainder of this paper "real algebraic variety" means an *oriented* quasiprojective variety $X_{\mathbb{R}}$ over $\mathbb{R}$, $X(\mathbb{R})$ is the set of its $\mathbb{R}$-points with the Euclidean topology, $X_{\mathbb{C}} = X_{\mathbb{R}} \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C}$ is the complexification and $X(\mathbb{C})$ the analytic space of its complex points. We also assume that $\dim_{\mathbb{R}} X(\mathbb{R}) = \dim_{\mathbb{C}} X(\mathbb{C})$.

DEFINITION 3.2. A real algebraic variety $X_{\mathbb{R}}$ as above is called $\mathbb{Q}$-Gorenstein log-terminal if the analytic space $X(\mathbb{C})$ is $\mathbb{Q}$-Gorenstein log-terminal.

EXAMPLE 3.3. *The affine variety*

$$x_1^2 - x_2^2 + x_3^2 - x_4^2 = 0 \tag{3-2}$$

in $\mathbb{R}^4$ *is three-dimensional Gorenstein log-terminal and admits a crepant resolution*.

Indeed, it is well known that the complexification of the Gorenstein singularity (3-2) admits a small (and hence crepant) resolution having $\mathbb{P}^1$ as its exceptional set.

EXAMPLE 3.4. *The three-dimensional complex cone in* $\mathbb{C}^4$ *given by* $z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0$ *considered as a codimension-two subvariety of* $\mathbb{R}^8$ *is a* $\mathbb{Q}$*-Gorenstein log-terminal variety over* $\mathbb{R}$ *and its complexification admits a crepant resolution*.

Indeed, this codimension-two subvariety is a real analytic space which is the intersection of two quadrics in $\mathbb{R}^8$ given by

$$a_1^2+a_2^2+a_3^2+a_4^2-b_1^2-b_2^2-b_3^2-b_4^2 = 0 = a_1b_1+a_2b_2+a_3b_3+a_4b_4, \tag{3-3}$$

where $a_i = \operatorname{Re} z_i, b_i = \operatorname{Im} z_i$. The complexification is the cone over complete intersection of two quadrics in $\mathbb{P}^7$. Moreover, the defining equations of this complete intersection become, after the change of coordinates $x_i = a_i + \sqrt{-1}\, b_i$, $y_i = a_i - \sqrt{-1}\, b_i$,

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0 \quad \text{and} \quad y_1^2 + y_2^2 + y_3^2 + y_4^2 = 0. \tag{3-4}$$

The singular locus is the union of two disjoint two-dimensional quadrics and the singularity along each is A_1 (i.e., the intersection of the transversal to it in $\mathbb{P}^7$ has an A_1 singularity). To resolve (3-3), one can blow up $\mathbb{C}^8$ at the origin, which results in a $\mathbb{C}$-fibration over the complete intersection (3-4). It can be resolved by small resolutions along two nonsingular components of the singular locus of (3-4). A direct calculation (considering, for example, the order of the pole of the form $dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_2 \wedge dy_3 \wedge dy_4/(x_1 y_1)$ along the intersection of the exceptional locus of the blow-up $\tilde{\mathbb{C}}^8$ of $\mathbb{C}^8$ with the proper preimage of (3-4) in $\tilde{\mathbb{C}}^8$) shows that we have a log-terminal resolution of the Gorenstein singularity which is the complexification of (3-3).

3B. The elliptic genus of resolutions of real varieties with $\mathbb{Q}$-Gorenstein log-terminal singularities. Let X be a real algebraic manifold and let $D = \sum \alpha_k D_k$, with each α_k in $\mathbb{Q}$, be a divisor on the complexification $X_{\mathbb{C}}$ of X (i.e., the D_k are irreducible components of D). Let x_i denote the Chern roots of the tangent bundle of $X_{\mathbb{C}}$ and denote by d_k the classes corresponding to D_k (Section 1).

DEFINITION 3.5. Let X be a real algebraic manifold and D a divisor on the complexification $X_{\mathbb{C}}$ of X. The Ochanine class $\mathcal{E}LL_{\mathcal{O}}(X, D)$ of the pair (X, D) is the specialization

$$\mathcal{E}LL\big(X_{\mathbb{C}}, D, q, z = \tfrac{1}{2}\big)$$

of the two-variable elliptic class of the pair $\mathcal{E}LL(X_{\mathbb{C}}, D, q, z)$ given by

$$\left(\prod_l \frac{\big(\frac{x_l}{2\pi \mathrm{i}}\big)\theta\big(\frac{x_l}{2\pi \mathrm{i}} - z\big)\theta'(0)}{\theta(-z)\theta\big(\frac{x_l}{2\pi \mathrm{i}}\big)}\right) \times \left(\prod_k \frac{\theta\big(\frac{d_k}{2\pi \mathrm{i}} - (\alpha_k+1)z\big)\theta(-z)}{\theta\big(\frac{d_k}{2\pi \mathrm{i}} - z\big)\theta(-(\alpha_k+1)z)}\right). \tag{3-5}$$

The Ochanine elliptic genus of the pair (X, D) as above is

$$Ell(X_{\mathbb{R}}, D) = \sqrt{\mathcal{E}LL\big(X_{\mathbb{C}}, D, q, \tfrac{1}{2}\big)} \cup \mathrm{cl}(X(\mathbb{R}))[X(\mathbb{C})]. \tag{3-6}$$

Here $\sqrt{\mathcal{E}LL}$ denotes the class corresponding to the unique series with constant term 1 and having $\mathcal{E}LL$ as its square.

The class of pairs above is the class (1-3) considered in Definition 1.4 in the case where group G is trivial. One can define an orbifold version of this class as well, specializing (1-3) to $z = \frac{1}{2}$. See [8] for further discussion of the class $\mathcal{E}LL(X, D)$.

The relation with Ochanine's definition is as follows: if D is the trivial divisor on $X_{\mathbb{C}}$, the result coincides with the genus [43]. More precisely:

LEMMA 3.6. *Let $X_{\mathbb{R}}$ be a real algebraic manifold with nonsingular complexification $X_{\mathbb{C}}$. Then*

$$Ell(X_{\mathbb{R}}) = \sqrt{\mathcal{E}LL(T_{X(\mathbb{C})})} \cup \mathrm{cl}(X(\mathbb{R}))[X(\mathbb{C})].$$

PROOF. Indeed, we have

$$0 \to T_{X(\mathbb{R})} \to T_{X(\mathbb{C})}|_{X(\mathbb{R})} \to T_{X(\mathbb{R})} \to 0, \tag{3-7}$$

with the identification of the normal bundle to $X_{\mathbb{R}}$ with its tangent bundle given by multiplication by $\sqrt{-1}$. Hence $\mathcal{E}LL(X_{\mathbb{R}})^2 = i^*(\mathcal{E}LL X_{\mathbb{C}})$, where $i : X_{\mathbb{R}} \to X_{\mathbb{C}}$ is the canonical embedding. Now the lemma follows from the identification of the characteristic series (3-1) and specialization $z = \frac{1}{2}$ of the series in (1-1) (see [6]) and the identification which is just a definition of the class $\mathrm{cl}_Z \in$

$H^{\dim_{\mathbb{R}} Y - \dim_{\mathbb{R}} Z}$ of a submanifold Z of a manifold Y: $\mathrm{cl}_Z \cup \alpha[Y] = i^*(\alpha) \cap [Z]$ for any $\alpha \in H^{\dim_{\mathbb{R}} Z}(Y)$. Indeed, we have

$$Ell(X_{\mathbb{R}}) = \mathcal{E}LL(T_{X(\mathbb{R})})[X(\mathbb{R})] = \sqrt{\mathcal{E}LL(T_{X(\mathbb{C})})}\,|_{X(\mathbb{R})}[X(\mathbb{R})]$$
$$= \sqrt{\mathcal{E}LL(T_{X(\mathbb{C})})} \cup \mathrm{cl}(X(\mathbb{R}))[X(\mathbb{C})]. \quad \square$$

Our main result in this section is the following:

THEOREM 3.7. *Let $\pi : (\tilde{X}, \tilde{D}) \to (X, D)$ be a resolution of singularities of a real algebraic pair with $\mathbb{Q}$-Gorenstein log-terminal singularities; i.e., $K_{\tilde{X}} + \tilde{D} = \pi^*(K_X + D)$. Then the elliptic genus of the pair $(\tilde{X}, \tilde{D})$ is independent of the resolution. In particular, if a real algebraic variety X has a crepant resolution, its elliptic genus is independent of a choice of crepant resolution.*

PROOF. Indeed for a blowup $f : (\tilde{X}, \tilde{D}) \to (X, D)$ we have

$$f_*\Big(\sqrt{\mathcal{E}LL\big(\tilde{X}, \tilde{D}, q, \tfrac{1}{2}\big)}\Big) = \sqrt{\mathcal{E}LL\big(X, D, q, \tfrac{1}{2}\big)} \tag{3-8}$$

This is a special case of the push-forward formula (1-4) in theorem 1.5, with G being the trivial group. Hence

$$\mathcal{E}LL_{\mathcal{O}}(X_{\mathbb{R}}, D) = \sqrt{\mathcal{E}LL\big(X_{\mathbb{C}}, D, q, \tfrac{1}{2}\big)} \cup \mathrm{cl}(X_{\mathbb{R}})[X_{\mathbb{C}}]$$
$$= \sqrt{\mathcal{E}LL(\tilde{X}_{\mathbb{C}}, \tilde{D}, q, \tfrac{1}{2})} \cup f^*([X_{\mathbb{R}}] \cap [X_{\mathbb{C}}]) = \mathcal{E}LL(\tilde{X}_{\mathbb{R}}, \tilde{D}),$$

as follows from projection formula since $f^*(\mathrm{cl}\,[X_{\mathbb{R}}]) = [\mathrm{cl}\,\tilde{X}_{\mathbb{R}}]$ and since f_* is the identity on H_0.

For a crepant resolution one has $D = 0$ and hence by Lemma 3.6 the elliptic genus of $X_{\mathbb{R}}$ is the Ochanine genus of the real manifold, which is its crepant resolution. $\square$

REMARK 3.8. Examples 3.3 and 3.4 show that singularities admitting a crepant resolution include real three-dimensional cones and real points of complex three-dimensional cones.

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