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The weight filtration for real algebraic varieties

CLINT MCCRORY AND ADAM PARUSIŃSKI

ABSTRACT. Using the work of Guillén and Navarro Aznar we associate to each real algebraic variety a filtered chain complex, the weight complex, which is well-defined up to a filtered quasi-isomorphism, and induces on Borel–Moore homology with $\mathbb{Z}_2$ coefficients an analog of the weight filtration for complex algebraic varieties.

The weight complex can be represented by a geometrically defined filtration on the complex of semialgebraic chains. To show this we define the weight complex for Nash manifolds and, more generally, for arc-symmetric sets, and we adapt to Nash manifolds the theorem of Mikhalkin that two compact connected smooth manifolds of the same dimension can be connected by a sequence of smooth blowups and blowdowns.

The weight complex is acyclic for smooth blowups and additive for closed inclusions. As a corollary we obtain a new construction of the virtual Betti numbers, which are additive invariants of real algebraic varieties, and we show their invariance by a large class of mappings that includes regular homeomorphisms and Nash diffeomorphisms.

The weight filtration of the homology of a real variety was introduced by Totaro [37]. He used the work of Guillén and Navarro Aznar [15] to show the existence of such a filtration, by analogy with Deligne's weight filtration for complex varieties [10] as generalized by Gillet and Soulé [14]. There is also earlier unpublished work on the real weight filtration by M. Wodzicki, and more recent unpublished work on weight filtrations by Guillén and Navarro Aznar [16].

Totaro's weight filtration for a compact variety is associated to the spectral sequence of a cubical hyperresolution. (For an introduction to cubical hyperresolutions of complex varieties see [34], Chapter 5.) For complex varieties

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this spectral sequence collapses with rational coefficients, but for real varieties, where it is defined with $\mathbb{Z}_2$ coefficients, the spectral sequence does not collapse in general. We show, again using the work of Guillén and Navarro Aznar, that the weight spectral sequence is itself a natural invariant of a real variety. There is a functor that assigns to each real algebraic variety a filtered chain complex, the *weight complex*, that is unique up to filtered quasi-isomorphism, and functorial for proper regular morphisms. The weight spectral sequence is the spectral sequence associated to this filtered complex, and the weight filtration is the corresponding filtration of Borel–Moore homology with coefficients in $\mathbb{Z}_2$.

Using the theory of Nash constructible functions we give an independent construction of a functorial filtration on the complex of semialgebraic chains in Kurdyka's category of arc-symmetric sets [19; 21], and we show that the filtered complex obtained in this way represents the weight complex of a real algebraic variety. We obtain in particular that the weight complex is invariant under regular rational homeomorphisms of real algebraic sets in the sense of Bochnak, Coste and Roy [5].

The characteristic properties of the weight complex describe how it behaves with respect to generalized blowups (acyclicity) and inclusions of open subvarieties (additivity). The initial term of the weight spectral sequence yields additive invariants for real algebraic varieties, the virtual Betti numbers [24]. Thus we obtain that the virtual Betti numbers are invariants of regular homeomorphisms of real algebraic sets. For real toric varieties, the weight spectral sequence is isomorphic to the toric spectral sequence introduced by Bihan, Franz, McCrory, and van Hamel [4].

In Section 1 we prove the existence and uniqueness of the filtered weight complex of a real algebraic variety. The weight complex is the unique acyclic additive extension to all varieties of the functor that assigns to a nonsingular projective variety the complex of semialgebraic chains with the canonical filtration. To apply the extension theorems of Guillén and Navarro Aznar [15], we work in the category of schemes over $\mathbb{R}$, for which one has resolution of singularities, the Chow–Hironaka Lemma (see [15, (2.1.3)]), and the compactification theorem of Nagata [28]. We obtain the weight complex as a functor of schemes and proper regular morphisms.

In Section 2 we characterize the weight filtration of the semialgebraic chain complex using resolution of singularities. In Section 3 we introduce the Nash constructible filtration of semialgebraic chains, following Pennaneac'h [32], and we show that it gives the weight filtration. A key tool is Mikhalkin's theorem [26] that any two connected closed C^∞ manifolds of the same dimension can be connected by a sequence of blowups and blowdowns. Section 4 we present several applications to real geometry.

In Section 5 we show that for a real toric variety the Nash constructible filtration is the same as the filtration on cellular chains defined by Bihan *et al.* using toric topology.

1. The homological weight filtration

We begin with a brief discussion of the extension theorem of Guillén and Navarro Aznar. Suppose that G is a functor defined for smooth varieties over a field of characteristic zero. The main theorem of [15] gives a criterion for the extension of G to a functor G' defined for all (possibly singular) varieties. This criterion is a relation between the value of G on a smooth variety X and the value of G on the blowup of X along a smooth center. The extension G' satisfies a generalization of this blowup formula for any morphism $f : \tilde{X} \to X$ of varieties that is an isomorphism over the complement of a subvariety Y of X. If one requires an even stronger additivity formula for $G'(X)$ in terms of $G'(Y)$ and $G'(X \setminus Y)$, then one can assume that the original functor G is defined only for smooth projective varieties.

The structure of the target category of the functor G is important in this theory. The prototype is the derived category of chain complexes in an abelian category. That is, the objects are chain complexes, and the set of morphisms between two complexes is expanded to include the inverses of quasi-isomorphisms (morphisms that induce isomorphisms on homology). Guillén and Navarro introduce a generalization of the category of chain complexes called a *descent category*, which has a class of morphisms E that are analogous to quasi-isomorphisms, and a functor **s** from diagrams to objects that is analogous to the total complex of a diagram of chain complexes.

In our application we consider varieties over the field of real numbers, and the target category is the derived category of filtered chain complexes of vector spaces over $\mathbb{Z}_2$. Since this category is closely related to the classical category of chain complexes, it is not hard to check that it is a descent category. Our starting functor G is rather simple: It assigns to a smooth projective variety the complex of semialgebraic chains with the canonical filtration. The blowup formula follows from a short exact sequence (1-3) for the homology groups of a blowup.

Now we turn to a precise statement and proof of Theorem 1.1, which is our main result.

By a *real algebraic variety* we mean a reduced separated scheme of finite type over $\mathbb{R}$. By a *compact* variety we mean a scheme that is complete (proper over $\mathbb{R}$). We adopt the following notation of Guillén and Navarro Aznar [15]. Let $\mathbf{Sch}_c(\mathbb{R})$ be the category of real algebraic varieties and proper regular morphisms, *i. e.* proper morphisms of schemes. By Reg we denote the subcategory of compact nonsingular varieties, and by $\mathbf{V}(\mathbb{R})$ the category of projective nonsingular varieties. A proper morphism or a compactification of varieties will always be understood in the scheme-theoretic sense.

In this paper we are interested in the topology of the set of real points of a real algebraic variety X. Let $\underline{X}$ denote the set of real points of X. The set $\underline{X}$, with its sheaf of regular functions, is a real algebraic variety in the sense of Bochnak, Coste and Roy [5]. For a variety X we denote by $C_*(X)$ the complex of semialgebraic chains of $\underline{X}$ with coefficients in $\mathbb{Z}_2$ and closed supports. The homology of $C_*(X)$ is the Borel–Moore homology of $\underline{X}$ with $\mathbb{Z}_2$ coefficients, and will be denoted by $H_*(X)$.

1A. Filtered complexes. Let $\mathcal{C}$ be the category of bounded complexes of $\mathbb{Z}_2$ vector spaces with increasing bounded filtration,

$$K_* = \cdots \leftarrow K_0 \leftarrow K_1 \leftarrow K_2 \leftarrow \cdots, \quad \cdots \subset F_{p-1}K_* \subset F_p K_* \subset F_{p+1}K_* \subset \cdots.$$

Such a filtered complex defines a spectral sequence $\{E^r, d^r\}$, $r = 1, 2, \dots$, with

$$E^0_{p,q} = \frac{F_p K_{p+q}}{F_{p-1}K_{p+q}}, \qquad E^1_{p,q} = H_{p+q}\left(\frac{F_p K_*}{F_{p-1}K_*}\right),$$

that converges to the homology of K_*,

$$E^\infty_{p,q} = \frac{F_p(H_{p+q}K_*)}{F_{p-1}(H_{p+q}K_*)},$$

where $F_p(H_n K_*) = \mathrm{Image}[H_n(F_p K_*) \to H_n(K_*)]$; see [22, Thm. 3.1]. A *quasi-isomorphism* in $\mathcal{C}$ is a filtered quasi-isomorphism, that is, a morphism of filtered complexes that induces an isomorphism on E^1. Thus a quasi-isomophism induces an isomorphism of the associated spectral sequences.

Following (1.5.1) in [15], we denote by $\mathrm{Ho}\,\mathcal{C}$ the category $\mathcal{C}$ localized with respect to filtered quasi-isomorphisms.

Every bounded complex K_* has a *canonical filtration* [8] given by

$$F^{\mathrm{can}}_p K_* = \begin{cases} K_q & \text{if } q > -p, \\ \ker \partial_q & \text{if } q = -p, \\ 0 & \text{if } q < -p. \end{cases}$$

We have

$$E^1_{p,q} = H_{p+q}\left(\frac{F^{\mathrm{can}}_p K_*}{F^{\mathrm{can}}_{p-1}K_*}\right) = \begin{cases} H_{p+q}(K_*) & \text{if } p+q = -p, \\ 0 & \text{otherwise.} \end{cases} \tag{1-1}$$

Thus a quasi-isomorphism of complexes induces a filtered quasi-isomorphism of complexes with canonical filtration.

To certain types of diagrams in $\mathcal{C}$ we can associate an element of $\mathcal{C}$, the *simple filtered complex* of the given diagram. We use notation from [15]. For $n \geq 0$ let

$\square_n^+$ be the partially ordered set of subsets of $\{0, 1, \ldots, n\}$. A *cubical diagram* of type $\square_n^+$ in a category $\mathcal{X}$ is a contravariant functor from $\square_n^+$ to $\mathcal{X}$. If $\mathcal{K}$ is a cubical diagram in $\mathcal{C}$ of type $\square_n^+$, let $K_{*,S}$ be the complex labeled by the subset $S \subset \{0, 1, \ldots n\}$, and let $|S|$ denote the number of elements of S. The simple complex $\mathbf{s}\,\mathcal{K}$ is defined by

$$\mathbf{s}\,\mathcal{K}_k = \bigoplus_{i+|S|-1=k} \mathcal{K}_{i,S}$$

with differentials $\partial : \mathbf{s}\,\mathcal{K}_k \to \mathbf{s}\,\mathcal{K}_{k-1}$ defined as follows. For each S let $\partial' : K_{i,S} \to K_{i-1,S}$ be the differential of $K_{*,S}$. If $T \subset S$ and $|T| = |S| - 1$, let $\partial_{T,S} : K_{*,S} \to K_{*,T}$ be the chain map corresponding to the inclusion of T in S. If $a \in K_{i,S}$, let

$$\partial''(a) = \sum \partial_{T,S}(a),$$

where the sum is over all $T \subset S$ such that $|T| = |S| - 1$, and

$$\partial(a) = \partial'(a) + \partial''(a).$$

The filtration of $\mathbf{s}\,\mathcal{K}$ is given by $F_p\,\mathbf{s}\,\mathcal{K} = \mathbf{s}\,F_p\mathcal{K}$,

$$(F_p\,\mathbf{s}\,\mathcal{K})_k = \bigoplus_{i+|S|-1=k} F_p(\mathcal{K}_{i,S}).$$

The simple complex functor $\mathbf{s}$ is defined for cubical diagrams in the category $\mathcal{C}$, but not for diagrams in the derived category $\operatorname{Ho}\mathcal{C}$, since a diagram in $\operatorname{Ho}\mathcal{C}$ does not necessarily correspond to a diagram in $\mathcal{C}$. However, for each $n \geq 0$, the functor $\mathbf{s}$ is defined on the derived category of cubical diagrams of type $\square_n^+$. (A quasi-isomorphism in the category of cubical diagrams of type $\square_n^+$ is a morphism of diagrams that is a quasi-isomorphism on each object in the diagram.)

To address this technical problem, Guillén and Navarro Aznar introduce the *Φ-rectification* of a functor with values in a derived category [15, (1.6)], where Φ is the category of finite orderable diagrams [15, (1.1.2)]. A (Φ-)rectification of a functor G with values in a derived category $\operatorname{Ho}\mathcal{C}$ is an extension of G to a functor of diagrams, with values in the derived category of diagrams, satisfying certain naturality properties [15, (1.6.5)]. A factorization of G through the category $\mathcal{C}$ determines a canonical rectification of G. One says that G is *rectified* if a rectification of G is given.

1B. The weight complex. To state the next theorem, we only need to consider diagrams in of type $\square_0^+$ or type $\square_1^+$. The inclusion of a closed subvariety $Y \subset X$ is a $\square_0^+$-diagram in $\mathbf{Sch}_c(\mathbb{R})$. An *acyclic square* ([15], (2.1.1)) is a $\square_1^+$-diagram in $\mathbf{Sch}_c(\mathbb{R})$,

$$\begin{array}{ccc} \tilde{Y} & \longrightarrow & \tilde{X} \\ \big\downarrow & & \big\downarrow \pi \\ Y & \xrightarrow{\ i\ } & X \end{array} \tag{1-2}$$

where i is the inclusion of a closed subvariety, $\tilde{Y} = \pi^{-1}(Y)$, and the restriction of π is an isomorphism $\tilde{X} \setminus \tilde{Y} \to X \setminus Y$. An *elementary acyclic square* is an acyclic square such that X is compact and nonsingular, Y is nonsingular, and π is the blowup of X along Y.

For a real algebraic variety X, let $F^{\text{can}}C_*(X)$ denote the complex $C_*(X)$ of semialgebraic chains with the canonical filtration.

THEOREM 1.1. *The functor*

$$F^{\text{can}}C_* : \mathbf{V}(\mathbb{R}) \to \operatorname{Ho}\mathcal{C}$$

that associates to a nonsingular projective variety M the semialgebraic chain complex of M with canonical filtration admits an extension to a functor defined for all real algebraic varieties and proper regular morphisms,

$$\mathcal{W}C_* : \mathbf{Sch}_c(\mathbb{R}) \to \operatorname{Ho}\mathcal{C},$$

such that $\mathcal{W}C_*$ is rectified and has the following properties:

(i) *Acyclicity. For an acyclic square* (1-2) *the simple filtered complex of the diagram*

$$\begin{array}{ccc} \mathcal{W}C_*(\tilde{Y}) & \longrightarrow & \mathcal{W}C_*(\tilde{X}) \\ \big\downarrow & & \big\downarrow \\ \mathcal{W}C_*(Y) & \longrightarrow & \mathcal{W}C_*(X) \end{array}$$

is acyclic (*quasi-isomorphic to the zero complex*).

(ii) *Additivity. For a closed inclusion $Y \subset X$, the simple filtered complex of the diagram*

$$\mathcal{W}C_*(Y) \to \mathcal{W}C_*(X)$$

is naturally quasi-isomorphic to $\mathcal{W}C_*(X \setminus Y)$.

Such a functor $\mathcal{W}C_*$ is unique up to a unique quasi-isomorphism.

PROOF. This theorem follows from [15], Theorem $(2.2.2)^{\mathrm{op}}$. By Proposition $(1.7.5)^{\mathrm{op}}$ of [15], the category $\mathcal{C}$, with the class of quasi-isomorphisms and the operation of simple complex $\mathbf{s}$ defined above, is a category of homological descent. Since it factors through $\mathcal{C}$, the functor $F^{\mathrm{can}}C_*$ is Φ-rectified ([15], (1.6.5), (1.1.2)). Clearly $F^{\mathrm{can}}C_*$ is additive for disjoint unions (condition (F1) of [15]). It remains to check condition (F2) for $F^{\mathrm{can}}C_*$, that the simple filtered complex associated to an elementary acyclic square is acyclic.

Consider the elementary acyclic square (1-2). Let $\mathcal{K}$ be the simple complex associated to the $\square_1^+$-diagram

$$\begin{array}{ccc} F^{\mathrm{can}}C_*(\tilde{Y}) & \longrightarrow & F^{\mathrm{can}}C_*(\tilde{X}) \\ \downarrow & & \downarrow \\ F^{\mathrm{can}}C_*(Y) & \longrightarrow & F^{\mathrm{can}}C_*(X) \end{array}$$

By definition of the canonical filtration, for each p we have

$$(F_p\,\mathbf{s}\,\mathcal{K})_k/(F_{p-1}\,\mathbf{s}\,\mathcal{K})_k \neq 0 \quad \text{only for } -p+2 \geq k \geq -p-1,$$

and the complex $(E^0_{p,*}, d^0)$ has the form

$$0 \to \frac{(F_p\,\mathbf{s}\,\mathcal{K})_{-p+2}}{(F_{p-1}\,\mathbf{s}\,\mathcal{K})_{-p+2}} \to \frac{(F_p\,\mathbf{s}\,\mathcal{K})_{-p+1}}{(F_{p-1}\,\mathbf{s}\,\mathcal{K})_{-p+1}} \to \frac{(F_p\,\mathbf{s}\,\mathcal{K})_{-p}}{(F_{p-1}\,\mathbf{s}\,\mathcal{K})_{-p}} \to \frac{(F_p\,\mathbf{s}\,\mathcal{K})_{-p-1}}{(F_{p-1}\,\mathbf{s}\,\mathcal{K})_{-p-1}} \to 0.$$

A computation gives

$$H_{-p+2}(E^0_{p,*}) = 0,$$
$$H_{-p+1}(E^0_{p,*}) = \mathrm{Ker}[H_{-p}(\tilde{Y}) \to H_{-p}(Y) \oplus H_{-p}(\tilde{X})],$$
$$H_{-p}(E^0_{p,*}) = \mathrm{Ker}[H_{-p}(Y) \oplus H_{-p}(\tilde{X}) \to H_{-p}(X)]/\mathrm{Im}[H_{-p}(\tilde{Y}) \to H_{-p}(Y) \oplus H_{-p}(\tilde{X})],$$
$$H_{-p-1}(E^0_{p,*}) = H_{-p}(X)/\mathrm{Im}[H_{-p}(Y) \oplus H_{-p}(\tilde{X}) \to H_{-p}(X)].$$

These groups are zero because for all k we have the short exact sequence of an elementary acyclic square,

$$0 \to H_k(\tilde{Y}) \to H_k(Y) \oplus H_k(\tilde{X}) \to H_k(X) \to 0; \tag{1-3}$$

see [25], proof of Proposition 2.1. □

REMARK 1.2. This above argument shows that the functor F^{can} is acyclic on any acyclic square (1-2), provided the varieties $X, Y, \tilde{X}, \tilde{Y}$ are nonsingular and compact.

REMARK 1.3. In Section 3 below, we show that the functor $\mathcal{W}C_*$ factors through the category of filtered chain complexes. This explains why $\mathcal{W}C_*$ is rectified.

If X is a real algebraic variety, the *weight complex* of X is the filtered complex $\mathcal{W}C_*(X)$. A stronger version of the uniqueness of $\mathcal{W}C_*$ is given by the following naturality theorem.

THEOREM 1.4. *Let $A_*, B_* : \mathbf{V}(\mathbb{R}) \to \mathcal{C}$ be functors whose localizations $\mathbf{V}(\mathbb{R}) \to \operatorname{Ho}\mathcal{C}$ satisfy the disjoint additivity condition* (F1) *and the elementary acyclicity condition* (F2) *of* [15]. *If $\tau : A_* \to B_*$ is a morphism of functors, then the localization of τ extends uniquely to a morphism $\tau' : \mathcal{W}A_* \to \mathcal{W}B_*$.*

PROOF. This follows from $(2.1.5)^{\mathrm{op}}$ and $(2.2.2)^{\mathrm{op}}$ of [15]. □

Thus if $\tau : A_*(M) \to B_*(M)$ is a quasi-isomorphism for all nonsingular projective varieties M, then $\tau' : \mathcal{W}A_*(X) \to \mathcal{W}B_*(X)$ is a quasi-isomorphism for all varieties X.

PROPOSITION 1.5. *For all real algebraic varieties X, the homology of the complex $\mathcal{W}C_*(X)$ is the Borel–Moore homology of X with $\mathbb{Z}_2$ coefficients,*

$$H_n(\mathcal{W}C_*(X)) = H_n(X).$$

PROOF. Let $\mathcal{D}$ be the category of bounded complexes of $\mathbb{Z}_2$ vector spaces. The forgetful functor $\mathcal{C} \to \mathcal{D}$ induces a functor $\varphi : \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{D}$. To see this, let A'_*, B'_* be filtered complexes, and let $A_* = \varphi(A'_*)$ and $B_* = \varphi(B'_*)$. A quasi-isomorphism $f : A'_* \to B'_*$ induces an isomorphism of the corresponding spectral sequences, which implies that f induces an isomorphism $H_*(A_*) \to H_*(B_*)$; in other words $f : A_* \to B_*$ is a quasi-isomorphism.

Let $C_* : \mathbf{Sch}_c(\mathbb{R}) \to \operatorname{Ho}\mathcal{D}$ be the functor that assigns to every real algebraic variety X the complex of semialgebraic chains $C_*(X)$. Then C_* satisfies properties (1) and (2) of Theorem 1.1. Acyclicity of C_* for an acyclic square (1-2) follows from the short exact sequence of chain complexes

$$0 \to C_*(\tilde{Y}) \to C_*(Y) \oplus C_*(\tilde{X}) \to C_*(X) \to 0.$$

The exactness of this sequence follows immediately from the definition of semialgebraic chains. Similarly, additivity of C_* for a closed embedding $Y \to X$ follows from the short exact sequence of chain complexes

$$0 \to C_*(Y) \to C_*(X) \to C_*(X \setminus Y) \to 0.$$

Now consider the functor $\mathcal{W}C_* : \mathbf{Sch}_c(\mathbb{R}) \to \operatorname{Ho}\mathcal{C}$ given by Theorem 1.1. The functors $\varphi \circ \mathcal{W}C_*$ and $C_* : \mathbf{Sch}_c(\mathbb{R}) \to \operatorname{Ho}\mathcal{D}$ are extensions of $C_* : \mathbf{V}(\mathbb{R}) \to \operatorname{Ho}\mathcal{D}$, so by [15] Theorem $(2.2.2)^{\mathrm{op}}$ we have that $\varphi(\mathcal{W}C_*(X))$ is quasi-isomorphic to $C_*(X)$ for all X. Thus $H_*(\mathcal{W}C_*(X)) = H_*(X)$, as desired. □

1C. The weight spectral sequence. If X is a real algebraic variety, the *weight spectral sequence* of X, $\{E^r, d^r\}$, $r = 1, 2, \ldots$, is the spectral sequence of the weight complex $\mathcal{W}C_*(X)$. It is well-defined by Theorem 1.1, and it converges to the homology of X by Proposition 1.5. The associated filtration of the homology of X is the *weight filtration*:

$$0 = \mathcal{W}_{-k-1} H_k(X) \subset \mathcal{W}_{-k} H_k(X) \subset \cdots \subset \mathcal{W}_0 H_k(X) = H_k(X),$$

where $H_k(X)$ is the homology with closed supports (Borel–Moore homology) with coefficients in $\mathbb{Z}_2$. (We show that $\mathcal{W}_{-k-1} H_k(X) = 0$ in Corollary 1.10.) The dual weight filtration on cohomology with compact supports is discussed in [25].

REMARK 1.6. We do not know the relation of the weight filtration of a real algebraic variety X to Deligne's weight filtration [10] on $H_*(X_{\mathbb{C}}; \mathbb{Q})$, the Borel–Moore homology with rational coefficients of the complex points $X_{\mathbb{C}}$. By analogy with Deligne's weight filtration, there should also be a weight filtration on the homology of X with classical compact supports and coefficients in $\mathbb{Z}_2$ (dual to cohomology with closed supports). We plan to study this filtration in subsequent work.

The weight spectral sequence $E^r_{p,q}$ is a second quadrant spectral sequence. (We will show in Corollary 1.10 that if $E^1_{p,q} \neq 0$ then (p, q) lies in the closed triangle with vertices $(0, 0)$, $(0, d)$, $(-d, 2d)$, where $d = \dim X$.) The reindexing

$$p' = 2p + q, \quad q' = -p, \quad r' = r + 1$$

gives a standard first quadrant spectral sequence, with

$$\tilde{E}^2_{p',q'} = E^1_{-q',p'+2q'}.$$

(If $\tilde{E}^2_{p',q'} \neq 0$ then (p', q') lies in the closed triangle with vertices $(0, 0)$, $(d, 0)$, $(0, d)$, where $d = \dim X$.) Note that the total grading is preserved: $p' + q' = p + q$.

The virtual Betti numbers [25] are the Euler characteristics of the rows of $\tilde{E}^2$, that is,

$$\beta_q(X) = \sum_p (-1)^p \dim_{\mathbb{Z}_2} \tilde{E}^2_{p,q}. \tag{1-4}$$

To prove this assertion we will show that the numbers $\beta_q(X)$ defined by (1-4) are additive and equal to the classical Betti numbers for X compact and nonsingular.

For each $q \geq 0$ consider the chain complex defined by the q-th row of the $\tilde{E}^1$ term,

$$C_*(X, q) = (\tilde{E}^1_{*,q}, \tilde{d}^1_{*,q}),$$

where $\tilde{d}^1_{p,q} : \tilde{E}^1_{p,q} \to \tilde{E}^1_{p-1,q}$. This chain complex is well-defined up to quasi-isomorphism, and its Euler characteristic is $\beta_q(X)$.

The additivity of $\mathcal{W}C_*$ implies that if Y is a closed subvariety of X then the chain complex $C_*(X \setminus Y, q)$ is quasi-isomorphic to the mapping cone of the chain map $C_*(Y,q) \to C_*(X,q)$, and hence there is a long exact sequence of homology groups

$$\cdots \to \tilde{E}^2_{p,q}(Y) \to \tilde{E}^2_{p,q}(X) \to \tilde{E}^2_{p,q}(X \setminus Y) \to \tilde{E}^2_{p-1,q}(Y) \cdots.$$

Therefore for each q we have

$$\beta_q(X) = \beta_q(X \setminus Y) + \beta_q(Y).$$

This is the additivity property of the virtual Betti numbers.

REMARK 1.7. Navarro Aznar pointed out to us that $C_*(X,q)$ is actually well-defined up to chain homotopy equivalence. One merely applies [15], Theorem $(2.2.2)^{\mathrm{op}}$, to the functor that assigns to a nonsingular projective variety M the chain complex

$$C_k(M,q) = \begin{cases} H_q(M) & \text{if } k = 0, \\ 0 & \text{if } k \neq 0, \end{cases}$$

in the category of bounded complexes of $\mathbb{Z}_2$ vector spaces localized with respect to chain homotopy equivalences. This striking application of the theorem of Guillén and Navarro Aznar led to our proof of the existence of the weight complex.

We say the weight complex is *pure* if the reindexed weight spectral sequence has $\tilde{E}^2_{p,q} = 0$ for $p \neq 0$. In this case the numbers $\beta_q(X)$ equal the classical Betti numbers of X.

PROPOSITION 1.8. *If X is a compact nonsingular variety, the weight complex $\mathcal{W}C_*(X)$ is pure. In other words, if $k \neq -p$ then*

$$H_k\left(\frac{\mathcal{W}_p C_*(X)}{\mathcal{W}_{p-1} C_*(X)}\right) = 0.$$

PROOF. For X projective and nonsingular, the filtered complex $\mathcal{W}C_*(X)$ is quasi-isomorphic to $C_*(X)$ with the canonical filtration. The inclusion $\mathbf{V}(\mathbb{R}) \to$ Reg has the extension property in (2.1.10) of [15]; the proof is similar to that in (2.1.11) of the same reference. Therefore by Theorem $(2.1.5)^{\mathrm{op}}$ [15], the functor $F^{\mathrm{can}}C_* : \mathbf{V}(\mathbb{R}) \to \operatorname{Ho}\mathcal{C}$ extends to a functor $\mathrm{Reg} \to \operatorname{Ho}\mathcal{C}$ that is additive for disjoint unions and acyclic, and this extension is unique up to quasi-isomorphism. But $F^{\mathrm{can}}C_* : \mathrm{Reg} \to \operatorname{Ho}\mathcal{C}$ is such an extension, since $F^{\mathrm{can}}C_*$ is additive for disjoint unions in Reg and acyclic for acyclic squares in Reg. (Compare the proof of Theorem 1.1 and Remark 1.2.) □

If X is compact, we will show that the reindexed weight spectral sequence $\tilde{E}^r_{p,q}$ is isomorphic to the spectral sequence of a *cubical hyperresolution* of X [15]. (The definition of cubical hyperresolution given in Chapter 5 of [34] is too weak for our purposes; see Example 1.12 below.)

A cubical hyperresolution of X is a special type of $\square^+_n$-diagram with final object X and all other objects compact and nonsingular. Removing X gives a $\square_n$-diagram, which is the same thing as a $\triangle_n$-diagram, *i.e.* a diagram labeled by the simplices contained in the standard n-simplex $\triangle_n$. (Subsets of $\{0, 1, \dots, n\}$ of cardinality $i+1$ correspond to i-simplices.)

The spectral sequence of a cubical hyperresolution is the spectral sequence of the filtered complex $(C_*, \hat{F})$, with $C_k = \bigoplus_{i+j=k} C_j X^{(i)}$, where $X^{(i)}$ is the disjoint union of the objects labeled by i-simplices of $\triangle_n$, and the filtration $\hat{F}$ is by skeletons,

$$\hat{F}_p C_k = \bigoplus_{i \le p} C_{k-i} X^{(i)}$$

The resulting first quadrant spectral sequence $\hat{E}^r_{p,q}$ converges to the homology of X, and the associated filtration is the weight filtration defined by Totaro [37].

Let $\partial = \partial' + \partial''$ be the boundary operator of the complex C_*, where $\partial'_i : C_j X^{(i)} \to C_j X^{(i-1)}$ is the simplicial boundary operator, and $\partial''_j : C_j X^{(i)} \to C_{j-1} X^{(i)}$ is $(-1)^i$ times the boundary operator on semialgebraic chains.

PROPOSITION 1.9. *If X is a compact variety, the weight spectral sequence E of X is isomorphic to the spectral sequence $\hat{E}$ of a cubical hyperresolution of X*:

$$E^r_{p,q} \cong \hat{E}^{r+1}_{2p+q,-p}.$$

Thus $\hat{E}^r_{p,q} \cong \tilde{E}^r_{p,q}$, the reindexed weight spectral sequence introduced above.

PROOF. The acyclicity property of the weight complex — condition (1) of Theorem 1.1 — implies that $\mathcal{W}C_*$ is acyclic for cubical hyperresolutions (see [15], proof of Theorem (2.1.5)). In other words, if the functor WC_* is applied to a cubical hyperresolution of X, the resulting $\square^+_n$-diagram in $\mathcal{C}$ is acyclic. This says that $\mathcal{W}C_*(X)$ is filtered quasi-isomorphic to the total filtered complex of the double complex $\mathcal{W}C_{i,j} = \mathcal{W}C_j X^{(i)}$. Since the varieties $X^{(i)}$ are compact and nonsingular, this filtered complex is quasi-isomorphic to the total complex $C_k = \bigoplus_{i+j=k} C_j X^{(i)}$ with the canonical filtration,

$$F^{\text{can}}_p C_k = \operatorname{Ker} \partial''_{-p} \oplus \bigoplus_{j > -p} C_j X^{(k-j)}.$$

Thus the spectral sequence of this filtered complex is the weight spectral sequence $E^r_{p,q}$.

We now compare the two increasing filtrations F^{can} and $\hat{F}$ on the complex C_*. The weight spectral sequence E is associated to the filtration F^{can}, and the cubical hyperresolution spectral sequence $\hat{E}$ is associated to the filtration $\hat{F}$. We show that $F^{\mathrm{can}} = \mathrm{Dec}\,(\hat{F})$, the *Deligne shift* of $\hat{F}$; for this notion see [8, (1.3.3)] or [34, A.49].

Let $\hat{F}'$ be the filtration

$$\hat{F}'_p C_k = \hat{Z}^1_{p,k-p} = \mathrm{Ker}[\partial : \hat{F}_p C_k \to C_{k-1}/\hat{F}_{p-1}C_{k-1}]$$

and $\hat{E}'$ the associated spectral sequence. By definition of the Deligne shift,

$$\hat{F}'_p C_k = \mathrm{Dec}\,\hat{F}_{p-k} C_k.$$

Now since $\partial = \partial' + \partial''$ it follows that

$$\hat{F}'_p C_k = F^{\mathrm{can}}_{p-k} C_k,$$

and $F^{\mathrm{can}}_{p-k} C_k = F^{\mathrm{can}}_{-q} C_k$, where $p+q=k$. Thus we can identify the spectral sequences

$$(\hat{E}')^{r+1}_{p,q} = E^r_{-q,p+2q} \quad \text{for } r \geq 1.$$

On the other hand, the inclusion $\hat{F}'_p C_k \to \hat{F}_p C_k$ induces an isomorphism of spectral sequences

$$(\hat{E}')^r_{p,q} \cong \hat{E}^r_{p,q} \quad \text{for } r \geq 2. \qquad \square$$

COROLLARY 1.10. *Let X be a real algebraic variety of dimension d, with weight spectral sequence E and weight filtration $\mathcal{W}$. For all p,q,r, if $E^r_{p,q} \neq 0$ then $p \leq 0$ and $-2p \leq q \leq d-p$. Thus for all k we have $\mathcal{W}_{-k-1} H_k(X) = 0$.*

PROOF. For X compact this follows from Proposition 1.9 and the fact that $\hat{E}^r_{p,q} \neq 0$ implies $p \geq 0$ and $0 \leq q \leq d-p$. If U is a noncompact variety, let X be a real algebraic compactification of U, and let $Y = X \setminus U$. We can assume that $\dim Y < d$. The corollary now follows from the additivity property of the weight complex (condition (2) of Theorem 1.1). $\square$

EXAMPLE 1.11. If X is a compact divisor with normal crossings in a nonsingular variety, a cubical hyperresolution of X is given by the decomposition of X into irreducible components. (The corresponding simplicial diagram associates to an i-simplex the disjoint union of the intersections of $i+1$ distinct irreducible components of X.) The spectral sequence of such a cubical hyperresolution is the Mayer–Vietoris (or Čech) spectral sequence associated to the decomposition. Example 3.3 of [25] is an algebraic surface X in affine 3-space such that X is the union of three compact nonsingular surfaces with normal crossings and the weight spectral sequence of X does not collapse: $\tilde{E}^2 \neq \tilde{E}^\infty$. The variety $U = \mathbb{R}^3 \setminus X$ is an example of a nonsingular noncompact variety with noncollapsing weight spectral sequence. (The additivity property (2) of Theorem 1.1 can be used to compute the spectral sequence of U.)

EXAMPLE 1.12. For a compact complex variety the Deligne weight filtration can be computed from the skeletal filtration of a simplicial smooth resolution of *cohomological descent* (see [9, (5.3)] or [34, (5.1.3)]). In particular, a rational homology class α has maximal weight if and only if α is in the image of the homology of the zero-skeleton of the resolution.

The following example shows that for real varieties the cohomological descent condition on a resolution is too weak to recover the weight filtration.

We construct a simplicial smooth variety $X_\bullet \to X$ of cohomological descent such that X is compact and the weight filtration of X does not correspond to the skeletal filtration of $X_\bullet$. Let $X = X_0 = S^1$, the unit circle in the complex plane, and let $f : X_0 \to X$ be the double cover $f(z) = z^2$. Let $X_\bullet$ be the Gabrielov–Vorobjov–Zell resolution associated to the map f [13] . Thus

$$X_n = X_0 \times_X X_0 \times_X \cdots (n+1) \cdots \times_X X_0,$$

a compact smooth variety of dimension 1. This resolution is of cohomological descent since the fibers of the geometric realization $|X_\bullet| \to X$ are contractible (see [13] or [34, (5.1.3)]).

Let $\alpha \in H_1(X)$ be the nonzero element ($\mathbb{Z}_2$ coefficients). Now $\alpha \in \mathcal{W}_{-1} H(X)$ since X is compact and nonsingular. Therefore, for every cubical hyperresolution of X, α lies in the image of the homology of the zero-skeleton (*i.e.*, the filtration of α with respect to the spectral sequence $\hat{E}$ is 0). But the filtration of α with respect to the skeletons of the resolution $X_\bullet \to X$ is greater than 0 since $\alpha \notin \operatorname{Im}[f_* : H_1(X_0) \to H_1(X)]$. In fact α has filtration 1 with respect to the skeletons of this resolution.

2. A geometric filtration

We define a functor

$$\mathcal{G}C_* : \mathbf{Sch}_c(\mathbb{R}) \to \mathcal{C}$$

that assigns to each real algebraic variety X the complex $C_*(X)$ of semialgebraic chains of X (with coefficients in $\mathbb{Z}_2$ and closed supports), together with a filtration

$$0 = \mathcal{G}_{-k-1} C_k(X) \subset \mathcal{G}_{-k} C_k(X) \subset \mathcal{G}_{-k+1} C_k(X) \subset \cdots \subset \mathcal{G}_0 C_k(X) = C_k(X). \quad \text{(2-1)}$$

We prove in Theorem 2.8 that the functor $\mathcal{G}C_*$ realizes the weight complex functor $\mathcal{W}C_* : \mathbf{Sch}_c(\mathbb{R}) \to \operatorname{Ho}\mathcal{C}$ given by Theorem 1.1. Thus the filtration $\mathcal{G}_*$ of chains gives the weight filtration of homology.

2A. Definition of the filtration $\mathcal{G}_*$. The filtration will first be defined for compact varieties. Recall that $\underline{X}$ denotes the set of real points of the real algebraic variety X.

THEOREM 2.1. *There exists a unique filtration* (2-1) *on semialgebraic* $\mathbb{Z}_2$-*chains of compact real algebraic varieties with the following properties. Let* X *be a compact real algebraic variety and let* $c \in C_k(X)$. *Then*

(1) *If* $Y \subset X$ *is an algebraic subvariety such that* $\operatorname{Supp} c \subset \underline{Y}$, *then*

$$c \in \mathcal{G}_p C_k(X) \iff c \in \mathcal{G}_p C_k(Y).$$

(2) *Let* $\dim X = k$ *and let* $\pi : \tilde{X} \to X$ *be a resolution of* X *such that there is a normal crossing divisor* $D \subset \tilde{X}$ *with* $\operatorname{Supp} \partial(\pi^{-1}c) \subset \underline{D}$. *Then for* $p \geq -k$,

$$c \in \mathcal{G}_p C_k(X) \iff \partial(\pi^{-1}c) \in \mathcal{G}_p C_{k-1}(D).$$

We call a resolution $\pi : \tilde{X} \to X$ *adapted* to $c \in C_k(X)$ if it satisfies condition (2) above. For the definition of the support $\operatorname{Supp} c$ and the pullback $\pi^{-1}c$ see the Appendix.

PROOF. We proceed by induction on k. If $k = 0$ then $0 = \mathcal{G}_{-1}C_0(X) \subset \mathcal{G}_0 C_0(X) = C_0(X)$. In the rest of this subsection we assume the existence and uniqueness of the filtration for chains of dimension $< k$, and we prove the statement for chains of dimension k.

LEMMA 2.2. *Let* $X = \bigcup_{i=1}^{s} X_i$ *where* X_i *are subvarieties of* X. *Then for* $m < k$,

$$c \in \mathcal{G}_p C_m(X) \iff c|_{X_i} \in \mathcal{G}_p C_m(X_i) \text{ for all } i.$$

PROOF. By (1) we may assume that $\dim X = m$ and then that all X_i are distinct of dimension m. Thus an adapted resolution of X is a collection of adapted resolutions of each component of X. □

See the Appendix for the definition of the restriction $c|_{X_i}$.

PROPOSITION 2.3. *The filtration* $\mathcal{G}_p$ *given by Theorem* 2.1 *is functorial*; *that is, for a regular morphism* $f : X \to Y$ *of compact real algebraic varieties,* $f_*(\mathcal{G}_p C_m(X)) \subset \mathcal{G}_p C_m(Y)$, *for* $m < k$.

PROOF. We prove that if the filtration satisfies the statement of Theorem 2.1 for chains of dimension $< k$ and is functorial on chains of dimension $< k-1$ then it is functorial on chains of dimension $k-1$.

Let $c \in C_{k-1}(X)$, and let $f : X \to Y$ be a regular morphism of compact real algebraic varieties. By (1) of Theorem 2.1 we may assume $\dim X = \dim Y =$

$k-1$ and by Lemma 2.2 that X and Y are irreducible. We may assume that f is dominant; otherwise $f_* c = 0$. Then there exists a commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \tilde{Y} \\ {\scriptstyle \pi_X}\big\downarrow & & \big\downarrow{\scriptstyle \pi_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

where π_X is a resolution of X adapted to c and π_Y a resolution of Y adapted to $f_* c$. Then

$$\begin{gathered} c \in \mathcal{G}_p(X) \iff \partial(\pi_X^{-1} c) \in \mathcal{G}_p(\tilde{X}) \Rightarrow \tilde{f}_* \, \partial(\pi_X^{-1} c) \in \mathcal{G}_p(\tilde{Y}), \\ \tilde{f}_* \, \partial(\pi_X^{-1} c) = \partial \tilde{f}_*(\pi_X^{-1} c) = \partial(\pi_Y^{-1} f_* c), \\ \partial(\pi_Y^{-1} f_* c) \in \mathcal{G}_p(\tilde{Y}) \iff f_* c \in \mathcal{G}_p(Y), \end{gathered}$$

where the implication in the first line follows from the inductive assumption. □

COROLLARY 2.4. *The boundary operator ∂ preserves the filtration $\mathcal{G}_p$:*

$$\partial \mathcal{G}_p C_m(X)) \subset \mathcal{G}_p C_{m-1}(X) \quad \textit{for } m < k.$$

PROOF. Let $\pi : \tilde{X} \to X$ be a resolution of X adapted to c. Let $\tilde{c} = \pi^{-1} c$. Then $c = \pi_* \tilde{c}$ and

$$c \in \mathcal{G}_p \iff \partial \tilde{c} \in \mathcal{G}_p \Rightarrow \partial c = \partial \pi_* \tilde{c} = \pi_* \partial \tilde{c} \in \mathcal{G}_p. \qquad \square$$

Let $c \in C_k(X)$, dim $X = k$. In order to show that condition (2) of Theorem 2.1 is independent of the choice of $\tilde{\pi}$ we need the following lemma.

LEMMA 2.5. *Let X be a nonsingular compact real algebraic variety of dimension k and let $D \subset X$ be a normal crossing divisor. Let $c \in C_k(X)$ satisfy* Supp $\partial c \subset D$. *Let $\pi : \tilde{X} \to X$ be the blowup of a nonsingular subvariety $C \subset X$ that has normal crossings with D. Then*

$$\partial c \in \mathcal{G}_p C_{k-1}(X) \iff \partial(\pi^{-1}(c)) \in \mathcal{G}_p C_{k-1}(\tilde{X}).$$

PROOF. Let $\tilde{D} = \pi^{-1}(D)$. Then $\tilde{D} = E \cup \bigcup \tilde{D}_i$, where $E = \pi^{-1}(C)$ is the exceptional divisor and $\tilde{D}_i$ denotes the strict transform of D_i. By Lemma 2.2,

$$\partial c \in \mathcal{G}_p C_{k-1}(X) \iff \partial c|_{D_i} \in \mathcal{G}_p C_{k-1}(D_i) \ \text{ for all } i.$$

Let $\partial_i c = \partial c|_{D_i}$. The restriction $\pi_i = \pi|_{\tilde{D}_i} : \tilde{D}_i \to D_i$ is the blowup with smooth center $C \cap D_i$. Hence, by the inductive assumption,

$$\partial(\partial_i c) \in \mathcal{G}_p C_{k-2}(D_i) \iff \partial \pi_i^{-1}(\partial_i c) = \partial\big(\partial(\pi^{-1}(c))|_{\tilde{D}_i}\big) \in \mathcal{G}_p C_{k-2}(\tilde{D}_i)$$

By the inductive assumption of Theorem 2.1,

$$\partial(\partial_i c) \in \mathcal{G}_p C_{k-2}(D_i) \iff \partial_i c \in \mathcal{G}_p C_{k-1}(D_i),$$

and we have similar properties for $\partial(\pi^{-1}(c))|_{\tilde{D}_i}$ and $\partial(\pi^{-1}(c))|_E$.

Thus, to complete the proof it suffices to show that if $\partial\big(\partial(\pi^{-1}(c))|_{\tilde{D}_i}\big)$ lies in $\mathcal{G}_p C_{k-2}(\tilde{D}_i)$ for all i, then $\partial\big(\partial(\pi^{-1}(c))|_E\big) \in \mathcal{G}_p C_{k-2}(E)$. This follows from

$$0 = \partial(\partial\pi^{-1}(c)) = \partial\Big(\sum_i \partial(\pi^{-1}(c))|_{\tilde{D}_i} + \partial(\pi^{-1}(c))|_E\Big). \qquad \square$$

Let $\pi_i : X_i \to X,\ i = 1, 2$, be two resolutions of X adapted to c. Then there exists $\sigma : \tilde{X}_1 \to X_1$, the composition of finitely many blowups with smooth centers that have normal crossings with the strict transforms of all exceptional divisors, such that $\pi_1 \circ \sigma$ factors through X_2,

$$\begin{array}{ccc} \tilde{X}_1 & \xrightarrow{\sigma} & X_1 \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\pi_1} \\ X_2 & \xrightarrow{\pi_2} & X \end{array}$$

By Lemma 2.5,

$$\partial(\pi_1^{-1}(c)) \in \mathcal{G}_p C_{k-1}(X_1) \iff \partial(\sigma^{-1}(\pi_1^{-1}(c))) \in \mathcal{G}_p C_{k-1}(\tilde{X}_1).$$

On the other hand,

$$\rho_* \partial(\sigma^{-1}(\pi_1^{-1}(c))) = \rho_* \partial(\rho^{-1}(\pi_2^{-1}(c))) = \partial(\pi_2^{-1}(c)),$$

and consequently by Proposition 2.3 we have

$$\partial(\pi_1^{-1}(c)) \in \mathcal{G}_p C_{k-1}(X_1) \implies \partial(\pi_2^{-1}(c)) \in \mathcal{G}_p C_{k-1}(X_2).$$

By symmetry, $\partial(\pi_2^{-1}(c)) \in \mathcal{G}_p(X)$ implies $\partial(\pi_1^{-1}(c)) \in \mathcal{G}_p(X)$. This completes the proof of Theorem 2.1. $\square$

2B. Properties of the filtration $\mathcal{G}_*$. Let U be a (not necessarily compact) real algebraic variety and let X be a real algebraic compactification of U. We extend the filtration $\mathcal{G}_p$ to U as follows. If $c \in C_*(U)$, let $\bar{c} \in C_*(X)$ be its closure. We define

$$c \in \mathcal{G}_p C_k(U) \iff \bar{c} \in \mathcal{G}_p C_k(X).$$

See the Appendix for the definition of the closure of a chain.

PROPOSITION 2.6. *$\mathcal{G}_p C_k(U)$ is well-defined; that is, for two compactifications X_1 and X_2 of U, we have*

$$c_1 \in \mathcal{G}_p C_k(X_1) \iff c_2 \in \mathcal{G}_p C_k(X_2),$$

where c_i denotes the closure of c in X_i, $i = 1, 2$.

PROOF. We may assume that $k = \dim U$. By a standard argument, any two compactifications can be dominated by a third one. Indeed, denote the inclusions by $i_i : U \hookrightarrow X_i$. Then the Zariski closure X of the image of (i_1, i_2) in $X_1 \times X_2$ is a compactification of U.

Thus we may assume that there is a morphism $f : X_2 \to X_1$ that is the identity on U. Then, by functoriality, $c_2 \in \mathcal{G}_p C_k(X_2)$ implies $c_1 = f_*(c_2) \in \mathcal{G}_p C_k(X_1)$. By the Chow–Hironaka lemma there is a resolution $\pi_1 : \tilde{X}_1 \to X_1$, adapted to c_1, that factors through f: $\pi_1 = f \circ g$. Then $c_1 \in \mathcal{G}_p C_k(X_1)$ is equivalent to $\pi_1^{-1}(c_1) \in \mathcal{G}_p C_k(\tilde{X}_1)$; but this implies that $c_2 = g_*(\pi_1^{-1}(c_1)) \in \mathcal{G}_p C_k(X_2)$, as needed. □

THEOREM 2.7. *The filtration $\mathcal{G}_*$ defines a functor $\mathcal{G}C_* : \mathbf{Sch}_c(\mathbb{R}) \to \mathcal{C}$ with the following properties*:

(1) *For an acyclic square* (1-2) *the following sequences are exact*:

$$0 \to \mathcal{G}_p C_k(\tilde{Y}) \to \mathcal{G}_p C_k(Y) \oplus \mathcal{G}_p C_k(\tilde{X}) \to \mathcal{G}_p C_k(X) \to 0,$$

$$0 \to \frac{\mathcal{G}_p C_k(\tilde{Y})}{\mathcal{G}_{p-1} C_k(\tilde{Y})} \to \frac{\mathcal{G}_p C_k(Y)}{\mathcal{G}_{p-1} C_k(Y)} \oplus \frac{\mathcal{G}_p C_k(\tilde{X})}{\mathcal{G}_{p-1} C_k(\tilde{X})} \to \frac{\mathcal{G}_p C_k(X)}{\mathcal{G}_{p-1} C_k(X)} \to 0.$$

(2) *For a closed inclusion $Y \subset X$, with $U = X \setminus Y$, the following sequences are exact*:

$$0 \to \mathcal{G}_p C_k(Y) \to \mathcal{G}_p C_k(X) \to \mathcal{G}_p C_k(U) \to 0,$$

$$0 \to \frac{\mathcal{G}_p C_k(Y)}{\mathcal{G}_{p-1} C_k(Y)} \to \frac{\mathcal{G}_p C_k(X)}{\mathcal{G}_{p-1} C_k(X)} \to \frac{\mathcal{G}_p C_k(U)}{\mathcal{G}_{p-1} C_k(U)} \to 0.$$

PROOF. The exactness of the first sequence of (2) follows directly from the definitions (moreover, this sequence splits via $c \mapsto \bar{c}$). The exactness of the second sequence of (2) now follows by a diagram chase. Similarly, the exactness of the first sequence of (1) follows from the definitions, and the exactness of the second sequence of (1) is proved by a diagram chase. □

For any variety X, the filtration $\mathcal{G}_*$ is contained in the canonical filtration,

$$\mathcal{G}_p C_k(X) \subset F_p^{\mathrm{can}} C_k(X), \tag{2-2}$$

since $\partial_k(\mathcal{G}_{-k} C_k(X)) = 0$. Thus on the category of nonsingular projective varieties we have a morphism of functors

$$\sigma : \mathcal{G}C_* \to F^{\mathrm{can}} C_*.$$

THEOREM 2.8. *For every nonsingular projective real algebraic variety M*,

$$\sigma(M) : \mathcal{G}C_*(M) \to F^{\mathrm{can}} C_*(M)$$

is a filtered quasi-isomorphism. Hence, for every real algebraic variety X the localization of σ induces a quasi-isomorphism $\sigma'(X) : \mathcal{G}C_*(X) \to \mathcal{W}C_*(X)$.

Theorem 2.8 follows from Corollary 3.11 and Corollary 3.12, which will be shown in the next section.

3. The Nash constructible filtration

In this section we introduce the *Nash constructible filtration*

$$0 = \mathcal{N}_{-k-1}C_k(X) \subset \mathcal{N}_{-k}C_k(X) \subset \mathcal{N}_{-k+1}C_k(X) \subset \cdots \subset \mathcal{N}_0 C_k(X) = C_k(X) \quad (3\text{-}1)$$

on the semialgebraic chain complex $C_*(X)$ of a real algebraic variety X. We show that this filtration induces a functor

$$\mathcal{N}C_* : \mathbf{Sch}_c(\mathbb{R}) \to \mathcal{C}$$

that realizes the weight complex functor $\mathcal{W}C_* : \mathbf{Sch}_c(\mathbb{R}) \to \operatorname{Ho}\mathcal{C}$. In order to prove this assertion in Theorem 3.11, we have to extend $\mathcal{N}C_*$ to a wider category of sets and morphisms. The objects of this category are certain semialgebraic subsets of the set of real points of a real algebraic variety, and they include in particular all connected components of real algebraic subsets of $\mathbb{RP}^n$. The morphisms are certain proper continuous semialgebraic maps between these sets. This extension is crucial for the proof. As a corollary we show that for real algebraic varieties the Nash constructible filtration $\mathcal{N}_*$ coincides with the geometric filtration $\mathcal{G}_*$ of Section 2A. In this way we complete the proof of Theorem 2.8.

For real algebraic varieties, the Nash constructible filtration was first defined in an unpublished paper of H. Pennaneac'h [32], by analogy with the algebraically constructible filtration [31; 33]. Theorem 3.11 implies, in particular, that the Nash constructible filtration of a compact variety is the same as the filtration given by a cubical hyperresolution; this answers affirmatively a question of Pennaneac'h [32, (2.9)].

3A. Nash constructible functions on $\mathbb{RP}^n$ and arc-symmetric sets. In real algebraic geometry it is common to work with real algebraic subsets of the affine space $\mathbb{R}^n \subset \mathbb{RP}^n$ instead of schemes over $\mathbb{R}$, and with (entire) regular rational mappings as morphisms; see for instance [3] or [5]. Since $\mathbb{RP}^n$ can be embedded in $\mathbb{R}^N$ by a biregular rational map ([3], [5] (3.4.4)), this category also contains algebraic subsets of $\mathbb{RP}^n$.

A *Nash constructible function* on $\mathbb{RP}^n$ is a function $\varphi : \mathbb{RP}^n \to \mathbb{Z}$ such that there exist a finite family of regular rational mappings $f_i : Z_i \to \mathbb{RP}^n$ defined on projective real algebraic sets Z_i, connected components Z_i' of Z_i, and integers m_i, such that for all $x \in \mathbb{RP}^n$,

$$\varphi(x) = \sum_i m_i \chi(f_i^{-1}(x) \cap Z_i'), \tag{3-2}$$

where χ is the Euler characteristic. Nash constructible functions were introduced in [24]. Nash constructible functions on $\mathbb{RP}^n$ form a ring.

EXAMPLE 3.1.

(1) If $Y \subset \mathbb{RP}^n$ is Zariski constructible (a finite set-theoretic combination of algebraic subsets), then its characteristic function $\mathbf{1}_Y$ is Nash constructible.

(2) A subset $S \subset \mathbb{RP}^n$ is called *arc-symmetric* if every real analytic arc $\gamma : (a, b) \to \mathbb{RP}^n$ either meets S at isolated points or is entirely included in S. Arc-symmetric sets were first studied by K. Kurdyka in [19]. As shown in [24], a semialgebraic set $S \subset \mathbb{RP}^n$ is arc-symmetric if and only if it is closed in $\mathbb{RP}^n$ and $\mathbf{1}_S$ is Nash constructible. By the existence of arc-symmetric closure [19; 21], for a set $S \subset \mathbb{RP}^n$ the function $\mathbf{1}_S$ is Nash constructible and only if S is a finite set-theoretic combination of semialgebraic arc-symmetric subsets of $\mathbb{RP}^n$. If $\mathbf{1}_S$ is Nash constructible we say that S is an $\mathcal{AS}$ *set*.

(3) Any connected component of a compact algebraic subset of $\mathbb{RP}^n$ is arc-symmetric. So is any compact real analytic and semialgebraic subset of $\mathbb{RP}^n$.

(4) Every Nash constructible function on $\mathbb{RP}^n$ is in particular *constructible* (constant on strata of a finite semialgebraic stratification of $\mathbb{RP}^n$). Not all constructible functions are Nash constructible. By [24], every constructible function $\varphi : \mathbb{RP}^n \to 2^n\mathbb{Z}$ is Nash constructible.

Nash constructible functions form the smallest family of constructible functions that contains characteristic functions of connected components of compact real algebraic sets, and that is stable under the natural operations inherited from sheaf theory: pullback by regular rational morphisms, pushforward by proper regular rational morphisms, restriction to Zariski open sets, and duality; see [24]. In terms of the *pushforward* (fiberwise integration with respect to the Euler characteristic) the formula (3-2) can be expressed as $\varphi = \sum_i m_i f_{i*}\mathbf{1}_{Z_i'}$. Duality is closely related to the *link operator*, an important tool for studying the topological properties of real algebraic sets. For more on Nash constructible function see [7] and [21].

If $S \subset \mathbb{RP}^n$ is an $\mathcal{AS}$ set (*i.e.* $\mathbf{1}_S$ is Nash constructible), we say that a function on S is *Nash constructible* if it is the restriction of a Nash constructible function on $\mathbb{RP}^n$. In particular, this defines Nash constructible functions on affine real algebraic sets. (In the non-compact case this definition is more restrictive than that of [24].)

3B. Nash constructible functions on real algebraic varieties. Let X be a real algebraic variety and let $\underline{X}$ denote the set of real points on X. We call a function $\varphi : \underline{X} \to \mathbb{Z}$ *Nash constructible* if its restriction to every affine chart is Nash constructible. The following lemma shows that this extends our definition of Nash constructible functions on affine real algebraic sets.

LEMMA 3.2. *If X_1 and X_2 are projective compactifications of the affine real algebraic variety U, then $\varphi : \underline{U} \to \mathbb{Z}$ is the restriction of a Nash constructible function on $\underline{X}_1$ if and only if φ is the restriction of a Nash constructible function on $\underline{X}_2$.*

PROOF. We may suppose that there is a regular projective morphism $f : X_1 \to X_2$ that is an isomorphism on U; *cf.* the proof of Proposition 2.6. Then the statement follows from the following two properties of Nash constructible functions. If $\varphi_2 : \underline{X}_2 \to \mathbb{Z}$ is Nash constructible, so is its pullback $f^*\varphi_2 = \varphi_2 \circ f : \underline{X}_1 \to \mathbb{Z}$. If $\varphi_1 : \underline{X}_1 \to \mathbb{Z}$ is Nash constructible, so is its pushforward $f_*\varphi_1 : \underline{X}_2 \to \mathbb{Z}$. □

PROPOSITION 3.3. *Let X be a real algebraic variety and let $Y \subset X$ be a closed subvariety. Let $U = X \setminus Y$. Then $\varphi : \underline{X} \to \mathbb{Z}$ is Nash constructible if and only if the restrictions of φ to $\underline{Y}$ and $\underline{U}$ are Nash constructible.*

PROOF. It suffices to check the assertion for X affine; this case is easy. □

THEOREM 3.4. *Let X be a complete real algebraic variety. The function $\varphi : \underline{X} \to \mathbb{Z}$ is Nash constructible if and only if there exist a finite family of regular morphisms $f_i : Z_i \to X$ defined on complete real algebraic varieties Z_i, connected components Z_i' of $\underline{Z}_i$, and integers m_i, such that for all $x \in \underline{X}$,*

$$\varphi = \sum_i m_i \, f_{i\,*} \mathbf{1}_{Z_i'}. \tag{3-3}$$

PROOF. If X is complete but not projective, then X can be dominated by a birational regular morphism $\pi : \tilde{X} \to X$, with $\tilde{X}$ projective (Chow's Lemma). Let $Y \subset X$, $\dim Y < \dim X$, be a closed subvariety such that π induces an isomorphism $\tilde{X} \setminus \pi^{-1}(Y) \to X \setminus Y$. Then, by Proposition 3.3, $\varphi : \underline{X} \to \mathbb{Z}$ is Nash constructible if and only if $\pi^*\varphi$ and φ restricted to $\underline{Y}$ are Nash constructible.

Let Z be a complete real algebraic variety and let $f : Z \to X$ be a regular morphism. Let Z' be a connected component of $\underline{Z}$. We show that $\varphi = f_* \mathbf{1}_{Z'}$ is Nash constructible. This is obvious if both X and Z are projective. If they are not, we may dominate both X and Z by projective varieties, using Chow's Lemma, and reduce to the projective case by induction on dimension.

Let $\varphi : \underline{X} \to \mathbb{Z}$ be Nash constructible. Suppose first that X is projective. Then $\underline{X} \subset \mathbb{RP}^n$ is a real algebraic set. Let $A \subset \mathbb{RP}^m$ be a real algebraic set and let $f : A \to \underline{X}$ be a regular rational morphism $f = g/h$, where h does not vanish on A, cf. [3]. Then the graph of f is an algebraic subset $\Gamma \subset \mathbb{RP}^n \times \mathbb{RP}^m$

and the set of real points of a projective real variety Z. Let A' be a connected component of A, and Γ' the graph of f restricted to A'. Then $f_* \mathbf{1}_{A'} = \pi_* \mathbf{1}_{\Gamma'}$, where π denotes the projection on the second factor.

If X is complete but not projective, we again dominate it by a birational regular morphism $\pi : \tilde{X} \to X$, with $\tilde{X}$ projective. Let $\varphi : \underline{X} \to \mathbb{Z}$ be Nash constructible. Then $\tilde{\varphi} = \varphi \circ \pi : \underline{\tilde{X}} \to \mathbb{Z}$ is Nash constructible. Thus, by the case considered above, there are regular morphisms $\tilde{f}_i : \tilde{Z}_i \to \tilde{X}$, and connected components $\tilde{Z}_i'$ such that

$$\tilde{\varphi}(x) = \sum_i m_i \, \tilde{f}_{i\,*} \mathbf{1}_{\tilde{Z}_i'}.$$

Then $\pi_* \tilde{\varphi} = \sum_i m_i \, \tilde{\pi}_* f_{i\,*} \mathbf{1}_{\tilde{Z}_i'}$ and differs from φ only on the set of real points of a variety of dimension smaller than $\dim X$. We complete the argument by induction on dimension. □

If X is a real algebraic variety, we again say that $S \subset \underline{X}$ is an $\mathcal{AS}$ *set* if $\mathbf{1}_S$ is Nash constructible, and $\varphi : S \to \mathbb{Z}$ is *Nash constructible* if the extension of φ to $\underline{X}$ by zero is a Nash constructible function on $\underline{X}$.

COROLLARY 3.5. *Let X, Y be complete real algebraic varieties and let S be an $\mathcal{AS}$ subset of $\underline{X}$, and T an $\mathcal{AS}$ subset of $\underline{Y}$. Let $\varphi : S \to \mathbb{Z}$ and $\psi : T \to \mathbb{Z}$ be Nash constructible. Let $f : S \to T$ be a map with $\mathcal{AS}$ graph $\Gamma \subset \underline{X} \times \underline{Y}$ and let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ denote the standard projections. Then*

$$f_*(\varphi) = (\pi_Y)_*(\mathbf{1}_\Gamma \cdot \pi_X^* \varphi) \tag{3-4}$$

and

$$f^*(\psi) = (\pi_X)_*(\mathbf{1}_\Gamma \cdot \pi_Y^* \psi) \tag{3-5}$$

are Nash constructible.

3C. Definition of the Nash constructible filtration. Denote by $\mathcal{X}_{\mathcal{AS}}$ the category of locally compact $\mathcal{AS}$ subsets of real algebraic varieties as objects and continuous proper maps with $\mathcal{AS}$ graphs as morphisms.

Let $T \in \mathcal{X}_{\mathcal{AS}}$. We say that $\varphi : T \to \mathbb{Z}$ is *generically Nash constructible on T in dimension k* if φ coincides with a Nash constructible function everywhere on T except on a semialgebraic subset of T of dimension $< k$. We say that φ is *generically Nash constructible on T* if φ is Nash constructible in dimension $d = \dim T$.

Let $c \in C_k(T)$, and let $-k \le p \le 0$. We say that c is *p-Nash constructible*, and write $c \in \mathcal{N}_p C_k(T)$, if there exists $\varphi_{c,p} : T \to 2^{k+p}\mathbb{Z}$, generically Nash constructible in dimension k, such that

$$c = \{x \in T \; ; \; \varphi_{c,p}(x) \notin 2^{k+p+1}\mathbb{Z}\} \text{ up to a set of dimension } < k. \tag{3-6}$$

up to a set of dimension less than k. The choice of $\varphi_{c,p}$ is not unique. Let Z denote the Zariski closure of $\operatorname{Supp} c$. By multiplying $\varphi_{c,p}$ by $\mathbf{1}_Z$, we may always assume that $\operatorname{Supp}\varphi \subset Z$ and hence, in particular, that $\dim \operatorname{Supp}\varphi_{c,p} \le k$.

We say that $c \in C_k(T)$ is *pure* if $c \in \mathcal{N}_{-k} C_k(T)$. By Theorem 3.9 of [21] and the existence of arc-symmetric closure [19; 21], $c \in C_k(T)$ is pure if and only if $\operatorname{Supp} c$ coincides with an $\mathcal{AS}$ set (up to a set of dimension smaller than k). For T compact this means that c is pure if and only if c can be represented by an arc-symmetric set. By [24], if $\dim T = k$ then every semialgebraically constructible function $\varphi : T \to 2^k\mathbb{Z}$ is Nash constructible. Hence $\mathcal{N}_0 C_k(T) = C_k(T)$.

The boundary operator preserves the Nash constructible filtration:

$$\partial \mathcal{N}_p C_k(T) \subset \mathcal{N}_p C_{k-1}(T).$$

Indeed, if $c \in C_k(T)$ is given by (3-6) and $\dim \operatorname{Supp}\varphi_{c,p} \le k$, then

$$\partial c = \{x \in Z \,;\, \varphi_{\partial c,p}(x) \notin 2^{k+p}\mathbb{Z}\}, \tag{3-7}$$

where $\varphi_{\partial c,p}$ equals $\frac{1}{2}\Lambda\varphi_{c,p}$ for k odd and $\frac{1}{2}\Omega\varphi_{c,p}$ for k even [24]. A geometric interpretation of this formula is as follows; see [7]. Let Z be the Zariski closure of $\operatorname{Supp} c$, so $\dim Z = k$ if $c \neq 0$. Let W be an algebraic subset of Z such that $\dim W < k$ and $\varphi_{c,p}$ is locally constant on $Z \setminus W$. At a generic point x of W, we define $\partial_W \varphi_{c,p}(x)$ as the average of the values of $\varphi_{c,p}$ on the local connected components of $Z \setminus W$ at x. It can be shown that $\partial_W \varphi_{c,p}(x)$ is generically Nash constructible in dimension $k-1$. (For k odd it equals $(\frac{1}{2}\Lambda\varphi_{c,p})|_W$ and for k even it equals $(\frac{1}{2}\Omega\varphi_{c,p})|_W$; see [24].)

We say that a square in $\mathcal{X}_{\mathcal{AS}}$

$$\begin{array}{ccc} \tilde S & \longrightarrow & \tilde T \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ S & \xrightarrow{\ i\ } & T \end{array} \tag{3-8}$$

is acyclic if i is a closed inclusion, $\tilde S = \pi^{-1}(Y)$ and the restriction of π is a homeomorphism $\tilde T \setminus \tilde S \to T \setminus S$.

THEOREM 3.6. *The functor $\mathcal{N}C_* : \mathcal{X}_{\mathcal{AS}} \to \mathcal{C}$, defined on the category $\mathcal{X}_{\mathcal{AS}}$ of locally compact $\mathcal{AS}$ sets and continuous proper maps with $\mathcal{AS}$ graphs, satisfies*:

(1) *For an acyclic square* (3-8) *the sequences*

$$0 \to \mathcal{N}_p C_k(\tilde S) \to \mathcal{N}_p C_k(S) \oplus \mathcal{N}_p C_k(\tilde T) \to \mathcal{N}_p C_k(T) \to 0,$$

$$0 \to \frac{\mathcal{N}_p C_k(\tilde S)}{\mathcal{N}_{p-1} C_k(\tilde S)} \to \frac{\mathcal{N}_p C_k(S)}{\mathcal{N}_{p-1} C_k(S)} \oplus \frac{\mathcal{N}_p C_k(\tilde T)}{\mathcal{N}_{p-1} C_k(\tilde T)} \to \frac{\mathcal{N}_p C_k(T)}{\mathcal{N}_{p-1} C_k(T)} \to 0,$$

are exact.

(2) *For a closed inclusion $S \subset T$, the restriction to $U = T \setminus S$ induces a morphism of filtered complexes $\mathcal{N}C_*(T) \to \mathcal{N}C_*(U)$, and the sequences*

$$0 \to \mathcal{N}_p C_k(S) \to \mathcal{N}_p C_k(T) \to \mathcal{N}_p C_k(U) \to 0,$$
$$0 \to \frac{\mathcal{N}_p C_k(S)}{\mathcal{N}_{p-1} C_k(S)} \to \frac{\mathcal{N}_p C_k(T)}{\mathcal{N}_{p-1} C_k(T)} \to \frac{\mathcal{N}_p C_k(U)}{\mathcal{N}_{p-1} C_k(U)} \to 0,$$

are exact.

PROOF. We first show that $\mathcal{N}C_*$ is a functor; that is, for a proper morphism $f : T \to S$, $f_*\mathcal{N}_p C_k(T) \subset \mathcal{N}_p C_k(S)$. Let $c \in \mathcal{N}_p C_k(T)$ and let $\varphi = \varphi_{c,p}$ be a Nash constructible function on T satisfying (3-6) (up to a set of dimension $< k$). Then

$$f_* c = \{y \in S \ ; \ f_*(\psi)(y) \notin 2^{k+p+1}\mathbb{Z}\};$$

that is, $\varphi_{f_*c,p} = f_*\varphi_{c,p}$.

For a closed inclusion $S \subset T$, the restriction to $U = T \setminus S$ of a Nash constructible function on T is Nash constructible. Therefore the restriction defines a morphism $\mathcal{N}C_*(T) \to \mathcal{N}C_*(U)$. The exactness of the first sequence of (2) can be verified easily by direct computation. We note, moreover, that for fixed k the morphism

$$\mathcal{N}_* C_k(T) \to \mathcal{N}_* C_k(U)$$

splits (the splitting does not commute with the boundary), by assigning to $c \in \mathcal{N}_p C_k(U)$ its closure $\bar{c} \in C_k(T)$. Let $\varphi : T \to 2^{k+p}\mathbb{Z}$ be a Nash constructible function such that $\varphi|_{T\setminus S} = \varphi_{c,p}$. Then $\bar{c} = \{x \in T \ ; \ (\mathbf{1}_T - \mathbf{1}_S)\varphi(x) \notin 2^{k+p+1}\mathbb{Z}\}$ up to a set of dimension $< k$.

The exactness of the second sequence of (2) and the sequences of (1) now follow by standard arguments. (See the proof of Theorem 2.7.) □

3D. The Nash constructible filtration for Nash manifolds. A *Nash function* on an open semialgebraic subset U of $\mathbb{R}^N$ is a real analytic semialgebraic function. Nash morphisms and Nash manifolds play an important role in real algebraic geometry. In particular a connected component of compact nonsingular real algebraic subset of $\mathbb{R}^n$ is a Nash submanifold of $\mathbb{R}^N$ in the sense of [5] (2.9.9). Since $\mathbb{RP}^n$ can be embedded in $\mathbb{R}^N$ by a rational diffeomorphism ([3], [5] (3.4.2)) the connected components of nonsingular projective real algebraic varieties can be considered as Nash submanifolds of affine space. By the Nash Theorem [5, 14.1.8], every compact C^∞ manifold is C^∞-diffeomorphic to a Nash submanifold of an affine space, and moreover such a model is unique up to Nash diffeomorphism [5, Corollary 8.9.7]. In what follows by a *Nash manifold* we mean a compact Nash submanifold of an affine space.

Compact Nash manifolds and the graphs of Nash morphisms on them are $\mathcal{AS}$ sets. If N is a Nash manifold, the Nash constructible filtration is contained in the canonical filtration,

$$\mathcal{N}_p C_k(N) \subset F_p^{\mathrm{can}} C_k(N), \tag{3-9}$$

since $\partial_k(\mathcal{N}_{-k} C_k(N)) = 0$. Thus on the category of Nash manifolds and Nash maps have a morphism of functors

$$\tau : \mathcal{N}C_* \to F^{\mathrm{can}} C_*.$$

THEOREM 3.7. *For every Nash manifold* N,

$$\tau(N) : \mathcal{N}C_*(N) \to F^{\mathrm{can}} C_*(N)$$

is a filtered quasi-isomorphism.

PROOF. We show that for all p and k, $\tau(N)$ induces an isomorphism

$$\tau_* : H_k(\mathcal{N}_p C_*(N)) \cong H_k(F_p^{\mathrm{can}} C_*(N)). \tag{3-10}$$

Then, by the long exact homology sequences of $(\mathcal{N}_p C_*(N), \mathcal{N}_{p-1} C_*(N))$ and $(F_p^{\mathrm{can}} C_*(N),\ F_{p-1}^{\mathrm{can}} C_*(N))$,

$$\tau_* : H_k\left(\frac{\mathcal{N}_p C_*(N)}{\mathcal{N}_{p-1} C_*(N)}\right) \to H_k\left(\frac{F_p^{\mathrm{can}} C_*(N)}{F_{p-1}^{\mathrm{can}} C_*(N)}\right)$$

is an isomorphism, which shows the claim of the theorem.

We proceed by induction on the dimension of N. We call a Nash morphism $\pi : \tilde{N} \to N$ a *Nash multi-blowup* if π is a composition of blowups along nowhere dense Nash submanifolds.

PROPOSITION 3.8. *Let* N, N' *be compact connected Nash manifolds of the same dimension*. *Then there exist multi-blowups* $\pi : \tilde{N} \to N$, $\sigma : \tilde{N}' \to N'$ *such that* $\tilde{N}$ *and* $\tilde{N}'$ *are Nash diffeomorphic*.

PROOF. By a theorem of Mikhalkin (see [26] and Proposition 2.6 in [27]), any two connected closed C^∞ manifolds of the same dimension can be connected by a sequence of C^∞ blowups and and then blowdowns with smooth centers. We show that this C^∞ statement implies an analogous statement in the Nash category.

Let M be a closed C^∞ manifold. By the Nash–Tognoli Theorem there is a nonsingular real algebraic set X, *a fortiori* a Nash manifold, that is C^∞-diffeomorphic to M. Moreover, by approximation by Nash mappings, any two Nash models of M are Nash diffeomorphic; see Corollary 8.9.7 in [5]. Thus in order to show Proposition 3.8 we need only the following lemma.

LEMMA 3.9. *Let $C \subset M$ be a C^∞ submanifold of a closed C^∞ manifold M. Suppose that M is C^∞-diffeomorphic to a Nash manifold N. Then there exists a Nash submanifold $D \subset N$ such that the blowups $Bl(M, C)$ of M along C and $Bl(N, D)$ of N along D are C^∞-diffeomorphic.*

Proof. By the relative version of Nash–Tognoli Theorem proved by Akbulut and King, as well as Benedetti and Tognoli (see for instance Remark 14.1.15 in [5]), there is a nonsingular real algebraic set X and a C^∞ diffeomorphism $\varphi : M \to X$ such that $Y = \varphi(C)$ is a nonsingular algebraic set. Then the blowups $Bl(M, C)$ of M along C and $Bl(X, Y)$ of X along Y are C^∞-diffeomorphic. Moreover, since X and N are C^∞-diffeomorphic, they are Nash diffeomorphic by a Nash diffeomorphism $\psi : X \to N$. Then $Bl(X, Y)$ and $Bl(N, \psi(Y))$ are Nash diffeomorphic. This proves the lemma and the proposition. □

LEMMA 3.10. *Let N be a compact connected Nash manifold and let $\pi : \tilde{N} \to N$ denote the blowup of N along a nowhere dense Nash submanifold Y. Then $\tau(N)$ is a quasi-isomorphism if and only if $\tau(\tilde{N})$ is a quasi-isomorphism.*

PROOF. Let $\tilde{Y} = \pi^{-1}(Y)$ denote the exceptional divisor of π. For each p consider the diagram

$$\begin{array}{ccccccc}
\to & H_{k+1}(\mathcal{N}_p C_*(N)) & \to & H_k(\mathcal{N}_p C_*(\tilde{Y})) & \to & H_k(\mathcal{N}_p C_*(Y)) \oplus H_k(\mathcal{N}_p C_*(\tilde{N})) & \to \\
& \downarrow & & \downarrow & & \downarrow & \\
\to & H_{k+1}(F_p^{\mathrm{can}} C_*(N)) & \to & H_k(F_p^{\mathrm{can}} C_*(\tilde{Y})) & \to & H_k(F_p^{\mathrm{can}} C_*(Y)) \oplus H_k(F^{\mathrm{can}} C_*(\tilde{N})) & \to
\end{array}$$

The top row is exact by Theorem 3.6. For all manifolds N and for all p and k, we have

$$H_k(F_p^{\mathrm{can}} C_*(N)) = \begin{cases} H_k(N) & \text{if } k \geq -p, \\ 0 & \text{if } k < -p, \end{cases}$$

so the short exact sequences (1-3) give that the bottom row is exact. The lemma now follows from the inductive assumption and the Five Lemma. □

Consequently it suffices to show that $\tau(N)$ is a quasi-isomorphism for a single connected Nash manifold of each dimension n. We check this assertion for the standard sphere S^n by showing that

$$H_k(\mathcal{N}_p C_*(S^n)) = \begin{cases} H_k(S^n) & \text{if } k = 0 \text{ or } n \text{ and } p \geq -k, \\ 0 & \text{otherwise.} \end{cases}$$

Let $c \in \mathcal{N}_p C_k(S^n)$, $k < n$, be a cycle described as in (3-6) by the Nash constructible function $\varphi_{c,p} : Z \to 2^{k+p}\mathbb{Z}$, where Z is the Zariski closure of $\operatorname{Supp} c$. Then c can be contracted to a point. More precisely, choose $p \in S^n \setminus Z$. Then $S^n \setminus \{p\}$ and $\mathbb{R}^n$ are isomorphic. Define a Nash constructible function

$\Phi : Z \times \mathbb{R} \to 2^{k+p+1}\mathbb{Z}$ by the formula

$$\Phi(x,t) = \begin{cases} 2\varphi_{c,p}(x) & \text{if } t \in [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$c \times [0,1] = \{(x,t) \in Z \times \mathbb{R}\ ;\ \Phi(x,t) \notin 2^{k+p+2}\mathbb{Z}\};$$

so $c \times [0,1] \in \mathcal{N}_p C_{k+1}(Z \times \mathbb{R})$. The morphism $f : Z \times \mathbb{R} \to \mathbb{R}^n$, $f(x,t) = tx$, is proper and for $k > 0$

$$\partial f_*(c \times [0,1]) = f_*(\partial c \times [0,1]) = c,$$

which shows that c is a boundary in $\mathcal{N}_p C_*(S^n)$. If $k = 0$ then $\partial f_*(c \times [0,1]) = c - (\deg c)[0]$.

If $c \in \mathcal{N}_p C_n(S^n)$ is a cycle, then c is a cycle in $C_n(S^n)$; that is, either $c = 0$ or $c = [S^n]$. This completes the proof of Theorem 3.7. □

3E. Consequences for the weight filtration.

COROLLARY 3.11. *For every real algebraic variety X the localization of τ induces a quasi-isomorphism $\tau'(X) : \mathcal{N}C_*(X) \to \mathcal{W}C_*(X)$.*

PROOF. Theorem 3.6 yields that the functor $\mathcal{N}C_* : \mathbf{Sch}_c(\mathbb{R}) \to \mathrm{Ho}\,\mathcal{C}$ satisfies properties (1) and (2) of Theorem 1.1. Hence Theorem 3.7 and Theorem 1.4 give the desired result. □

COROLLARY 3.12. *Let X be a real algebraic variety. Then for all p and k, $\mathcal{N}_p C_k(X) = \mathcal{G}_p C_k(X)$.*

PROOF. We show that the Nash constructible filtration satisfies properties (1) and (2) of Theorem 2.1. This is obvious for property (1). We show property (2). Let $\tilde{c} = \pi^{-1}(c)$. First we note that

$$c \in \mathcal{N}_p C_k(X) \iff \tilde{c} \in \mathcal{N}_p C_k(\tilde{X}).$$

Indeed, ($\Leftarrow$) follows from functoriality, since $c = \pi_*(\tilde{c})$. If c is given by (3-1) then $\pi^*(\varphi_{c,p})$ is Nash constructible and describes $\tilde{c}$. Thus it suffices to show

$$\tilde{c} \in \mathcal{N}_p C_k(\tilde{X}) \iff \partial\tilde{c} \in \mathcal{N}_p C_{k-1}(\tilde{X})$$

for $p \geq -k$, with the implication ($\Rightarrow$) being obvious. If $p = -k$ then each cycle is arc-symmetric. (Such a cycle is a union of connected components of $\tilde{X}$, since $\tilde{X}$ is nonsingular and compact.) For $p > -k$ suppose, contrary to our claim, that

$$\tilde{c} \in \mathcal{N}_p C_k(\tilde{X}) \setminus \mathcal{N}_{p-1} C_k(\tilde{X}) \quad \text{and} \quad \partial\tilde{c} \in \mathcal{N}_{p-1} C_{k-1}(\tilde{X}).$$

By Corollary 3.11 and Proposition 1.8

$$H_k\left(\frac{\mathcal{N}_p C_*(\tilde{X})}{\mathcal{N}_{p-1} C_*(\tilde{X})}\right) = 0,$$

and $\tilde{c}$ has to be a relative boundary. But dim $\tilde{X} = k$ and $C_{k+1}(\tilde{X}) = 0$. This completes the proof. □

4. Applications to real algebraic and analytic geometry

Algebraic subsets of affine space, or more generally Z-open or Z-closed affine or projective sets in the sense of Akbulut and King [3], are $\mathcal{AS}$ sets. So are the graphs of regular rational mappings. Therefore Theorems 3.6 and 3.7 give the following result.

THEOREM 4.1. *The Nash constructible filtration of closed semialgebraic chains defines a functor from the category of affine real algebraic sets and proper regular rational mappings to the category of bounded chain complexes of* $\mathbb{Z}_2$ *vector spaces with increasing bounded filtration*.

This functor is additive and acyclic; *that is, it satisfies properties* (1) *and* (2) *of Theorem* 3.6; *and it induces the weight spectral sequence and the weight filtration on Borel–Moore homology with coefficients in* $\mathbb{Z}_2$.

For compact nonsingular algebraic sets, the reindexed weight spectral sequence is pure: $\tilde{E}^2_{p,q} = 0$ *for* $p > 0$.

For the last claim of the theorem we note that every compact affine real algebraic set that is nonsingular in the sense of [3] and [5] admits a compact nonsingular complexification. Thus the claim follows from Theorem 3.7.

The purity of $\tilde{E}^2$ implies the purity of $\tilde{E}^\infty$: $\tilde{E}^\infty_{p,q} = 0$ for $p > 0$. Consequently every nontrivial homology class of a nonsingular compact affine or projective real algebraic variety can be represented by a semialgebraic arc-symmetric set, a result proved directly in [18] and [21].

REMARK 4.2. Theorem 3.6 and Theorem 3.7 can be used in more general contexts. A compact real analytic semialgebraic subset of a real algebraic variety is an $\mathcal{AS}$ set. A compact semialgebraic set that is the graph of a real analytic map, or more generally the graph of an arc-analytic mapping (*cf.* [21]), is arc-symmetric. In Section 3E we have already used that compact affine Nash manifolds and graphs of Nash morphisms defined on compact Nash manifolds are arc-symmetric.

The weight filtration of homology is an isomorphism invariant but not a homeomorphism invariant; this is discussed in [25] for the dual weight filtration of cohomology.

PROPOSITION 4.3. *Let X and Y be locally compact $\mathcal{AS}$ sets, and let $f : X \to Y$ be a homeomorphism with $\mathcal{AS}$ graph. Then $f_* : \mathcal{N}C_*(X) \to \mathcal{N}C_*(Y)$ is an isomorphism of filtered complexes.*

Consequently, f_* induces an isomorphism of the weight spectral sequences of X and Y and of the weight filtrations of $H_*(X)$ and $H_*(Y)$. Thus the virtual Betti numbers (1-4) *of X and Y are equal.*

PROOF. The first claim follows from the fact that $\mathcal{N}C_* : \mathcal{X}_{\mathcal{AS}} \to \mathcal{C}$ is a functor; see the proof of Theorem 3.6. The rest of the proposition then follows from Theorem 3.6 and Theorem 3.7. □

REMARK 4.4. Proposition 4.3 applies, for instance, to regular homeomorphisms such as $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$. The construction of the virtual Betti numbers of [25] was extended to $\mathcal{AS}$ sets by G. Fichou in [11], where their invariance by Nash diffeomorphism was shown. The arguments of [25] and [11] use the weak factorization theorem of [1].

4A. The virtual Poincaré polynomial. Let X be a locally compact $\mathcal{AS}$ set. The virtual Betti numbers give rise to the *virtual Poincaré polynomial*

$$\beta(X) = \sum_i \beta_i(X)\, t^i. \tag{4-1}$$

For real algebraic varieties the virtual Poincaré polynomial was first introduced in [25]. For $\mathcal{AS}$ sets, not necessarily locally compact, it was defined in [11]. It satisfies the following properties [25; 11]:

(i) *Additivity:* For finite disjoint union $X = \bigsqcup X_i$, we have $\beta(X) = \sum \beta(X_i)$.
(ii) *Multiplicativity:* $\beta(X \times Y) = \beta(X) \cdot \beta(Y)$.
(iii) *Degree:* For $X \neq \varnothing$, $\deg \beta(X) = \dim X$ and the leading coefficient $\beta(X)$ is strictly positive.

(If X is not locally compact we can decompose it into a finite disjoint union of locally compact $\mathcal{AS}$ sets $X = \bigsqcup X_i$ and define $\beta(X) = \sum \beta(X_i)$.)

We say that a function $X \to e(X)$ defined on real algebraic sets is an *invariant* if it an isomorphism invariant, that is $e(X) = e(Y)$ if X and Y are isomorphic (by a biregular rational mapping). We say that e is additive if e takes values in an abelian group and $e(X \setminus Y) = e(X) - e(Y)$ for all $Y \subset X$. We say e is multiplicative if e takes values in a ring and $e(X \times Y) = e(X)e(Y)$ for all X, Y. The following theorem states that the virtual Betti polynomial is a universal additive, or additive and multiplicative, invariant defined on real algebraic sets (or real points of real algebraic varieties in general), among those invariants that do not distinguish Nash diffeomorphic compact nonsingular real algebraic sets.

THEOREM 4.5. *Let e be an additive invariant defined on real algebraic sets. Suppose that for every pair X, Y of Nash diffeomorphic nonsingular compact real algebraic sets we have $e(X) = e(Y)$. Then there exists a unique group homomorphism $h_e : \mathbb{Z}[t] \to G$ such that $e = h_e \circ \beta$. If, moreover, e is multiplicative then h_e is a ring homomorphism.*

PROOF. Define $h(t^n) = e(\mathbb{R}^n)$. We claim that the additive invariant $\varphi(X) = h(\beta(X)) - e(X)$ vanishes for every real algebraic set X. This is the case for $X = \mathbb{R}^n$ since $\beta(\mathbb{R}^n) = t^n$. By additivity, this is also the case for $S^n = \mathbb{R}^n \sqcup pt$. By the existence of an algebraic compactification and resolution of singularities, it suffices to show the claim for compact nonsingular real algebraic sets.

Let X be a compact nonsingular real algebraic set and let $\tilde{X}$ be the blowup of X along a smooth nowhere dense center. Then, using induction on $\dim X$, we see that $\varphi(X) = 0$ if and only if $\varphi(\tilde{X}) = 0$. By the relative version of the Nash–Tognoli Theorem, the same result holds if we have that $\tilde{X}$ is Nash diffeomorphic to the blowup of a nowhere dense Nash submanifold of X. Thus the claim and hence the first statement follows from Mikhalkin's Theorem. □

Following earlier results of Ax and Borel, K. Kurdyka showed in [20] that any regular injective self-morphism $f : X \to X$ of a real algebraic variety is surjective. It was then showed in [29] that an injective continuous self-map $f : X \to X$ of a locally compact $\mathcal{AS}$ set, such that the graph of f is an $\mathcal{AS}$ set, is a homeomorphism. The arguments of both [20] and [29] are topological and use the continuity of f in essential way. The use of additive invariants allows us to handle the non-continuous case.

THEOREM 4.6. *Let X be an $\mathcal{AS}$ set and let $f : X \to X$ be a map with $\mathcal{AS}$ graph. If f is injective then it is surjective.*

PROOF. It suffices to show that there exists a finite decomposition $X = \bigsqcup X_i$ into locally compact $\mathcal{AS}$ sets such that for each i, f restricted to X_i is a homeomorphism onto its image. Then, by Corollary 4.3,

$$\beta\big(X \setminus \textstyle\bigsqcup_i f(X_i)\big) = \beta(X) - \sum_i \beta(X_i) = 0,$$

and hence, by the degree property, $X \setminus \bigsqcup_i f(X_i) = \varnothing$.

To get the required decomposition first we note that by classical theory there exists a semialgebraic stratification of $X = \bigsqcup S_j$ such that f restricted to each stratum is real analytic. We show that we may choose strata belonging to the class $\mathcal{AS}$. (We do not require the strata to be connected.) By [20] and [29], each semialgebraic subset A of a real algebraic variety V has a minimal $\mathcal{AS}$ closure in V, denoted $\bar{A}^{\mathcal{AS}}$. Moreover if A is $\mathcal{AS}$ then $\dim \bar{A}^{\mathcal{AS}} \setminus A < \dim A$. Therefore, we may take as the first subset of the decomposition the complement in X of the $\mathcal{AS}$ closure of the union of strata S_j of dimension $< \dim X$, and then proceed by induction on dimension.

Let $X = \bigsqcup S_j$ be a stratification with $\mathcal{AS}$ strata and such that f is analytic on each stratum. Then, for each stratum S_j, we apply the above argument to f^{-1} defined on $f(S_j)$. The induced subdivision of $f(S_j)$, and hence of S_j, satisfies the required property. □

Of course, in general, surjectivity does not apply injectivity for a self-map. Nevertheless we have the following result.

THEOREM 4.7. *Let X be an $\mathcal{AS}$ set and let $f : X \to X$ be a surjective map with $\mathcal{AS}$ graph. Suppose that there exist a finite $\mathcal{AS}$ decomposition $X = \bigsqcup Y_i$ and $\mathcal{AS}$ sets F_i such that for each i, $f^{-1}(Y_i)$ is homeomorphic to $Y_i \times F_i$ by a homeomorphism with $\mathcal{AS}$ graph. Then f is injective.*

PROOF. We have

$$0 = \beta(X) - \beta(f(X)) = \sum \beta(Y_i)(\beta(F_i) - 1).$$

Therefore $\beta(F_i) - 1 = 0$ for each i; otherwise the polynomial on the right-hand side would be nonzero with strictly positive leading coefficient. □

4B. Application to spaces of orderings. Let V be an irreducible real algebraic subset of $\mathbb{R}^N$. A function $\varphi : V \to \mathbb{Z}$ is called *algebraically constructible* if it satisfies one of the following equivalent properties [24; 30]:

(i) There exist a finite family of proper regular morphisms $f_i : Z_i \to V$, and integers m_i, such that for all $x \in V$,

$$\varphi(x) = \sum_i m_i \chi(f_i^{-1}(x) \cap Z_i). \tag{4-2}$$

(ii) There are finitely many polynomials $P_i \in \mathbb{R}[x_1, \dots, x_N]$ such that for all $x \in V$,

$$\varphi(x) = \sum_i \operatorname{sgn} P_i(x).$$

Let $K = K(V)$ denote the field of rational functions of V. A function $\varphi : V \to \mathbb{Z}$ is generically algebraically constructible if and only if can be identified, up to a set of dimension smaller $\dim V$, with the signature of a quadratic form over K. Denote by $\mathcal{X}$ the real spectrum of K. A (semialgebraically) constructible function on V, up to a set of dimension smaller $\dim V$, can be identified with a continuous function $\varphi : \mathcal{X} \to \mathbb{Z}$; see [5, Chapter 7], [23], and [6]. The representation theorem of Becker and Bröcker gives a fan criterion for recognizing generically algebraically constructible function on V. The following two theorems are due to I. Bonnard.

THEOREM 4.8 [6]. *A constructible function $\varphi : V \to \mathbb{Z}$ is generically algebraically constructible if and only for any finite fan F of $\mathcal{X}$*

$$\sum_{\sigma \in F} \varphi(\sigma) \equiv 0 \mod |F|. \tag{4-3}$$

For the notion of a fan see [5, Chapter 7], [23], and [6]. The number of elements $|F|$ of a finite fan F is always a power of 2. It is known that for every finite fan F of $\mathcal{X}$ there exists a valuation ring B_F of K compatible with F, and on whose residue field the fan F induces exactly one or two distinct orderings. Denote by $\mathcal{F}$ the set of these fans of K for which the residue field induces only one ordering.

THEOREM 4.9 [6]. *A constructible function $\varphi : V \to \mathbb{Z}$ is generically Nash constructible if and only if* (4-3) *holds for every fan $F \in \mathcal{F}$.*

The following question is due to M. Coste and M. A. Marshall [23, Question 2]:

Suppose that a constructible function $\varphi : V \to \mathbb{Z}$ satisfies (4-3) *for every fan F of K with $|F| \le 2^n$. Does there exists a generically algebraically constructible function $\psi : V \to \mathbb{Z}$ such that for each $x \in V$, $\varphi(x) - \psi(x) \equiv 0 \mod 2^n$?*

We give a positive answer to the Nash constructible analog of this question.

THEOREM 4.10. *Suppose that a constructible function $\varphi : V \to \mathbb{Z}$ satisfies* (4-3) *for every fan $F \in \mathcal{F}$ with $|F| \le 2^n$. Then there exists a generically Nash constructible function $\psi : V \to \mathbb{Z}$ such that for each $x \in V$, $\varphi(x) - \psi(x) \equiv 0 \mod 2^n$.*

PROOF. We proceed by induction on n and on $k = \dim V$. The case $n = 0$ is trivial.

Suppose $\varphi : V \to \mathbb{Z}$ satisfies (4-3) for every fan $F \in \mathcal{F}$ with $|F| \le 2^n$, $n \ge 1$. By the inductive assumption, φ is congruent modulo 2^{n-1} to a generically Nash constructible function ψ_{n-1}. By replacing φ by $\varphi - \psi_{n-1}$, we may suppose 2^{n-1} divides φ.

We may also suppose V compact and nonsingular, just choosing a model for $K = K(V)$. Moreover, by resolution of singularities, we may assume that φ is constant in the complement of a normal crossing divisor $D = \bigcup D_i \subset V$.

Let c be given by (3-6) with $\varphi_{c,p} = \varphi$ and $p = n - k - 1$. At a generic point x of D_i define $\partial_{D_i} \varphi(x)$ as the average of the values of φ on the local connected components of $V \setminus D$ at x. Then $\partial c = \sum_i \partial_i c$, where $\partial_i c$ is described by $\partial_{D_i} \varphi$ as in (3-7) (see [7]). Note that the constructible functions $\partial_{D_i} \varphi$ satisfy the inductive assumption for $n-1$. Hence each $\partial_{D_i} \varphi$ is congruent to a generically Nash constructible function modulo 2^{n-1}. In other words $\partial c \in \mathcal{N}_p C_{k-1}(V)$.

Then by Corollary 3.12 we have $c \in \mathcal{N}_p C_k(V)$, which implies the statement of the theorem. □

Using Corollary 3.12 we obtain the following result. The original proof was based on the fan criterion (Theorem 4.9).

PROPOSITION 4.11 [7]. *Let $V \subset \mathbb{R}^N$ be compact, irreducible, and nonsingular. Suppose that the constructible function $\varphi : V \to \mathbb{Z}$ is locally constant in the complement of a normal crossing divisor $D = \bigcup D_i \subset V$. Then φ is generically Nash constructible if and only if $\partial_D \varphi$ is generically Nash constructible.*

PROOF. We show only ($\Leftarrow$). Suppose $2^{k+p} | \varphi$ generically, where $k = \dim V$, and let c be given by (3-6) with $\varphi_{c,p} = \varphi$. Then by our assumption $\partial c \in \mathcal{N}_p C_{k-1}(V)$. By Corollary 3.12 we have $c \in \mathcal{N}_p C_k(V)$, which shows that, modulo 2^{k+p+1}, φ coincides with a generically Nash constructible function ψ. Then we apply the same argument to $\varphi - \psi$. □

REMARK 4.12. We note that Proposition 4.11 implies neither Theorem 4.10 nor Corollary 3.12. Similarly the analog of this proposition proved in [6] does not give an answer to Coste and Marshall's question.

5. The toric filtration

In their investigation of the relation between the homology of the real and complex points of a toric variety [4], Bihan *et al.* define a filtration on the cellular chain complex of a real toric variety. We prove that this filtered complex is quasi-isomorphic to the semialgebraic chain complex with the Nash constructible filtration. Thus the toric filtered chain complex realizes the weight complex, and the real toric spectral sequence of [4] is isomorphic to the weight spectral sequence.

For background on toric varieties see [12]. We use a simplified version of the notation of [4]. Let Δ be a rational fan in $\mathbb{R}^n$, and let X_Δ be the real toric variety defined by Δ. The group $\mathbb{T} = (\mathbb{R}^*)^n$ acts on X_Δ, and the k-dimensional orbits $\mathcal{O}_\sigma$ of this action correspond to the codimension k cones σ of Δ.

The positive part X_Δ^+ of X_Δ is a closed semialgebraic subset of X_Δ, and there is a canonical retraction $r : X_\Delta \to X_\Delta^+$ that can be identified with the orbit map of the action of the finite group $T = (S^0)^n$ on X_Δ, where $S^0 = \{-1, +1\} \subset \mathbb{R}^*$. The T-quotient of the k-dimensional $\mathbb{T}$-orbit $\mathcal{O}_\sigma$ is a semialgebraic k-cell c_σ of X_Δ^+, and $\mathcal{O}_\sigma$ is a disjoint union of k-cells, each of which maps homeomorphically onto c_σ by the quotient map. This decomposition defines a cell structure on X_Δ such that X_Δ^+ is a subcomplex and the quotient map is cellular. Let $C_*(\Delta)$ be the cellular chain complex of X_Δ with coefficients in $\mathbb{Z}_2$. The closures of the cells of X_Δ are not necessarily compact, but they are semialgebraic subsets of X_Δ. Thus we have a chain map

$$\alpha : C_*(\Delta) \to C_*(X_\Delta) \tag{5-1}$$

from cellular chains to semialgebraic chains.

The *toric filtration* of the cellular chain complex $C_*(\Delta)$ is defined as follows [4]. For each $k \geq 0$ we define vector subspaces

$$0 = \mathcal{T}_{-k-1}C_k(\Delta) \subset \mathcal{T}_{-k}C_k(\Delta) \subset \mathcal{T}_{-k+1}C_k(\Delta) \subset \cdots \subset \mathcal{T}_0 C_k(\Delta) = C_k(\Delta), \tag{5-2}$$

such that $\partial_k(\mathcal{T}_p C_k(\Delta)) \subset \mathcal{T}_p C_{k-1}(\Delta)$ for all k and p.

Let σ be a cone of the fan Δ, with $\operatorname{codim}\sigma = k$. Let $C_k(\sigma)$ be the subspace of $C_k(\Delta)$ spanned by the k-cells of $\mathcal{O}_\sigma$. Then

$$C_k(\Delta) = \bigoplus_{\operatorname{codim}\sigma = k} C_k(\sigma).$$

The orbit $\mathcal{O}_\sigma$ has a distinguished point $x_\sigma \in c_\sigma \subset X_\Delta^+$. Let $T_\sigma = T/T^{x_\sigma}$, where T^{x_σ} is the T-stabilizer of x_σ. We identify the orbit $T \cdot x_\sigma$ with the multiplicative group T_σ. Each k-cell of $\mathcal{O}_\sigma$ contains a unique point of the orbit $T \cdot x_\sigma$. Thus we can make the identification $C_k(\sigma) = C_0(T_\sigma)$, the set of formal sums $\sum_i a_i[g_i]$, where $a_i \in \mathbb{Z}_2$ and $g_i \in T_\sigma$. The multiplication of T_σ defines a multiplication on $C_0(T_\sigma)$, so that $C_0(T_\sigma)$ is just the group algebra of T_σ over $\mathbb{Z}_2$.

Let $\mathcal{I}_\sigma$ be the augmentation ideal of the algebra $C_0(T_\sigma)$, that is,

$$\mathcal{I}_\sigma = \operatorname{Ker}[\varepsilon : C_0(T_\sigma) \to \mathbb{Z}_2] \quad \text{with } \varepsilon \sum_i a_i[g_i] = \sum_i a_i.$$

For $p \leq 0$ we define $\mathcal{T}_p C_k(\sigma)$ to be the subspace corresponding to the ideal $(\mathcal{I}_\sigma)^{-p} \subset C_0(T_\sigma)$, and we let

$$\mathcal{T}_p C_k(\Delta) = \sum_{\operatorname{codim}\sigma = k} \mathcal{T}_p C_k(\sigma).$$

If $\sigma < \tau$ in Δ and $\operatorname{codim}\tau = \operatorname{codim}\sigma - 1$, the geometry of Δ determines a group homomorphism $\varphi_{\tau\sigma} : T_\sigma \to T_\tau$ (see [4]). Let $\partial_{\tau\sigma} : C_k(\sigma) \to C_{k-1}(\tau)$ be the induced algebra homomorphism. We have $\partial_{\tau\sigma}(\mathcal{I}_\sigma) \subset \mathcal{I}_\tau$. The boundary map $\partial_k : C_k(\Delta) \to C_{k-1}(\Delta)$ is given by $\partial_k(\sigma) = \sum_\tau \partial_{\tau\sigma}(\tau)$, and $\partial_k(\mathcal{T}_p C_k(\Delta)) \subset \mathcal{T}_p C_{k-1}(\Delta)$, so $\mathcal{T}_p C_*(\Delta)$ is a subcomplex of $C_*(\Delta)$.

PROPOSITION 5.1. *For all $k \geq 0$ and $p \leq 0$, the chain map α* (5-1) *takes the toric filtration* (5-2) *to the Nash filtration* (3-1),

$$\alpha(\mathcal{T}_p C_k(\Delta)) \subset \mathcal{N}_p C_k(X_\Delta).$$

PROOF. It suffices to show that for every cone $\sigma \in \Delta$ with $\operatorname{codim} \sigma = k$,

$$\alpha(\mathcal{T}_p C_k(\sigma)) \subset \mathcal{N}_p C_k(\mathcal{O}_\sigma).$$

The variety $\mathcal{O}_\sigma$ is isomorphic to $(\mathbb{R}^*)^k$, the toric variety of the trivial fan $\{0\}$ in $\mathbb{R}^k$, and the action of T_σ on $\mathcal{O}_\sigma$ corresponds to the action of $T_k = \{-1, +1\}^k$ on $(\mathbb{R}^*)^k$. The k-cells of $(\mathbb{R}^*)^k$ are its connected components. Let $\mathcal{I}_k \subset C_0(T_k)$ be the augmentation ideal. Let $q = -p$, so $0 \le q \le k$. The vector space $C_0(T_k)$ has dimension 2^k, and for each q the quotient $\mathcal{I}^q/\mathcal{I}^{q+1}$ has dimension $\binom{k}{q}$. A basis for $\mathcal{I}^q/\mathcal{I}^{q+1}$ can be defined as follows. Let $t_1, \dots, t_k$ be the standard generators of the multiplicative group T_k,

$$t_i = (t_{i1}, \dots, t_{ik}),\ t_{ij} = \begin{cases} -1 & \text{if } i = j, \\ \ \ 1 & \text{if } i \neq j. \end{cases}$$

If $S \subset \{1, \dots, k\}$, let T_S be the subgroup of T_k generated by $\{t_i\ ;\ i \in S\}$, and define $[T_S] \in C_0(T_k)$ by

$$[T_S] = \sum_{t \in T_S} [t].$$

Then $\{[T_S]\ ; |S| = q\}$ is a basis for $\mathcal{I}^q/\mathcal{I}^{q+1}$ (see [4]).

To prove that $\alpha((\mathcal{I}_k)^q) \subset \mathcal{N}_{-q} C_k((\mathbb{R}^*)^k)$ we just need to show that if $|S| = q$ then $\alpha([T_S]) \in \mathcal{N}_{-q} C_k((\mathbb{R}^*)^k)$. Now the chain $\alpha([T_S]) \in C_k((\mathbb{R}^*)^k)$ is represented by the semialgebraic set $A_S \subset (\mathbb{R}^*)^k$,

$$A_S = \{(x_1, \dots, x_k)\ ;\ x_i > 0,\ i \notin S\},$$

and $\varphi = 2^{k-q} \mathbf{1}_{A_S}$ is Nash constructible. To see this consider the compactification $(\mathbb{P}^1(\mathbb{R}))^k$ of $(\mathbb{R}^*)^k$. We have $\varphi = \tilde{\varphi}|(\mathbb{R}^*)^k$, where $\tilde{\varphi} = f_* \mathbf{1}_{(\mathbb{P}^1(\mathbb{R}))^k}$, with $f : (\mathbb{P}^1(\mathbb{R}))^k \to (\mathbb{P}^1(\mathbb{R}))^k$ defined as follows. If $z = (u : v) \in \mathbb{P}^1(\mathbb{R})$, let $f_1(z) = (u : v)$, and $f_2(z) = (u^2 : v^2)$. Then

$$f(z_1, \dots, z_k) = (w_1, \dots, w_k),\ w_i = \begin{cases} f_1(z_i) & \text{if } i \in S, \\ f_2(z_i) & \text{if } i \notin S. \end{cases}$$

This completes the proof. □

LEMMA 5.2. *Let σ be a codimension k cone of Δ, and let*

$$C_i(\sigma) = \begin{cases} C_k(\sigma) & \text{if } i = k, \\ 0 & \text{if } i \neq k. \end{cases}$$

For all $p \le 0$,

$$\alpha_* : H_*(\mathcal{T}_p C_*(\sigma)) \to H_*(\mathcal{N}_p C_*(\mathcal{O}_\sigma))$$

is an isomorphism.

PROOF. Again we only need to consider the case $\mathcal{O}_\sigma = (\mathbb{R}^*)^k$, where σ is the trivial cone 0 in $\mathbb{R}^n$. Now

$$H_i(C_*(0)) = \begin{cases} C_k(0) & \text{if } i = k, \\ 0 & \text{if } i \neq k, \end{cases}$$

and

$$H_i(C_*((\mathbb{R}^*)^k)) = \begin{cases} \operatorname{Ker} \partial_k & \text{if } i = k, \\ 0 & \text{if } i \neq k, \end{cases}$$

where $\partial_k : C_k((\mathbb{R}^*)^k) \to C_{k-1}((\mathbb{R}^*)^k)$. The vector space $\operatorname{Ker} \partial_k$ has basis the cycles represented by the components of $(\mathbb{R}^*)^k$, and $\alpha : C_k(0) \to C_k((\mathbb{R}^*)^k)$ is a bijection from the cells of $C_k(0)$ to the components of $(\mathbb{R}^*)^k$. Thus $\alpha : C_k(0) \to \operatorname{Ker} \partial_k$ is an isomorphism of vector spaces. Therefore α takes the basis $\{A_S \ ; \ |S| = q\}_{q=0,\dots,k}$ to a basis of $\operatorname{Ker} \partial_k$. The proof of Proposition 5.1 shows that if $|S| \geq q$ then $A_S \in \mathcal{N}_{-q} C_k((\mathbb{R}^*)^k)$. We claim further that if $|S| < q$ then $A_S \notin \mathcal{N}_{-q} C_k((\mathbb{R}^*)^k)$. It follows that $\{A_S \ ; \ |S| \geq q\}$ is a basis for $H_k(N_{-q} C_*((\mathbb{R}^*)^k)$, and so

$$\alpha_* : H_*(\mathcal{T}_{-q} C_*(0)) \to H_*(\mathcal{N}_{-q} C_*((\mathbb{R}^*)^k))$$

is an isomorphism, as desired.

To prove the claim, it suffices to show that if $\bar{A}_S$ is the closure of A_S in $\mathbb{R}^n$, then $\bar{A}_S \notin \mathcal{N}_{-q} C_k((\mathbb{R}^*)^k)$. We show this by induction on k. The case $k = 1$ is clear: If $\bar{A} = \{x \ ; \ x \geq 0\}$ then $\bar{A} \notin \mathcal{N}_{-1} C_1(\mathbb{R})$ because $\partial \bar{A} \neq 0$. In general $\bar{A}_S = \{(x_1, \dots, x_k) \ ; \ x_i \geq 0, i \notin S\}$. Suppose $\bar{A}_S$ is $(-q)$-Nash constructible for some $q > |S|$. Then there exists $\varphi : \mathbb{R}^k \to 2^{k-q}\mathbb{Z}$ generically Nash constructible in dimension k such that $\bar{A}_S = \{x \in \mathbb{R}^k \ ; \ \varphi(x) \notin 2^{k-q+1}\mathbb{Z}\}$, up to a set of dimension $< k$. Let $j \notin S$, and let $W_j = \{(x_1, \dots, x_k) \ ; \ x_j = 0\} \cong \mathbb{R}^{k-1}$. Then $\partial_{W_j} \varphi : W_j \to 2^{k-q-1}\mathbb{Z}$, and $\bar{A}_S \cap W_j = \{x \in W_j \ ; \ \partial_{W_j} \varphi(x) \notin 2^{k-q}\mathbb{Z}\}$, up to a set of dimension $< k-1$. Hence $\bar{A}_S \cap W_j \in \mathcal{N}_{-q} C_{k-1}(W_j)$. But

$$\bar{A}_S \cap W_j = \{(x, \dots, x_k) \ ; \ x_j = 0, x_i \geq 0, i \notin S\},$$

and so by the inductive hypothesis $\bar{A}_S \cap W_j \notin \mathcal{N}_{-q} C_{k-1}(W_j)$, which is a contradiction. □

LEMMA 5.3. *For every toric variety X_Δ and every $p \leq 0$,*

$$\alpha_* : H_*(\mathcal{T}_p C_*(\Delta)) \to H_*(\mathcal{N}_p C_*(X_\Delta))$$

is an isomorphism.

PROOF. We show by induction on orbits that the lemma is true for every variety Z that is a union of orbits in the toric variety X_Δ. Let Σ be a subset of Δ, and let $\Sigma' = \Sigma \setminus \{\sigma\}$, where $\sigma \in \Sigma$ is a minimal cone, *i. e.* there is no $\tau \in \Sigma$ with $\tau < \sigma$. Let Z and Z' be the unions of the orbits corresponding to cones in Σ

and Σ', respectively. Then Z' is closed in Z, and $Z \setminus Z' = \mathcal{O}_\sigma$. We have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H_i(\mathcal{T}_p C_*(\Sigma')) & \longrightarrow & H_i(\mathcal{T}_p C_*(\Sigma)) & \longrightarrow & H_i(\mathcal{T}_p C_*(\sigma)) & \longrightarrow & H_{i-1}(\mathcal{T}_p C_*(\Sigma')) & \longrightarrow \cdots \\ & \downarrow \beta_i & & \downarrow \gamma_i & & \downarrow \alpha_i & & \downarrow \beta_{i-1} & \\ \cdots \longrightarrow & H_i(\mathcal{N}_p C_*(\Sigma')) & \longrightarrow & H_i(\mathcal{N}_p C_*(\Sigma)) & \longrightarrow & H_i(\mathcal{N}_p C_*(\sigma)) & \longrightarrow & H_{i-1}(\mathcal{N}_p C_*(\Sigma')) & \longrightarrow \cdots \end{array}$$

By Lemma 5.3 α_i is an isomorphism for all i. By inductive hypothesis β_i is an isomorphism for all i. Therefore γ_i is an isomorphism for all i. □

THEOREM 5.4. *For every toric variety X_Δ and every $p \leq 0$,*

$$\alpha_* : H_*\left(\frac{\mathcal{T}_p C_*(\Delta)}{\mathcal{T}_{p-1} C_*(\Delta)}\right) \to H_*\left(\frac{\mathcal{N}_p C_*(X_\Delta)}{\mathcal{N}_{p-1} C_*(X_\Delta)}\right)$$

is an isomorphism.

PROOF. This follows from Lemma 5.3 and the long exact homology sequences of the pairs $(\mathcal{T}_p C_*(\Delta), \mathcal{T}_{p-1} C_*(\Delta))$ and $(\mathcal{N}_p C_*(X_\Delta), \mathcal{N}_{p-1} C_*(X_\Delta))$. □

Thus for every toric variety X_Δ the toric filtered complex $\mathcal{T}C_*(\Delta)$ is quasi-isomorphic to the Nash constructible filtered complex $\mathcal{N}C_*(X_\Delta)$, and so the toric spectral sequence [4] is isomorphic to the weight spectral sequence.

EXAMPLE 5.5. For toric varieties of dimension at most 4, the toric spectral sequence collapses [4; 35]. V. Hower [17] discovered that the spectral sequence does not collapse for the 6-dimensional projective toric variety associated to the matroid of the Fano plane.

Appendix: Semialgebraic chains

In this appendix we denote by X a locally compact semialgebraic set (*i.e.* a semialgebraic subset of the set of real points of a real algebraic variety) and by $C_*(X)$ the complex of semialgebraic chains of X with closed supports and coefficients in $\mathbb{Z}_2$. The complex $C_*(X)$ has the following geometric description, which is equivalent to the usual definition using a semialgebraic triangulation [5, 11.7].

A *semialgebraic chain* c of X is an equivalence class of closed semialgebraic subsets of X. For $k \geq 0$, let $S_k(X)$ be the $\mathbb{Z}_2$ vector space generated by the closed semialgebraic subsets of X of dimension $\leq k$. Then $C_k(X)$ is the $\mathbb{Z}_2$ vector space obtained as the quotient of $S_k(X)$ by the following relations:

(i) If A and B are closed semialgebraic subsets of X of dimension at most k, then

$$A + B \sim \operatorname{cl}(A \div B),$$

where $A \div B = (A \cup B) \setminus (A \cap B)$ is the symmetric difference of A and B, and cl denotes closure.

(ii) If A is a closed semialgebraic subset of X and $\dim A < k$, then $A \sim 0$.

If the chain c is represented by the semialgebraic set A, we write $c = [A]$. If $c \in C_k(X)$, the *support* of c, denoted $\operatorname{Supp} c$, is the smallest closed semialgebraic set representing c. If $c = [A]$ then $\operatorname{Supp} c = \{x \in A \ ; \ \dim_x A = k\}$.

The *boundary* operator $\partial_k : C_k(X) \to C_{k-1}(X)$ can be defined using the link operator Λ on constructible functions [24]. If $c \in C_k(X)$ with $c = [A]$, then $\partial_k c = [\partial A]$, where $\partial A = \{x \in A \ ; \ \Lambda \mathbf{1}_A(x) \equiv 1 \pmod 2\}$. The operator ∂_k is well-defined, and $\partial_{k-1}\partial_k = 0$, since $\Lambda \circ \Lambda = 2\Lambda$.

If $f : X \to Y$ is a proper continuous semialgebraic map, the *pushforward* homomorphism $f_* : C_k(X) \to C_k(Y)$ is defined as follows. Let A be a representative of c. Then $f(A) \sim B_1 + \cdots + B_l$, where each closed semialgebraic set B_i has the property that $\#(A \cap f^{-1}(y))$ is constant mod 2 on $B_i \setminus B_i'$ for some closed semialgebraic set $B_i \subset B_i$ with $\dim B_i' < k$. For each i let $n_i \in \mathbb{Z}_2$ be this constant value. Then $f_*(c) = n_1[B_1] + \cdots + n_l[B_l]$.

Alternately, $f_*(c) = [B]$, where $B = \mathrm{cl}\{y \in Y \ ; \ f_*\mathbf{1}_A(y) \equiv 1 \pmod 2\}$, and f_* is pushforward for constructible functions [24]. From this definition it is easy to prove the standard properties $g_* f_* = (gf)_*$ and $\partial_k f_* = f_* \partial_k$.

We use two basic operations on semialgebraic chains: restriction and closure. These operations do not commute with the boundary operator in general.

Let $c \in C_k(X)$ and let $Z \subset X$ be a locally closed semialgebraic subset. If $c = [A]$, we define the *restriction* by $c|_Z = [A \cap Z] \in C_k(Z)$. This operation is well-defined. If U is an open semialgebraic subset of X, then $\partial_k(c|_U) = (\partial_k c)|_U$.

Now let $c \in C_k(Z)$ with $Z \subset X$ locally closed semialgebraic. If $c = [A]$ we define the *closure* by $\bar{c} = [\mathrm{cl}(A)] \in C_k(X)$, where $\mathrm{cl}(A)$ is the closure of A in X. Closure is a well-defined operation on semialgebraic chains.

By means of the restriction and closure operations, we define the pullback of a chain in the following situation, which can be applied to an acyclic square (1-2) of real algebraic varieties. Consider a square of locally closed semialgebraic sets,

$$\begin{array}{ccc} \tilde{Y} & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{\ i\ } & X \end{array}$$

such that $\pi : \tilde{X} \to X$ is a proper continuous semialgebraic map, i is the inclusion of a closed semialgebraic subset, $\tilde{Y} = \pi^{-1}(Y)$, and the restriction of π is a homeomorphism $\pi' : \tilde{X} \setminus \tilde{Y} \to X \setminus Y$. Let $c \in C_k(X)$. We define the *pullback*

$\pi^{-1}c \in C_k(\tilde{X})$ by the formula

$$\pi^{-1}c = \overline{((\pi')^{-1})_*(c|_{X\setminus Y})}.$$

Pullback does not commute with the boundary operator in general.

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CLINT MCCRORY
MATHEMATICS DEPARTMENT
UNIVERSITY OF GEORGIA
ATHENS, GA 30602
UNITED STATES
clint@math.uga.edu

ADAM PARUSIŃSKI
LABORATOIRE J.-A. DIEUDONNÉ
U.M.R. Nº 6621 DU C.N.R.S.
UNIVERSITÉ DE NICE - SOPHIA ANTIPOLIS
PARC VALROSE
06108 NICE CEDEX 02
FRANCE
adam.parusinski@unice.fr