Topology of Stratified Spaces
MSRI Publications
Volume **58**, 2011

An exponential history of functions with logarithmic growth

MATT KERR AND GREGORY PEARLSTEIN

ABSTRACT. We survey recent work on normal functions, including limits and singularities of admissible normal functions, the Griffiths–Green approach to the Hodge conjecture, algebraicity of the zero locus of a normal function, Néron models, and Mumford–Tate groups. Some of the material and many of the examples, especially in Sections 5 and 6, are original.

Introduction

CONTENTS

In a talk on the theory of motives, A. A. Beilinson remarked that according to his time-line of results, advances in the (relatively young) field were apparently a logarithmic function of t; hence, one could expect to wait 100 years for the next significant milestone. Here we allow ourselves to be more optimistic: following on a drawn-out history which begins with Poincaré, Lefschetz, and Hodge, the theory of *normal functions* reached maturity in the programs of Bloch, Griffiths,

The second author was supported by NSF grant DMS-0703956.

Zucker, and others. But the recent blizzard of results and ideas, inspired by works of M. Saito on admissible normal functions, and Green and Griffiths on the Hodge Conjecture, has been impressive indeed. Besides further papers of theirs, significant progress has been made in work of P. Brosnan, F. Charles, H. Clemens, H. Fang, J. Lewis, R. Thomas, Z. Nie, C. Schnell, C. Voisin, A. Young, and the authors — much of this in the last 4 years. This seems like a good time to try to summarize the state of the art and speculate about the future, barring (say) 100 more results between the time of writing and the publication of this volume.

In the classical algebraic geometry of curves, Abel's theorem and Jacobi inversion articulate the relationship (involving rational integrals) between configurations of points with integer multiplicities, or zero-cycles, and an abelian variety known as the Jacobian of the curve: the latter algebraically parametrizes the cycles of degree 0 modulo the subgroup arising as divisors of meromorphic functions. Given a family $\mathcal{X}$ of algebraic curves over a complete base curve S, with smooth fibers over S^* (S minus a finite point set Σ over which fibers have double point singularities), Poincaré [P1; P2] defined *normal functions* as holomorphic sections of the corresponding family of Jacobians over S which behave normally (or logarithmically) in some sense near the boundary. His main result, which says essentially that they parametrize 1-dimensional cycles on $\mathcal{X}$, was then used by Lefschetz (in the context where $\mathcal{X}$ is a pencil of hyperplane sections of a projective algebraic surface) to prove his famous $(1,1)$ theorem for algebraic surfaces [L]. This later became the basis for the Hodge conjecture, which says that certain *topological-analytic* invariants of an *algebraic* variety must come from *algebraic* subvarieties:

CONJECTURE 1. *For a smooth projective complex algebraic variety X, with $\mathrm{Hg}^m(X)_{\mathbb{Q}}$ the classes in $H^{2m}_{\mathrm{sing}}(X^{\mathrm{an}}_{\mathbb{C}},\mathbb{Q})$ of type (m,m), and $CH^m(X)$ the Chow group of codimension-m algebraic cycles modulo rational equivalence, the fundamental class map $CH^m(X)\otimes\mathbb{Q}\to\mathrm{Hg}^m(X)_{\mathbb{Q}}$ is surjective.*

Together with a desire to learn more about the structure of Chow groups (the Bloch–Beilinson conjectures reviewed in §5), this can be seen as the primary motivation behind all the work described (as well as the new results) in this paper. In particular, in §1 (after mathematically fleshing out the Poincaré–Lefschetz story) we describe the attempts to directly generalize Lefschetz's success to higher-codimension cycles which led to Griffiths' Abel–Jacobi map (from the codimension m cycle group of a variety X to its m-th "intermediate" Jacobian), horizontality and variations of mixed Hodge structure, and S. Zucker's Theorem on Normal Functions. As is well-known, the breakdown (beyond codimension 1) of the relationship between cycles and (intermediate)

Jacobians, and the failure of the Jacobians to be algebraic, meant that the same game played in 1 parameter would not work outside very special cases.

It has taken nearly three decades to develop the technical underpinnings for a study of normal functions over a *higher*-dimensional base S: Kashiwara's work on admissible variations of mixed Hodge structure [K], M. Saito's introduction of mixed Hodge modules [S4], multivariable nilpotent and SL_2-orbit theorems ([KNU1],[Pe2]), and so on. And then in 2006, Griffiths and Green had a fundamental idea tying the Hodge conjecture to the presence of *nontorsion singularities* — nontrivial invariants in local intersection cohomology — for multiparameter normal functions arising from Hodge classes on algebraic varieties [GG]. We describe their main result and the follow-up work [BFNP] in § 3. Prior to that the reader will need some familiarity with the boundary behavior of "admissible" normal functions arising from higher codimension algebraic cycles. The two principal invariants of this behavior are called *limits* and *singularities*, and we have tried in § 2 to give the reader a geometric feel for these through several examples and an explanation of the precise sense in which the limit of Abel–Jacobi invariants (for a family of cycles) is again some kind of Abel–Jacobi invariant. In general throughout §§ 1–2 (and § 4.5–6) normal functions are "of geometric origin" (arise from cycles), whereas in the remainder the formal Hodge-theoretic point of view dominates (though Conjecture 1 is always in the background). We should emphasize that the first two sections are intended for a broad audience, while the last four are of a more specialized nature; one might say that the difficulty level increases exponentially.

The transcendental (nonalgebraic) nature of intermediate Jacobians means that even for a normal function of geometric origin, algebraicity of its vanishing locus (as a subset of the base S), let alone its sensitivity to the field of definition of the cycle, is not a foreordained conclusion. Following a review of Schmid's nilpotent and SL_2-orbit theorems (which lie at the heart of the limit mixed Hodge structures introduced in § 2), in § 4 we explain how generalizations of those theorems to mixed Hodge structures (and multiple parameters) have allowed complex algebraicity to be proved for the zero loci of "abstract" admissible normal functions [BP1; BP2; BP3; S5]. We then address the field of definition in the geometric case, in particular the recent result of Charles [Ch] under a hypothesis on the VHS underlying the zero locus, the situation when the family of cycles is algebraically equivalent to zero, and what all this means for filtrations on Chow groups. Another reason one would want the zero locus to be algebraic is that the Griffiths–Green normal function attached to a nontrivial Hodge class can then be shown, by an observation of C. Schnell, to have a singularity in the intersection of the zero locus with the boundary $\Sigma \subset S$ (though this intersection could very well be empty).

Now, *a priori*, admissible normal functions (ANFs) are only horizontal and holomorphic sections of a Jacobian bundle over $S\setminus\Sigma$ which are highly constrained along the boundary. Another route (besides orbit theorems) that leads to algebraicity of their zero loci is the construction of a "Néron model" — a partial compactification of the Jacobian bundle satisfying a Hausdorff property (though not a complex analytic space in general) and graphing admissible normal functions over all of S. Néron models are taken up in §5; as they are better understood they may become useful in defining global invariants of (one or more) normal functions. However, unless the underlying variation of Hodge structure (VHS) is a nilpotent orbit the group of components of the Néron model (i.e., the possible singularities of ANFs at that point) over a codimension≥ 2 boundary point remains mysterious. Recent examples of M. Saito [S6] and the second author [Pe3] show that there are analytic obstructions which prevent ANFs from surjecting onto (or even mapping nontrivially to) the putative singularity group for ANFs (rational $(0,0)$ classes in the local intersection cohomology). At first glance this appears to throw the existence of singularities for Griffiths–Green normal functions (and hence the Hodge conjecture) into serious doubt, but in §5.5 we show that this concern is probably ill-founded.

The last section is devoted to a discussion of Mumford–Tate groups of mixed Hodge structures (introduced by Y. André [An]) and variations thereof, in particular those attached to admissible normal functions. The motivation for writing this section was again to attempt to "force singularities to exist" via conditions on the normal function (e.g., involving the zero locus) which maximize the monodromy of the underlying local system inside the M-T group; we were able to markedly improve André's maximality result (but not to produce singularities). Since the general notion of (non)singularity of a VMHS at a boundary point is defined here (in §6.3), which generalizes the notion of singularity of a normal function, we should point out that there is another sense in which the word "singularity" is used in this paper. The "singularities" of a *period mapping associated to a VHS or VMHS* are points where the connection has poles or the local system has monodromy (Σ in the notation above), and at which one must compute a limit mixed Hodge structure (LMHS). These contain the "singularities of the VMHS", nearly always as a *proper* subset; indeed, pure VHS never have singularities (in the sense of §6.3), though their corresponding period mappings do.

This paper has its roots in the first author's talk at a conference in honor of Phillip Griffiths' 70th birthday at the IAS, and the second author's talk at MSRI during the conference on the topology of stratified spaces to which this volume is dedicated. The relationship between normal functions and stratifications occurs in the context of mixed Hodge modules and the Decomposition Theorem

[BBD], and is most explicitly on display in the construction of the multivariable Néron model in §5 as a topological group whose restrictions to the strata of a Whitney stratification are complex Lie groups. We want to thank the conference organizers and Robert Bryant for doing an excellent job at putting together and hosting a successful interdisciplinary meeting blending (amongst other topics) singularities and topology of complex varieties, L^2 and intersection cohomology, and mixed Hodge theory, all of which play a role below. We are indebted to Patrick Brosnan, Phillip Griffiths, and James Lewis for helpful conversations and sharing their ideas. We also want to thank heartily both referees as well as Chris Peters, whose comments and suggestions have made this a better paper.

One observation on notation is in order, mainly for experts: to clarify the distinction in some places between monodromy weight filtrations arising in LMHS and weight filtrations postulated as part of the data of an admissible variation of mixed Hodge structure (AVMHS), the former are always denoted $M_\bullet$ (and the latter $W_\bullet$) in this paper. In particular, for a degeneration of (pure) weight n HS with monodromy logarithm N, the weight filtration on the LMHS is written $M(N)_\bullet$ (and centered at n). While perhaps nontraditional, this is consistent with the notation $M(N, W)_\bullet$ for relative weight monodromy filtrations for (admissible) degenerations of MHS. That is, when W is "trivial" ($W_n = \mathcal{H}$, $W_{n-1} = \{0\}$) it is simply omitted.

Finally, we would like to draw attention to the interesting recent article [Gr4] of Griffiths which covers ground related to our §§2–5, but in a complementary fashion that may also be useful to the reader.

1. Prehistory and classical results

The present chapter is not meant to be heroic, but merely aims to introduce a few concepts which shall be used throughout the paper. We felt it would be convenient (whatever one's background) to have an up-to-date, "algebraic" summary of certain basic material on normal functions and their invariants in one place. For background or further (and better, but much lengthier) discussion of this material the reader may consult the excellent books [Le1] by Lewis and [Vo2] by Voisin, as well as the lectures of Green and Voisin from the "Torino volume" [GMV] and the papers [Gr1; Gr2; Gr3] of Griffiths.

Even experts may want to glance this section over since we have included some bits of recent provenance: the relationship between log-infinitesimal and topological invariants, which uses work of M. Saito; the result on inhomogeneous Picard–Fuchs equations, which incorporates a theorem of Müller-Stach and del Angel; the important example of Morrison and Walcher related to open mirror symmetry; and the material on K-motivation of normal functions (see §1.3 and §1.7), which will be used in Sections 2 and 4.

Before we begin, a word on the *currents* that play a rôle in the bullet-train proof of Abel's Theorem in § 1.1. These are differential forms with distribution coefficients, and may be integrated against C^∞ forms, with exterior derivative d defined by "integration by parts". They form a complex computing $\mathbb{C}$-cohomology (of the complex manifold on which they lie) and include C^∞chains and log-smooth forms. For example, for a C^∞ chain Γ, the delta current δ_Γ has the defining property $\int \delta_\Gamma \wedge \omega = \int_\Gamma \omega$ for any C^∞ form ω. (For more details, see Chapter 3 of [GH].)

1.1. Abel's Theorem. Our (historically incorrect) story begins with a divisor D of degree zero on a smooth projective algebraic curve $X/\mathbb{C}$; the associated analytic variety X^{an} is a Riemann surface. (Except when explicitly mentioned, we continue to work over $\mathbb{C}$.) Writing $D = \sum_{\text{finite}} n_i p_i \in Z^1(X)_{\text{hom}}$ ($n_i \in \mathbb{Z}$ such that $\sum n_i = 0$, $p_i \in X(\mathbb{C})$), by Riemann's existence theorem one has a meromorphic 1-form $\hat{\omega}$ with $\text{Res}_{p_i}(\hat{\omega}) = n_i$ $(\forall i)$. Denoting by $\{\omega_1, \ldots, \omega_g\}$ a basis for $\Omega^1(X)$, consider the map

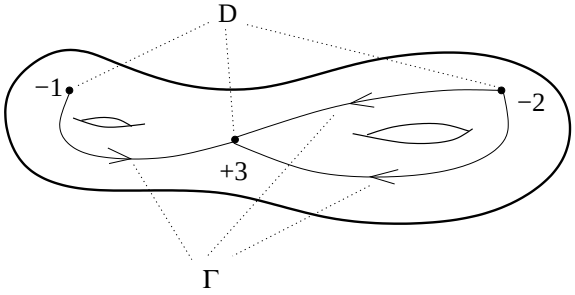
$$\begin{array}{ccccc} & & \widetilde{AJ} & & \\ & \overset{\frown}{\longrightarrow} & & & \\ Z^1(X)_{\text{hom}} & \longrightarrow & \dfrac{\Omega^1(X)^\vee}{\int_{H_1(X,\mathbb{Z})}(\cdot)} & \xrightarrow[\cong]{\text{ev}_{\{\omega_i\}}} & \dfrac{\mathbb{C}^g}{\Lambda^{2g}} =: J^1(X) \end{array} \tag{1-1}$$

$$D \longmapsto \int_\Gamma \longmapsto \left(\int_\Gamma \omega_1, \ldots, \int_\Gamma \omega_g\right)$$

where $\Gamma \in C_1(X^{\text{an}})$ is any chain with $\partial\Gamma = D$ and $J^1(X)$ is the *Jacobian* of X. The 1-current $\kappa := \hat{\omega} - 2\pi i \delta_\Gamma$ is closed; moreover, if $\widetilde{AJ}(D) = 0$ then Γ may be chosen so that all $\int_\Gamma \omega_i = 0$ implies $\int_X \kappa \wedge \omega_i = 0$. We can therefore smooth κ in its cohomology class to $\omega = \kappa - d\eta$ ($\omega \in \Omega^1(X)$; $\eta \in D^0(X)$ =0-currents), and

$$f := \exp\left\{\int (\hat{\omega} - \omega)\right\} \tag{1-2}$$

$$= \quad e^{2\pi i \int \delta_\Gamma} e^\eta \tag{1-3}$$



is single-valued — though possibly discontinuous — by (1-3), while being meromorphic — though possibly multivalued — by (1-2). Locally at p_i, $e^{\int (n_i/z)\,dz} = Cz^{n_i}$ has the right degree; and so the divisor of f is precisely D. Conversely, if $D = (f) = f^{-1}(0) - f^{-1}(\infty)$ for $f \in \mathbb{C}(X)^*$, then

$$t \mapsto \int_{f^{-1}\overrightarrow{(0.t)}} (\cdot)$$

induces a holomorphic map $\mathbb{P}^1 \to J^1(X)$. Such a map is necessarily constant (say, to avoid pulling back a nontrivial holomorphic 1-form), and by evaluating at $t = 0$ one finds that this constant is zero. So we have proved part (i) of

THEOREM 2. (i) [Abel] *Writing $Z^1(X)_{\text{rat}}$ for the divisors of functions $f \in \mathbb{C}(X)^*$, $\widetilde{AJ}$ descends to an injective homomorphism of abelian groups*

$$CH^1(X)_{\text{hom}} := \frac{Z^1(X)_{\text{hom}}}{Z^1(X)_{\text{rat}}} \xrightarrow{AJ} J^1(X).$$

(ii) [Jacobi inversion] *AJ is surjective; in particular, fixing $q_1, \ldots, q_g \in X(\mathbb{C})$ the morphism* $\text{Sym}^g X \to J^1(X)$ *induced by $p_1 + \cdots + p_g \mapsto \int_{\partial^{-1}(\sum p_i - q_i)}(\cdot)$ is birational.*

Here $\partial^{-1} D$ means any 1-chain bounding on D. Implicit in (ii) is that $J^1(X)$ is an (abelian) algebraic variety; this is a consequence of ampleness of the theta line bundle (on $J^1(X)$) induced by the polarization

$$Q : H^1(X, \mathbb{Z}) \times H^1(X, \mathbb{Z}) \to \mathbb{Z}$$

(with obvious extensions to $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$) defined equivalently by cup product, intersection of cycles, or integration $(\omega, \eta) \mapsto \int_X \omega \wedge \eta$. The ampleness boils down to the *second Riemann bilinear relation*, which says that $iQ(\cdot, \bar{\cdot})$ is positive definite on $\Omega^1(X)$.

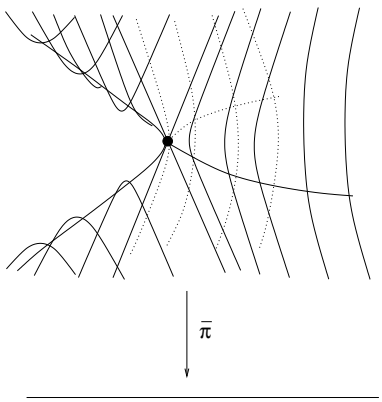
1.2. Normal functions. We now wish to vary the Abel–Jacobi map in families. Until §2, all our normal functions shall be over a curve S. Let $\mathcal{X}$ be a smooth projective surface, and $\bar{\pi} : \mathcal{X} \to S$ a (projective) morphism which is

(a) smooth off a finite set $\Sigma = \{s_1, \ldots, s_e\} \subset S$, and

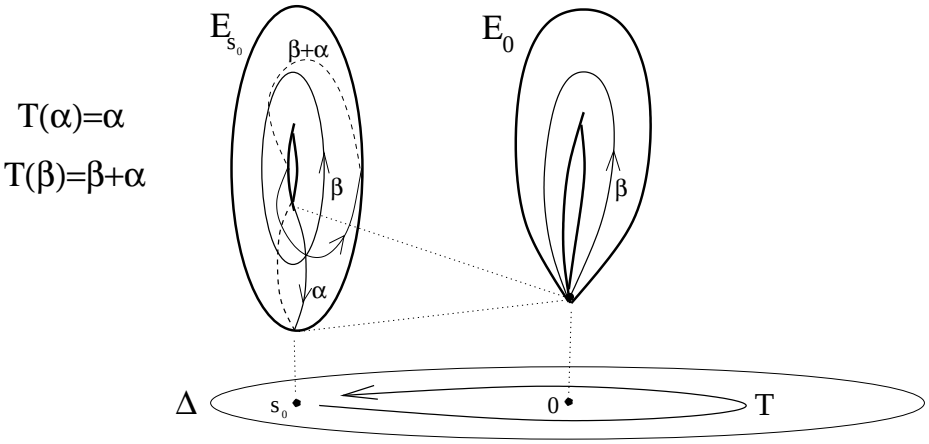
(b) locally of the form $(x_1, x_2) \mapsto x_1 x_2$ at singularities (of $\bar{\pi}$).

Write $X_s := \bar{\pi}^{-1}(s)$ ($s \in S$) for the fibers. The singular fibers X_{s_i} ($i = 1, \ldots, e$) then have only nodal (ordinary double point) singularities, and writing $\mathcal{X}^*$ for their complement we have $\pi : \mathcal{X}^* \to S^* := S \backslash \Sigma$. Fixing a general $s_0 \in S^*$, the local monodromies $T_{s_i} \in \text{Aut}\left(H^1(X_{s_0}, \mathbb{Z}) =: \mathbb{H}_{\mathbb{Z}, s_0}\right)$ of the local system $\mathbb{H}_{\mathbb{Z}} := R^1 \pi_* \mathbb{Z}_{\mathcal{X}^*}$ are then computed by the Picard–Lefschetz formula

$$(T_{s_i} - I)\gamma = \sum_j (\gamma \cdot \delta_j)\delta_j. \tag{1-4}$$



Here $\{\delta_j\}$ are the Poincaré duals of the (possibly nondistinct) vanishing cycle classes $\in \ker\{H_1(X_{s_0},\mathbb{Z}) \to H_1(X_{s_i},\mathbb{Z})\}$ associated to each node on X_{s_i}; we note $(T_{s_i} - I)^2 = 0$. For a family of elliptic curves, (1-4) is just the familiar *Dehn twist:*



(For the reader new to such pictures, the two crossing segments in the "local real" picture at the top of the page become the two touching "thimbles", i.e., a small neighborhood of the singularity in E_0, in this diagram.)

Now, in our setting, the bundle of Jacobians $\mathcal{J} := \bigcup_{s\in S^*} J^1(X_s)$ is a complex (algebraic) manifold. It admits a partial compactification to a fiber space of complex abelian Lie groups, by defining

$$J^1(X_{s_i}) := \frac{H^0(\omega_{X_{s_i}})}{\mathrm{im}\{H^1(X_{s_i},\mathbb{Z})\}}$$

(with ω_{X_s} the dualizing sheaf) and $\mathcal{J}_e := \bigcup_{s\in S} J^1(X_s)$. (How this is topologized will be discussed in a more general context in § 5.) The same notation will

denote their sheaves of sections,

$$0 \to \mathbb{H}_{\mathbb{Z}} \to \mathcal{F}^{\vee} \to \mathcal{J} \to 0 \qquad (\text{on } S^*) \tag{1-5}$$

$$0 \to \mathbb{H}_{\mathbb{Z},e} \to (\mathcal{F}_e)^{\vee} \to \mathcal{J}_e \to 0 \qquad (\text{on } S), \tag{1-6}$$

with $\mathcal{F} := \pi_* \omega_{\mathcal{X}/S}$, $\mathcal{F}_e := \bar{\pi}_* \omega_{\mathcal{X}/S}$, $\mathbb{H}_{\mathbb{Z}} = R^1 \pi_* \mathbb{Z}$, $\mathbb{H}_{\mathbb{Z},e} = R^1 \bar{\pi}_* \mathbb{Z}$.

DEFINITION 3. A *normal function* (NF) is a holomorphic section (over S^*) of $\mathcal{J}$. An *extended* (*or Poincaré*) *normal function* (ENF) is a holomorphic section (over S) of $\mathcal{J}_e$. An NF is *extendable* if it lies in $\operatorname{im}\{H^0(S, \mathcal{J}_e) \to H^0(S^*, \mathcal{J})\}$.

Next consider the long exact cohomology sequence (sections over S^*)

$$0 \to H^0(\mathbb{H}_{\mathbb{Z}}) \to H^0(\mathcal{F}^{\vee}) \to H^0(\mathcal{J}) \to H^1(\mathbb{H}_{\mathbb{Z}}) \to H^1(\mathcal{F}^{\vee}); \tag{1-7}$$

the *topological invariant* of a normal function $\nu \in H^0(\mathcal{J})$ is its image $[\nu] \in H^1(S^*, \mathbb{H}_{\mathbb{Z}})$. It is easy to see that the restriction of $[\nu]$ to $H^1(\Delta_i^*, \mathbb{H}_{\mathbb{Z}})$ (Δ_i a punctured disk about s_i) computes the local monodromy $(T_{s_i} - I)\tilde{\nu}$ (where $\tilde{\nu}$ is a multivalued local lift of ν to $\mathcal{F}^{\vee}$), modulo the monodromy of topological cycles. We say that ν is locally liftable if all these restrictions vanish, i.e., if

$$(T_{s_i} - I)\tilde{\nu} \in \operatorname{im}\{(T_{s_i} - I)\mathbb{H}_{\mathbb{Z},s_0}\}.$$

Together with the assumption that as a (multivalued, singular) "section" of $\mathcal{F}_e^{\vee}$, $\tilde{\nu}_e$ has at worst logarithmic divergence at s_i (the "logarithmic growth" in the title), this is equivalent to extendability.

1.3. Normal functions of geometric origin. Let $\mathfrak{Z} \in Z^1(\mathcal{X})_{\text{prim}}$ be a divisor properly intersecting fibers of $\bar{\pi}$ and avoiding its singularities, and which is *primitive* in the sense that each $Z_s := \mathfrak{Z} \cdot X_s$ ($s \in S^*$) is of degree 0. (In fact, the intersection conditions can be done away with, by moving the divisor in a rational equivalence.) Then $s \mapsto AJ(Z_s)$ defines a section $\nu_{\mathfrak{Z}}$ of $\mathcal{J}$, and it can be shown that a multiple $N\nu_{\mathfrak{Z}} = \nu_{N\mathfrak{Z}}$ of $\nu_{\mathfrak{Z}}$ is always extendable. One says that $\nu_{\mathfrak{Z}}$ itself is *admissible*.

Now assume $\bar{\pi}$ has a section $\sigma : S \to \mathcal{X}$ (also avoiding singularities) and consider the analog of (1-7) for $\mathcal{J}_e$

$$0 \to \frac{H^0(\mathcal{F}_e^{\vee})}{H^0(\mathbb{H}_{\mathbb{Z},e})} \to H^0(\mathcal{J}_e) \to \ker\left\{H^1(\mathbb{H}_{\mathbb{Z},e}) \to H^1(\mathcal{F}_e^{\vee})\right\} \to 0.$$

With a bit of work, this becomes

$$0 \to J^1(\mathcal{X}/S)_{\text{fix}} \longrightarrow ENF \xrightarrow{[\cdot]} \frac{\operatorname{Hg}^1(\mathcal{X})_{\text{prim}}}{\mathbb{Z}\langle [X_{s_0}] \rangle} \to 0, \tag{1-8}$$

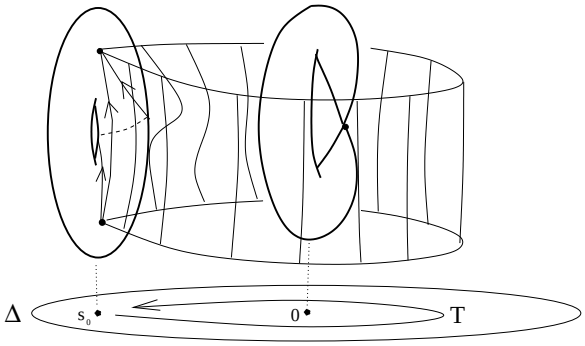
where the Jacobian of the fixed part $J^1(\mathcal{X}/S)_{\text{fix}} \hookrightarrow J^1(X_s)$ ($\forall s \in S$) gives a constant subbundle of $\mathcal{J}_e$ and the primitive Hodge classes $\operatorname{Hg}^1(\mathcal{X})_{\text{prim}}$ are the

Q-orthogonal complement of a general fiber X_{s_0} of $\bar{\pi}$ in $\mathrm{Hg}^1(\mathcal{X}) := H^2(\mathcal{X},\mathbb{Z}) \cap H^{1,1}(\mathcal{X},\mathbb{C})$.

PROPOSITION 4. *Let ν be an ENF.*

(i) *If $[\nu] = 0$ then ν is a constant section of $\mathcal{J}_{\mathrm{fix}} := \bigcup_{s\in S} J^1(\mathcal{X}/S)_{\mathrm{fix}} \subset \mathcal{J}_e$.*
(ii) *If $(\nu =)\nu_\mathfrak{Z}$ is of geometric origin, then $[\nu_\mathfrak{Z}] = \overline{[\mathfrak{Z}]}$ ($[\mathfrak{Z}]$ = fundamental class).*
(iii) [Poincaré Existence Theorem] *Every ENF is of geometric origin.*

We note that (i) follows from considering sections $\{\omega_1,\dots,\omega_g\}(s)$ of $\mathcal{F}_e^\vee$ whose restrictions to general X_s are linearly independent (such do exist), evaluating a lift $\tilde{\nu} \in H^0(\mathcal{F}_e^\vee)$ against them, and applying Liouville's Theorem. The resulting constancy of the abelian integrals, by a result in Hodge Theory (cf. end of § 1.6), implies the membership of $\nu(s) \in \mathcal{J}_{\mathrm{fix}}$. To see (iii), apply "Jacobi inversion with parameters" and $q_i(s) = \sigma(s)$ $(\forall i)$ over S^* (really, over the generic point of S), and then take Zariski closure.[1] Finally, when ν is geometric, the monodromies of a lift $\tilde{\nu}$ (to $\mathcal{F}_e^\vee$) around each loop in S (which determine $[\nu]$) are just the corresponding monodromies of a bounding 1-chain Γ_s ($\partial\Gamma_s = Z_s$), which identify with the Leray $(1,1)$ component of $[\mathfrak{Z}]$ in $H^2(\mathcal{X})$; this gives the gist of (ii).



A normal function is said to be *motivated over K* ($K \subset \mathbb{C}$ a subfield) if it is of geometric origin as above, and if the coefficients of the defining equations of $\mathfrak{Z}$, $\mathcal{X}$, $\bar{\pi}$, and S belong to K.

1.4. Lefschetz (1,1) Theorem. Now take $X \subset \mathbb{P}^N$ to be a smooth projective surface of degree d, and $\{X_s := X \cdot H_s\}_{s\in\mathbb{P}^1}$ a *Lefschetz pencil* of hyperplane sections: the singular fibers have exactly one (nodal) singularity. Let $\beta : \mathcal{X} \twoheadrightarrow X$ denote the blow-up at the base locus $B := \bigcap_{s\in\mathbb{P}^1} X_s$ of the pencil, and $\bar{\pi} : \mathcal{X} \to \mathbb{P}^1 =: S$ the resulting fibration. We are now in the situation considered above, with $\sigma(S)$ replaced by d sections $E_1 \amalg \cdots \amalg E_d = \beta^{-1}(B)$, and fibers of genus $g = \binom{d-1}{2}$; and with the added bonus that there is no torsion in any

[1] Here the $q_i(s)$ are as in Theorem 2(ii) (but varying with respect to a parameter). If at a generic point $\nu(\eta)$ is a special divisor then additional argument is needed.

$H^1(\Delta_i^*, \mathbb{H}_\mathbb{Z})$, so that admissible $\Rightarrow$ extendable. Hence, given $Z \in Z^1(X)_{\text{prim}}$ ($\deg(Z \cdot X_{s_0}) = 0$): $\beta^* Z$ is primitive, $v_Z := v_{\beta^* Z}$ is an ENF, and $[v_Z] = \beta^*[Z]$ under $\beta^* : \mathrm{Hg}^1(X)_{\text{prim}} \hookrightarrow \mathrm{Hg}^1(\mathcal{X})_{\text{prim}}/\mathbb{Z}\langle[X_{s_0}]\rangle$.

If, on the other hand, we start with a Hodge class $\xi \in \mathrm{Hg}^1(X)_{\text{prim}}$, $\beta^*\xi$ is (by (1-8) + Poincaré existence) the class of a geometric ENF $v_\mathfrak{Z}$; and $[\mathfrak{Z}] \equiv [v_\mathfrak{Z}] \equiv \beta^*\xi$ mod $\mathbb{Z}\langle[X_{s_0}]\rangle$ implies $\xi \equiv \beta_*\beta^*\xi \equiv [\beta_*\mathfrak{Z} =: Z]$ in $\mathrm{Hg}^1(X)/\mathbb{Z}\langle[X_{s_0}]\rangle$, which implies $\xi = [Z']$ for some $Z' \in Z^1(X)_{(\text{prim})}$. This is the gist of Lefschetz's original proof [L] of

THEOREM 5. *Let X be a* (*smooth projective algebraic*) *surface. The fundamental class map $CH^1(X) \overset{[\cdot]}{\to} \mathrm{Hg}^1(X)$ is* (*integrally*) *surjective.*

This continues to hold in higher dimension, as can be seen from an inductive treatment with ENF's or (more easily) from the "modern" treatment of Theorem 5 using the exponential exact sheaf sequence

$$0 \to \mathbb{Z}_X \longrightarrow \mathcal{O}_X \overset{e^{2\pi i(\cdot)}}{\longrightarrow} \mathcal{O}_X^* \to 0.$$

One simply puts the induced long exact sequence in the form

$$0 \to \frac{H^1(X, \mathcal{O})}{H^1(X, \mathbb{Z})} \to H^1(X, \mathcal{O}^*) \to \ker\left\{H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O})\right\} \to 0,$$

and interprets it as

$$\begin{array}{ccccccc} 0 & \longrightarrow & J^1(X) & \longrightarrow & \left\{\begin{array}{c}\text{holomorphic}\\ \text{line bundles}\end{array}\right\} & \longrightarrow & \mathrm{Hg}^1(X) \longrightarrow 0 \\ & & & & \Big\downarrow & \nearrow & \\ & & & & CH^1(X) & & \end{array} \qquad (1\text{-}9)$$

where the dotted arrow takes the divisor of a meromorphic section of a given bundle. Existence of the section is a standard but nontrivial result.

We note that for $\mathcal{X} \to \mathbb{P}^1$ a Lefschetz pencil of X, in (1-8) we have

$$J^1(\mathcal{X}/\mathbb{P}^1)_{\text{fix}} = J^1(X) := \frac{H^1(X, \mathbb{C})}{F^1H^1(X, \mathbb{C}) + H^1(X, \mathbb{Z})},$$

which is zero if X is a complete intersection; in that case *ENF* is finitely generated and β^* embeds $\mathrm{Hg}^1(\mathcal{X})_{\text{prim}}$ in *ENF*.

EXAMPLE 6. For X a cubic surface $\subset \mathbb{P}^3$, divisors with support on the 27 lines already surject onto $\mathrm{Hg}^1(X) = H^2(X, \mathbb{Z}) \cong \mathbb{Z}^7$. Differences of these lines generate all primitive classes, hence all of $\mathrm{im}(\beta^*)$ ($\cong \mathbb{Z}^6$) in *ENF* ($\cong \mathbb{Z}^8$).

Note that $\mathcal{J}_e$ is essentially an elliptic surface and *ENF* comprises the (holomorphic) sections passing through the $\mathbb{C}^*$'s over points of Σ. There are no torsion sections.

1.5. Griffiths' AJ map. A $\mathbb{Z}$-Hodge structure (HS) of weight m comprises a finitely generated abelian group $H_\mathbb{Z}$ together with a descending filtration $F^\bullet$ on $H_\mathbb{C} := H_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{C}$ satisfying $F^p H_\mathbb{C} \oplus \overline{F^{m-p+1} H_\mathbb{C}} = H_\mathbb{C}$, the *Hodge filtration*; we denote the lot by H. Examples include the m-th (singular/Betti + de Rham) cohomology groups of smooth projective varieties over $\mathbb{C}$, with $F^p H^m_{dR}(X, \mathbb{C})$ being that part of the de Rham cohomology represented by C^∞ forms on X^{an} with *at least* p holomorphic differentials wedged together in each monomial term. (These are forms of *Hodge type* $(p, m-p) + (p+1, m-p-1) + \cdots$; note that $H^{p,m-p}_\mathbb{C} := F^p H_\mathbb{C} \cap \overline{F^{m-p} H_\mathbb{C}}$.) To accommodate H^m of nonsmooth or incomplete varieties, the notion of a ($\mathbb{Z}$-)mixed Hodge structure (MHS) V is required: in addition to $F^\bullet$ on $V_\mathbb{C}$, introduce a decreasing *weight filtration* $W_\bullet$ on $V_\mathbb{Q}$ such that the $\left(\text{Gr}^W_i V_\mathbb{Q}, (\text{Gr}^W_i (V_\mathbb{C}, F^\bullet))\right)$ are $\mathbb{Q}$-HS of weight i. Mixed Hodge structures have Hodge group

$$\text{Hg}^p(V) := \ker\{V_\mathbb{Z} \oplus F^p W_{2p} V_\mathbb{C} \to V_\mathbb{C}\}$$

(for for $V_\mathbb{Z}$ torsion-free becomes $V_\mathbb{Z} \cap F^p W_{2p} V_\mathbb{C}$) and Jacobian group

$$J^p(V) := \frac{W_{2p} V_\mathbb{C}}{F^p W_{2p} V_\mathbb{C} + W_{2p} V_\mathbb{Q} \cap V_\mathbb{Z}},$$

with special cases $\text{Hg}^m(X) := \text{Hg}^m(H^{2m} X))$ and $J^m(X) := J^m(H^{2m-1}(X))$. Jacobians of HS yield complex tori, and subtori correspond bijectively to sub-HS.

A *polarization* of a Hodge structure H is a morphism Q of HS (defined over $\mathbb{Z}$; complexification respects $F^\bullet$) from $H \times H$ to the trivial HS $\mathbb{Z}(-m)$ of weight $2m$ (and type (m, m)), such that viewed as a pairing Q is nondegenerate and satisfies a positivity constraint generalizing that in § 1.1 (the *second Hodge–Riemann bilinear relation*). A consequence of this definition is that under Q, F^p is the annihilator of F^{m-p+1} (the *first Hodge–Riemann bilinear relation* in abstract form). If X is a smooth projective variety of dimension d, $[\Omega]$ the class of a hyperplane section, write (for $k \le d$, say)

$$H^m(X, \mathbb{Q})_{\text{prim}} := \ker\{H^m(X, \mathbb{Q}) \overset{\cup \Omega^{d-k+1}}{\longrightarrow} H^{2d-m+2}(X, \mathbb{Q})\}.$$

This Hodge structure is then polarized by $Q(\xi, \eta) := (-1)^{\binom{m}{2}} \int_X \xi \wedge \eta \wedge \Omega^{d-k}$, $[\Omega]$ the class of a hyperplane section (obviously since this is a $\mathbb{Q}$-HS, the polarization is only defined over $\mathbb{Q}$).

Let X be a smooth projective $(2m-1)$-fold; we shall consider some equivalence relations on algebraic cycles of codimension m on X. Writing $Z^m(X)$ for the free abelian group on irreducible (complex) codimension p subvarieties of X, two cycles $Z_1, Z_2 \in Z^m(X)$ are homologically equivalent if their difference bounds a C^∞ chain $\Gamma \in C^{\text{top}}_{2m-1}(X^{\text{an}};\mathbb{Z})$ (of real dimension $2m-1$). Algebraic equivalence is generated by (the projection to X of) differences of the form $W\cdot(X\times\{p_1\}) - W\cdot(X\times\{p_2\})$ where C is an algebraic curve, $W \in Z^m(X\times C)$, and $p_1, p_2 \in C(\mathbb{C})$ (or $C(K)$ if we are working over a subfield $K \subset \mathbb{C}$). Rational equivalence is obtained by taking C to be rational ($C \cong \mathbb{P}^1$), and for $m = 1$ is generated by divisors of meromorphic functions. We write $Z^m(X)_{\text{rat}}$ for cycles $\equiv_{\text{rat}} 0$, etc. Note that

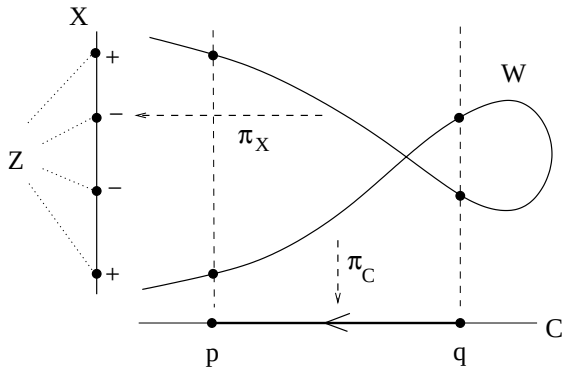
$$CH^m(X) := \frac{Z^m(X)}{Z^m(X)_{\text{rat}}} \supset CH^m(X)_{\text{hom}} := \frac{Z^m(X)_{\text{hom}}}{Z^m(X)_{\text{rat}}}$$

and

$$CH^m(X)_{\text{hom}} \supset CH^m(X)_{\text{alg}} := \frac{Z^m(X)_{\text{alg}}}{Z^m(X)_{\text{rat}}}$$

are proper inclusions in general.

Now let $W \subset X\times C$ be an irreducible subvariety of codimension m, with π_X and π_C the projections from a desingularization of W to X and C. If we put $Z_i := \pi_{X_*}\pi_C^*\{p_i\}$, then $Z_1 \equiv_{\text{alg}} Z_2$ implies $Z_1 \equiv_{\text{hom}} Z_2$, which can be seen explicitly by setting $\Gamma := \pi_{X_*}\pi_C^*(\overrightarrow{q.p})$ (so that $Z_1 - Z_2 = \partial\Gamma$).



Let ω be a d-closed form of Hodge type $(j, 2m-j-1)$ on X, for j at least m. Consider $\int_\Gamma \omega = \int_p^q \kappa$, where $\kappa := \pi_{C_*}\pi_X^*\omega$ is a d-closed 1-current of type $(j-m+1, m-j)$ as integration along the $(m-1)$-dimensional fibers of π_C eats up $(m-1, m-1)$. So $\kappa = 0$ unless $j = m$, and by a standard regularity theorem in that case κ is holomorphic. In particular, if C is rational, we have $\int_\Gamma \omega = 0$. This is essentially the reasoning behind the following result:

PROPOSITION 7. *The Abel–Jacobi map*

$$CH^m(X)_{\text{hom}} \xrightarrow{AJ} \frac{\left(F^m H^{2m-1}(X,\mathbb{C})\right)^\vee}{\int_{H_{2m-1}(X,\mathbb{Z})}(\cdot)} \cong J^m(X) \tag{1-10}$$

induced by $Z = \partial\Gamma \mapsto \int_\Gamma(\cdot)$, *is well-defined and restricts to*

$$CH^m(X)_{\text{alg}} \xrightarrow{AJ_{\text{alg}}} \frac{F^m H^{2m-1}_{\text{hdg}}(X,\mathbb{C})}{\int_{H_{2m-1}(X,\mathbb{Z})}(\cdot)} \cong J^m(H^{2m-1}_{\text{hdg}}(X)) =: J^m_h(X), \tag{1-11}$$

where $H^{2m-1}_{\text{hdg}}(X)$ *is the largest sub-HS of* $H^{2m-1}(X)$ *contained* (*after tensoring with* $\mathbb{C}$) *in* $H^{m-1,m}(X,\mathbb{C}) \oplus H^{m,m-1}(X,\mathbb{C})$. *While* $J^m(X)$ *is in general only a complex torus* (*with respect to the complex structure of Griffiths*), $J^m_h(X)$ *is an abelian variety. Further, assuming a special case of the generalized Hodge conjecture, if* X *is defined over* k *then* $J^m_h(X)$ *and* $AJ_{\text{alg}}(Z)$ *are defined over* $\bar{k}$.

REMARK 8. (i) To see that $J^m_h(X)$ is an abelian variety, one uses the Kodaira embedding theorem: by the Hodge–Riemann bilinear relations, the polarization of $H^{2m-1}(X)$ induces a Kähler metric $h(u,v) = -iQ(u,\bar{v})$ on $J^m_h(X)$ with rational Kähler class.

(ii) The mapping (1-10) is neither surjective nor injective in general, and (1-11) is not injective in general; however, (1-11) is conjectured to be surjective, and regardless of this $J^m_{\text{alg}}(X) := \text{im}(AJ_{\text{alg}}) \subseteq J^m_h(X)$ is in fact a subabelian variety.

(iii) A point in $J^m(X)$ is naturally the invariant of an extension of MHS

$$0 \to (H =) H^{2m-1}(X,\mathbb{Z}(m)) \to E \to \mathbb{Z}(0) \to 0$$

(where the "twist" $\mathbb{Z}(m)$ reduces weight by $2m$, to (-1)). The invariant is evaluated by taking two lifts $\nu_F \in F^0W_0E_{\mathbb{C}}$, $\nu_{\mathbb{Z}} \in W_0E_{\mathbb{Z}}$ of $1 \in \mathbb{Z}(0)$, so that $\nu_F - \nu_{\mathbb{Z}} \in W_0H_{\mathbb{C}}$ is well-defined modulo the span of $F^0W_0H_{\mathbb{C}}$ and $W_0H_{\mathbb{Z}}$ hence is in $J^0(H) \cong J^m(X)$. The resulting isomorphism

$$J^m(X) \cong \text{Ext}^1_{\text{MHS}}(\mathbb{Z}(0), H^{2m-1}(X,\mathbb{Z}(m)))$$

is part of an extension-class approach to AJ maps (and their generalizations) due to Carlson [Ca].

(iv) The Abel–Jacobi map appears in [Gr3].

1.6. Horizontality. Generalizing the setting of § 1.2, let $\mathcal{X}$ be a smooth projective $2m$-fold fibered over a curve S with singular fibers $\{X_{s_i}\}$ each of either

(i) NCD (normal crossing divisor) type: locally $(x_1,\ldots,x_{2m}) \overset{\pi}{\mapsto} \prod_{j=1}^k x_j$; or

(ii) ODP (ordinary double point) type: locally $(\underline{x}) \mapsto \sum_{j=1}^{2m} x_j^2$.

An immediate consequence is that all $T_{s_i} \in \mathrm{Aut}\big(H^{2m-1}(X_{s_0}, \mathbb{Z})\big)$ are *unipotent*: $(T_{s_i} - I)^n = 0$ for $n \geq 2m$ in case (i) or $n \geq 2$ in case (ii). (If all fibers are of NCD type, then we say the family $\{X_s\}$ of $(2m-1)$-folds is *semistable*.)

The Jacobian bundle of interest is $\mathcal{J} := \bigcup_{s \in S^*} J^m(X_s)$ $(\supset \mathcal{J}_{\mathrm{alg}})$. Writing

$$\big\{\mathcal{F}^{(m)} := \mathbb{R}^{2m-1}\pi_* \Omega^{\bullet \geq m}_{\mathcal{X}^*/S^*}\big\} \subset \big\{\mathcal{H} := \mathbb{R}^{2m-1}\pi_* \Omega^{\bullet}_{\mathcal{X}^*/S^*}\big\} \supset \big\{\mathbb{H}_{\mathbb{Z}} := R^{2m-1}\pi_* \mathbb{Z}_{\mathcal{X}^*}\big\},$$

and noting $\mathcal{F}^\vee \cong \mathcal{H}/\mathcal{F}$ via $Q : \mathcal{H}^{2m-1} \times \mathcal{H}^{2m-1} \to \mathcal{O}_{S^*}$, the sequences (1-5) and (1-7), as well as the definitions of NF and topological invariant $[\,\cdot\,]$, all carry over. A normal function of geometric origin, likewise, comes from $\mathfrak{Z} \in Z^m(\mathcal{X})_{\mathrm{prim}}$ with $Z_{s_0} := \mathfrak{Z} \cdot X_{s_0} \equiv_{\mathrm{hom}} 0$ (on X_{s_0}), but now has an additional feature known as *horizontality*, which we now explain.

Working locally over an analytic ball $U \subset S^*$ containing s_0, let

$$\tilde{\omega} \in \Gamma(\mathcal{X}_U, F^{m+1}\Omega^{2m-1}_{\mathcal{X}^\infty})$$

be a "lift" of $\omega(s) \in \Gamma(U, \mathcal{F}^{m+1})$, and $\Gamma_s \in C^{\mathrm{top}}_{2m-1}(X_s; \mathbb{Z})$ be a continuous family of chains with $\partial \Gamma_s = Z_s$. Let P^ε be a path from s_0 to $s_0 + \varepsilon$; then $\hat{\Gamma}^\varepsilon := \bigcup_{s \in P^\varepsilon} \Gamma_s$ has boundary $\Gamma_{s_0+\varepsilon} - \Gamma_{s_0} + \bigcup_{s \in P^\varepsilon} Z_s$, and

$$\begin{aligned}
\left(\frac{\partial}{\partial s}\int_{\Gamma_s} \omega(s)\right)_{s=s_0} &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\Gamma_{s_0+\varepsilon} - \Gamma_{s_0}} \tilde{\omega} \\
&= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\int_{\partial \hat{\Gamma}^\varepsilon} \tilde{\omega} - \int_{s_0}^{s_0+\varepsilon} \int_{Z_s} \omega(s) \right) \\
&= \int_{\Gamma_{s_0}} \big\langle \widetilde{d/dt}, d\tilde{\omega} \big\rangle - \int_{Z_{s_0}} \omega(s_0), \qquad (1\text{-}12)
\end{aligned}$$

where $\pi_* \widetilde{d/dt} = d/dt$ (with $\widetilde{d/dt}$ tangent to $\hat{\Gamma}^\varepsilon$, $\hat{Z}^\varepsilon$).

The Gauss–Manin connection $\nabla : \mathcal{H} \to \mathcal{H} \otimes \Omega^1_{S^*}$ differentiates the periods of cohomology classes (against topological cycles) in families, satisfies Griffiths transversality $\nabla(\mathcal{F}^m) \subset \mathcal{F}^{m-1} \otimes \Omega^1_{S^*}$, and is computed by

$$\nabla\omega = \big[\big\langle \widetilde{d/dt}, d\tilde{\omega} \big\rangle\big] \otimes dt.$$

Moreover, the pullback of any form of type F^m to Z_{s_0} (which is of dimension $m-1$) is zero, so that $\int_{Z_{s_0}} \omega(s_0) = 0$ and $\int_{\Gamma_{s_0}} \nabla\omega$ is well-defined. If $\tilde{\Gamma} \in \Gamma(U, \mathcal{H})$ is any lift of $AJ(\Gamma_s) \in \Gamma(U, \mathcal{J})$, we therefore have

$$Q\big(\nabla_{d/dt}\tilde{\Gamma}, \omega\big) = \frac{d}{ds} Q(\tilde{\Gamma}, \omega) - Q(\tilde{\Gamma}, \nabla\omega) = \frac{d}{ds}\int_{\Gamma_s} \omega - \int_{\Gamma_s} \nabla\omega,$$

which is zero by (1-12) and the remarks just made. We have shown that $\nabla_{d/dt}\tilde{\Gamma}$ kills $\mathcal{F}^{m+1}$, and so $\nabla_{d/dt}\tilde{\Gamma}$ is a local section of $\mathcal{F}^{m-1}$.

DEFINITION 9. A normal function $\nu \in H^0(S^*, \mathcal{J})$ is *horizontal* if $\nabla\tilde{\nu} \in \Gamma(U, \mathcal{F}^{m-1} \otimes \Omega^1_U)$ for any local lift $\tilde{\nu} \in \Gamma(U, \mathcal{H})$. Equivalently, if we set $\mathcal{H}_{\text{hor}} := \ker\big(\mathcal{H} \xrightarrow{\nabla} \frac{\mathcal{H}}{\mathcal{F}^{m-1}} \otimes \Omega^1_{S^*}\big) \supset \mathcal{F}^m =: \mathcal{F}$, $(\mathcal{F}^\vee)_{\text{hor}} := \frac{\mathcal{H}_{\text{hor}}}{\mathcal{F}}$, and $\mathcal{J}_{\text{hor}} := \frac{(\mathcal{F}^\vee)_{\text{hor}}}{\mathbb{H}_\mathbb{Z}}$, then $\text{NF}_{\text{hor}} := H^0(S, \mathcal{J}_{\text{hor}})$.

Much as an AJ image was encoded in a MHS in Remark 8(ii), we may encode horizontal normal functions in terms of variations of MHS. A VMHS $\mathcal{V}/S^*$ consists of a $\mathbb{Z}$-local system $\mathbb{V}$ with an increasing filtration of $\mathbb{V}_\mathbb{Q} := \mathbb{V}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Q}$ by sub- local systems $W_i\mathbb{V}_\mathbb{Q}$, a decreasing filtration of $\mathcal{V}_{(\mathcal{O})} := \mathbb{V}_\mathbb{Q} \otimes_\mathbb{Q} \mathcal{O}_{S^*}$ by holomorphic vector bundles $\mathcal{F}^j (= \mathcal{F}^j\mathcal{V})$, and a connection $\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_{S^*}$ such that $\nabla(\mathbb{V}) = 0$, the fibers $(\mathbb{V}_s, W_\bullet, V_s, F^\bullet_s)$ yield $\mathbb{Z}$-MHS, and $\nabla(\mathcal{F}^j) \subset \mathcal{F}^{j-1} \otimes \Omega^1_{S^*}$ (transversality). (Of course, a VHS is just a VMHS with one nontrivial $\text{Gr}^W_i\,\mathbb{V}_\mathbb{Q}$, and $((\mathbb{H}_\mathbb{Z}, \mathcal{H}, \mathcal{F}^\bullet), \nabla)$ in the geometric setting above gives one.) A horizontal normal function corresponds to an extension

$$0 \to \overbrace{\mathcal{H}(m)}^{\substack{\text{wt.}-1\\ \text{VHS}}} \to \mathcal{E} \to \mathbb{Z}(0)_{S^*} \to 0 \tag{1-13}$$

"varying" the setup of Remark 8(iii), with the transversality of the lift of $\nu_F(s)$ (together with flatness of $\nu_\mathbb{Z}(s)$) reflecting horizontality.

REMARK 10. Allowing the left-hand term of (1-13) to have weight less than -1 yields "higher" normal functions related to families of *generalized* ("higher") algebraic cycles. These have been studied in [DM1; DM2; DK], and will be considered in later sections.

An important result on VHS over a smooth quasiprojective base is that the global sections $H^0(S^*, \mathbb{V})$ (resp. $H^0(S^*, \mathbb{V}_\mathbb{R})$, $H^0(S^*, \mathbb{V}_\mathbb{C})$) span the $\mathbb{Q}$-local system (resp. its tensor product with $\mathbb{R}$, $\mathbb{C}$) of a (necessarily constant) sub-VMHS $\subset \mathcal{V}$, called the *fixed part* $\mathcal{V}_{\text{fix}}$ (with constant Jacobian bundle $\mathcal{J}_{\text{fix}}$).

1.7. Infinitesimal invariant. Given $\nu \in \text{NF}_{\text{hor}}$, the "$\nabla\tilde{\nu}$" for various local liftings patch together after going modulo $\nabla\mathcal{F}^m \subset \mathcal{F}^{m-1} \otimes \Omega^1_{S^*}$. If $\nabla\tilde{\nu} = \nabla f$ for $f \in \Gamma(U, \mathcal{F}^m)$, then the alternate lift $\tilde{\nu} - f$ is flat, i.e., equals $\sum_i c_i\gamma_i$ where $\{\gamma_i\} \subset \Gamma(U, \mathbb{V}_\mathbb{Z})$ is a basis and the c_i are complex constants. Since the composition ($s \in S^*$) $H^{2m-1}(X_s, \mathbb{R}) \hookrightarrow H^{2m-1}(X_s, \mathbb{C}) \twoheadrightarrow \frac{H^{2m-1}(X_s,\mathbb{C})}{F^m}$ is an isomorphism, we may take the $c_i \in \mathbb{R}$, and then they are unique in $\mathbb{R}/\mathbb{Z}$. This implies that $[\nu]$ lies in the torsion group $\ker\big(H^1(\mathbb{H}_\mathbb{Z}) \to H^1(\mathbb{H}_\mathbb{R})\big)$, so that a multiple $N\nu$ lifts to $H^0(S^*, \mathbb{H}_\mathbb{R}) \subset \mathcal{H}_{\text{fix}}$. This motivates the definition of an

infinitesimal invariant

$$\delta\nu \in \mathbb{H}^1\big(S^*, \mathcal{F}^m \xrightarrow{\nabla} \mathcal{F}^{m-1}\otimes\Omega^1_{S^*}\big) \underset{\text{affine}}{\overset{\text{if } S^*}{=\!=}} H^0\left(S, \tfrac{\mathcal{F}^{m-1}\otimes\Omega^1}{\mathcal{F}^m}\right) \tag{1-14}$$

as the image of $\nu \in H^0\big(S^*, \frac{\mathcal{H}_{\text{hor}}}{\mathcal{F}}\big)$ under the connecting homomorphism induced by

$$0 \to \text{Cone}\big(\mathcal{F}^m \xrightarrow{\nabla} \mathcal{F}^{m-1}\otimes\Omega^1\big)[-1] \to \text{Cone}\big(\mathcal{H} \xrightarrow{\nabla} \mathcal{H}\otimes\Omega^1\big)[-1] \to \frac{\mathcal{H}_{\text{hor}}}{\mathcal{F}} \to 0. \tag{1-15}$$

PROPOSITION 11. *If $\delta\nu = 0$, then up to torsion, $[\nu] = 0$ and ν is a (constant) section of $\mathcal{J}_{\text{fix}}$.*

An interesting application to the differential equations satisfied by normal functions is essentially due to Manin [Ma]. For simplicity let $S = \mathbb{P}^1$, and suppose $\mathcal{H}$ is generated by $\omega \in H^0(S^*, \mathcal{F}^{2m-1})$ as a D-module, with monic *Picard–Fuchs operator* $F(\nabla_{\delta_s := s\frac{d}{ds}}) \in \mathbb{C}(\mathbb{P}^1)^*[\nabla_{\delta_s}]$ killing ω. Then its periods satisfy the homogeneous P-F equation $F(\delta_s)\int_{\gamma_i}\omega = 0$, and one can look at the multivalued holomorphic function $Q(\tilde{\nu}, \omega)$ (where Q is the polarization, and $\tilde{\nu}$ is a multivalued lift of ν to $\mathcal{H}_{\text{hor}}/\mathcal{F}$), which in the geometric case is just $\int_{\Gamma_s}\omega(s)$. The resulting equation

$$(2\pi i)^m F(\delta_s) Q(\tilde{\nu}, \omega) =: G(s) \tag{1-16}$$

is called the *inhomogeneous Picard–Fuchs equation* of ν.

PROPOSITION 12. (i) [DM1] *$G \in \mathbb{C}(\mathbb{P}^1)^*$ is a rational function holomorphic on S^*; in the K-motivated setting (taking also $\omega \in H^0(\mathbb{P}^1, \bar{\pi}_*\omega_{\mathcal{X}/\mathbb{P}^1})$, and hence F, over K), $G \in K(\mathbb{P}^1)^*$.*

(ii) [Ma; Gr1] $G \equiv 0 \iff \delta\nu = 0$.

EXAMPLE 13. [MW] The solutions to

$$(2\pi i)^2\Big\{\delta_z^4 - 5z\prod_{\ell=1}^{4}(5\delta_z + \ell)\Big\}(\cdot) = -\frac{15}{4}\sqrt{z}$$

are the membrane integrals $\int_{\Gamma_s}\omega(s)$ for a family of 1-cycles on the mirror quintic family of Calabi–Yau 3-folds. (The family of cycles is actually only well-defined on the double-cover of this family, as reflected by the $\sqrt{z}$.) What makes this example particularly interesting is the "mirror dual" interpretation of the solutions as generating functions of open Gromov–Witten invariants of a fixed Fermat quintic 3-fold.

The horizontality relation $\nabla\tilde{\nu} \in \mathcal{F}^{m-1} \otimes \Omega^1$ is itself a differential equation, and the constraints it puts on ν over higher-dimensional bases will be studied in §5.4–5.

Returning to the setting described in §1.6, there are *canonical extensions* $\mathcal{H}_e$ and $\mathcal{F}_e^\bullet$ of $\mathcal{H}, \mathcal{F}^\bullet$ across the s_i as holomorphic vector bundles or subbundles (reviewed in §2 below); for example, if all fibers are of NCD type then $\mathcal{F}_e^p \cong \mathbb{R}^{2m-1}\bar{\pi}_*\Omega^{\bullet\geq p}_{\mathcal{X}/S}(\log(\mathcal{X}\backslash\mathcal{X}^*))$. Writing[2]

$$\mathbb{H}_{\mathbb{Z},e} := R^{2m-1}\bar{\pi}_*\mathbb{Z}_{\mathcal{X}} \quad \text{and} \quad \mathcal{H}_{e,\text{hor}} := \ker\left\{\mathcal{H}_e \overset{\nabla}{\to} \frac{\mathcal{H}_e}{\mathcal{F}_e^{m-1}} \otimes \Omega^1_S(\log\Sigma)\right\},$$

we have short exact sequences

$$0 \to \mathbb{H}_{\mathbb{Z},e} \to \frac{\mathcal{H}_{e(,\text{hor})}}{\mathcal{F}_e^m} \to \mathcal{J}_{e(,\text{hor})} \to 0 \tag{1-17}$$

and set $ENF_{(hor)} := H^0(S, \mathcal{J}_{e(,\text{hor})})$.

THEOREM 14. (i) $\mathfrak{Z} \in Z^m(\mathcal{X})_{\text{prim}}$ *implies* $N\nu_{\mathfrak{Z}} \in ENF_{\text{hor}}$ *for some* $N \in \mathbb{N}$.
(ii) $\nu \in ENF_{\text{hor}}$ *with* $[\nu]$ *torsion implies* $\delta\nu = 0$.

REMARK 15. (ii) is essentially a consequence of the proof of Corollary 2 in [S2]. For $\nu \in ENF_{\text{hor}}$, $\delta\nu$ lies in the subspace

$$\mathbb{H}^1\big(S, \mathcal{F}^m \overset{\nabla}{\to} \mathcal{F}_e^{m-1} \otimes \Omega^1_S(\log\Sigma)\big),$$

the restriction of

$$\mathbb{H}^1\big(S^*, \mathcal{F}^m \overset{\nabla}{\to} \mathcal{F}^{m-1} \otimes \Omega^1_{S^*}\big) \to H^1(S^*, \mathbb{H}_{\mathbb{C}})$$

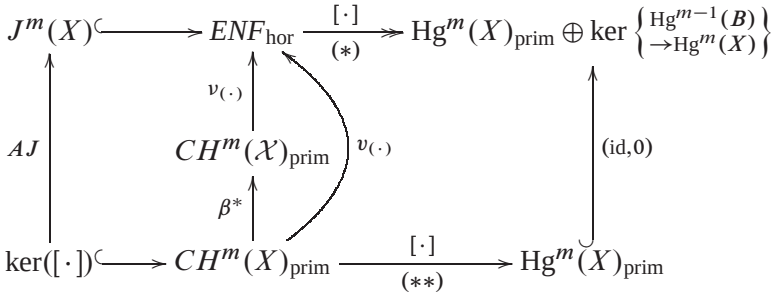
to which is injective.

1.8. The Hodge Conjecture? Putting together Theorem 14(ii) and Proposition 12, we see that a horizontal ENF with trivial topological invariant lies in $H^0(S, \mathcal{J}_{\text{fix}}) =: J^m(\mathcal{X}/S)_{\text{fix}}$ (constant sections). In fact, the long exact sequence associated to (17) yields

$$0 \to J^m(\mathcal{X}/S)_{\text{fix}} \to ENF_{\text{hor}} \overset{[\cdot]}{\to} \frac{\text{Hg}^m(\mathcal{X})_{\text{prim}}}{\text{im}\{\text{Hg}^{m-1}(X_{s_0})\}} \to 0,$$

with $[\nu_{\mathfrak{Z}}] = \overline{[\mathfrak{Z}]}$ (if $\nu_{\mathfrak{Z}} \in ENF$) as before. If $\mathcal{X} \overset{\bar{\pi}}{\to} \mathbb{P}^1 = S$ is a Lefschetz pencil on a $2m$-fold X, this becomes

[2]Warning: while $\mathcal{H}_e$ has no jumps in rank, the stalk of $\mathbb{H}_{\mathbb{Z},e}$ at $s_i \in \Sigma$ is of strictly smaller rank than at $s \in S^*$.



(1-18)

where the surjectivity of $(*)$ is due to Zucker (compare Theorems 31 and 32 in §3 below; his result followed on work of Griffiths and Bloch establishing the surjectivity for *sufficiently ample* Lefschetz pencils). What we are after (modulo tensoring with $\mathbb{Q}$) is surjectivity of the fundamental class map $(**)$. This would clearly follow from surjectivity of $\nu_{(\cdot)}$, i.e., a Poincaré existence theorem, as in §1.4. By Remark 8(ii) this cannot work in most cases; however we have this:

THEOREM 16. *The Hodge Conjecture HC(m,m) is true for X if $J^m(X_{s_0}) = J^m(X_{s_0})_{\text{alg}}$ for a general member of the pencil.*

EXAMPLE 17 [Zu1]. As $J^2 = J^2_{\text{alg}}$ is true for cubic threefolds by the work of Griffiths and Clemens [GC], HC$(2,2)$ holds for cubic fourfolds in $\mathbb{P}^5$.

The Lefschetz paradigm, of taking a 1-parameter family of slices of a primitive Hodge class to get a normal function and constructing a cycle by Jacobi inversion, appears to have led us (for the most part) to a dead end in higher codimension. A beautiful new idea of Griffiths and Green, to be described in §3, replaces the Lefschetz pencil by a complete linear system (of higher degree sections of X) so that $\dim(S) \gg 1$, and proposes to recover algebraic cycles *dual* to the given Hodge class from features of the (admissible) normal function in codimension ≥ 2 on S.

1.9. Deligne cycle-class. This replaces the fundamental and AJ classes by one object. Writing $\mathbb{Z}(m) := (2\pi i)^m\mathbb{Z}$, define the Deligne cohomology of X (smooth projective of any dimension) by

$$H^*_{\mathcal{D}}(X^{\text{an}}, \mathbb{Z}(m)) := H^*\big(\text{Cone}\big\{C^\bullet_{\text{top}}(X^{\text{an}}; \mathbb{Z}(m)) \oplus F^m\mathcal{D}^\bullet(X^{\text{an}}) \to D^\bullet(X^{\text{an}})\big\}[-1]\big),$$

and $c_{\mathcal{D}} : CH^m(X) \to H^{2m}_{\mathcal{D}}(X, \mathbb{Z}(m))$ by $Z \mapsto (2\pi i)^m(Z_{\text{top}}, \delta_Z, 0)$. One easily derives the exact sequence

$$0 \to J^m(X) \to H^{2m}_{\mathcal{D}}(X, \mathbb{Z}(m)) \to \text{Hg}^m(X) \to 0,$$

which invites comparison to the top row of (1-18).

2. Limits and singularities of normal functions

Focusing on the geometric case, we now wish to give the reader a basic intuition for many of the objects — singularities, Néron models, limits of NF's and VHS — which will be treated from a more formal Hodge-theoretic perspective in later sections.[3] The first part of this section (§§2.2–8) considers a cohomologically trivial cycle on a 1-parameter semistably degenerating family of odd-dimensional smooth projective varieties. Such a family has two invariants "at" the central singular fiber:

- the limit of the Abel–Jacobi images of the intersections of the cycle with the smooth fibers, and
- the Abel–Jacobi image of the intersection of the cycle with the singular fiber.

We define what these mean and explain the precise sense in which they agree, which involves limit mixed Hodge structures and the Clemens–Schmid exact sequence, and links limits of AJ maps to the Bloch–Beilinson regulator on higher K-theory.

In the second part, we consider what happens if the cycle is only assumed to be homologically trivial *fiberwise*. In this case, just as the fundamental class of a cycle on a variety must be zero to define its AJ class, the family of cycles has a singularity class which must be zero in order to define the limit AJ invariant. Singularities are first introduced for normal functions arising from families of cycles, and then in the abstract setting of admissible normal functions (and higher normal functions). At the end we say a few words about the relation of singularities to the Hodge conjecture, their rôle in multivariable Néron models, and the analytic obstructions to singularities discovered by M. Saito, topics which §3, §5.1–2, and §5.3–5, respectively, will elaborate extensively upon.

We shall begin by recasting $c_{\mathcal{D}}$ from §1.9 in a more formal vein, which works $\otimes\mathbb{Q}$. The reader should note that henceforth in this paper, we have to introduce appropriate Hodge twists (largely suppressed in §1) into VHS, Jacobians, and related objects.

2.1. AJ map. As we saw earlier (Section 1), the AJ map is the basic Hodge-theoretic invariant attached to a cohomologically trivial algebraic cycle on a smooth projective algebraic variety $X/\mathbb{C}$; say $\dim(X) = 2m-1$. In the diagram that follows, if $cl_{X,\mathbb{Q}}(Z) = 0$ then $Z = \partial\Gamma$ for Γ (say) a rational C^∞ $(2m-1)$-chain on X^{an}, and $\int_\Gamma \in (F^m H^{2m-1}(X,\mathbb{C}))^\vee$ induces $AJ_{X,\mathbb{Q}}(Z)$.

[3] Owing to our desire to limit preliminaries and/or notational complications here, there are a few unavoidable inconsistencies of notation between this and later sections.

$$\begin{array}{ccccc}
 & & \mathrm{Hom}_{\mathrm{MHS}}\big(\mathbb{Q}(0), H^{2m}(X,\mathbb{Q}(m))\big) & = & (H^{2m}(X))^{(m,m)}_{\mathbb{Q}} \\
 & \overset{cl_X}{\nearrow} & \uparrow & & \\
CH^m(X) & \longrightarrow & \mathrm{Ext}^1_{D^b\mathrm{MHS}}\big(\mathbb{Q}(0), \mathcal{K}^\bullet[2m](m)\big) & & \\
\uparrow & & \uparrow & & \\
\ker(cl_X) & \xrightarrow{AJ_X} & \mathrm{Ext}^1_{\mathrm{MHS}}\big(\mathbb{Q}(0), H^{2m-1}(X,\mathbb{Q}(m))\big) & = & J^m(X)_{\mathbb{Q}} \cong \dfrac{\big(F^m H^{2m-1}_{\mathbb{C}}\big)^\vee}{H^{2m-1}_{\mathbb{Q}(m)}}
\end{array} \tag{2-1}$$

The middle term in the vertical short-exact sequence is isomorphic to Deligne cohomology and Beilinson's absolute Hodge cohomology $H^{2m}_{\mathcal{H}}(X^{\mathrm{an}},\mathbb{Q}(m))$, and can be regarded as the ultimate strange fruit of Carlson's work on extensions of mixed Hodge structures. Here $\mathcal{K}^\bullet$ is a canonical complex of MHS quasi-isomorphic (noncanonically) to $\bigoplus_i H^i(X)[-i]$, constructed from two general configurations of hyperplane sections $\{H_i\}_{i=0}^{2m-1}$, $\{\tilde{H}_j\}_{j=0}^{2m-1}$ of X. More precisely, looking (for $|I|, |J| > 0$) at the corresponding "cellular" cohomology groups

$$C^{I,J}_{H,\tilde{H}}(X) := H^{2m-1}\Big(X\setminus \bigcup_{i\in I} H_i,\ \bigcup_{j\in J} H_j \setminus \cdots; \mathbb{Q}\Big),$$

one sets

$$\mathcal{K}^\ell := \bigoplus_{\substack{I,J \\ |I|-|J|=\ell-2m+1}} C^{I,J}_{H,\tilde{H}}(X);$$

refer to [RS]. (Ignoring the description of $J^m(X)$ and AJ, and the comparisons to $c_{\mathcal{D}}$, $H_{\mathcal{D}}$, all of this works for smooth quasiprojective X as well; the vertical short-exact sequence is true even without smoothness.)

The reason for writing AJ in this way is to make plain the analogy to (2-9) below. We now pass back to $\mathbb{Z}$-coefficients.

2.2. AJ in degenerating families. To let $AJ_X(Z)$ vary with respect to a parameter, consider a semistable degeneration (SSD) over an analytic disk

$$\begin{array}{ccccc}
\mathcal{X}^* & \hookrightarrow & \mathcal{X} & \hookleftarrow_{\iota_0} & X_0 = \bigcup_i Y_i \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \bar{\pi}} & & \downarrow \\
\Delta^* & \xrightarrow{J} & \Delta & \hookleftarrow & \{0\}
\end{array} \tag{2-2}$$

where X_0 is a reduced NCD with smooth irreducible components Y_i, $\mathcal{X}$ is smooth of dimension $2m$, $\bar{\pi}$ is proper and holomorphic, and π is smooth. An algebraic cycle $\mathcal{Z} \in Z^m(\mathcal{X})$ properly intersecting fibers gives rise to a family

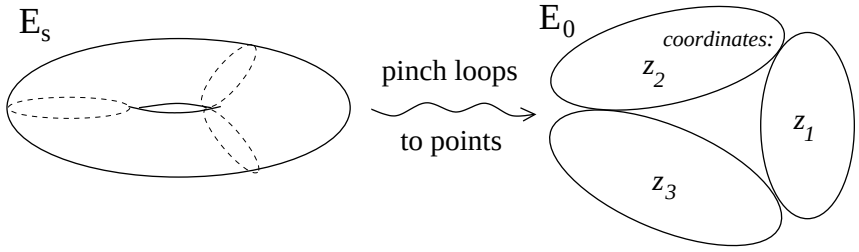
$$Z_s := \mathcal{Z}\cdot X_s \in Z^m(X_s), \qquad s\in\Delta.$$

Assume $0 = [\mathfrak{Z}] \in H^{2m}(\mathcal{X})$ [which implies $0 = [Z_s] \in H^{2m}(X_s)$]; then is there a sense in which

$$\lim_{s\to 0} AJ_{X_s}(Z_s) = AJ_{X_0}(Z_0)? \tag{2-3}$$

(Of course, we have yet to say what either side means.)

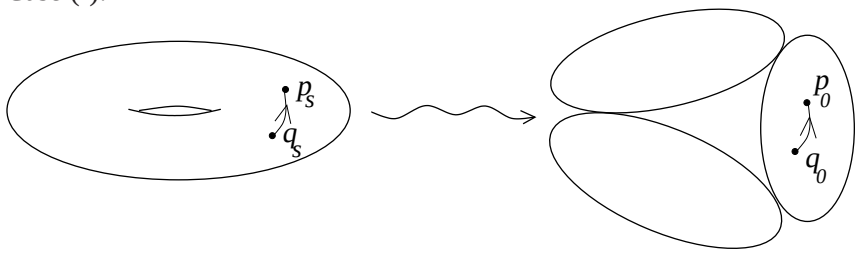
2.3. Classical example. Consider a degeneration of elliptic curves E_s which pinches 3 loops in the same homology class to points, yielding for E_0 three $\mathbb{P}^1$'s joined at 0 and ∞ (called a "Néron 3-gon" or "Kodaira type I_3" singular fiber).



Denote the total space by $\mathcal{E} \xrightarrow{\bar{\pi}} \Delta$. One has a family of holomorphic 1-forms $\omega_s \in \Omega^1(E_s)$ limiting to $\{\mathrm{dlog}(z_j)\}_{j=1}^3$ on E_0; this can be thought of as a holomorphic section of $R^0\bar{\pi}_*\Omega^1_{\mathcal{E}/\Delta}(\log E_0)$.

There are two distinct possibilities for limiting behavior when $Z_s = p_s - q_s$ is a difference of points. (These do not include the case where one or both of p_0, q_0 lies in the intersection of two of the $\mathbb{P}^1$'s, since in that case $\mathfrak{Z}$ is not considered to properly intersect X_0.)

Case (I):



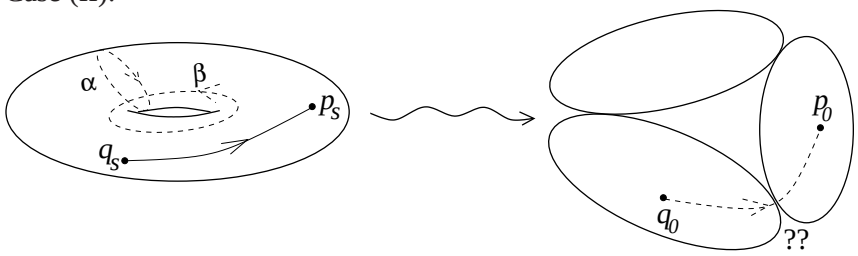
Here p_0 and q_0 lie in the same $\mathbb{P}^1$ (the $j = 1$ component, say): in which case

$$AJ_{E_s}(Z_s) = \int_{q_s}^{p_s} \omega_s \in \mathbb{C}/\mathbb{Z}\left\langle \textstyle\int_{\alpha_s}\omega_s, \int_{\beta_s}\omega_s \right\rangle$$

limits to

$$\int_{p_0}^{q_0} \mathrm{dlog}(z_1) = \log\frac{z_1(p_0)}{z_1(q_0)} \in \mathbb{C}/2\pi i\mathbb{Z}.$$

Case (II):



In this case, p_0 and q_0 lie in different $\mathbb{P}^1$ components, in which case $0 \neq [Z_0] \in H^2(X_0)$ [which implies $[\mathfrak{Z}] \neq 0$] and we say that $AJ(Z_0)$ is "obstructed".

2.4. Meaning of the LHS of (2-3**).** If we assume only that $0 = [\mathfrak{Z}^*] \in H^{2m}(\mathcal{X}^*)$, then

$$AJ_{X_s}(Z_s) \in J^m(X_s) \tag{2-4}$$

is defined for each $s \in \Delta^*$. We can make this into a horizontal, holomorphic section of a bundle of intermediate Jacobians, which is what we shall mean henceforth by a *normal function* (on Δ^* in this case).

Recall the ingredients of a variation of Hodge structure (VHS) over Δ^*:

$$\mathcal{H} = ((\mathbb{H}, \mathcal{H}_{\mathcal{O}}, \mathcal{F}^\bullet), \nabla), \quad \nabla \mathcal{F}^p \subset \mathcal{F}^{p-1} \otimes \Omega^1_S, \quad 0 \to \mathbb{H} \to \frac{\mathcal{H}_{\mathcal{O}}}{\mathcal{F}^m} \to \mathcal{J} \to 0,$$

where $\mathbb{H} = R^{2m-1}\pi_*\mathbb{Z}(m)$ is a local system, $\mathcal{H}_{\mathcal{O}} = \mathbb{H} \otimes_{\mathbb{Z}} \mathcal{O}_{\Delta^*}$ is [the sheaf of sections of] a holomorphic vector bundle with holomorphic subbundles $\mathcal{F}^\bullet$, and these yield HS's H_s fiberwise (notation: $H_s = (\mathbb{H}_s, H_{s(,\mathbb{C})}, F_s^\bullet)$). Henceforth we shall abbreviate $\mathcal{H}_{\mathcal{O}}$ to $\mathcal{H}$.

Then (2-4) yields a section of the intermediate Jacobian bundle

$$\nu_{\mathfrak{Z}} \in \Gamma(\Delta^*, \mathcal{J}).$$

Any holomorphic vector bundle over Δ^* is trivial, each trivialization inducing an extension to Δ. The extensions we want are the "canonical" or "privileged" ones (denoted $(\cdot)_e$); as in § 1.7, we define an extended Jacobian bundle $\mathcal{J}_e$ by

$$0 \to \jmath_*\mathbb{H} \to \frac{\mathcal{H}_e}{\mathcal{F}_e^m} \to \mathcal{J}_e \to 0. \tag{2-5}$$

THEOREM 18 [EZ]. *There exists a holomorphic* $\bar{\nu}_{\mathfrak{Z}} \in \Gamma(\Delta, \mathcal{J}_e)$ *extending* $\nu_{\mathfrak{Z}}$.

Define $\lim_{s\to 0} AJ_{X_S}(Z_s) := \bar{\nu}_{\mathfrak{Z}}(0)$ in $(\mathcal{J}_e)_0$, the fiber over 0 of the Jacobian bundle. To be precise: since $H^1(\Delta, \jmath_*\mathbb{H}) = \{0\}$, we can lift the $\bar{\nu}_{\mathfrak{Z}}$ to a section of the middle term of (2-5), i.e., of a vector bundle, evaluate at 0, then quotient by $(\jmath_*\mathbb{H})_0$.

2.5. Meaning of the RHS of (2-3**).** Higher Chow groups

$$CH^p(X,n) := \frac{\left\{\begin{array}{c}\text{“admissible, closed” codimension p}\\ \text{algebraic cycles on } X\times\mathbb{A}^n\end{array}\right\}}{\text{“higher” rational equivalence}}$$

were introduced by Bloch to compute algebraic K_n-groups of X, and come with “regulator maps” $\mathrm{reg}^{p,n}$ to generalized intermediate Jacobians

$$J^{p,n}(X) := \frac{H^{2p-n-1}(X,\mathbb{C})}{F^pH^{2p-n-1}(X,\mathbb{C}) + H^{2p-n-1}(X,\mathbb{Z}(p))}.$$

(Explicit formulas for $\mathrm{reg}^{p,n}$ have been worked out by the first author with J. Lewis and S. Müller-Stach in [KLM].) The singular fiber X_0 has motivic cohomology groups $H^*_{\mathcal{M}}(X_0,\mathbb{Z}(\cdot))$ built out of higher Chow groups on the substrata

$$Y^{[\ell]} := \amalg_{|I|=\ell+1}Y_I := \amalg_{|I|=\ell+1}(\bigcap_{i\in I}Y_i),$$

(which yield a semi-simplicial resolution of X_0). Inclusion induces

$$\iota_0^*:\ CH^m(\mathcal{X})_{\mathrm{hom}} \to H^{2m}_{\mathcal{M}}(X_0,\mathbb{Z}(m))_{\mathrm{hom}}$$

and we define $Z_0 := \iota_0^*\mathfrak{Z}$. The AJ map

$$AJ_{X_0}:\ H^{2m}_{\mathcal{M}}(X_0,\mathbb{Z}(m))_{\mathrm{hom}} \to J^m(X_0) := \frac{H^{2m-1}(X_0,\mathbb{C})}{\left\{\begin{array}{c}F^mH^{2m-1}(X_0,\mathbb{C})+\\ H^{2m-1}(X_0,\mathbb{Z}(m))\end{array}\right\}}$$

is built out of regulator maps on substrata, in the sense that the semi-simplicial structure of X_0 induces “weight” filtrations $M_\bullet$ on both sides[4] and

$$\mathrm{Gr}^M_{-\ell}\,H^{2m}_{\mathcal{M}}(X_0,\mathbb{Z}(m))_{\mathrm{hom}} \overset{\mathrm{Gr}^M_{-\ell}AJ}{\longrightarrow} \mathrm{Gr}^M_{-\ell}\,J^m(X_0)$$

boils down to

$$\{\text{subquotient of } CH^m(Y^{[\ell]},\ell)\} \overset{\mathrm{reg}^{m,\ell}}{\longrightarrow} \{\text{subquotient of } J^{m,\ell}(Y^{[\ell]})\}.$$

[4]For the advanced reader, we note that if $M_\bullet$ is Deligne's weight filtration on $H^{2m-1}(X_0,\mathbb{Z}(m))$, then $M_{-\ell}J^m(X_0) := \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(0), M_{-\ell-1}H^{2m-1}(X_0,\mathbb{Z}(m)))$. The definition of the $M_\bullet$ filtration on motivic cohomology is much more involved, and we must refer the reader to [GGK, sec. III.A].

2.6. Meaning of equality in (2-3**).** Specializing (2-5) to 0, we have

$$(\bar{\nu}_3(0) \in)\ J^m_{\lim}(X_s) := (\mathcal{J}_e)_0 = \frac{(\mathcal{H}_e)_0}{(\mathcal{F}^m_e)_0 + (_J*\mathbb{H})_0},$$

where $(_J*\mathbb{H})_0$ are the monodromy invariant cycles (and we are thinking of the fiber $(\mathcal{H}_e)_0$ over 0 as the limit MHS of $\mathcal{H}$, see next subsection). H. Clemens [Cl1] constructed a retraction map $\mathfrak{r} : \mathcal{X} \twoheadrightarrow X_0$ inducing

$$\begin{array}{ccc} H^{2m-1}(X_0,\mathbb{Z}) & \xrightarrow{\mathfrak{r}^*} & H^{2m-1}(\mathcal{X},\mathbb{Z}) \\ & \searrow^{\mu} & \downarrow \\ & & \Gamma(\Delta^*,\mathbb{H}) \\ & & \downarrow \\ & & \Gamma(\Delta,{}_J*\mathbb{H}) \\ & & \downarrow \\ & & (_J*\mathbb{H})_0 \\ & & \downarrow \\ & & H^{2m-1}_{\lim}(X_s,\mathbb{Z}) \end{array} \tag{2-6}$$

(where μ is a morphism of MHS), which in turn induces

$$J(\mu) :\ J^m(X_0) \to J^m_{\lim}(X_s).$$

THEOREM 19 [GGK]. $\lim_{s\to 0} AJ_{X_s}(Z_s) = J(\mu)\left(AJ_{X_0}(Z_0)\right)$.

2.7. Graphing normal functions. On Δ^*, let $T : \mathbb{H} \to \mathbb{H}$ be the counterclockwise monodromy transformation, which is unipotent since the degeneration is semistable. Hence the monodromy logarithm

$$N := \log(T) = \sum_{k=1}^{2m-1} \frac{(-1)^{k-1}}{k}(T-I)^k$$

is defined, and we can use it to "untwist" the local system $\otimes\mathbb{Q}$:

$$\mathbb{H}_{\mathbb{Q}} \mapsto \tilde{\mathbb{H}}_{\mathbb{Q}} := \exp\left(-\frac{\log s}{2\pi i}N\right)\mathbb{H}_{\mathbb{Q}} \hookrightarrow \mathcal{H}_e.$$

In fact, this yields a basis for, and defines, the privileged extension $\mathcal{H}_e$. Moreover, since N acts on $\tilde{\mathbb{H}}_{\mathbb{Q}}$, it acts on $\mathcal{H}_e$, and therefore on $(\mathcal{H}_e)_0 = H^{2m-1}_{\lim}(X_s)$, inducing a "weight monodromy filtration" $M_\bullet$. Writing $H = H^{2m-1}_{\lim}(X_s,\mathbb{Q}(m))$, this is the unique filtration $\{0\} \subset M_{-2m} \subset \cdots \subset M_{2m-2} = H$ satisfying

$N(M_k) \subset M_{k-2}$ and $N^k : \mathrm{Gr}^M_{-1+k} H \overset{\cong}{\to} \mathrm{Gr}^M_{-1-k} H$ for all k. In general it is centered about the weight of the original variation (cf. the convention in the Introduction).

EXAMPLE 20. In the "Dehn twist" example of § 1.2, $N = T - I$ (with $N(\alpha) = 0$, $N(\beta) = \alpha$) so that $\tilde{\alpha} = \alpha$, $\tilde{\beta} = \beta - \frac{\log s}{2\pi i}\alpha$ are monodromy free and yield an $\mathcal{O}_\Delta$-basis of $\mathcal{H}_e$. We have $M_{-3} = \{0\}$, $M_{-2} = M_{-1} = \langle\alpha\rangle$, $M_0 = H$.

REMARK 21. Rationally, $\ker(N) = \ker(T - I)$ even when $N \neq T - I$.

By [Cl1], μ maps $H^{2m-1}(X_0)$ onto $\ker(N) \subset H^{2m-1}_{\lim}(X_s)$ and is compatible with the two $M_\bullet$'s; together with Theorem 19 this implies

THEOREM 22. $\lim_{s\to 0} AJ_{X_s}(Z_s) \in J^m(\ker(N))$ $(\subset J^m_{\lim}(X_s))$. (Here we really mean $\ker(T - I)$ so that J^m is defined integrally.)

Two remarks:

- This was not visible classically for curves ($J^1(\ker(N)) = J^1_{\lim}(X_s)$).
- Replacing $(\mathcal{J}_e)_0$ by $J^m(\ker(N))$ yields $\mathcal{J}'_e$, which is a "slit-analytic[5] Hausdorff topological space" ($\mathcal{J}_e$ is non-Hausdorff because in the quotient topology there are nonzero points in $(\mathcal{J}_e)_0$ that look like limits of points in the zero-section of $\mathcal{J}_e$, hence cannot be separated from $0 \in (\mathcal{J}_e)_0$.[6]) This is the correct extended Jacobian bundle for graphing "unobstructed" (in the sense of the classical example) or "singularity-free" normal functions. Call this the "pre-Néron-model".

2.8. Nonclassical example. Take a degeneration of Fermat quintic 3-folds

$$\mathcal{X} \;=\; \text{semistable reduction of } \left\{ s \sum_{j=1}^{4} z_j^5 = \prod_{k=0}^{4} z_k \right\} \;\subset\; \mathbb{P}^4 \times \Delta,$$

so that X_0 is the union of 5 $\mathbb{P}^3$'s blown up along curves isomorphic to $C = \{x^5 + y^5 + z^5 = 0\}$. Its motivic cohomology group $H^4_{\mathcal{M}}(X_0, \mathbb{Q}(2))_{\mathrm{hom}}$ has Gr^M_0 isomorphic to 10 copies of $\mathrm{Pic}^0(C)$, Gr^M_{-1} isomorphic to 40 copies of $\mathbb{C}^*$, $\mathrm{Gr}^W_{-2} = \{0\}$, and $\mathrm{Gr}^M_{-3} \cong K_3^{\mathrm{ind}}(\mathbb{C})$. One has a commuting diagram

$$\begin{array}{ccccc} H^4_{\mathcal{M}}(X_0, \mathbb{Q}(2))_{\mathrm{hom}} & \xrightarrow{AJ_{X_0}} & J^2(X_0)_{\mathbb{Q}} & = & J^2(\ker(N))_{\mathbb{Q}} \\ \uparrow & & \uparrow & & \\ K_3^{\mathrm{ind}}(\mathbb{C}) & \xrightarrow{\mathrm{reg}^{2,3}} & \mathbb{C}/(2\pi i)^2\mathbb{Q} & \xrightarrow{\mathrm{Im}} & \mathbb{R} \end{array} \tag{2-7}$$

[5] That is, each point has a neighborhood of the form: open ball about $\underline{0}$ in $\mathbb{C}^{a+b}$ intersected with $((\mathbb{C}^a \backslash \{\underline{0}\}) \times \mathbb{C}^b) \cup (\{\underline{0}\} \times \mathbb{C}^c)$, where $c \leq b$.

[6] See the example before Theorem II.B.9 in [GGK].

and explicit computations with higher Chow precycles in [GGK, § 4] lead to the result:

THEOREM 23. *There exists a family of* 1*-cycles* $\mathfrak{Z} \in CH^2(\mathcal{X})_{\hom,\mathbb{Q}}$ *such that* $Z_0 \in M_{-3}H^4_{\mathcal{M}}$ *and* $\operatorname{Im}(AJ_{X_0}(Z_0)) = D_2(\sqrt{-3})$, *where* D_2 *is the Bloch–Wigner function*.

Hence, $\lim_{s\to 0} AJ_{X_s}(Z_s) \neq 0$ and so the general Z_s in this family is not rationally equivalent to zero. The main idea is that the family of cycles limits to a (nontrivial) higher cycle in a substratum of the singular fiber.

2.9. Singularities in 1 parameter. If only $[Z_s] = 0$ ($s \in \Delta^*$), and $[\mathfrak{Z}^*] = 0$ *fails*, then

$$\lim_{s\to 0} AJ \text{ is obstructed}$$

and we say $\bar{\nu}_{\mathfrak{Z}}(s)$ has a singularity (at $s = 0$), measured by the finite group

$$G \cong \frac{\operatorname{Im}(T_{\mathbb{Q}} - I) \cap \mathbb{H}_{\mathbb{Z}}}{\operatorname{Im}(T_{\mathbb{Z}} - I)} = \begin{cases} \mathbb{Z}/3\mathbb{Z} \text{ in the classical example,} \\ (\mathbb{Z}/5\mathbb{Z})^3 \text{ in the nonclassical one.} \end{cases}$$

(The $(\mathbb{Z}/5\mathbb{Z})^3$ is generated by differences of lines limiting to distinct components of X_0.) The Néron model is then obtained by replacing $J(\ker(N))$ (in the pre-Néron-model) by its product with G (this will graph *all* admissible normal functions, as defined below).

The next example demonstrates the "finite-group" (or torsion) nature of singularities in the 1-parameter case. In § 2.10 we will see how this feature disappears when there are many parameters.

EXAMPLE 24. Let $\xi \in \mathbb{C}$ be general and fixed. Then

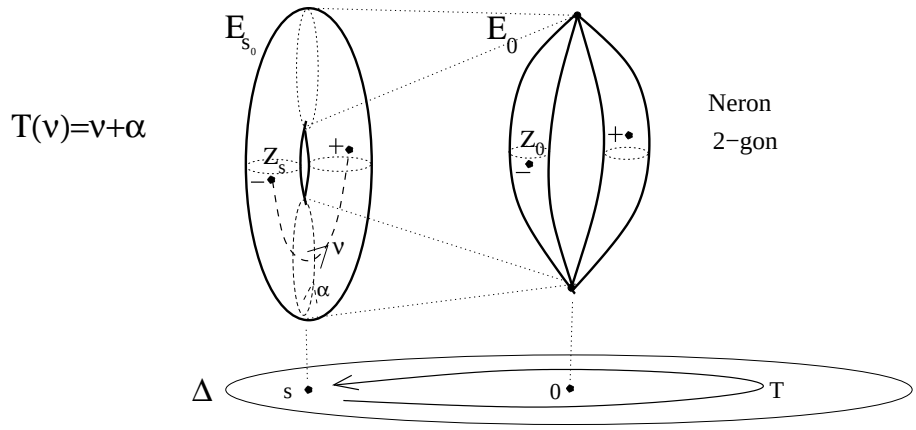
$$C_s = \{x^2 + y^2 + s(x^2y^2 + \xi) = 0\}$$

defines a family of elliptic curves (in $\mathbb{P}^1 \times \mathbb{P}^1$) over Δ^* degenerating to a Néron 2-gon at $s = 0$. The cycle

$$Z_s := \left(i\sqrt{\frac{1+\xi s}{1+s}}, 1\right) - \left(-i\sqrt{\frac{1+\xi s}{1+s}}, 1\right)$$

is nontorsion, with points limiting to distinct components. (See figure on next page.)

Hence, $AJ_{C_s}(Z_s) =: \nu(s)$ limits to the nonidentity component ($\cong \mathbb{C}^*$) of the Néron model. The presence of the nonidentity component removes the obstruction (observed in case (II) of § 2.3) to graphing ANFs with singularities.



Two remarks:

- By tensoring with $\mathbb{Q}$, we can "correct" this: write α, β for a basis for $H^1(C_s)$ and N for the monodromy log about 0, which sends $\alpha \mapsto 0$ and $\beta \mapsto 2\alpha$. Since $N(\nu) = \alpha = N(\frac{1}{2}\beta)$, $\nu - \frac{1}{2}\beta$ will pass through the identity component (which becomes isomorphic to $\mathbb{C}/\mathbb{Q}(1)$ after tensoring with $\mathbb{Q}$, however).
- Alternately, to avoid tensoring with $\mathbb{Q}$, one can add a 2-torsion cycle like

$$T_s := (i\xi^{\frac{1}{4}}, \xi^{\frac{1}{4}}) - (-i\xi^{\frac{1}{4}}, -\xi^{\frac{1}{4}}).$$

2.10. Singularities in 2 parameters.

EXAMPLE 25. Now we will effectively allow ξ (from the last example) to vary: consider the smooth family

$$C_{s,t} := \{x^2 + y^2 + sx^2y^2 + t = 0\}$$

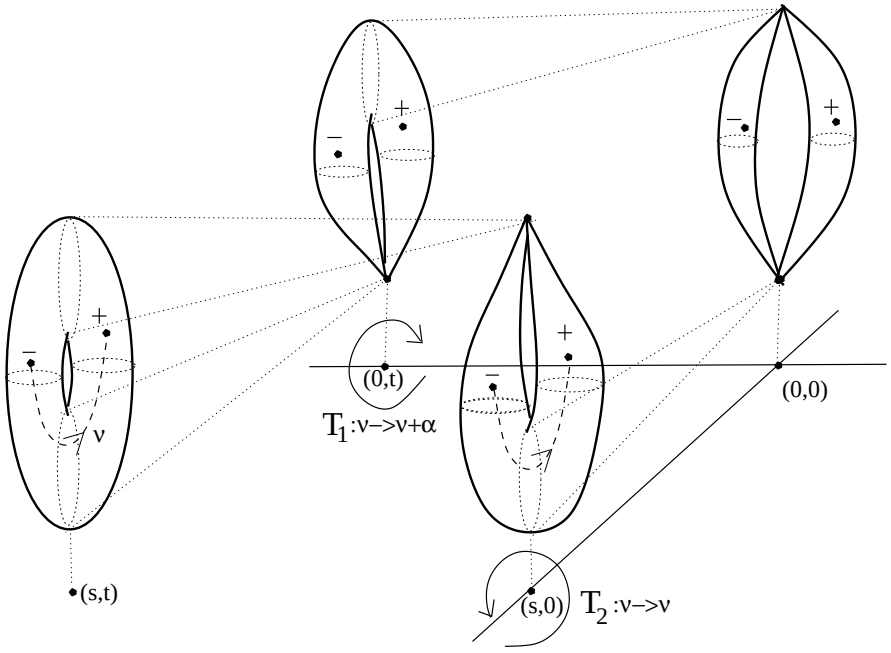
over $(\Delta^*)^2$. The degenerations $t \to 0$ and $s \to 0$ pinch physically distinct cycles in the same homology class to zero, so that $C_{0,0}$ is an I_2; we have obviously that $N_1 = N_2$ (both send $\beta \mapsto \alpha \mapsto 0$). Take

$$Z_{s,t} := \left(i\sqrt{\frac{1+t}{1+s}}, 1\right) - \left(-i\sqrt{\frac{1+t}{1+s}}, 1\right)$$

for our family of cycles, which splits between the two components of the I_2 at $(0,0)$. See figure at top of next page.

Things go much more wrong here. Here are 3 ways to see this:

- try to correct monodromy (as we did in Example 24 with $-\frac{1}{2}\beta$): $N_1(\nu) = \alpha$, $N_1(\beta) = \alpha$, $N_2(\nu) = 0$, $N_2(\beta) = \alpha$ implies an impossibility;
- in T_s (from Example 1), $\xi^{1/4}$ becomes (here) $(t/s)^{1/4}$ — so its obvious extension isn't well-defined. In fact, there is *no* 2-torsion family of cycles with fiber over $(0,0)$ a difference of two points in the two distinct components of



$C_{0,0}$ (that is, one that limits to have the same cohomology class in $H^2(C_{0,0})$ as $Z_{0,0}$).

- take the "motivic limit" of AJ at $t = 0$: under the uniformization of $C_{s,0}$ by

$$\mathbb{P}^1 \ni z \longmapsto \left(\frac{2z}{1-sz^2}, \frac{2iz}{1+sz^2}\right),$$
$$\left(\frac{i}{s}(1+\sqrt{1+s})\right) - \left(\frac{i}{s}(1-\sqrt{1+s})\right) \longmapsto Z_{s,0}.$$

Moreover, the isomorphism $\mathbb{C}^* \cong K_1(\mathbb{C}) \cong M_{-1}H^2_{\mathcal{M}}(C_{s,0}, \mathbb{Z}(1))$ $(\ni Z_{s,0})$ sends

$$\frac{1+\sqrt{1+s}}{1-\sqrt{1+s}} \in \mathbb{C}^*$$

to $Z_{s,0}$, and at $s = 0$ (considering it as a precycle in $Z^1(\Delta, 1)$) this obviously has a residue.

The upshot is that *nontorsion* singularities appear in codimension 2 and up.

2.11. Admissible normal functions. We now pass to the abstract setting of a complex analytic manifold $\bar{S}$ (for example a polydisk or smooth projective variety) with Zariski open subset S, writing $D = \bar{S} \setminus S$ for the complement. Throughout, we shall assume that $\pi_0(S)$ is finite and $\pi_1(S)$ is finitely generated. Let $\mathcal{V} = (\mathbb{V}, \mathcal{V}_{(\mathcal{O})}, \mathcal{F}^\bullet, W_\bullet)$ be a variation of MHS over S.

Admissibility is a condition which guarantees (at each $x \in D$) a well-defined limit MHS for $\mathcal{V}$ up to the action $\mathcal{F}^\bullet \mapsto \exp(\lambda \log T)\mathcal{F}^\bullet$ ($\lambda \in \mathbb{C}$) of local unipotent monodromies $T \in \rho(\pi_1(U_x \cap S))$. If D is a divisor with local normal crossings at x, and $\mathcal{V}$ is admissible, then a choice of coordinates $s_1, \ldots, s_m$ on an analytic neighborhood $U = \Delta^k$ of x (with $\{s_1 \cdots s_m = 0\} = D$) produces the LMHS $(\psi_{\underline{s}}\mathcal{V})_x$. Here we shall only indicate what admissibility, and this LMHS, is in two cases: variations of pure HS, and generalized normal functions (cf. Definition 26).

As a consequence of Schmid's nilpotent- and SL_2-orbit theorems, pure variation is always admissible. If $\mathcal{V} = \mathcal{H}$ is a pure variation in one parameter, we have (at least in the unipotent case) already defined "$H_{\lim}$" and now simply replace that notation by "$(\psi_{\underline{s}}\mathcal{H})_x$". In the multiple parameter (or nonunipotent) setting, simply pull the variation back to an analytic curve $\Delta^* \to (\Delta^*)^m \times \Delta^{k-m} \subset S$ whose closure passes through x, and take the LMHS of that. The resulting $(\psi_{\underline{s}}\mathcal{H})_x$ is independent of the choice of curve (up to the action of local monodromy mentioned earlier). In particular, letting $\{N_i\}$ denote the local monodromy logarithms, the weight filtration $M_\bullet$ on $(\psi_{\underline{s}}\mathcal{H})_x$ is just the weight monodromy filtration attached to their sum $N := \sum a_i N_i$ (where the $\{a_i\}$ are arbitrary positive integers).

Now let $r \in \mathbb{N}$.

DEFINITION 26. A (*higher*) *normal function* over S is a VMHS of the form $\mathcal{V}$ in (the short-exact sequence)

$$0 \to \mathcal{H} \longrightarrow \mathcal{V} \longrightarrow \mathbb{Z}_S(0) \to 0 \tag{2-8}$$

where $\mathcal{H}$ is a [pure] VHS of weight $(-r)$ and the [trivial, constant] variation $\mathbb{Z}_S(0)$ has trivial monodromy. (The terminology "higher" only applies when $r > 1$.) This is equivalent to a holomorphic, horizontal section of the generalized Jacobian bundle

$$J(\mathcal{H}) := \frac{\mathcal{H}}{\mathcal{F}^0\mathcal{H} + \mathbb{H}_{\mathbb{Z}}}.$$

EXAMPLE 27. Given a smooth proper family $\mathcal{X} \xrightarrow{\pi} S$, with $x_0 \in S$. A higher algebraic cycle $\mathfrak{Z} \in CH^p(\mathcal{X}, r-1)_{\mathrm{prim}} := \ker\{CH^p(\mathcal{X}, r-1) \to CH^p(X_{x_0}, r-1) \to \mathrm{Hg}^{p,r-1}(X_{x_0})\}$ yields a section of $J(R^{2p-r}\pi_*\mathbb{C} \otimes \mathcal{O}_S) =: \mathcal{J}^{p,r-1}$; this is what we shall mean by a *(higher) normal function of geometric origin.*[7] (The notion of *motivation over K* likewise has an obvious extension from the classical 1-parameter case in §1.)

[7] Note that $\mathrm{Hg}^{p,r-1}(X_{x_0})_{\mathbb{Q}} := H^{2p-r+1}(X_{x_0}, \mathbb{Q}(p)) \cap F^p H^{2p-r+1}(X_{x_0}, \mathbb{C})$ is actually zero for $r > 1$, so that the "prim" comes for free for some multiple of $\mathfrak{Z}$.

We now give the definition of admissibility for VMHS of the form in Definition 26 (but simplifying to $D = \{s_1 \cdots s_k = 0\}$), starting with the local unipotent case. For this we need Deligne's definition [De1] of the $I^{p,q}(H)$ of a MHS H, for which the reader may refer to Theorem 68 (in §4) below. To simplify notation, we shall abbreviate $I^{p,q}(H)$ to $H^{(p,q)}$, so that, for instance, $H_{\mathbb{Q}}^{(p,p)} = I^{p,p}(H) \cap H_{\mathbb{Q}}$, and drop the subscript x for the LMHS notation.

DEFINITION 28. Let $S = (\Delta^*)^k$, $\mathcal{V} \in \mathrm{NF}^r(S, \mathcal{H})_{\mathbb{Q}}$ (i.e., as in Definition 26, $\otimes\mathbb{Q}$), and $x = (\underline{0})$.

(I) [unipotent case] Assume the monodromies T_i of $\mathbb{H}$ are unipotent, so that the logarithms N_i and associated monodromy weight filtrations $M_\bullet^{(i)}$ are defined. (Note that the $\{N_i\}$ resp. $\{T_i\}$ automatically commute, since any local system must be a representation of $\pi_1((\Delta^*)^k)$, an abelian group.) We may "untwist" the local system $\otimes\mathbb{Q}$ via

$$\tilde{\mathbb{V}} := \exp\left(\frac{-1}{2\pi\sqrt{-1}}\sum_i \log(s_i)N_i\right)\mathbb{V}_{(\mathbb{Q})},$$

and set $\mathcal{V}_e := \tilde{\mathbb{V}} \otimes \mathcal{O}_{\Delta^k}$ for the Deligne extension. Then $\mathcal{V}$ is ($\bar{S}$-)admissible if and only if

(a) $\mathcal{H}$ is polarizable,

(b) there exists a lift $\nu_{\mathbb{Q}} \in (\tilde{\mathbb{V}})_0$ of $1 \in \mathbb{Q}(0)$ such that $N_i\nu_{\mathbb{Q}} \in M_{-2}^{(i)}(\psi_{\underline{s}}\mathcal{H})_{\mathbb{Q}}$ $(\forall i)$, and

(c) there exists a lift $\nu_F(s) \in \Gamma(\overline{S}, \mathcal{V}_e)$ of $1 \in \mathbb{Q}_S(0)$ such that $\nu_F|_S \in \Gamma(S, \mathcal{F}^0)$.

(II) In general there exists a minimal finite cover $\zeta : (\Delta^*)^k \to (\Delta^*)^k$ (sending $\underline{s} \mapsto \underline{s}^{\underline{\mu}}$) such that the $T_i^{\mu_i}$ are unipotent. $\mathcal{V}$ is admissible if and only if $\zeta^*\mathcal{V}$ satisfies (a), (b), and (c).

The main result [K; SZ] is then that $\mathcal{V} \in \mathrm{NF}^r(S, \mathcal{H})^{\mathrm{ad}}_{\bar{S}}$ has well-defined $\psi_{\underline{s}}\mathcal{V}$, given as follows. On the underlying rational structure $(\tilde{\mathbb{V}})_0$ we put the weight filtration $M_i = M_i\psi_{\underline{s}}\mathcal{H} + \mathbb{Q}\langle\nu_{\mathbb{Q}}\rangle$ for $i \geq 0$ and $M_i = M_i\psi_{\underline{s}}\mathcal{H}$ for $i < 0$; while on its complexification ($\cong (\mathcal{V}_e)_0$) we put the Hodge filtration $F^j = F^j\psi_{\underline{s}}\mathcal{H}_{\mathbb{C}} + \mathbb{C}\langle\nu_F(0)\rangle$ for $j \leq 0$ and $F^j = F^j\psi_{\underline{s}}\mathcal{H}$ for $j > 0$. (Here we are using the inclusion $\tilde{\mathbb{H}} \subset \tilde{\mathbb{V}}$, and the content of the statement is that this actually does define a MHS.)

We can draw some further conclusions from (a)–(c) in case (I). With some work, it follows from (c) that

(c′) $\nu_F(0)$ gives a lift of $1 \in \mathbb{Q}(0)$ satisfying $N_i\nu_F(0) \in (\psi_{\underline{s}}\mathcal{H})^{(-1,-1)}$;

and one can also show that $N_i\nu_{\mathbb{Q}} \in M_{-2}(\psi_{\underline{s}}\mathcal{H})_{\mathbb{Q}}$ $(\forall i)$. Furthermore, if $r = 1$ then each $N_i\nu_{\mathbb{Q}}$ [resp. $N_i\nu_F(0)$] belongs to the image under $N_i : \psi_{\underline{s}}\mathcal{H} \to \psi_{\underline{s}}\mathcal{H}(-1)$

of a rational [resp. type-$(0,0)$] element. To see this, use the properties of N_i to deduce that $\mathrm{im}(N_i) \supseteq M^{(i)}_{-r-1}$; then for $r = 1$ we have, from (b) and (c), $N_i v_F(0),\ N_i v_{\mathbb{Q}} \in M^{(i)}_{-2}$.

(III) The definition of admissibility over an arbitrary smooth base S together with good compactification $\bar{S}$ is then local, i.e., reduces to the $(\Delta^*)^k$ setting. Another piece of motivation for the definition of admissibility is this, for which we refer the reader to [BZ, Theorem 7.3]:

THEOREM 29. *Any* (*higher*) *normal function of geometric origin is admissible*.

2.12. Limits and singularities of ANFs. Now the idea of the "limit of a normal function" should be to interpret $\psi_{\underline{s}}\mathcal{V}$ as an extension of $\mathbb{Q}(0)$ by $\psi_{\underline{s}}\mathcal{H}$. The obstruction to being able to do this is the singularity, as we now explain. All MHS in this section are $\mathbb{Q}$-MHS.

According to [BFNP, Corollary 2.9], we have

$$\mathrm{NF}^r(S,\mathcal{H})^{\mathrm{ad}}_{\bar{S}} \otimes \mathbb{Q} \cong \mathrm{Ext}^1_{\mathrm{VMHS}(S)^{\mathrm{ad}}_{\bar{S}}}(\mathbb{Q}(0),\mathcal{H}),$$

as well as an equivalence of categories $\mathrm{VMHS}(S)^{\mathrm{ad}}_{\bar{S}} \simeq \mathrm{MHM}(S)^{\mathrm{ps}}_{\bar{S}}$. We want to push (in a sense canonically extend) our ANF $\mathcal{V}$ into $\bar{S}$ and restrict the result to x. Of course, writing $\jmath : S \hookrightarrow \bar{S}$, $\jmath_*$ is not right exact; so to preserve our extension, we take the derived functor $R\jmath_*$ and land in the derived category $D^b\mathrm{MHM}(\bar{S})$. Pulling back to $D^b\mathrm{MHM}(\{x\}) \cong D^b\mathrm{MHS}$ by $\imath_x^*$, we have defined an invariant $(\imath_x^* R\jmath_*)^{\mathrm{Hdg}}$:

$$\begin{array}{ccc}
 & & \mathrm{Hom}_{\mathrm{MHS}}\left(\mathbb{Q}(0), H^1\mathcal{K}^\bullet\right) \\
 & \overset{\mathrm{sing}_x}{\nearrow} & \uparrow \\
\mathrm{NF}^r(S,\mathcal{H}) & \xrightarrow[(\imath_x^* R\jmath_*)^{\mathrm{Hdg}}]{} & \mathrm{Ext}^1_{D^b\mathrm{MHS}}\left(\mathbb{Q}(0), \mathcal{K}^\bullet := \imath_x^* R\jmath_* \mathcal{H}\right) \\
\uparrow & & \uparrow \\
\ker(\mathrm{sing}_x) & \xrightarrow{\lim_x} & \mathrm{Ext}^1_{\mathrm{MHS}}\left(\mathbb{Q}(0), H^0\mathcal{K}^\bullet\right)
\end{array} \tag{2-9}$$

where the diagram makes a clear analogy to (2-1).

For $S = (\Delta^*)^k$ and $\mathbb{H}_{\mathbb{Z}}$ unipotent we have

$$\mathcal{K}^\bullet \simeq \left\{\psi_{\underline{s}}\mathcal{H} \xrightarrow{\oplus N_i} \bigoplus_i \psi_{\underline{s}}\mathcal{H}(-1) \longrightarrow \bigoplus_{i<j} \psi_{\underline{s}}\mathcal{H}(-2) \longrightarrow \cdots\right\},$$

and the map

$$\mathrm{sing}_x : \mathrm{NF}^r((\Delta^*)^k,\mathcal{H})^{\mathrm{ad}}_{\Delta^k} \to (H^1\mathcal{K}^\bullet)^{(0,0)}_{\mathbb{Q}} \quad (\cong \mathrm{coker}(N)(-1) \text{ for } k = 1)$$

is induced by $\mathcal{V} \mapsto \{N_i \nu_{\mathbb{Q}}\} \equiv \{N_i \nu_F(0)\}$. The limits, which are computed by

$$\lim_x : \ker(\mathrm{sing}_x) \to J\big(\textstyle\bigcap_i \ker(N_i)\big),$$

more directly generalize the 1-parameter picture. The target $J(\bigcap \ker(N_i))$ is exactly what to put in over $\underline{0}$ to get the multivariable pre-Néron-model.

We have introduced the general case $r \geq 1$ because of interesting applications of higher normal functions to irrationality proofs, local mirror symmetry [DK]. In case $r = 1$ — we are dealing with classical normal functions — we can replace R_{J*} in the above by perverse intermediate extension ${}_{J!*}$ (which by a lemma in [BFNP] preserves the extension in this case: see Theorem 46 below). Correspondingly, $\mathcal{K}^\bullet$ is replaced by the local intersection cohomology complex

$$\mathcal{K}^\bullet_{\mathrm{red}} \simeq \Big\{\psi_{\underline{s}}\mathcal{H} \xrightarrow{\oplus N_i} \bigoplus_i \mathrm{Im}(N_i)(-1) \longrightarrow \bigoplus_{i<j} \mathrm{Im}(N_i N_j)(-2) \to \cdots\Big\};$$

while the target for $\lim_x$ is unchanged, the one for sing_x is reduced to 0 if $k = 1$ and to

$$\left(\frac{\ker(N_1) \cap \mathrm{im}(N_2)}{N_2(\ker N_1)}\right)^{(-1,-1)}_{\mathbb{Q}} \tag{2-10}$$

if $k = 2$.

2.13. Applications of singularities. We hint at some good things to come:

(i) Replacing the sing_x-target (e.g., (2-10)) by actual *images* of ANFs, and using their differences to glue pre-Néron components together yields a generalized Néron model (over Δ^r, or $\bar{S}$ more generally) graphing ANFs. Again over x one gets an extension of a discrete (but not necessarily finite) singularity group by the torus $J(\cap \ker(N_i))$. A. Young [Yo] did this for abelian varieties, then [BPS] for general VHS. This will be described more precisely in § 5.2.

(ii) (Griffiths and Green [GG]) The Hodge conjecture (HC) on a $2p$-dimensional smooth projective variety X is equivalent to the following statement for *each* primitive Hodge (p, p) class ζ and very ample line bundle $\mathcal{L} \to X$: there exists $k \gg 0$ such that the natural normal function[8] ν_ζ over $|\mathcal{L}^k| \setminus \hat{X}$ (the complement of the dual variety in the linear system) has a nontorsion singularity at some point of $\hat{X}$. So, in a *sense*, the analog of HC for $(\Delta^*)^k$ is surjectivity of sing_x onto $(H^1\mathcal{K}^\bullet_{\mathrm{red}})^{(0,0)}_{\mathbb{Q}}$, and this *fails*:

(iii) (M. Saito [S6], Pearlstein [Pe3]) Let $\mathcal{H}_0/\Delta^*$ be a VHS of weight 3 rank 4 with nontrivial Yukawa coupling. Twisting it into weight -1, assume the LMHS is of type II_1: $N^2 = 0$, with Gr^M_{-2} of rank 1. Take for $\mathcal{H}/(\Delta^*)^2$ the pullback of $\mathcal{H}_0$ by $(s, t) \mapsto st$. Then (2-10) $\neq \{0\} = \mathrm{sing}_{\underline{0}}\{\mathrm{NF}^1((\Delta^*)^2, \mathcal{H})^{\mathrm{ad}}_{\Delta^2}\}$. The

[8] cf. § 3.2–3, especially (3-5).

obstruction to the existence of normal functions with nontrivial singularity is analytic; and comes from a differential equation produced by the horizontality condition (see §5.4–5).

(iv) One can explain the meaning of the residue of the limit K_1 class in Example 25 above: writing $\jmath^1 : (\Delta^*)^2 \hookrightarrow \Delta^* \times \Delta$, $\jmath^2 : \Delta^* \times \Delta \hookrightarrow \Delta^2$, factor $(\imath_X^* R j_{!*})^{\mathrm{Hdg}}$ by $(\imath_X^* R \jmath_*^2)^{\mathrm{Hdg}} \circ (\imath_{\Delta^*}^* \jmath_{!*}^1)^{\mathrm{Hdg}}$ (where the $\imath^* R j_*^2$ corresponds to the residue). That is, limit a normal function (or family of cycles) to a higher normal function (or family of higher Chow cycles) over a codimension-1 boundary component; the latter can then have (unlike normal functions) a singularity in codimension 1 — i.e., in codimension 2 with respect to the original normal function.

This technique gives a quick proof of the existence of singularities for the Ceresa cycle by limiting it to an Eisenstein symbol (see [Co] and the Introduction to [DK]). Additionally, one gets a geometric explanation of why one does not expect the singularities in (ii) to be supported in high-codimension substrata of $\hat{X}$ (supporting very degenerate hypersurfaces of X): along these substrata one may reach (in the sense of (iv)) higher Chow cycles with rigid AJ invariants, hence no residues. For this reason codimension 2 tends to be a better place to look for singularities than in much higher codimension. These "shallow" substrata correspond to hypersurfaces with ordinary double points, and it was the original sense of [GG] that such points should trace out an algebraic cycle "dual" to the original Hodge class, giving an *effective* proof of the HC.

3. Normal functions and the Hodge conjecture

In this section, we discuss the connection between normal functions and the Hodge conjecture, picking up where §1 left off. We begin with a review of some properties of the Abel–Jacobi map. Unless otherwise noted, all varieties are defined over $\mathbb{C}$.

3.1. Zucker's Theorem on Normal Functions. Let X be a smooth projective variety of dimension d_X. Recall that $J_h^p(X)$ is the intermediate Jacobian associated to the maximal rationally defined Hodge substructure H of $H^{2p-1}(X)$ such that $H_{\mathbb{C}} \subset H^{p,p-1}(X) \oplus H^{p-1,p}(X)$, and that (by a result of Lieberman [Li])

$$J^p(X)_{\mathrm{alg}} = \mathrm{im}\left\{AJ_X : CH^p(X)_{\mathrm{alg}} \to J^p(X)\right\} \quad \text{is a subabelian variety of } J^p(X)_h. \tag{3-1}$$

NOTATION 30. If $f : X \to Y$ is a projective morphism then f^{sm} denotes the restriction of f to the largest Zariski open subset of Y over which f is smooth. Also, unless otherwise noted, in this section, the underlying lattice $\mathbb{H}_{\mathbb{Z}}$ of every variation of Hodge structure is assumed to be torsion free, and

hence for a geometric family $f : X \to Y$, we are really considering $\mathbb{H}_{\mathbb{Z}} = (R^k f_*^{\mathrm{sm}}\mathbb{Z})/\{\text{torsion}\}$.

As reviewed in § 1, Lefschetz proved that every integral $(1,1)$ class on a smooth projective surface is algebraic by studying Poincaré normal functions associated to such cycles. We shall begin here by revisiting Griffiths' program (also recalled in § 1) to prove the Hodge conjecture for higher codimension classes by extending Lefschetz's methods: By induction on dimension, the Hodge conjecture can be reduced to the case of middle-dimensional Hodge classes on even-dimensional varieties [Le1, Lecture 14]. Suppose therefore that $X \subseteq \mathbb{P}^k$ is a smooth projective variety of dimension $2m$. Following [Zu2, § 4], let us pick a Lefschetz pencil of hyperplane sections of X, i.e., a family of hyperplanes $H_t \subseteq \mathbb{P}^k$ of the form $t_0 w_0 + t_1 w_1 = 0$ parametrized by $t = [t_0, t_1] \in \mathbb{P}^1$ relative to a suitable choice of homogeneous coordinates $w = [w_0, \ldots, w_k]$ on $\mathbb{P}^k$ such that:

- for all but finitely many points $t \in \mathbb{P}^1$, the corresponding hyperplane section of $X_t = X \cap H_t$ is smooth;
- the base locus $B = X \cap \{w \in \mathbb{P}^k \mid w_0 = w_1 = 0\}$ is smooth; and
- each singular hyperplane section of X has exactly one singular point, which is an ordinary double point.

Given such a Lefschetz pencil, let

$$Y = \{\, (x,t) \in X \times \mathbb{P}^1 \mid x \in H_t \,\}$$

and let $\pi : Y \to \mathbb{P}^1$ denote projection onto the second factor. Let U denote the set of points $t \in \mathbb{P}^1$ such that X_t is smooth and $\mathcal{H}$ be the variation of Hodge structure over U with integral structure $\mathbb{H}_{\mathbb{Z}} = R^{2m-1}\pi_*^{\mathrm{sm}}\mathbb{Z}(m)$. Furthermore, by Schmid's nilpotent orbit theorem [Sc], the Hodge bundles $\mathcal{F}^\bullet$ have a canonical extension to a system of holomorphic bundles $\mathcal{F}_e^\bullet$ over $\mathbb{P}^1$. Accordingly, we have a short exact sequence of sheaves

$$0 \to j_*\mathbb{H}_{\mathbb{Z}} \to \mathcal{H}_e/\mathcal{F}_e^m \to \mathcal{J}_e^m \to 0, \tag{3-2}$$

where $j : U \to \mathbb{P}^1$ is the inclusion map. As before, let us call an element $\nu \in H^0(\mathbb{P}^1, \mathcal{J}_e^m)$ a Poincaré normal function. Then, we have the following two results [Zu2, Thms. 4.57, 4.17], the second of which is known as the Theorem on Normal Functions:

THEOREM 31. *Every Poincaré normal function satisfies Griffiths horizontality.*

THEOREM 32. *Every primitive integral Hodge class on X is the cohomology class of a Poincaré normal function.*

The next step in the proof of the Hodge conjecture via this approach is to show that for $t \in U$, the Abel–Jacobi map

$$AJ : CH^m(X_t)_{\text{hom}} \to J^m(X_t)$$

is surjective. However, for $m > 1$ this is rarely true (even granting the conjectural equality of $J^m(X)_{\text{alg}}$ and $J^m_h(X)$) since $J^m(X_t) \neq J^m_h(X_t)$ unless $H^{2m-1}(X_t, \mathbb{C}) = H^{m,m-1}(X_t) \oplus H^{m-1,m}(X_t)$. In plenty of cases of interest $J^m_h(X)$ is in fact trivial; Theorem 33 and Example 35 below give two different instances of this.

THEOREM 33 [Le1, Example 14.18]. *If $X \subseteq \mathbb{P}^k$ is a smooth projective variety of dimension $2m$ such that $H^{2m-1}(X) = 0$ and $\{X_t\}$ is a Lefschetz pencil of hyperplane sections of X such that $F^{m+1}H^{2m-1}(X_t) \neq 0$ for every smooth hyperplane section, then for generic $t \in U$, $J^m_h(X_t) = 0$.*

THEOREM 34. *If $J^p_h(X) = 0$, then the image of $CH^m(W)_{\text{hom}}$ in $J^p(X)$ under the Abel–Jacobi map is countable.*

SKETCH OF PROOF. As a consequence of (3-1), if $J^p_h(X) = 0$ the Abel–Jacobi map vanishes on $CH^p(X)_{\text{alg}}$. Therefore, the cardinality of the image of the Abel–Jacobi map on $CH^p(X)_{\text{hom}}$ is bounded by the cardinality of the Griffiths group $CH^p(X)_{\text{hom}}/CH^p(X)_{\text{alg}}$, which is known to be countable. □

EXAMPLE 35. Specific hypersurfaces with $J^p_h(X) = 0$ were constructed by Shioda [Sh]: Let Z^n_m denote the hypersurface in $\mathbb{P}^{n+1}$ defined by the equation

$$\sum_{i=0}^{n+1} x_i x_{i+1}^{m-1} = 0 \qquad (x_{n+2} = x_0).$$

Suppose that $n = 2p - 1 > 1$, $m \geq 2 + 3/(p-1)$ and

$$d_0 = \{(m-1)^{n+1} + (-1)^{n+1}\}/m$$

is prime. Then $J^p_h(Z^n_m) = 0$.

3.2. Singularities of admissible normal functions. In [GG], Griffiths and Green proposed an alternative program for proving the Hodge conjecture by studying the singularities of normal functions over higher-dimensional parameter spaces. Following [BFNP], let S a complex manifold and $\mathcal{H} = (\mathbb{H}_{\mathbb{Z}}, \mathcal{F}^\bullet\mathcal{H}_{\mathcal{O}})$ be a variation of polarizable Hodge structure of weight -1 over S. Then, we have the short exact sequence

$$0 \to \mathbb{H}_{\mathbb{Z}} \to \mathcal{H}/\mathcal{F}^0 \to J(\mathcal{H}) \to 0$$

of sheaves and hence an associated long exact sequence in cohomology. In particular, the cohomology class $\mathrm{cl}(\nu)$ of a normal function $\nu \in H^0(S, J(\mathcal{H}))$ is just the image of ν under the connecting homomorphism

$$\partial : H^0(S, J(\mathcal{H})) \to H^1(S, \mathbb{H}_\mathbb{Z}).$$

Suppose now that S is a Zariski open subset of a smooth projective variety $\bar{S}$. Then the singularity of ν at $p \in \bar{S}$ is the quantity

$$\sigma_{\mathbb{Z},p}(\nu) = \varinjlim_{p\in U} \mathrm{cl}(\nu|_{U\cap S}) \in \varinjlim_{p\in U} H^1(U \cap S, \mathbb{H}_\mathbb{Z}) = (R^1 j_* \mathbb{H}_\mathbb{Z})_p$$

where the limit is taken over all analytic open neighborhoods U of p, and $j : S \to \bar{S}$ is the inclusion map. The image of $\sigma_{\mathbb{Z},p}(\nu)$ in cohomology with rational coefficients will be denoted by $\mathrm{sing}_p(\nu_\zeta)$.

REMARK 36. If $p \in S$ then $\sigma_{\mathbb{Z},p}(\nu) = 0$.

THEOREM 37 [S1]. *Let ν be an admissible normal function on a Zariski open subset of a curve $\bar{S}$. Then, $\sigma_{\mathbb{Z},p}(\nu)$ is of finite order for each point $p \in \bar{S}$.*

PROOF. By [S1], an admissible normal function $\nu : S \to J(\mathcal{H})$ is equivalent to an extension

$$0 \to \mathcal{H} \to \mathcal{V} \to \mathbb{Z}(0) \to 0 \tag{3-3}$$

in the category of admissible variations of mixed Hodge structure. By the monodromy theorem for variations of pure Hodge structure, the local monodromy of $\mathcal{V}$ about any point $p \in \bar{S} - S$ is always quasi-unipotent. Without loss of generality, let us assume that it is unipotent and that $T = e^N$ is the local monodromy of $\mathcal{V}$ at p acting on some fixed reference fiber with integral structure $V_\mathbb{Z}$. Then, due to the length of the weight filtration W, the existence of the relative weight filtration of W and N is equivalent to the existence of an N-invariant splitting of W [SZ, Proposition 2.16]. In particular, let $e_\mathbb{Z} \in V_\mathbb{Z}$ project to $1 \in \mathrm{Gr}_0^W \cong \mathbb{Z}(0)$. Then, by admissibility, there exists an element $h_\mathbb{Q} \in H_\mathbb{Q} = W_{-1} \cap V_\mathbb{Q}$ such that

$$N(e_\mathbb{Z} + h_\mathbb{Q}) = 0$$

and hence $(T - I)(e_\mathbb{Z} + h_\mathbb{Q}) = 0$.[9] Any two such choices of $e_\mathbb{Z}$ differ by an element $h_\mathbb{Z} \in W_{-1} \cap V_\mathbb{Z}$. Therefore, an admissible normal function ν determines a class

$$[\nu] = [(T - I)e_\mathbb{Z}] \in \frac{(T - I)(H_\mathbb{Q})}{(T - I)(H_\mathbb{Z})}$$

Tracing through the definitions, one finds that the left-hand side of this equation can be identified with $\sigma_{\mathbb{Z},p}(\nu)$, whereas the right-hand side is exactly the torsion subgroup of $(R^1 j_* \mathbb{H}_\mathbb{Z})_p$. □

[9] Alternatively, one can just derive this from Definition 28(I).

DEFINITION 38 [BFNP]. An admissible normal function ν defined on a Zariski open subset of $\bar{S}$ is singular on $\bar{S}$ if there exists a point $p \in \bar{S}$ such that $\mathrm{sing}_p(\nu) \neq 0$.

Let S be a complex manifold and $f : X \to S$ be a family of smooth projective varieties over S. Let $\mathcal{H}$ be the variation of pure Hodge structure of weight -1 over S with integral structure $\mathbb{H}_{\mathbb{Z}} = R^{2p-1} f_* \mathbb{Z}(p)$. Then, an element $w \in J^p(X)\,(= J^0(H^{2p-1}(X, \mathbb{Z}(p))))$ defines a normal function $\nu_w : S \to J(\mathcal{H})$ by the rule

$$\nu_w(s) = i_s^*(w), \tag{3-4}$$

where i_s denotes inclusion of the fiber $X_s = f^{-1}(s)$ into X. More generally, let $H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p))$ denote the Deligne cohomology of X, and recall that we have a short exact sequence

$$0 \to J^p(X) \to H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p)) \to H^{p,p}(X, \mathbb{Z}(p)) \to 0.$$

Call a Hodge class

$$\zeta \in H^{p,p}(X, \mathbb{Z}(p)) := H^{p,p}(X, \mathbb{C}) \cap H^{2p}(X, \mathbb{Z}(p))$$

primitive with respect to f if $i_s^*(\zeta) = 0$ for all $s \in S$, and let $H^{p,p}_{\mathrm{prim}}(X, \mathbb{Z}(p))$ denote the group of all such primitive Hodge classes. Then, by the functoriality of Deligne cohomology, a choice of lifting $\tilde{\zeta} \in H^{2p}_{\mathcal{D}}(X, \mathbb{Z}(p))$ of a primitive Hodge class ζ determines a map $\nu_{\tilde{\zeta}} : S \to J(\mathcal{H})$. A short calculation (cf. [CMP, Ch. 10]) shows that $\nu_{\tilde{\zeta}}$ is a (horizontal) normal function over S. Furthermore, in the algebraic setting (meaning that X, S, f are algebraic), $\nu_{\tilde{\zeta}}$ is an admissible normal function [S1]. Let $\mathrm{ANF}(S, \mathcal{H})$ denote the group of admissible normal functions with underlying variation of Hodge structure $\mathcal{H}$. By abuse of notation, let $J^p(X) \subset \mathrm{ANF}(S, \mathcal{H})$ denote the image of the intermediate Jacobian $J^p(X)$ in $\mathrm{ANF}(S, \mathcal{H})$ under the map $w \mapsto \nu_w$. Then, since any two lifts $\tilde{\zeta}$ of ζ to Deligne cohomology differ by an element of the intermediate Jacobian $J^p(X)$, it follows that we have a well-defined map

$$AJ : H^{p,p}_{\mathrm{prim}}(X, \mathbb{Z}(p)) \to \mathrm{ANF}(S, \mathcal{H})/J^p(X). \tag{3-5}$$

REMARK 39. We are able to drop the notation $\mathrm{NF}(S, \mathcal{H})^{\mathrm{ad}}_{\bar{S}}$ used in §2, because in the global algebraic case it can be shown that admissibility is independent of the choice of compactification $\bar{S}$.

3.3. The Main Theorem. Returning to the program of Griffiths and Green, let X be a smooth projective variety of dimension $2m$ and $L \to X$ be a very ample line bundle. Let $\bar{P} = |L|$ and

$$\mathcal{X} = \{(x, s) \in X \times \bar{P} \mid s(x) = 0\} \tag{3-6}$$

be the incidence variety associated to the pair (X, L). Let $\pi : \mathcal{X} \to \bar{P}$ denote projection on the second factor, and let $\hat{X} \subset \bar{P}$ denote the dual variety of X (the points $s \in \bar{P}$ such that $X_s = \pi^{-1}(s)$ is singular). Let $\mathcal{H}$ be the variation of Hodge structure of weight -1 over $P = \bar{P} - \hat{X}$ attached to the local system $R^{2m-1}\pi_*^{\mathrm{sm}}\mathbb{Z}(m)$.

For a pair (X, L) as above, an integral Hodge class ζ of type (m, m) on X is primitive with respect to π^{sm} if and only if it is primitive in the usual sense of being annihilated by cup product with $c_1(L)$. Let $H^{m,m}_{\mathrm{prim}}(X, \mathbb{Z}(m))$ denote the group of all such primitive Hodge classes, and note that $H^{m,m}_{\mathrm{prim}}(X, \mathbb{Z}(m))$ is unchanged upon replacing L by $L^{\otimes d}$ for $d > 0$. Given $\zeta \in H^{m,m}_{\mathrm{prim}}(X, \mathbb{Z}(m))$, let

$$\nu_\zeta = AJ(\zeta) \in \mathrm{ANF}(P, \mathcal{H})/J^m(X)$$

be the associated normal function (3-5).

LEMMA 40. *If $\nu_w : P \to J(\mathcal{H})$ is the normal function* (3-4) *associated to an element $w \in J^m(X)$ then* $\mathrm{sing}_p(\nu_w) = 0$ *at every point $p \in \hat{X}$.*

Accordingly, for any point $p \in \hat{X}$ we have a well defined map

$$\overline{\mathrm{sing}_p} : \mathrm{ANF}(P, \mathcal{H})/J^m(X) \to (R^1 j_* \mathbb{H}_{\mathbb{Q}})_p$$

which sends the element $[\nu] \in \mathrm{ANF}(P, \mathcal{H})/J^m(X)$ to $\mathrm{sing}_p(\nu)$. In keeping with our prior definition, we say that ν_ζ is singular on $\bar{P}$ if there exists a point $p \in \hat{X}$ such that $\mathrm{sing}_p(\nu) \neq 0$.

CONJECTURE 41 [GG; BFNP]. *Let L be a very ample line bundle on a smooth projective variety X of dimension $2m$. Then, for every nontorsion class ζ in $H^{m,m}_{\mathrm{prim}}(X, \mathbb{Z}(m))$ there exists an integer $d > 0$ such that $AJ(\zeta)$ is singular on $\bar{P} = |L^{\otimes d}|$.*

THEOREM 42 [GG; BFNP; dCM]. *Conjecture* 41 *holds* (*for every even-dimensional smooth projective variety*) *if and only if the Hodge conjecture is true.*

To outline the proof of Theorem 42, observe that for any point $p \in \hat{X}$, we have the diagram

$$\begin{array}{ccc} H^{m,m}_{\mathrm{prim}}(X, \mathbb{Z}(m)) & \xrightarrow{AJ} & \mathrm{ANF}(P, \mathcal{H})/J^m(X) \\ \Big\downarrow{\scriptstyle \alpha_p} & & \Big\downarrow{\scriptstyle \overline{\mathrm{sing}_p}} \\ H^{2m}(X_p, \mathbb{Q}(m)) & \overset{\beta_p}{\underset{??}{\dashrightarrow}} & (R^1 j_* \mathbb{H}_{\mathbb{Q}})_p \end{array} \tag{3-7}$$

where $\alpha_p : H^{m,m}_{\mathrm{prim}}(X, \mathbb{Z}(m)) \to H^{2m}(X_p, \mathbb{Q}(m))$ is the restriction map.

Suppose that there exists a map

$$\beta_p : H^{2m}(X_p, \mathbb{Q}(m)) \to (R^1 j_* \mathbb{H}_{\mathbb{Q}})_p, \tag{3-8}$$

which makes the diagram (3-7) commute, and that after replacing L by $L^{\otimes d}$ for some $d > 0$ the restriction of β_p to the image of α_p is injective. Then, existence of a point $p \in \hat{X}$ such that $\mathrm{sing}_p(\nu_\zeta) \neq 0$ implies that the Hodge class ζ restricts nontrivially to X_p. Now recall that by Poincaré duality and the Hodge–Riemann bilinear relations, the Hodge conjecture for a smooth projective variety Y is equivalent to the statement that for every rational (q, q) class on Y there exists an algebraic cycle W of dimension $2q$ on Y such that $\gamma \cup [W] \neq 0$.

Let $f : \tilde{X}_p \to X_p$ be a resolution of singularities of X_p and $g = i \circ f$, where $i : X_p \to X$ is the inclusion map. By a weight argument $g^*(\zeta) \neq 0$, and so there exists a class $\xi \in \mathrm{Hg}^{m-1}(\tilde{X}_p)$ with $\xi \cup \zeta \neq 0$. Embedding $\tilde{X}_p$ in some projective space, and inducing on *even* dimension, we can assume that the Hodge conjecture holds for a general hyperplane section $\mathcal{I} : \mathcal{Y} \hookrightarrow \tilde{X}_p$. This yields an algebraic cycle $\mathcal{W}$ on $\mathcal{Y}$ with $[\mathcal{W}] = \mathcal{I}^*(\xi)$. Varying $\mathcal{Y}$ in a pencil, and using weak Lefschetz, $\mathcal{W}$ traces out[10] a cycle $W = \sum_j a_j W_j$ on $\tilde{X}_p$ with $[W] = \xi$, so that $g^*(\zeta) \cup [W] \neq 0$; in particular, $\zeta \cup g_*[W_j] \neq 0$ for some j.

Conversely, by the work of Thomas [Th], if the Hodge conjecture is true then the Hodge class ζ must restrict nontrivially to some singular hyperplane section of X (again for some $L^{\otimes d}$ for d sufficiently large). Now one uses the injectivity of β_p on $\mathrm{im}(\alpha_p)$ to conclude that ν_ζ has a singularity.

EXAMPLE 43. Let $X \subset \mathbb{P}^3$ be a smooth projective surface. For every $\zeta \in H^{1,1}_{\mathrm{prim}}(X, \mathbb{Z}(1))$, there is a reducible hypersurface section $X_p \subset X$ and component curve W of X_p such that $\deg(\zeta|_W) \neq 0$. (Note that $\deg(\zeta|_{X_p})$ is necessarily 0.) As the reader should check, this follows easily from Lefschetz (1,1). Moreover (writing d for the degree of X_p), p is a point in a codimension ≥ 2 substratum S' of $\hat{X} \subset \mathbb{P}H^0(\mathcal{O}(d))$ (since fibers over codimension-one substrata are irreducible), and $\mathrm{sing}_q(\nu_\zeta) \neq 0 \ \forall q \in S'$.

REMARK 44. There is a central geometric issue lurking in Conjecture 41:

If the HC holds, and $L = \mathcal{O}_X(1)$ (for some projective embedding of X), is there some minimum d_0 — uniform in some sense — for which $d \geq d_0$ implies that ν_ζ is singular?

In [GG] it is established that, at best, such a d_0 could only be uniform in moduli of the pair (X, ζ). (For example, in the case $\dim(X) = 2$, d_0 is of the form $C \times |\zeta \cdot \zeta|$, for C a constant. Since the self-intersection numbers of integral

[10] More precisely, one uses here a spread or Hilbert scheme argument. See for example the beginning of Chapter 14 of [Le1].

classes becoming Hodge in various Noether–Lefschetz loci increase without bound, there is certainly not any d_0 uniform in moduli of X.) Whether there is some such "lower bound" of this form remains an open question in higher dimension.

3.4. Normal functions and intersection cohomology. The construction of the map β_p depends on the decomposition theorem of Beilinson, Bernstein, and Deligne [BBD] and Morihiko Saito's theory of mixed Hodge modules [S4]. As first step in this direction, recall [CKS2] that if $\mathcal{H}$ is a variation of pure Hodge structure of weight k defined on the complement $S = \bar{S} - D$ of a normal crossing divisor on a smooth projective variety $\bar{S}$ then

$$H^\ell_{(2)}(S, \mathbb{H}_\mathbb{R}) \cong IH^\ell(\bar{S}, \mathbb{H}_\mathbb{R}),$$

where the left-hand side is L^2-cohomology and the right-hand side is intersection cohomology. Furthermore, via this isomorphism $IH^\ell(\bar{S}, \mathbb{H}_\mathbb{C})$ inherits a canonical Hodge structure of weight $k + \ell$.

REMARK 45. If Y is a complex algebraic variety, $\mathrm{MHM}(Y)$ is the category of mixed Hodge modules on Y. The category $\mathrm{MHM}(Y)$ comes equipped with a functor

$$\mathrm{rat} : \mathrm{MHM}(Y) \to \mathrm{Perv}(Y)$$

to the category of perverse sheaves on Y. If Y is smooth and $\mathcal{V}$ is a variation of mixed Hodge structure on Y then $\mathcal{V}[d_Y]$ is a mixed Hodge module on Y, and $\mathrm{rat}(\mathcal{V}[d_Y]) \cong \mathbb{V}[d_Y]$ is just the underlying local system $\mathbb{V}$ shifted into degree $-d_Y$.

If Y° is a Zariski open subset of Y and $\mathcal{P}$ is a perverse sheaf on Y° then

$$IH^\ell(Y, \mathcal{P}) = \mathbb{H}^{\ell - d_Y}(Y, j_{!*}\mathcal{P}[d_Y])$$

where $j_{!*}$ is the middle extension functor [BBD] associated to the inclusion map $j : Y^\circ \to Y$. Likewise, for any point $y \in Y$, the local intersection cohomology of $\mathcal{P}$ at y is defined to be

$$IH^\ell(Y, \mathcal{P})_y = \mathbb{H}^{k - d_Y}(\{y\}, i^* j_{!*}\mathcal{P}[d_Y])$$

where $i : \{y\} \to Y$ is the inclusion map. If $\mathcal{P}$ underlies a mixed Hodge module, the theory of MHM puts natural MHS on these groups, which in particular is how the pure HS on $IH^\ell(\bar{S}, \mathbb{H}_\mathbb{C})$ comes about.

THEOREM 46 [BFNP, Theorem 2.11]. *Let $\bar{S}$ be a smooth projective variety and $\mathcal{H}$ be a variation of pure Hodge structure of weight -1 on a Zariski open subset $S \subset \bar{S}$. Then, the group homomorphism*

$$\mathrm{cl} : \mathrm{ANF}(S, \mathcal{H}) \to H^1(S, \mathbb{H}_\mathbb{Q})$$

factors through $IH^1(\bar{S}, \mathbb{H}_{\mathbb{Q}})$.

SKETCH OF PROOF. Let $\nu \in \mathrm{ANF}(S, \mathcal{H})$ be represented by an extension

$$0 \to \mathcal{H} \to \mathcal{V} \to \mathbb{Z}(0) \to 0$$

in the category of admissible variations of mixed Hodge structure on S. Let $j : S \to \bar{S}$ be the inclusion map. Then, because $\mathcal{V}$ has only two nontrivial weight graded quotients which are adjacent, it follows by [BFNP, Lemma 2.18] that

$$0 \to j_{!*}\mathcal{H}[d_S] \to j_{!*}\mathcal{V}[d_S] \to \mathbb{Q}(0)[d_S] \to 0$$

is exact in $\mathrm{MHM}(\bar{S})$. □

REMARK 47. In this particular context, $j_{!*}\mathcal{V}[d_S]$ can be described as the unique prolongation of $\mathcal{V}[d_S]$ to $\bar{S}$ with no nontrivial sub or quotient object supported on the essential image of the functor $i : \mathrm{MHM}(Z) \to \mathrm{MHM}(\bar{S})$ where $Z = \bar{S} - S$ and $i : Z \to \bar{S}$ is the inclusion map.

In the local case of an admissible normal function on a product of punctured polydisks $(\Delta^*)^r$ with unipotent monodromy, the fact that $\mathrm{sing}_0(\nu)$ (where 0 is the origin of $\Delta^r \supseteq (\Delta^*)^r$) factors through the local intersection cohomology groups can be seen as follows: Such a normal function ν gives a short exact sequence of local systems

$$0 \to \mathbb{H}_{\mathbb{Q}} \to \mathbb{V}_{\mathbb{Q}} \to \mathbb{Q}(0) \to 0$$

over $(\Delta^*)^r$. Fix a reference fiber $V_{\mathbb{Q}}$ of $\mathbb{V}_{\mathbb{Q}}$ and let $N_j \in \mathrm{Hom}(V_{\mathbb{Q}}, V_{\mathbb{Q}})$ denote the monodromy logarithm of $\mathbb{V}_{\mathbb{Q}}$ about the j-th punctured disk. Then [CKS2], we get a complex of finite-dimensional vector spaces

$$B^p(V_{\mathbb{Q}}) = \bigoplus_{i_1 < i_2 < \cdots < i_p} N_{i_1} N_{i_2} \cdots N_{i_p}(V_{\mathbb{Q}})$$

with differential d, which acts on the summands of $B^p(V_{\mathbb{Q}})$ by the rule

$$N_{i_1} \cdots \hat{N}_{i_\ell} \cdots N_{i_{p+1}}(V_{\mathbb{Q}}) \overset{(-1)^{\ell-1} N_{i_\ell}}{\longrightarrow} N_{i_1} \cdots N_{i_\ell} \cdots N_{i_{p+1}}(V_{\mathbb{Q}})$$

(and taking the sum over all insertions). Let $B^*(H_{\mathbb{Q}})$ and $B^*(\mathbb{Q}(0))$ denote the analogous complexes attached to the local systems $\mathbb{H}_{\mathbb{Q}}$ and $\mathbb{Q}(0)$. By [GGM], the cohomology of the complex $B^*(H_{\mathbb{Q}})$ computes the local intersection cohomology of $\mathbb{H}_{\mathbb{Q}}$. In particular, since the complexes $B^*(\mathbb{Q}(0))$ and $B^*(H_{\mathbb{Q}})$ sit inside the standard Koszul complexes which compute the ordinary cohomology of $\mathbb{Q}(0)$ and $H_{\mathbb{Q}}$, in order show that sing_0 factors through $IH^1(\mathbb{H}_{\mathbb{Q}})$ it is sufficient

to show that $\partial \mathrm{cl}(v) \in H^1((\Delta^*)^r, \mathbb{H}_{\mathbb{Q}})$ is representable by an element of $B^1(H_{\mathbb{Q}})$. Indeed, let v be an element of $V_{\mathbb{Q}}$ which maps to $1 \in \mathbb{Q}(0)$. Then,

$$\partial\, \mathrm{cl}(v) = \partial 1 = [(N_1(v), \ldots, N_r(v))]$$

By admissibility and the short length of the weight filtration, for each j there exists an element $h_j \in H_{\mathbb{Q}}$ such that $N_j(h_j) = N_j(v)$, which is exactly the condition that

$$(N_1(v), \ldots, N_r(v)) \in B^1(V_{\mathbb{Q}}).$$

THEOREM 48 [BFNP, Theorem 2.11]. *Under the hypothesis of Theorem* 46, *for any point* $p \in \bar{S}$ *the group homomorphism* $\mathrm{sing}_p : \mathrm{ANF}(S, \mathcal{H}) \to (R^1 j_* \mathbb{H}_{\mathbb{Q}})_p$ *factors through the local intersection cohomology group* $IH^1(\mathbb{H}_{\mathbb{Q}})_p$.

To continue, we need to pass from Deligne cohomology to absolute Hodge cohomology. Recall that MHM(Spec($\mathbb{C}$)) is the category MHS of graded-polarizable $\mathbb{Q}$ mixed Hodge structures. Let $\mathbb{Q}(p)$ denote the Tate object of type $(-p, -p)$ in MHS and $\mathbb{Q}_Y(p) = a_Y^* \mathbb{Q}(p)$ where $a_Y : Y \to \mathrm{Spec}(\mathbb{C})$ is the structure morphism. Let $\mathbb{Q}_Y = \mathbb{Q}_Y(0)$.

DEFINITION 49. Let M be an object of MHM(Y). Then,

$$H^n_{\mathcal{AH}}(Y, M) = \mathrm{Hom}_{D^b \mathrm{MHM}}(\mathbb{Q}_Y, M[n])$$

is the absolute Hodge cohomology of M.

The functor $\mathrm{rat} : \mathrm{MHM}(Y) \to \mathrm{Perv}(Y)$ induces a "cycle class map"

$$\mathrm{rat} : H^n_{\mathcal{AH}}(Y, M) \to \mathbb{H}^n(Y, \mathrm{rat}(M))$$

from the absolute Hodge cohomology of M to the hypercohomology of $\mathrm{rat}(M)$. In the case where Y is smooth and projective, $H^{2p}_{\mathcal{AH}}(Y, \mathbb{Q}_Y(p))$ is the Deligne cohomology group $H^{2p}_{\mathcal{D}}(Y, \mathbb{Q}(p))$ and rat is the cycle class map on Deligne cohomology.

DEFINITION 50. Let $\bar{S}$ be a smooth projective variety and $\mathcal{V}$ be an admissible variation of mixed Hodge structure on a Zariski open subset S of $\bar{S}$. Then,

$$IH^n_{\mathcal{AH}}(\bar{S}, \mathcal{V}) = \mathrm{Hom}_{D^b \mathrm{MHM}(\bar{S})}(\mathbb{Q}_{\bar{S}}[d_S - n], j_{!*}\mathcal{V}[d_S]),$$
$$IH^n_{\mathcal{AH}}(\bar{S}, \mathcal{V})_s = \mathrm{Hom}_{D^b \mathrm{MHS}}(\mathbb{Q}[d_S - n], i^* j_{!*}\mathcal{V}[d_S]),$$

where $j : S \to \bar{S}$ and $i : \{s\} \to \bar{S}$ are inclusion maps.

The following lemma links absolute Hodge cohomology and admissible normal functions:

LEMMA 51. [BFNP, Proposition 3.3] *Let* $\mathcal{H}$ *be a variation of pure Hodge structure of weight* -1 *defined on a Zariski open subset* S *of a smooth projective variety* $\bar{S}$. *Then,* $IH^1_{\mathcal{AH}}(\bar{S}, \mathcal{H}) \cong \mathrm{ANF}(S, \mathcal{H}) \otimes \mathbb{Q}$.

3.5. Completion of the diagram (3-7**).** Let $f : X \to Y$ be a projective morphism between smooth algebraic varieties. Then, by the work of Morihiko Saito [S4], there is a direct sum decomposition

$$f_*\mathbb{Q}_X[d_X] = \bigoplus_i H^i\,(f_*\mathbb{Q}_X[d_X])\,[-i] \tag{3-9}$$

in $\mathrm{MHM}(Y)$. Furthermore, each summand $H^i(f_*\mathbb{Q}_X[d_X])$ is pure of weight $d_X + i$ and admits a decomposition according to codimension of support:

$$H^i\,(f_*\mathbb{Q}_X[d_X])\,[-i] = \textstyle\bigoplus_j E_{ij}[-i]; \tag{3-10}$$

i.e., $E_{ij}[-i]$ is a sum of Hodge modules supported on codimension j subvarieties of Y. Accordingly, we have a system of projection operators (inserting arbitrary twists)

$$\begin{aligned}
\textstyle\bigoplus \Pi_{ij} &: H^n_{\mathcal{AH}}(X, \mathbb{Q}(\ell)[d_X]) \quad \overset{\cong}{\to} \textstyle\bigoplus_{ij}\ H^{n-i}_{\mathcal{AH}}(Y, E_{ij}(\ell)),\\
\textstyle\bigoplus \Pi_{ij} &: H^n_{\mathcal{AH}}(X_p, \mathbb{Q}(\ell)[d_X]) \quad \overset{\cong}{\to} \textstyle\bigoplus_{ij}\ H^{n-i}_{\mathcal{AH}}(Y, \iota^* E_{ij}(\ell)),\\
\textstyle\bigoplus \Pi_{ij} &: \mathbb{H}^n(X, \mathrm{rat}(\mathbb{Q}(\ell)[d_X])) \overset{\cong}{\to} \textstyle\bigoplus_{ij}\ \mathbb{H}^{n-i}(Y, \mathrm{rat}(E_{ij}(\ell))),\\
\textstyle\bigoplus \Pi_{ij} &: \mathbb{H}^n(X_p, \mathrm{rat}(\mathbb{Q}(\ell)[d_X])) \overset{\cong}{\to} \textstyle\bigoplus_{ij}\ \mathbb{H}^{n-i}(Y, \iota^*\mathrm{rat}(E_{ij}(\ell))),
\end{aligned}$$

where $p \in Y$ and $\iota : \{p\} \to Y$ is the inclusion map.

LEMMA 52 [BFNP, Equation 4.12]. *Let $\mathcal{H}^q = R^q f_*^{\mathrm{sm}}\mathbb{Q}_X$ and recall that we have a decomposition*

$$\mathcal{H}^{2k-1} = \mathcal{H}^{2k-1}_{\mathrm{van}} \oplus \mathcal{H}^{2k-1}_{\mathrm{fix}}$$

where $\mathcal{H}^{2k-1}_{\mathrm{fix}}$ is constant and $\mathcal{H}^{2k-1}_{\mathrm{van}}$ has no global sections. For any point $p \in Y$, we have a commutative diagram

$$\begin{array}{ccc}
H^{2k}_{\mathcal{AH}}(X, \mathbb{Q}(k)) & \xrightarrow[\Pi]{} & \mathrm{ANF}(Y^{\mathrm{sm}}, \mathcal{H}^{2k-1}_{\mathrm{van}}(k)) \\
\downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} \\
H^{2k}(X_p, \mathbb{Q}(k)) & \xrightarrow[\Pi]{} & IH^1(\mathcal{H}^{2k-1}(k))_p
\end{array} \tag{3-11}$$

where Y^{sm} is the largest Zariski open set over which f is smooth and Π is induced by Π_{r0} for $r = 2k - 1 - d_X + d_Y$.

We now return to the setting of Conjecture 41: X is a smooth projective variety of dimension $2m$, L is a very ample line bundle on X and $\mathcal{X}$ is the associated incidence variety (3-6), with projections $\pi : \mathcal{X} \to \check{P}$ and $\mathrm{pr} : \mathcal{X} \to X$. Then, we have the following "Perverse weak Lefschetz theorem":

THEOREM 53 [BFNP, Theorem 5.1]. *Let $\mathcal{X}$ be the incidence variety associated to the pair (X, L) and $\pi_*\mathbb{Q}_{\mathcal{X}} = \bigoplus_{ij} E_{ij}$ in accord with* (3-9) *and* (3-10). *Then*:

(i) $E_{ij} = 0$ unless $i \cdot j = 0$.
(ii) $E_{i0} = H^i(X, \mathbb{Q}_X[2m-1]) \otimes \mathbb{Q}_{\bar{P}}[d_{\bar{P}}]$. for $i < 0$.

Note that by hard Lefschetz, $E_{ij} \cong E_{-i,j}(-i)$ [S4].

To continue, recall that given a Lefschetz pencil $\Lambda \subset \bar{P}$ of hyperplane sections of X, we have an associated system of vanishing cycles $\{\delta_p\}_{p \in \Lambda \cap \hat{X}} \subset H^{2m-1}(X_t, \mathbb{Q})$ on the cohomology of the smooth hyperplane sections X_t of X with respect to Λ. As one would expect, the vanishing cycles of Λ are *nonvanishing* if for some (hence all) $p \in \Lambda \cap \hat{X}$, $\delta_p \neq 0$ (in $H^{2m-1}(X_t, \mathbb{Q})$). Furthermore, this property depends only on L and not the particular choice of Lefschetz pencil Λ. This property can always be arranged by replacing L by $L^{\otimes d}$ for some $d > 0$.

THEOREM 54. *If all vanishing cycles are nonvanishing then $E_{01} = 0$. Otherwise, E_{01} is supported on a dense open subset of $\hat{X}$.*

Using the Theorems 53 and 54, we now prove that the diagram

$$\begin{array}{ccc} H^{2m}_{\mathcal{D}}(X, \mathbb{Z}(m))_{\text{prim}} & \xrightarrow[AJ]{} & \text{ANF}(P, \mathcal{H})/J^m(X) \\ \Big\downarrow{\scriptstyle \text{pr}^*} & & \Big\downarrow{\scriptstyle \otimes\mathbb{Q}} \\ H^{2m}_{\mathcal{AH}}(\mathcal{X}, \mathbb{Q}(m)) & \xrightarrow[\Pi]{} & \text{ANF}(P, \mathcal{H}_{\text{van}}) \otimes \mathbb{Q} \end{array} \tag{3-12}$$

commutes, where $H^{2m}_{\mathcal{D}}(X, \mathbb{Z}(m))_{\text{prim}}$ is the subgroup of $H^{2m}_{\mathcal{D}}(X, \mathbb{Z}(m))$ whose elements project to primitive Hodge classes in $H^{2m}(X, \mathbb{Z}(m))$, and Π is induced by Π_{00} together with projection onto $\mathcal{H}_{\text{van}}$. Indeed, by the decomposition theorem,

$$\begin{aligned} H^{2m}_{\mathcal{AH}}(\mathcal{X}, \mathbb{Q}(m)) &= H^{1-d_{\bar{P}}}_{\mathcal{AH}}(\mathcal{X}, \mathbb{Q}(m)[2m + d_{\bar{P}} - 1]) \\ &= \bigoplus H^{1-d_{\bar{P}}}_{\mathcal{AH}}(\bar{P}, E_{ij}(m)[-i]). \end{aligned}$$

Let $\tilde{\zeta} \in H^{2m}_{\mathcal{D}}(X, \mathbb{Z}(m))$ be a primitive Deligne class and $\omega = \bigoplus_{ij} \omega_{ij}$ denote the component of $\omega = \text{pr}^*(\tilde{\zeta})$ with respect to $E_{ij}(m)[-i]$ in accord with the previous equation. Then, in order to prove the commutativity of (3-12) it is sufficient to show that $(\omega)_q = (\omega_{00})_q$ for all $q \in P$. By Theorem 53, we know that $\omega_{ij} = 0$ unless $ij = 0$. Furthermore, by [BFNP, Lemma 5.5], $(\omega_{0j})_q = 0$ for $j > 1$. Likewise, by Theorem 54, $(\omega_{01})_q = 0$ for $q \in P$ since E_{01} is supported on $\hat{X}$.

Thus, in order to prove the commutativity of (3-12), it is sufficient to show that $(\omega_{i0})_q = 0$ for $i > 0$. However, as a consequence of Theorem 53(ii), $E_{i0}(m) =$

$K[d_{\bar{P}}]$, where K is a constant variation of Hodge structure on $\bar{P}$; and hence

$$\begin{aligned} H^{1-d_{\bar{P}}}(\mathcal{X}, E_{i0}(m)[-i]) &= \mathrm{Ext}^{1-d_{\bar{P}}}_{D^b\mathrm{MHM}(\bar{P})}(\mathbb{Q}_{\bar{P}}, K[d_{\bar{P}} - i]) \\ &= \mathrm{Ext}^{1-i}_{D^b\mathrm{MHM}(\bar{P})}(\mathbb{Q}_{\bar{P}}, K). \end{aligned}$$

Therefore, $(\omega_{i0})_q = 0$ for $i > 1$ while $(\omega_{10})_q$ corresponds to an element of $Hom(\mathbb{Q}(0), K_q)$ where K is the constant variation of Hodge structure with fiber $H^{2m}(X_q, \mathbb{Q}(m))$ over $q \in P$. It therefore follows from the fact that $\tilde{\zeta}$ is primitive that $(\omega_{10})_q = 0$. Splicing diagram (3-12) together with (3-11) (and replacing $f : X \to Y$ by $\pi : \mathcal{X} \to \bar{P}$, etc.) now gives the diagram (3-7).

REMARK 55. The effect of passing from $\mathcal{H}$ to $\mathcal{H}^{\mathrm{van}}$ in the constructions above is to annihilate $J^m(X) \subseteq H^{2m}_{\mathcal{D}}(X, \mathbb{Z}(m))_{\mathrm{prim}}$. Therefore, in (3-12) we can replace $H^{2m}_{\mathcal{D}}(X, \mathbb{Z}(m))_{\mathrm{prim}}$ by $H^{m,m}_{\mathrm{prim}}(X, \mathbb{Z}(m))$.

Finally, if all the vanishing cycles are nonvanishing, $E_{01} = 0$. Using this fact, we then get the injectivity of β_p on the image of α_p.

Returning to the beginning of this section, we now see that although extending normal functions along Lefschetz pencils is insufficient to prove the Hodge conjecture for higher codimension cycles, the Hodge conjecture is equivalent to a statement about the behavior of normal functions on the complement of the dual variety of X inside $|L|$ for $L \gg 0$. We remark that an interpretation of the GHC along similar lines has been done recently by the authors in [KP].

4. Zeroes of normal functions

4.1. Algebraicity of the zero locus. Some of the deepest evidence to date in support of the Hodge conjecture is the following result of Cattani, Deligne and Kaplan on the algebraicity of the Hodge locus:

THEOREM 56 [CDK]. *Let $\mathcal{H}$ be a variation of pure Hodge structure of weight* 0 *over a smooth complex algebraic variety S. Let α_{s_o} be an integral Hodge class of type* $(0,0)$ *on the fiber of $\mathcal{H}$ at s_o. Let U be a simply connected open subset of S containing s_o and α be the section of $\mathbb{H}_{\mathbb{Z}}$ over U defined by parallel translation of α_{s_o}. Let T be the locus of points in U such that $\alpha(s)$ is of type* $(0,0)$ *on the fiber of $\mathcal{H}$ over s. Then, the analytic germ of T at p is the restriction of a complex algebraic subvariety of S.*

More precisely, as explained in the introduction of [CDK], in the case where $\mathcal{H}$ arises from the cohomology of a family of smooth projective varieties $f : X \to S$, the algebraicity of the germ of T follows from the Hodge conjecture. A natural analog of this result for normal functions is this:

THEOREM 57. *Let S be a smooth complex algebraic variety, and $\nu : S \to J(\mathcal{H})$ be an admissible normal function, where $\mathcal{H}$ is a variation of pure Hodge structure of weight -1. Then, the zero locus*

$$\mathcal{Z}(\nu) = \{\, s \in S \mid \nu(s) = 0 \,\}$$

is a complex algebraic subvariety of S.

This theorem was still a conjecture when the present article was submitted, and has just been proved by the second author in work with P. Brosnan [BP3]. It is of particular relevance to the Hodge conjecture, due to the following relationship between the algebraicity of $\mathcal{Z}(\nu)$ and the existence of singularities of normal functions. Say $\dim(X) = 2m$, and let (X, L, ζ) be a triple consisting of a smooth complex projective variety X, a very ample line bundle L on X and a primitive integral Hodge class ζ of type (m, m). Let ν_ζ (assumed nonzero) be the associated normal function on the complement of the dual variety $\hat{X}$ constructed in § 3, and $\mathcal{Z}$ be its zero locus. Then, assuming that $\mathcal{Z}$ is algebraic and positive-dimensional, the second author conjectured that ν should have singularities along the intersection of the closure of $\mathcal{Z}$ with $\hat{X}$.

THEOREM 58 [Sl1]. *Let (X, L, ζ) be a triple as above, and assume that L is sufficiently ample that, given any point $p \in \hat{X}$, the restriction of β_p to the image of α_p in diagram* (3-7) *is injective. Suppose that $\mathcal{Z}$ contains an algebraic curve. Then, ν_ζ has a nontorsion singularity at some point of the intersection of the closure of this curve with $\hat{X}$.*

SKETCH OF PROOF. Let C be the normalization of the closure of the curve in $\mathcal{Z}$. Let $\mathcal{X} \to \bar{P}$ be the universal family of hyperplane sections of X over $\bar{P} = |L|$ and W be the pullback of X to C. Let $\pi : W \to C$ be the projection map, and U the set of points $c \in C$ such that $\pi^{-1}(c)$ is smooth and $W_U = \pi^{-1}(U)$. Via the Leray spectral sequence for π, it follows that restriction of ζ to W_U is zero because $U \subseteq \mathcal{Z}$ and ζ is primitive. On the other hand, since $W \twoheadrightarrow X$ is finite, ζ must restrict (pull back) nontrivially to W, and hence ζ must restrict nontrivially to the fiber $\pi^{-1}(c)$ for some point $c \in C$ in the complement of U. □

Unfortunately, crude estimates for the expected dimension of the zero locus $\mathcal{Z}$ arising in this context appear to be negative. For instance, take X to be an abelian surface in the following:

THEOREM 59. *Let X be a surface and $L = \mathcal{O}_X(D)$ be an ample line bundle on X. Then, for n sufficiently large, the expected dimension of the zero locus of the normal function ν_ζ attached to the triple $(X, L^{\otimes n}, \zeta)$ as above is*

$$h^{2,0} - h^{1,0} - n(D.K) - 1,$$

where K is the canonical divisor of X.

SKETCH OF PROOF. Since Griffiths' horizontality is trivial in this setting, computing the expected dimension boils down to computing the dimension of $|L|$ and genus of a smooth hyperplane section of X with respect to L. □

REMARK 60. In Theorem 59, we construct ν_ξ from a choice of lift to Deligne cohomology (or an algebraic cycle) to get an element of $\mathrm{ANF}(P,\mathcal{H})$. But this is disingenuous, since we are starting with a Hodge class. It is more consistent to work with $\nu_\xi \in \mathrm{ANF}(P,\mathcal{H})/J^1(X)$ as in equation (3-5), and then the dimension estimate improves by $\dim(J^1(X)) = h^{1,0}$ to $h^{2,0} - n(D.K) - 1$. Notice that this salvages at least the abelian surface case (though it is still a crude estimate). For surfaces of general type, one is still in trouble without more information, like the constant C in Remark 44.

We will not attempt to describe the proof of Theorem 57 in general, but we will explain the following special case:

THEOREM 61 [BP2]. *Let S be a smooth complex algebraic variety which admits a projective completion $\bar{S}$ such that $D = \bar{S} - S$ is a smooth divisor. Let $\mathcal{H}$ be a variation of pure Hodge structure of weight -1 on S and $\nu : S \to J(\mathcal{H})$ be an admissible normal function. Then, the zero locus $\mathcal{Z}$ of ν is an complex algebraic subvariety of S.*

REMARK 62. This result was obtained contemporaneously by Morihiko Saito in [S5].

In analogy with the proof of Theorem 56 on the algebraicity of the Hodge locus, which depends heavily on the several variable SL_2-orbit theorem for nilpotent orbits of pure Hodge structure [CKS1], the proof of Theorem 57 depends upon the corresponding result for nilpotent orbits of mixed Hodge structure. For simplicity of exposition, we will now review the 1-variable SL_2-orbit theorem in the pure case (which is due to Schmid [Sc]) and a version of the SL_2-orbit theorem in the mixed case [Pe2] sufficient to prove Theorem 61. For the proof of Theorem 57, we need the full strength of the several variable SL_2-orbit theorem of Kato, Nakayama and Usui [KNU1].

4.2. The classical nilpotent and SL_2-orbit theorems. To outline the proof of Theorem 61, we now recall the theory of degenerations of Hodge structure: Let $\mathcal{H}$ be a variation of pure Hodge structure of weight k over a simply connected complex manifold S. Then, via parallel translation back to a fixed reference fiber $H = H_{s_o}$ we obtain a period map

$$\varphi : S \to \mathcal{D}, \tag{4-1}$$

where $\mathcal{D}$ is Griffiths' classifying space of pure Hodge structures on H with fixed Hodge numbers $\{h^{p,k-p}\}$ which are polarized by the bilinear form Q of H. The

set $\mathcal{D}$ is a complex manifold upon which the Lie group

$$G_{\mathbb{R}} = \mathrm{Aut}_{\mathbb{R}}(Q)$$

acts transitively by biholomorphisms, and hence $\mathcal{D} \cong G_{\mathbb{R}}/G_{\mathbb{R}}^{F_o}$, where $G_{\mathbb{R}}^{F_o}$ is the isotropy group of $F_o \in \mathcal{D}$. The compact dual of $\mathcal{D}$ is the complex manifold

$$\check{\mathcal{D}} \cong G_{\mathbb{C}}/G_{\mathbb{C}}^{F_o}$$

where F_o is any point in $\mathcal{D}$. (In general, $F = F^\bullet$ denotes a Hodge filtration.) If S is not simply connected, then the period map (4-1) is replaced by

$$\varphi : S \to \Gamma\backslash\mathcal{D} \tag{4-2}$$

where Γ is the monodromy group of $\mathbb{H} \to S$ acting on the reference fiber H.

For variations of Hodge structure of geometric origin, S will typically be a Zariski open subset of a smooth projective variety $\bar{S}$. By Hironaka's resolution of singularities theorem, we can assume $D = \bar{S} - S$ to be a divisor with normal crossings. The period map (4-2) will then have singularities at the points of D about which $\mathbb{H}$ has nontrivial local monodromy. A precise local description of the singularities of the period map of a variation of Hodge structure was obtained by Schmid [Sc]: Let $\varphi : (\Delta^*)^r \to \Gamma\backslash\mathcal{D}$ be the period map of variation of pure polarized Hodge structure over the product of punctured disks. First, one knows that φ is locally liftable with quasi-unipotent monodromy. After passage to a finite cover, we therefore obtain a commutative diagram

$$\begin{array}{ccc} U^r & \xrightarrow[F]{} & \mathcal{D} \\ \downarrow & & \downarrow \\ (\Delta^*)^r & \xrightarrow[\varphi]{} & \Gamma\backslash\mathcal{D} \end{array} \tag{4-3}$$

where U^r is the r-fold product of upper half-planes and $U^r \to (\Delta^*)^r$ is the covering map

$$s_j = e^{2\pi i z_j}, \qquad j = 1, \ldots, r$$

with respect to the standard Euclidean coordinates $(z_1, \ldots, z_r)$ on $U^r \subset \mathbb{C}^r$ and $(s_1, \ldots, s_r)$ on $(\Delta^*)^r \subset \mathbb{C}^r$.

Let $T_j = e^{N_j}$ denote the monodromy of $\mathcal{H}$ about $s_j = 0$. Then,

$$\psi(z_1, \ldots, z_r) = e^{-\sum_j z_j N_j} . F(z_1, \ldots, z_r)$$

is a holomorphic map from U^r into $\check{\mathcal{D}}$ which is invariant under the transformation $z_j \mapsto z_j + 1$ for each j, and hence drops to a map $(\Delta^*)^r \to \check{\mathcal{D}}$ which we continue to denote by ψ.

DEFINITION 63. Let $\mathcal{D}$ be a classifying space of pure Hodge structure with associated Lie group $G_{\mathbb{R}}$. Let $\mathfrak{g}_{\mathbb{R}}$ be the Lie algebra of $G_{\mathbb{R}}$. Then, a holomorphic, horizontal map $\theta : \mathbb{C}^r \to \check{\mathcal{D}}$ is a nilpotent orbit if

(a) there exists $\alpha > 0$ such that $\theta(z_1, \ldots, z_r) \in \mathcal{D}$ if $\mathrm{Im}(z_j) > \alpha \ \forall\, j$; and

(b) there exist commuting nilpotent endomorphisms $N_1, \ldots, N_r \in \mathfrak{g}_{\mathbb{R}}$ and a point $F \in \check{\mathcal{D}}$ such that $\theta(z_1, \ldots, z_r) = e^{\sum_j z_j N_j}.F$.

THEOREM 64 (NILPOTENT ORBIT THEOREM [Sc]). *Let $\varphi : (\Delta^*)^r \to \Gamma\backslash\mathcal{D}$ be the period map of a variation of pure Hodge structure of weight k with unipotent monodromy. Let $d_{\mathcal{D}}$ be a $G_{\mathbb{R}}$-invariant distance on $\mathcal{D}$. Then*:

(a) $F_\infty = \lim_{s\to 0} \psi(s)$ *exists, i.e., $\psi(s)$ extends to a map $\Delta^r \to \check{\mathcal{D}}$*;

(b) $\theta(z_1, \ldots, z_r) = e^{\sum_j z_j N_j}.F_\infty$ *is a nilpotent orbit*; *and*

(c) *there exist constants C, α and $\beta_1, \ldots, \beta_r$ such that if* $\mathrm{Im}(z_j) > \alpha \ \forall\, j$ *then* $\theta(z_1, \ldots, z_r) \in \mathcal{D}$ *and*

$$d_{\mathcal{D}}(\theta(z_1, \ldots, z_r), F(z_1, \ldots, z_r)) < C \sum_j \mathrm{Im}(z_j)^{\beta_j} e^{-2\pi \mathrm{Im}(z_j)}.$$

REMARK 65. Another way of stating part (a) of this theorem is that the Hodge bundles $\mathcal{F}^p$ of $\mathcal{H}_{\mathcal{O}}$ extend to a system of holomorphic subbundles of the canonical extension of $\mathcal{H}_{\mathcal{O}}$. Indeed, recall from §2.7 that one way of constructing a model of the canonical extension in the unipotent monodromy case is to take a flat, multivalued frame $\{\sigma_1, \ldots, \sigma_m\}$ of $\mathbb{H}_{\mathbb{Z}}$ and twist it to form a single valued holomorphic frame $\{\tilde{\sigma}_1, \ldots, \tilde{\sigma}_m\}$ over $(\Delta^*)^r$ where $\tilde{\sigma}_j = e^{-\frac{1}{2\pi i}\sum_j \log(s_j) N_j}\sigma_j$, and then declaring this twisted frame to define the canonical extension.

Let N be a nilpotent endomorphism of a finite-dimensional vector space over a field k. Then, N can be put into Jordan canonical form, and hence (by considering a Jordan block) it follows that there is a unique, increasing filtration $\mathsf{W}(N)$ of V, such that, for each index j,

(a) $N(\mathsf{W}(N)_j) \subseteq \mathsf{W}(N)_{j-2}$ and

(b) $N^j : \mathrm{Gr}_j^{\mathsf{W}(N)} \to \mathrm{Gr}_{-j}^{\mathsf{W}(N)}$ is an isomorphism.

If ℓ is an integer then $(\mathsf{W}(N)[\ell])_j = \mathsf{W}(N)_{j+\ell}$.

THEOREM 66. *Let $\varphi : \Delta^* \to \Gamma\backslash\mathcal{D}$ be the period map of a variation of pure Hodge structure of weight k with unipotent monodromy $T = e^N$. Then, the limit Hodge filtration F_∞ of φ pairs with the* weight monodromy filtration $M(N) := \mathsf{W}(N)[-k]$ *to define a mixed Hodge structure relative to which N is a $(-1,-1)$-morphism.*

REMARK 67. The limit Hodge filtration F_∞ depends upon the choice of local coordinate s, or more precisely on the value of $(ds)_0$. Therefore, unless one has a preferred coordinate system (say, if the field of definition matters), in order

to extract geometric information from the limit mixed Hodge structure $H_\infty = (F_\infty, M(N))$ one usually has to pass to the mixed Hodge structure induced by H_∞ on the kernel or cokernel of N. In particular, if $X \to \Delta$ is a semistable degeneration, the local invariant cycle theorem asserts that we have an exact sequence

$$H^k(X_0) \to H_\infty \xrightarrow{N} H_\infty,$$

where the map $H^k(X_0) \to H_\infty$ is obtained by first including the reference fiber X_{s_o} into X and then retracting X onto X_0.

The proof of Theorem 66 depends upon Schmid's SL_2-orbit theorem. Informally, this result asserts that any 1-parameter nilpotent orbit is asymptotic to a nilpotent orbit arising from a representation of $\mathrm{SL}_2(\mathbb{R})$. In order to properly state Schmid's results we need to discuss splittings of mixed Hodge structures.

THEOREM 68 (DELIGNE [De1]). *Let (F, W) be a mixed Hodge structure on V. There exists a unique, functorial bigrading*

$$V_{\mathbb{C}} = \bigoplus_{p,q} I^{p,q}$$

such that

(a) $F^p = \bigoplus_{a \geq p} I^{a,b}$;
(b) $W_k = \bigoplus_{a+b \leq k} I^{a,b}$;
(c) $\overline{I^{p,q}} = I^{q,p} \mod \bigoplus_{r<q, s<p} I^{r,s}$.

In particular, if (F, W) is a mixed Hodge structure on V then (F, W) induces a mixed Hodge structure on $gl(V) \cong V \otimes V^*$ with bigrading

$$gl(V_{\mathbb{C}}) = \bigoplus_{r,s} gl(V)^{r,s}$$

where $gl(V)^{r,s}$ is the subspace of $gl(V)$ which maps $I^{p,q}$ to $I^{p+r,q+s}$ for all (p, q). In the case where (F, W) is graded-polarized, we have an analogous decomposition $\mathfrak{g}_{\mathbb{C}} = \bigoplus_{r,s} \mathfrak{g}^{r,s}$ of the Lie algebra of $G_{\mathbb{C}} (= \mathrm{Aut}(V_{\mathbb{C}}, Q))$. For future use, we define

$$\Lambda^{-1,-1}_{(F,W)} = \bigoplus_{r,s<0} gl(V)^{r,s} \tag{4-4}$$

and note that by properties (a)–(c) of Theorem 68

$$\lambda \in \Lambda^{-1,-1}_{(F,W)} \quad \Rightarrow \quad I^{p,q}_{(e^\lambda . F, W)} = e^\lambda . I^{p,q}_{(F,W)}. \tag{4-5}$$

A mixed Hodge structure (F, W) is split over $\mathbb{R}$ if $\bar{I}^{p,q} = I^{q,p}$ for (p, q). In general, a mixed Hodge structure (F, W) is not split over $\mathbb{R}$. However, by a theorem of Deligne [CKS1], there is a functorial splitting operation

$$(F, W) \mapsto (\hat{F}_\delta, W) = (e^{-i\delta} . F, W)$$

which assigns to any mixed Hodge structure (F, W) a split mixed Hodge structure $(\hat{F}_\delta, W)$, such that

(a) $\delta = \bar{\delta}$,

(b) $\delta \in \Lambda^{-1,-1}_{(F,W)}$, and

(c) δ commutes with all (r, r)-morphisms of (F, W).

REMARK 69. $\Lambda^{-1,-1}_{(F,W)} = \Lambda^{-1,-1}_{(\hat{F}_\delta,W)}$.

A nilpotent orbit $\hat{\theta}(z) = e^{zN}.F$ is an SL_2-orbit if there exists a group homomorphism $\rho : \mathrm{SL}_2(\mathbb{R}) \to G_\mathbb{R}$ such that

$$\hat{\theta}(g.\sqrt{-1}) = \rho(g).\hat{\theta}(\sqrt{-1})$$

for all $g \in \mathrm{SL}_2(\mathbb{R})$. The representation ρ is equivalent to the data of an sl_2-triple (N, H, N^+) of elements in $G_\mathbb{R}$ such that

$$[H, N] = -2N, \qquad [N^+, N] = H, \qquad [H, N^+] = 2N^+$$

We also note that, for nilpotent orbits of pure Hodge structure, the statement that $e^{zN}.F$ is an SL_2-orbit is equivalent to the statement that the limit mixed Hodge structure $(F, M(N))$ is split over $\mathbb{R}$ [CKS1].

THEOREM 70 (SL_2-ORBIT THEOREM, [Sc]). *Let $\theta(z) = e^{zN}.F$ be a nilpotent orbit of pure Hodge structure. Then, there exists a unique SL_2-orbit $\hat{\theta}(z) = e^{zN}.\hat{F}$ and a distinguished real-analytic function*

$$g(y) : (a, \infty) \to G_\mathbb{R}$$

(for some $a \in \mathbb{R}$) such that

(a) *$\theta(iy) = g(y).\hat{\theta}(iy)$ for $y > a$, and*

(b) *both $g(y)$ and $g^{-1}(y)$ have convergent series expansions about ∞ of the form*

$$g(y) = 1 + \sum_{k>0} g_j\, y^{-k}, \qquad g^{-1}(y) = 1 + \sum_{k>0} f_k\, y^{-k}$$

with g_k, $f_k \in \ker(\mathrm{ad}\, N)^{k+1}$.

Furthermore, the coefficients g_k and f_k can be expressed in terms of universal Lie polynomials in the Hodge components of δ with respect to $(\hat{F}, M(N))$ and $\mathrm{ad}\, N^+$.

REMARK 71. The precise meaning of the statement that $g(y)$ is a distinguished real-analytic function, is that $g(y)$ arises in a specific way from the solution of a system of differential equations attached to θ.

REMARK 72. If θ is a nilpotent orbit of pure Hodge structures of weight k and $\hat{\theta} = e^{zN}.\hat{F}$ is the associated SL_2-orbit then $(\hat{F}, M(N))$ is split over $\mathbb{R}$. The map $(F, M(N)) \mapsto (\hat{F}, M(N))$ is called the sl_2-splitting of $(F, M(N))$. Furthermore, $\hat{F} = e^{-\xi}.F$ where ξ is given by universal Lie polynomials in the Hodge components of δ. In this way, one obtains an sl_2-splitting $(F, W) \mapsto (\hat{F}, W)$ for any mixed Hodge structure (F, W).

4.3. Nilpotent and SL_2-orbit theorems in the mixed case. In analogy to the theory of period domains for pure HS, one can form a classifying space of graded-polarized mixed Hodge structure $\mathcal{M}$ with fixed Hodge numbers. Its points are the decreasing filtrations F of the reference fiber V which pair with the weight filtration W to define a graded-polarized mixed Hodge structure (with the given Hodge numbers). Given a variation of mixed Hodge structure $\mathcal{V}$ of this type over a complex manifold S, one obtains a period map

$$\phi : S \to \Gamma \backslash \mathcal{M}.$$

$\mathcal{M}$ is a complex manifold upon which the Lie group G, consisting of elements of $\mathrm{GL}(V_{\mathbb{C}})$ which preserve W and act by real isometries on Gr^W, acts transitively. Next, let $G_{\mathbb{C}}$ denote the Lie group consisting of elements of $\mathrm{GL}(V_{\mathbb{C}})$ which preserve W and act by *complex* isometries on Gr^W. Then, in analogy with the pure case, the "compact dual" $\check{\mathcal{M}}$ of $\mathcal{M}$ is the complex manifold

$$\check{\mathcal{M}} \cong G_{\mathbb{C}}/G_{\mathbb{C}}^{F_o}$$

for any base point $F_o \in \mathcal{M}$. The subgroup $G_{\mathbb{R}} = G \cap \mathrm{GL}(V_{\mathbb{R}})$ acts transitively on the real-analytic submanifold $\mathcal{M}_{\mathbb{R}}$ consisting of points $F \in \mathcal{M}$ such that (F, W) is split over $\mathbb{R}$.

EXAMPLE 73. Let $\mathcal{M}$ be the classifying space of mixed Hodge structures with Hodge numbers $h^{1,1} = h^{0,0} = 1$. Then, $\mathcal{M} \cong \mathbb{C}$.

The proof of Schmid's nilpotent orbit theorem depends critically upon the fact that the classifying space $\mathcal{D}$ has negative holomorphic sectional curvature along horizontal directions [GS]. Thus, although one can formally carry out all of the constructions leading up to the statement of the nilpotent orbit theorem in the mixed case, in light of the previous example it follows that one can not have negative holomorphic sectional curvature in the mixed case, and hence there is no reason to expect an analog of Schmid's Nilpotent Orbit Theorem in the mixed case. Indeed, for this classifying space $\mathcal{M}$, the period map $\varphi(s) = \exp(s)$ gives an example of a period map with trivial monodromy which has an essential singularity at ∞. Some additional condition is clearly required, and this is where admissibility comes in.

In the geometric case of a degeneration of pure Hodge structure, Steenbrink [St] gave an alternative construction of the limit Hodge filtration that can be extended to variations of mixed Hodge structure of geometric origin [SZ]. More generally, given an *admissible* variation of mixed Hodge structure $\mathcal{V}$ over a smooth complex algebraic variety $S \subseteq \bar{S}$ such that $D = \bar{S} - S$ is a normal crossing divisor, and any point $p \in D$ about which $\mathbb{V}$ has unipotent local monodromy, one has an associated nilpotent orbit $(e^{\sum_j z_j N_j}.F_\infty, W)$ with limit mixed Hodge structure (F_∞, M) where M is the *relative weight filtration* of $N = \sum_j N_j$ and W.[11] Furthermore, one has the following "group theoretic" version of the nilpotent orbit theorem: As in the pure case, a variation of mixed Hodge structure $\mathcal{V} \to (\Delta^*)^r$ with unipotent monodromy gives a holomorphic map

$$\begin{aligned} \psi : (\Delta^*)^r &\to \check{\mathcal{M}}, \\ \underline{z} &\mapsto e^{-\sum z_j N_j} F(\underline{z}), \end{aligned}$$

and this extends to Δ^r if $\mathcal{V}$ is admissible. Let

$$\mathfrak{q}_\infty = \bigoplus_{r<0} \mathfrak{g}^{r,s}$$

where $\mathfrak{g}_\mathbb{C} = \mathrm{Lie}(G_\mathbb{C}) = \bigoplus_{r,s} \mathfrak{g}^{r,s}$ relative to the limit mixed Hodge structure (F_∞, M). Then $\mathfrak{q}_\infty$ is a nilpotent Lie subalgebra of $\mathfrak{g}_\mathbb{C}$ which is a vector space complement to the isotropy algebra $\mathfrak{g}_\mathbb{C}^{F_\infty}$ of F_∞. Consequently, there exists an open neighborhood $\mathcal{U}$ of zero in $\mathfrak{g}_\mathbb{C}$ such that

$$\begin{aligned} \mathcal{U} &\to \check{\mathcal{M}}, \\ u &\mapsto e^u.F_\infty \end{aligned}$$

is a biholomorphism, and hence after shrinking Δ^r as necessary we can write

$$\psi(s) = e^{\Gamma(s)}.F_\infty$$

relative to a unique $\mathfrak{q}_\infty$-valued holomorphic function Γ on Δ^r which vanishes at 0. Recalling the construction of ψ from the lifted period map F, it follows that

$$F(z_1, \ldots, z_r) = e^{\sum_j z_j N_j} e^{\Gamma(s)}.F_\infty.$$

This is called the *local normal form* of $\mathcal{V}$ at p and will be used in the calculations of §5.4–5.

There is also a version of Schmid's SL_2-orbit theorem for admissible nilpotent orbits. In the case of 1-variable and weight filtrations of short length, the is due to the second author in [Pe2]. More generally, Kato, Nakayama and Usui

[11] Recall [SZ] that in general the relative weight filtration $M = M(N, W)$ is the unique filtration (if it exists) such that $N(M_k) \subset M_{k-2}$ and M induces the monodromy weight filtration of N on each Gr_i^W (centered about i).

proved a several variable SL_2-orbit theorem with arbitrary weight filtration in [KNU1]. Despite the greater generality of [KNU1], in this paper we are going to stick with the version of the SL_2-orbit theorem from [Pe2] as it is sufficient for our needs and has the advantage that for normal functions, mutatis mutandis, it is identical to Schmid's result.

4.4. Outline of proof of Theorem 61. Let us now specialize to the case of an admissible normal function $\nu : S \to J(\mathcal{H})$ over a curve and outline the proof [BP1] of Theorem 61. Before proceeding, we do need to address one aspect of the SL_2-orbit theorem in the mixed case. Let $\hat{\theta} = (e^{zN}.F, W)$ be an admissible nilpotent orbit with limit mixed Hodge structure (F, M) which is split over $\mathbb{R}$. Then, $\hat{\theta}$ induces an SL_2-orbit on each Gr_k^W, and hence a corresponding sl_2-representation ρ_k.

DEFINITION 74. Let W be an increasing filtration, indexed by $\mathbb{Z}$, of a finite dimensional vector space V. A *grading* of W is a direct sum decomposition $W_k = V_k \oplus W_{k-1}$ for each index k.

In particular, a mixed Hodge structure (F, W) on V gives a grading of W by the rule $V_k = \bigoplus_{p+q=k} I^{p,q}$. Furthermore, if the ground field has characteristic zero, a grading of W is the same thing as a semisimple endomorphism Y of V which acts as multiplication by k on V_k. If (F, W) is a mixed Hodge structure we let $Y_{(F,W)}$ denote the grading of W which acts on $I^{p,q}$ as multiplication by $p + q$, the *Deligne grading* of (F, W).

Returning to the admissible nilpotent orbit $\hat{\theta}$ considered above, we now have a system of representations ρ_k on Gr_k^W. To construct an sl_2-representation on the reference fiber V, we need to pick a grading Y of W. Clearly for each Hodge flag $F(z)$ in the orbit we have the Deligne grading $Y_{(F(z),W)}$; but we are after something more canonical. Now we also have the Deligne grading $Y_{(\hat{F},M)}$ of M associated to the sl_2-splitting of the LMHS. In the unpublished letter [De3], Deligne observed that:

THEOREM 75. *There exists a unique grading Y of W which commutes with $Y_{(\hat{F},M)}$ and has the property that if (N_0, H, N_0^+) denote the liftings of the sl_2-triples attached to the graded representations ρ_k via Y then $[N - N_0, N_0^+] = 0$.*

With this choice of sl_2-triple, and $\hat{\theta}$ an admissible nilpotent orbit in 1-variable of the type arising from an admissible normal function, the main theorem of [Pe2] asserts that one has a direct analog of Schmid's SL_2-orbit theorem as stated above for $\hat{\theta}$.

REMARK 76. More generally, given an admissible nilpotent orbit $(e^{zN}F, W)$ with relative weight filtration $M = M(N, W)$, Deligne shows that there exists

a grading $Y = Y(N, Y_{(F,M)})$ with similar properties. See [BP1] for details and further references.

REMARK 77. In the case of a normal function, if we decompose N according to $\operatorname{ad} Y$ we have $N = N_0 + N_{-1}$ where N_{-1} must be either zero or a highest weight vector of weight -1 for the representation of $sl_2(\mathbb{R})$ defined by (N_0, H, N_0^+). Accordingly, since there are no vectors of highest weight -1, we have $N = N_0$ and hence $[Y, N] = 0$.

The next thing that we need to recall is that if $\nu : S \to J(\mathcal{H})$ is an admissible normal function which is represented by an extension

$$0 \to H \to \mathcal{V} \to \mathbb{Z}(0) \to 0$$

in the category of admissible variations of mixed Hodge structure on S then the zero locus $\mathcal{Z}$ of ν is exactly the set of points where the corresponding Deligne grading $Y_{(\mathcal{F},\mathcal{W})}$ is integral. In the case where $S \subset \bar{S}$ is a curve, in order to prove the algebraicity of $\mathcal{Z}$, all we need to do is show that $\mathcal{Z}$ cannot contain a sequence of points $s(m)$ which accumulate to a puncture $p \in \bar{S} - S$ unless ν is identically zero. The first step towards the proof of Theorem 61 is the following result [BP1]:

THEOREM 78. *Let $\varphi : \Delta^* \to \Gamma\backslash\mathcal{M}$ denote the period map of an admissible normal function $\nu : \Delta^* \to J(\mathcal{H})$ with unipotent monodromy, and Y be the grading of W attached to the nilpotent orbit θ of φ by Deligne's construction* (Theorem 75). *Let $F : U \to \mathcal{M}$ denote the lifting of φ to the upper half-plane. Then, for* $\operatorname{Re}(z)$ *restricted to an interval of finite length, we have*

$$\lim_{\operatorname{Im}(z)\to\infty} Y_{(F(z),W)} = Y$$

SKETCH OF PROOF. Using [De3], one can prove this result in the case where φ is a nilpotent orbit with limit mixed Hodge structure which is split over $\mathbb{R}$. Let $z = x + iy$. In general, one writes

$$F(z) = e^{zN} e^{\Gamma(s)}.F_\infty = e^{xN} e^{iyN} e^{\Gamma(s)} e^{-iyN} e^{iyN}.F_\infty$$

where e^{xN} is real, $e^{iyN}.F_\infty$ can be approximated by an SL_2-orbit and the function $e^{iyN} e^{\Gamma(s)} e^{-iyN}$ decays to 1 very rapidly. □

In particular, if there exists a sequence $s(m)$ which converges to p along which $Y_{(\mathcal{F},\mathcal{W})}$ is integral it then follows from the previous theorem that Y is integral. An explicit computation then shows that the equation of the zero locus near p is given by the equation

$$\operatorname{Ad}(e^{\Gamma(s)})Y = Y,$$

which is clearly holomorphic on a neighborhood of p in $\bar{S}$.

That concludes the proof for S a curve. In the case where S has a compactification $\bar{S}$ such that $\bar{S} - S$ is a smooth divisor, one can prove Theorem 61 by the same techniques by studying the dependence of the preceding constructions on holomorphic parameters, i.e., at a point in D we get a nilpotent orbit

$$\theta(z; s_2, \ldots, s_r) = e^{zN}.F_\infty(s_2, \ldots, s_r),$$

where $F_\infty(s_2, \ldots, s_r)$ depend holomorphically on the parameters $(s_2, \ldots, s_r)$.

4.5. Zero loci and filtrations on Chow groups. Returning now to the algebraicity of the Hodge locus discussed at the beginning of this section, the Hodge Conjecture would further imply that if $f : X \to S$ can be defined over an algebraically closed subfield of $\mathbb{C}$ then so can the germ of T. Claire Voisin [Vo1] gave sufficient conditions for T to be defined over $\bar{\mathbb{Q}}$ if $f : X \to S$ is defined over $\mathbb{Q}$. Very recently F. Charles [Ch] carried out an analogous investigation of the field of definition of the zero locus $\mathcal{Z}$ of a normal function motivated over $\mathbb{F}$. We reprise this last notion (from Sections 1 and 2):

DEFINITION 79. Let S be a smooth quasiprojective variety defined over a subfield $\mathbb{F}_0 \subset \mathbb{C}$, and let $\mathbb{F} \subset \mathbb{C}$ be a finitely generated extension of $\mathbb{F}_0$. An admissible normal function $\nu \in \mathrm{ANF}(S, \mathcal{H})$ is *motivated over* $\mathbb{F}$ if there exists a smooth quasiprojective variety $\mathcal{X}$, a smooth projective morphism $f : \mathcal{X} \to S$, and an algebraic cycle $\mathfrak{Z} \in Z^m(\mathcal{X})_{\mathrm{prim}}$, all defined over $\mathbb{F}$, such that $\mathcal{H}$ is a subVHS of $(R^{2m-1} f_* \mathbb{Z}) \otimes \mathcal{O}_S$ and $\nu = \nu_{\mathfrak{Z}}$.

REMARK 80. Here $Z^m(\mathcal{X})_{\mathrm{prim}}$ denotes algebraic cycles with homologically trivial restriction to fibers. One traditionally also assumes $\mathfrak{Z}$ is flat over S, but this can always be achieved by restricting to $U \subset S$ sufficiently small (Zariski open); and then by [S1] *(i)* $\nu_{\mathfrak{Z}_U}$ *is* $\bar{S}$ *admissible.* Next, for any $s_0 \in S$ one can move $\mathfrak{Z}$ by a rational equivalence to intersect X_{s_0} (hence the $\{X_s\}$ for s in an analytic neighborhood of s_0) properly, and then use the remarks at the beginning of [Ki] or [GGK, § III.B] to see that *(ii)* $\nu_{\mathfrak{Z}}$ *is defined and holomorphic over all of* S. Putting *(i)* and *(ii)* together with [BFNP, Lemma 7.1], we see that $\nu_{\mathfrak{Z}}$ is itself admissible.

Recall that the level of a VHS $\mathcal{H}$ is (for a generic fiber H_s) the maximum difference $|p_1 - p_2|$ for H^{p_1,q_1} and H^{p_2,q_2} both nonzero. A fundamental open question about motivic normal functions is then:

CONJECTURE 81. (i) $[\mathfrak{Z}\mathfrak{L}(D, E)]$ *For every* $\mathbb{F} \subset \mathbb{C}$ *finitely generated over* $\bar{\mathbb{Q}}$, $S/\mathbb{F}$ *smooth quasiprojective of dimension* D, *and* $\mathcal{H} \to S$ *VHS of weight* (-1) *and level* $\leq 2E-1$, *the following holds*: ν *motivated over* $\mathbb{F}$ *implies that* $\mathcal{Z}(\nu)$ *is an at most countable union of subvarieties of* S *defined over* (*possibly different*) *finite extensions of* $\mathbb{F}$.

(ii) [$\widetilde{\mathfrak{Z}\mathfrak{L}}(D,E)$] *Under the same hypotheses,* $\mathcal{Z}(\nu)$ *is an algebraic subvariety of* S *defined over an algebraic extension of* $\mathbb{F}$.

Clearly Theorem 57 and Conjecture $\mathfrak{Z}\mathfrak{L}(D,E)$ together imply $\widetilde{\mathfrak{Z}\mathfrak{L}}(D,E)$, but it is much more natural to phrase some statements (especially Proposition 86 below) in terms of $\mathfrak{Z}\mathfrak{L}(D,E)$. If true even for $D=1$ (but general E), Conjecture 81(i) would resolve a longstanding question on the structure of Chow groups of complex projective varieties. To wit, the issue is whether the second Bloch–Beilinson filtrand and the kernel of the AJ map must agree; we now wish to describe this connection. We shall write $\widetilde{\mathfrak{Z}\mathfrak{L}}(D,1)_{\text{alg}}$ for the case when ν is motivated by a family of cycles algebraically equivalent to zero.

Let X be smooth projective and $m\in\mathbb{N}$. Denoting "$\otimes\mathbb{Q}$" by a subscript $\mathbb{Q}$, we have the two "classical" invariants $cl_{X,\mathbb{Q}}: CH^m(X)_{\mathbb{Q}}\to \mathrm{Hg}^m(X)_{\mathbb{Q}}$ and $AJ_{X,\mathbb{Q}}$: $\ker(cl_{X,\mathbb{Q}})\to J^m(X)_{\mathbb{Q}}$. It is perfectly natural both to ask for further Hodge-theoretic invariants for cycle-classes in $\ker(AJ_{X,\mathbb{Q}})$, and inquire as to what sort of filtration might arise from their successive kernels. The idea of a (conjectural) *system* of decreasing filtrations on the rational Chow groups of *all* smooth projective varieties over $\mathbb{C}$, compatible with the intersection product, morphisms induced by correspondences, and the algebraic Künneth components of the diagonal Δ_X, was introduced by A. Beilinson [Be], and independently by S. Bloch. (One has to assume something like the Hard Lefschetz Conjecture so that these Künneth components exist; the compatibility roughly says that $\mathrm{Gr}^i\, CH^m(X)_{\mathbb{Q}}$ is controlled by $H^{2m-i}(X)$.) Such a filtration $F^\bullet_{\mathrm{BB}}$ is unique if it exists and is universally known as a *Bloch–Beilinson filtration* (BBF); there is a wide variety of constructions which yield a BBF under the assumption of various more-or-less standard conjectures. The one which is key for the filtration (due to Lewis [Le2]) we shall consider is the *arithmetic Bloch–Beilinson Conjecture* (BBC):

CONJECTURE 82. *If* $\mathcal{X}/\bar{\mathbb{Q}}$ *is a quasiprojective variety, the absolute-Hodge cycle-class map*

$$c_{\mathcal{H}}: CH^m(\mathcal{X})_{\mathbb{Q}}\to H^{2m}_{\mathcal{H}}(\mathcal{X}^{\mathrm{an}}_{\mathbb{C}},\mathbb{Q}(m)) \tag{4-6}$$

is injective. (Here $CH^m(\mathcal{X})$ denotes $\equiv_{\mathrm{rat}}$-classes of cycles over $\bar{\mathbb{Q}}$, and differs from $CH^m(\mathcal{X}_{\mathbb{C}})$.)

Now for $X/\mathbb{C}$, $c_{\mathcal{H}}$ on $CH^m(X)_{\mathbb{Q}}$ is far from injective (the kernel usually not even being parametrizable by an algebraic variety); but any given cycle $Z\in Z^m(X)$ (a priori defined over $\mathbb{C}$) is in fact defined over a subfield $K\subset\mathbb{C}$ finitely generated over $\bar{\mathbb{Q}}$, say of transcendence degree t. Considering X, Z over K, the $\bar{\mathbb{Q}}$-spread then provides

- a smooth projective variety $\bar{S}/\bar{\mathbb{Q}}$ of dimension t, with $\bar{\mathbb{Q}}(\bar{S})\xrightarrow{\cong} K$ and $s_0:$ $\mathrm{Spec}(K)\to\bar{S}$ the corresponding generic point;

- a smooth projective variety $\bar{\mathcal{X}}$ and projective morphism $\bar{\pi} : \bar{\mathcal{X}} \to \bar{S}$, both defined over $\bar{\mathbb{Q}}$, such that $X = X_{s_0} := \mathcal{X} \times_{s_0} \mathrm{Spec}(K)$; and
- an algebraic cycle $\bar{\mathcal{Z}} \in Z^m(\mathcal{X}_{(\bar{\mathbb{Q}})})$ with $Z = \bar{\mathcal{Z}} \times_{s_0} \mathrm{Spec}(K)$.

Writing $\bar{\pi}^{\mathrm{sm}} =: \pi : \mathcal{X} \to S$ (and $\mathcal{Z} := \bar{\mathcal{Z}} \cap \mathcal{X}$), we denote by $U \subset S$ any affine Zariski open subvariety defined over $\bar{\mathbb{Q}}$, and put $\mathcal{X}_U := \pi^{-1}(U)$, $\mathcal{Z}_U := \bar{\mathcal{Z}} \cap \mathcal{X}_U$; note that s_0 factors through all such U.

The point is that exchanging the field of definition for additional geometry allows $c_{\mathcal{H}}$ to "see" more; in fact, since we are over $\bar{\mathbb{Q}}$, it should now (by BBC) see everything. Now $c_{\mathcal{H}}(\mathcal{Z}_U)$ packages cycle-class and Abel–Jacobi invariants together, and the idea behind Lewis's filtration (and filtrations of M. Saito and Green/Griffiths) is to split the whole package up into Leray graded pieces with respect to π. Miraculously, the 0-th such piece turns out to agree with the fundamental class of Z, and the next piece is the normal function generated by $\mathcal{Z}_U$. The pieces after that define the so-called *higher* cycle-class and AJ maps.

More precisely, we have

$$\begin{array}{ccc} & & CH^m(X_{(K)})_{\mathbb{Q}} \\ & \overset{\Psi :=}{\nearrow} & \cong \big\downarrow \text{spread} \\ & & \mathrm{im}\{CH^m(\bar{\mathcal{X}})_{\mathbb{Q}} \to \varinjlim_U CH^m(\mathcal{X}_U)_{\mathbb{Q}}\} \\ & & \big\downarrow c_{\mathcal{H}} \\ \underline{H}^{2m}_{\mathcal{H}} : = & = & \mathrm{im}\Big(H^{2m}_{\mathcal{D}}(\bar{\mathcal{X}}^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q}(m)) \to \varinjlim_U H^{2m}_{\mathcal{H}}((\mathcal{X}_U)^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q}(m))\Big) \end{array} \tag{4-7}$$

with $c_{\mathcal{H}}$ (hence Ψ) conjecturally injective. Lewis [Le2] defines a Leray filtration $\mathcal{L}^\bullet \underline{H}^{2m}_{\mathcal{H}}$ with graded pieces

$$\begin{array}{c} 0 \\ \big\downarrow \\ \dfrac{J^0\Big(\varinjlim_U W_{-1} H^{i-1}(U, R^{2m-i}\pi_*\mathbb{Q}(m))\Big)}{\mathrm{im} \varinjlim_U \mathrm{Hg}^0\Big(\mathrm{Gr}^W_0 H^i(U, R^{2m-i}\pi_*\mathbb{Q}(m))\Big)} \\ \big\downarrow \beta \\ \mathrm{Gr}^i_{\mathcal{L}} \underline{H}^{2m}_{\mathcal{H}} \\ \big\downarrow \alpha \\ \mathrm{Hg}^0\Big(\varinjlim_U W_0 H^i(U, R^{2m-i}\pi_*\mathbb{Q}(m))\Big) \\ \big\downarrow \\ 0 \end{array} \tag{4-8}$$

and sets $\mathcal{L}^i CH^m(X_K)_{\mathbb{Q}} := \Psi^{-1}(\mathcal{L}^i \underline{H}^{2m}_{\mathcal{H}})$. For $Z \in \mathcal{L}^i CH^m(X_K)_{\mathbb{Q}}$, we put $cl^i_X(Z) := \alpha(\mathrm{Gr}^i_{\mathcal{L}} \Psi(Z))$; if this vanishes then $\mathrm{Gr}^i_{\mathcal{L}} \Psi(Z) =: \beta(aj^{i-1}_X(Z))$, and vanishing of $cl^i(Z)$ and $aj^{i-1}(Z)$ implies membership in $\mathcal{L}^{i+1}$. One easily finds that $cl^0_X(Z)$ identifies with $cl_{X,\mathbb{Q}}(Z) \in \mathrm{Hg}^0(X)_{\mathbb{Q}}$.

REMARK 83. The arguments of Hg^0 and J^0 in (4-8) have canonical and functorial MHS by [Ar]. One should think of the top term as $\mathrm{Gr}^{i-1}_{\mathcal{L}}$ of the lowest-weight part of $J^m(\mathcal{X}_U)$ and the bottom as $\mathrm{Gr}^i_{\mathcal{L}}$ of the lowest-weight part of $\mathrm{Hg}^m(\mathcal{X}_U)$ (both in the limit over U).

Now to get a candidate BBF, Lewis takes

$$\mathcal{L}^i CH^m(X_{\mathbb{C}})_{\mathbb{Q}} := \varinjlim_{\substack{K \subset \mathbb{C} \\ \text{f.g.}/\bar{\mathbb{Q}}}} \mathcal{L}^i CH^m(X_K)_{\mathbb{Q}}.$$

Some consequences of the definition of a BBF mentioned above, specifically the compatibility with the Künneth components of Δ_X, include these:

(a) $\begin{cases} F^0_{\mathrm{BB}} CH^m(X)_{\mathbb{Q}} = CH^m(X)_{\mathbb{Q}}, \\ F^1_{\mathrm{BB}} CH^m(X)_{\mathbb{Q}} = CH^m_{\mathrm{hom}}(X)_{\mathbb{Q}}, \\ F^2_{BB} CH^m(X)_{\mathbb{Q}} \subseteq \ker(AJ_{X,\mathbb{Q}}), \end{cases}$

(b) $F^{m+1}_{\mathrm{BB}} CH^m(X) = \{0\}$.

These are sometimes stated as additional requirements for a BBF.

THEOREM 84 [Le2]. *$\mathcal{L}^\bullet$ is intersection- and correspondence-compatible, and satisfies* (*a*). *Assuming BBC, $\mathcal{L}^\bullet$ satisfies* (*b*); *and additionally assuming HLC, $\mathcal{L}^\bullet$ is a BBF*.

The limits in (4-8) inside J^0 and Hg^0 stabilize for sufficiently small U; replacing S by such a U, we may consider the normal function $\nu_{\mathcal{Z}} \in \mathrm{ANF}(S, \mathcal{H}^{2m-1}_{\mathcal{X}/S})$ attached to the $\bar{\mathbb{Q}}$-spread of Z.

PROPOSITION 85. (i) *For $i = 1$,* (4-8) *becomes*

$$0 \to J^m_{\mathrm{fix}}(\mathcal{X}/S)_{\mathbb{Q}} \to \mathrm{Gr}^1_{\mathcal{L}} \underline{H}^{2m}_{\mathcal{H}} \to \left(H^1(S, R^{2m-1}\pi_*\mathbb{Q})\right)^{(0,0)} \to 0.$$

(ii) *For $Z \in CH^m_{\mathrm{hom}}(X_K)_{\mathbb{Q}}$, we have $cl^1_X(Z) = [\nu_{\mathcal{Z}}]_{\mathbb{Q}}$. If this vanishes, then $aj^0_X(Z) = AJ_X(Z)_{\mathbb{Q}} \in J^m_{\mathrm{fix}}(\mathcal{X}/S)_{\mathbb{Q}} \subset J^m(X)_{\mathbb{Q}}$* (implying $\mathcal{L}^2 \subset \ker AJ_{\mathbb{Q}}$).

So for $Z \in CH^m_{\mathrm{hom}}(X_K)$ with $\bar{\mathbb{Q}}$-spread $\mathcal{Z}$ over S, the information contained in $\mathrm{Gr}^1_{\mathcal{L}} \Psi(Z)$ is (up to torsion) precisely $\nu_{\mathcal{Z}}$. Working over $\mathbb{C}$, $\mathcal{Z} \cdot X_{s_0} = Z$ is the fiber of the spread at a *very general point* $s_0 \in S(\mathbb{C})$: $\mathrm{trdeg}(\bar{\mathbb{Q}}(s_0)/\bar{\mathbb{Q}})$ is maximal, i.e., equal to the dimension of S. Since AJ is a *transcendental* (rather than algebraic) invariant, there is no outright guarantee that vanishing of

$AJ_X(Z) \in J^m(X)$ — or equivalently, of the normal function at a very general point — implies the identical vanishing of $\nu_{\mathcal{Z}}$ or even $[\nu_{\mathcal{Z}}]$. To display explicitly the depth of the question:

PROPOSITION 86. (i) $\mathcal{Z}\mathfrak{L}(1,E)$ $(\forall E \in \mathbb{N}) \iff \mathcal{L}^2 CH^m(X)_{\mathbb{Q}} = \ker(AJ_{X,\mathbb{Q}})$ ($\forall$ *sm. proj.* $X/\mathbb{C}$).

(ii) $\mathcal{Z}\mathfrak{L}(1,1)_{\text{alg}} \iff \mathcal{L}^2 CH^m(X)_{\mathbb{Q}} \cap CH^m_{\text{alg}}(X)_{\mathbb{Q}} = \ker(AJ_{X,\mathbb{Q}}) \cap CH^m_{\text{alg}}(X)_{\mathbb{Q}}$ ($\forall$ *sm. proj.* $X/\mathbb{C}$).

Roughly speaking, these statements say that "sensitivity of the zero locus (of a cycle-generated normal function) to field of definition" is equivalent to "spreads of homologically and AJ-trivial cycles give trivial normal functions". In (ii), the cycles in both statements are assumed algebraically equivalent to zero.

PROOF. We first remark that for any variety S with field of definition $\mathbb{F}$ of minimal transcendence degree, no proper $\bar{\mathbb{F}}$-subvariety of S contains (in its complex points) a very general point of S.

(i) $(\Rightarrow)$: Let $\mathcal{Z}$ be the $\bar{\mathbb{Q}}$-spread of Z with $AJ(Z)_{\mathbb{Q}} = 0$, and suppose $\mathrm{Gr}^1_{\mathcal{L}} \Psi(Z) = \mathrm{Gr}^1_{\mathcal{L}} c_{\mathcal{H}}(\mathcal{Z})$ does not vanish. Taking a 1-dimensional very general multiple hyperplane section $S_0 \subset S$ through s_0 (S_0 is "minimally" defined over $k \overset{\text{trdeg.}1}{\subseteq} K$), the restriction $\mathrm{Gr}^1_{\mathcal{L}} c_{\mathcal{H}}(\mathcal{Z}_0) \neq 0$ by weak Lefschetz. Since each $\mathcal{Z}(\nu_{N\mathcal{Z}_0}) \subseteq S_0$ is a union of subvarieties defined over $\bar{k}$ and contains s_0 for some $N \in \mathbb{N}$, one of these is all of S_0 (which implies $\mathrm{Gr}^1_{\mathcal{L}} \Psi(Z) = 0$), a contradiction. So $Z \in \mathcal{L}^2$.

$(\Leftarrow)$: Let $\mathcal{X}_0 \to S_0$, $\mathcal{Z}_0 \in Z^m(\mathcal{X}_0)_{\text{prim}}$, $\dim(S_0) = 1$, all be defined over k and suppose $\mathcal{Z}(\nu_{\mathcal{Z}_0})$ contains a point s_0 not defined over $\bar{k}$. Spreading this out over $\bar{\mathbb{Q}}$ to $\mathcal{Z}, \mathcal{X}, S \supset S_0 \ni s_0$, we have: $s_0 \in S$ is very general, $\mathcal{Z}$ is the $\bar{\mathbb{Q}}$-spread of $Z = \mathcal{Z}_0 \cdot X_{s_0}$, and $AJ(Z)_{\mathbb{Q}} = 0$. So $Z \in \mathcal{L}^2$ implies $\nu_{\mathcal{Z}}$ is torsion, which implies $\nu_{\mathcal{Z}_0}$ is torsion. But then $\nu_{\mathcal{Z}_0}$ is zero since it is zero somewhere (at s_0). So $\mathcal{Z}(\nu_{\mathcal{Z}_0})$ is either S_0 or a (necessarily countable) union of $\bar{k}$-points of S_0.

(ii) The spread $\mathcal{Z}$ of $Z_{(s_0)} \equiv_{\text{alg}} 0$ has every fiber $Z_s \equiv_{\text{alg}} 0$, hence $\nu_{\mathcal{Z}}$ is a section of $J(\mathcal{H})$, $\mathcal{H} \subset (R^{2m-1}\pi_*\mathbb{Q}(m)) \otimes \mathcal{O}_S$ a subVHS of level one (which can be taken to satisfy $H_s = (H^{2m-1}(X_s))_h$ for a.e. $s \in S$). The rest is as in (i). □

REMARK 87. A related candidate BBF which occurs in work of the first author with J. Lewis [KL, §4], is defined via successive kernels of *generalized* normal functions (associated to the $\bar{\mathbb{Q}}$-spread $\mathcal{Z}$ of a cycle). These take values on very general $(i-1)$-dimensional subvarieties of S (rather than at points), and have the above $cl^i(Z)$ as their topological invariants.

4.6. Field of definition of the zero locus. We shall begin by showing that the equivalent conditions of Proposition 86(ii) are satisfied; the idea of the argument is due in part to S. Saito [Sa]. The first paragraph in the following in fact gives more:

THEOREM 88. *$\widetilde{\mathfrak{Z}\mathfrak{L}}(D,1)_{\text{alg}}$ holds for all $D \in \mathbb{N}$. That is, the zero locus of any normal function motivated by a family of cycles over $\mathbb{F}$ algebraically equivalent to zero, is defined over an algebraic extension of $\mathbb{F}$.*

Consequently, cycles algebraically and Abel–Jacobi-equivalent to zero on a smooth projective variety over $\mathbb{C}$, lie in the second Lewis filtrand.

PROOF. Consider $\mathfrak{Z} \in Z^m(\mathcal{X})_{\text{prim}}$ and $f : \mathcal{X} \to S$ defined over K (K being finitely generated over $\bar{\mathbb{Q}}$), with $Z_s \equiv_{\text{alg}} 0\ \forall s \in S$; and let $s_0 \in \mathcal{Z}(\nu_{\mathfrak{Z}})$. (Note: s_0 is just a complex point of S.) We need to show:

$$\exists N \in \mathbb{N} \text{ such that } \sigma(s_0) \in \mathcal{Z}(\nu_{N\mathfrak{Z}}) \text{ for any } \sigma \in \text{Gal}(\mathbb{C}/K). \tag{4-9}$$

Here is why (4-9) will suffice: the analytic closure of the set of all conjugate points is simply the point's K-spread $S_0 \subset S$, a (possibly reducible) algebraic subvariety defined over K. Clearly, on the s_0-connected component of S_0, $\nu_{\mathfrak{Z}}$ itself then vanishes; and this component is defined over an algebraic extension of K. Trivially, $\mathcal{Z}(\nu_{\mathfrak{Z}})$ is the union of such connected spreads of its points s_0; and since K is finitely generated over $\bar{\mathbb{Q}}$, there are only countably many subvarieties of S_0 defined over K or algebraic extensions thereof. This proves $\mathfrak{Z}\mathfrak{L}(D,1)_{\text{alg}}$, hence (by Theorem 57) $\widetilde{\mathfrak{Z}\mathfrak{L}}(D,1)_{\text{alg}}$.

To show (4-9), write $X = X_{s_0}$, $Z = Z_{s_0}$, and $L(/K)$ for their field of definition. There exist over L

- a smooth projective curve C and points $0, q \in C(L)$;
- an algebraic cycle $W \in Z^m(C \times X)$ such that $Z = W_*(q-0)$; and
- another cycle $\Gamma \in Z^1(J(C) \times C)$ defining Jacobi inversion.

Writing $\Theta := W \circ \Gamma \in Z^m(J(C) \times X)$, the induced map

$$[\Theta]_* : J(C) \to J^m(X)_{\text{alg}} \left(\subseteq J^m(X)_h\right)$$

is necessarily a morphism of abelian varieties over L; hence the identity connected component of $\ker([\Theta]_*)$ is a subabelian variety of $J(C)$ defined over an algebraic extension $L' \supset L$. Define $\theta := \Theta|_B \in Z^m(B \times X)$, and observe that $[\theta]_* : B \to J^m(X)_{\text{alg}}$ is zero by construction, so that $cl(\theta) \in \mathcal{L}^2 H^{2m}(B \times X)$.

Now, since $AJ_X(Z) = 0$, a multiple $b := N.AJ_C(q-0)$ belongs to B, and then $N.Z = \theta_* b$. This "algebraizes" the AJ-triviality of $N.Z$: conjugating the 6-tuple $(s_0, Z, X, B, \theta, b)$ to $(\sigma(s_0), Z^\sigma[= Z_{\sigma(s_0)}], X^\sigma[= X_{\sigma(s_0)}], B^\sigma, \theta^\sigma, b^\sigma)$, we still have $N.Z^\sigma = \theta^\sigma_* b^\sigma$ and $cl(\theta^\sigma) \in \mathcal{L}^2 H^{2m}(B^\sigma \times X^\sigma)$ by motivicity of the Leray filtration [Ar], and this implies $N.AJ(Z^\sigma) = [\theta^\sigma]_* b^\sigma = 0$ as desired. □

We now turn to the result of [Ch] indicated at the outset of §4.5. While interesting, it sheds no light on $\mathfrak{Z}\mathfrak{L}(1, E)$ or filtrations, since the hypothesis that the VHS $\mathcal{H}$ have no global sections is untenable over a point.

THEOREM 89 [Ch, Theorem 3]. *Let $\mathcal{Z}$ be the zero locus of a k-motivated normal function $\nu : S \to J(\mathcal{H})$. Assume that $\mathcal{Z}$ is algebraic and $\mathbb{H}_{\mathbb{C}}$ has no nonzero global sections over $\mathcal{Z}$. Then $\mathcal{Z}$ is defined over a finite extension of k.*

PROOF. Charles's proof of this result uses the ℓ-adic Abel–Jacobi map. Alternatively, we can proceed as follows (using, with $\mathbb{F} = k$, the notation of Definition 79): take $\mathcal{Z}_0 \subset \mathcal{Z}(\nu)$ to be an irreducible component (without loss of generality assumed smooth), and $\mathfrak{Z}_{\mathcal{Z}_0}$ the restriction of $\mathfrak{Z}$ to $\mathcal{Z}_0$. Let $[\mathfrak{Z}_{\mathcal{Z}_0}]$ and $[\mathfrak{Z}_{\mathcal{Z}_0}]_{dR}$ denote the Betti and de Rham fundamental classes of $\mathfrak{Z}_{\mathcal{Z}_0}$, and $\mathcal{L}$ the Leray filtration. Then, $\mathrm{Gr}^1_{\mathcal{L}}[\mathfrak{Z}_{\mathcal{Z}_0}]$ is the topological invariant of $[\mathfrak{Z}_{\mathcal{Z}_0}]$ in $H^1(\mathcal{Z}_0, R^{2m-1} f_* \mathbb{Z})$, whereas $\mathrm{Gr}^1_{\mathcal{L}}[\mathfrak{Z}_{\mathcal{Z}_0}]_{dR}$ is the infinitesimal invariant of $\nu_{\mathfrak{Z}}$ over $\mathcal{Z}_0$. In particular, since $\mathcal{Z}_0$ is contained in the zero locus of $\nu_{\mathfrak{Z}}$,

$$\mathrm{Gr}^j_{\mathcal{L}}[\mathfrak{Z}_{\mathcal{Z}_0}]_{dR} = 0, \quad j = 0, 1. \tag{4-10}$$

Furthermore, by the algebraicity of the Gauss–Manin connection, (4-10) is invariant under conjugation:

$$\mathrm{Gr}^j_{\mathcal{L}}[\mathfrak{Z}_{\mathcal{Z}_0^\sigma}]_{dR} = (\mathrm{Gr}^j_{\mathcal{L}}[\mathfrak{Z}_{\mathcal{Z}_0}]_{dR})^\sigma$$

and hence $\mathrm{Gr}^j_{\mathcal{L}}[\mathfrak{Z}_{\mathcal{Z}_0^\sigma}]_{dR} = 0$ for $j = 0, 1$. Therefore, $\mathrm{Gr}^j_{\mathcal{L}}[\mathfrak{Z}_{\mathcal{Z}_0^\sigma}] = 0$ for $j = 0, 1$, and hence $AJ(Z_s)$ takes values in the fixed part of $J(\mathcal{H})$ for $s \in \mathcal{Z}_0^\sigma$. By assumption, $\mathbb{H}_{\mathbb{C}}$ has no fixed part over $\mathcal{Z}_0$, and hence no fixed part over $\mathcal{Z}_0^\sigma$ (since conjugation maps ∇-flat sections to ∇-flat sections by virtue of the algebraicity of the Gauss–Manin connection). As such, conjugation must take us to another component of $\mathcal{Z}$, and hence (since $\mathcal{Z}$ is algebraic over $\mathbb{C}$ implies $\mathcal{Z}$ has only finitely many components), $\mathcal{Z}_0$ must be defined over a finite extension of k. □

We conclude with a more direct analog of Voisin's result [Vo1, Theorem 0.5(2)] on the algebraicity of the Hodge locus. If $\mathcal{V}$ is a variation of mixed Hodge structure over a complex manifold and

$$\alpha \in (\mathcal{F}^p \cap \mathcal{W}_{2p} \cap V_{\mathbb{Q}})_{s_o}$$

for some $s_o \in S$, then the Hodge locus T of α is the set of points in S where some parallel translate of α belongs to $\mathcal{F}^p$.

REMARK 90. If (F, W) is a mixed Hodge structure on V and $v \in F^p \cap W_{2p} \cap V_{\mathbb{Q}}$ then v is of type (p, p) with respect to Deligne's bigrading of (F, W).

THEOREM 91. *Let S be a smooth complex algebraic variety defined over a subfield k of $\mathbb{C}$, and $\mathcal{V}$ be an admissible variation of mixed Hodge structure of geometric origin over S. Suppose that T is an irreducible subvariety of S over $\mathbb{C}$ such that*:

(a) *T is an irreducible component of the Hodge locus of some*

$$\alpha \in (\mathcal{F}^p \cap \mathcal{W}_{2p} \cap \mathbb{V}_{\mathbb{Q}})_{t_o};$$

(b) *$\pi_1(T, t_o)$ fixes only the line generated by α.*

Then, T is defined over $\bar{k}$.

PROOF. If $\mathcal{V} \cong \mathbb{Q}(p)$ for some p then $T = S$. Otherwise, T cannot be an isolated point without violating (b). Assume therefore that $\dim T > 0$. Over T, we can extend α to a flat family of de Rham classes. By the algebraicity of the Gauss–Manin connection, the conjugate α^σ is flat over T^σ. Furthermore, if T^σ supports any additional flat families of de Rham classes, conjugation by σ^{-1} gives a contradiction to (b). Therefore, $\alpha^\sigma = \lambda\beta$, where β is a $\pi_1(T^\sigma)$-invariant Betti class on T^σ which is unique up to scaling. Moreover,

$$Q(\alpha, \alpha) = Q(\alpha^\sigma, \alpha^\sigma) = \lambda^2 Q(\beta, \beta)$$

and hence there are countably many Hodge classes that one can conjugate α to via $\mathrm{Gal}(\mathbb{C}/k)$. Accordingly, T must be defined over $\bar{k}$. □

5. The Néron model and obstructions to singularities

The unifying theme of the previous sections is the study of algebraic cycles via degenerations using the Abel–Jacobi map. In particular, in the case of a semistable degeneration $\pi : X \to \Delta$ and a *cohomologically trivial* cycle Z which properly intersects the fibers, we have

$$\lim_{s\to 0} AJ_{X_s}(Z_s) = AJ_{X_0}(Z_0)$$

as explained in detail in § 2. In general however, the existence of the limit Abel–Jacobi map is obstructed by the existence of the singularities of the associated normal function. Nonetheless, using the description of the asymptotic behavior provided by the nilpotent and SL_2-orbit theorems, we can define the limits of admissible normal functions along curves and prove the algebraicity of the zero locus.

5.1. Néron models in one parameter. In this section we consider the problem of geometrizing these constructions (ANFs and their singularities, limits and zeroes) by constructing a Néron model which graphs admissible normal functions. The quest to construct such objects has a long history which traces back to the work of Néron on minimal models for abelian varieties A_K defined over the

field of fractions K of a discrete valuation ring R. In [Na], Nakamura proved the existence of an analytic Néron model for a family of abelian varieties $\mathcal{A} \to \Delta^*$ arising from a variation of Hodge structure $\mathcal{H} \to \Delta^*$ of level 1 with unipotent monodromy. With various restrictions, this work was then extended to normal functions arising from higher codimension cycles in the work of Clemens [Cl2], El Zein and Zucker [EZ], and Saito [S1].

REMARK 92. Unless otherwise noted, throughout this section we assume that the local monodromy of the variation of Hodge structure $\mathcal{H}$ under consideration is unipotent, and the local system $\mathbb{H}_{\mathbb{Z}}$ is torsion free.

A common feature in all of these analytic constructions of Néron models for variations of Hodge structure over Δ^* is that the fiber over $0 \in \Delta$ is a complex Lie group which has only finitely many components. Furthermore, the component into which a given normal function ν extends is determined by the value of $\sigma_{\mathbb{Z},0}(\nu)$. Using the methods of the previous section, one way to see this is as follows: Let

$$0 \to \mathcal{H} \to \mathcal{V} \to \mathbb{Z}(0) \to 0$$

represent an admissible normal function $\nu : \Delta^* \to J(\mathcal{H})$ and $F : U \to \mathcal{M}$ denote the lifting of the period map of $\mathcal{V}$ to the upper half-plane, with monodromy $T = e^N$. Then, using the SL_2-orbit theorem of the previous section, it follows (cf. Theorem 4.15 of [Pe2]) that

$$Y_{\mathrm{Hodge}} = \lim_{\mathrm{Im}(z)\to\infty} e^{-zN}.Y_{(F(z),W)}$$

exists, and is equal to the grading $Y(N, Y_{(F_\infty,M)})$ constructed in the previous section; recall also that $Y(N, Y_{(F_\infty,M)}) \in \ker(\mathrm{ad}\, N)$ due to the short length of the weight filtration. Suppose further that there exists an integral grading $Y_{\mathrm{Betti}} \in \ker(\mathrm{ad}\, N)$ of the weight filtration W. Let $j : \Delta^* \to \Delta$ and $i : \{0\} \to \Delta$ denote the inclusion maps. Then, $Y_{\mathrm{Hodge}} - Y_{\mathrm{Betti}}$ defines an element in

$$J(H_0) = \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}(0), H^0(i^* R_{j_*}\mathcal{H})) \tag{5-1}$$

by simply applying $Y_{\mathrm{Hodge}} - Y_{\mathrm{Betti}}$ to any lift of $1 \in \mathbb{Z}(0) = \mathrm{Gr}^W_0$. Reviewing §2 and §3, we see that the obstruction to the existence of such a grading Y_{Betti} is exactly the class $\sigma_{\mathbb{Z},0}(\nu)$.

REMARK 93. More generally, if $\mathcal{H}$ is a variation of Hodge structure of weight -1 over a smooth complex algebraic variety S and $\bar{S}$ is a good compactification of S, given a point $s \in \bar{S}$ we define

$$J(H_s) = \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z}, H_s), \tag{5-2}$$

where $H_s = H^0(i_s^* R_{j*}\mathcal{H})$ and $j : S \to \bar{S}$, $i_s : \{s\} \to \bar{S}$ are the inclusion maps. In case $\bar{S}\backslash S$ is a NCD in a neighborhood of S, with $\{N_i\}$ the logarithms of the unipotent parts of the local monodromies, then $H_s \cong \bigcap_j \ker(N_j)$.

In general, except in the classical case of degenerations of Hodge structure of level 1, the dimension of $J(H_0)$ is usually strictly less than the dimension of the fibers of $J(\mathcal{H})$ over Δ^*. Therefore, any generalized Néron model $J_\Delta(\mathcal{H})$ of $J(\mathcal{H})$ which graphs admissible normal functions cannot be a complex analytic space. Rather, in the terminology of Kato and Usui [KU; GGK], we obtain a "slit analytic fiber space". In the case where the base is a curve, the observations above can be combined into the following result:

THEOREM 94. *Let $\mathcal{H}$ be a variation of pure Hodge structure of weight -1 over a smooth algebraic curve S with smooth projective completion $\bar{S}$. Let $j : S \to \bar{S}$ denote the inclusion map. Then, there exists a Néron model for $J(\mathcal{H})$, i.e., a topological group $J_{\bar{S}}(\mathcal{H})$ over $\bar{S}$ satisfying the following two conditions*:

(i) *$J_{\bar{S}}(\mathcal{H})$ restricts to $J(\mathcal{H})$ over S.*
(ii) *There is a one-to-one correspondence between the set of admissible normal functions $\nu : S \to J(\mathcal{H})$ and the set of continuous sections $\bar{\nu} : \bar{S} \to J_{\bar{S}}(\mathcal{H})$ which restrict to holomorphic, horizontal sections of $J(\mathcal{H})$ over S.*

Furthermore:

(iii) *There is a short exact sequence of topological groups*

$$0 \to J_{\bar{S}}(\mathcal{H})^0 \to J_{\bar{S}}(\mathcal{H}) \to G \to 0,$$

where G_s is the torsion subgroup of $(R^1_{j*}\mathcal{H}_\mathbb{Z})_s$ for any $s \in \bar{S}$.
(iv) *$J_{\bar{S}}(\mathcal{H})^0$ is a slit analytic fiber space, with fiber $J(H_s)$ over $s \in \bar{S}$.*
(v) *If $\nu : S \to J(\mathcal{H})$ is an admissible normal function with extension $\bar{\nu}$ then the image of $\bar{\nu}$ in G_s at the point $s \in \bar{S} - S$ is equal to $\sigma_{\mathbb{Z},s}(\nu)$. Furthermore, if $\sigma_{\mathbb{Z},s}(\nu) = 0$ then the value of $\bar{\nu}$ at s is given by the class of $Y_{\text{Hodge}} - Y_{\text{Betti}}$ as in* (5-1). *Equivalently, in the geometric setting, if $\sigma_{\mathbb{Z},s}(\nu) = 0$ then the value of $\bar{\nu}$ at s is given by the limit Abel–Jacobi map.*

Regarding the topology of the Néron model, let us consider more generally the case of smooth complex variety S with good compactification $\bar{S}$, and recall from § 2 that we have also have the Zucker extension $J^Z_{\bar{S}}(\mathcal{H})$ obtained by starting from the short exact sequence of sheaves

$$0 \to \mathbb{H}_\mathbb{Z} \to \mathcal{H}_\mathcal{O}/F^0 \to J(\mathcal{H}) \to 0$$

and replacing $\mathbb{H}_\mathbb{Z}$ by $j_*\mathbb{H}_\mathbb{Z}$ and $\mathcal{H}_\mathcal{O}/F^0$ by its canonical extension. Following [S5], let us suppose that $D = \bar{S} - S$ is a smooth divisor, and let $J^Z_{\bar{S}}(\mathcal{H})^{\text{Inv}}_D$ be the subset of $J^Z_{\bar{S}}(\mathcal{H})$ defined by the local monodromy invariants.

THEOREM 95 [S5]. *The Zucker extension $J^{Z}_{\bar{S}}(\mathcal{H})$ has the structure of a complex Lie group over $\bar{S}$, and it is a Hausdorff topological space on a neighborhood of $J^{Z}_{\bar{S}}(\mathcal{H})^{\mathrm{Inv}}_{D}$.*

Specializing this result to the case where S is a curve, we then recover the result of the first author together with Griffiths and Green that $J_{\bar{S}}(\mathcal{H})^0$ is Hausdorff, since in this case we can identify $J_{\bar{S}}(\mathcal{H})^0$ with $J^{Z}_{\bar{S}}(\mathcal{H})^{\mathrm{Inv}}_{D}$.

REMARK 96. Using this Hausdorff property, Saito was able to prove in [S5] the algebraicity of the zero locus of an admissible normal function in this setting (i.e., D smooth).

5.2. Néron models in many parameters. To extend this construction further, we have to come to terms with the fact that unless S has a compactification $\bar{S}$ such that $D = \bar{S} - S$ is a smooth divisor, the normal functions that we consider may have nontorsion singularities along the boundary divisor. This will be reflected in the fact that the fibers G_s of G need no longer be finite groups. The first test case is when $\mathcal{H}$ is a Hodge structure of level 1. In this case, a Néron model for $J(\mathcal{H})$ was constructed in the thesis of Andrew Young [Yo]. More generally, in joint work with Patrick Brosnan and Morihiko Saito, the second author proved the following result:

THEOREM 97 [BPS]. *Let S be a smooth complex algebraic variety and $\mathcal{H}$ be a variation of Hodge structure of weight -1 over S. Let $j : S \to \bar{S}$ be a good compactification of $\bar{S}$ and $\{S_\alpha\}$ be a Whitney stratification of $\bar{S}$ such that*

(a) *S is one of the strata of $\bar{S}$, and*

(b) *the $R^k j_* \mathbb{H}_{\mathbb{Z}}$ are locally constant on each stratum.*

Then, there exists a generalized Néron model for $J(\mathcal{H})$, i.e., a topological group $J_{\bar{S}}(\mathcal{H})$ over $\bar{S}$ which extends $J(\mathcal{H})$ and satisfies these two conditions:

(i) *The restriction of $J_{\bar{S}}(\mathcal{H})$ to S is $J(\mathcal{H})$.*

(ii) *Any admissible normal function $\nu : S \to J(\mathcal{H})$ has a unique extension to a continuous section $\bar{\nu}$ of $J_{\bar{S}}(\mathcal{H})$.*

Furthermore:

(iii) *There is a short exact sequence of topological groups*

$$0 \to J_{\bar{S}}(\mathcal{H})^0 \to J_{\bar{S}}(\mathcal{H}) \to G \to 0$$

over $\bar{S}$ such that G_s is a discrete subgroup of $(R^1 j_* \mathbb{H}_{\mathbb{Z}})_s$ for any point $s \in \bar{S}$.

(iv) *The restriction of $J_{\bar{S}}(\mathcal{H})^0$ to any stratum S_α is a complex Lie group over S_α with fiber $J(H_s)$ over $s \in \bar{S}$.*

(v) *If $\nu : S \to J(\mathcal{H})$ is an admissible normal function with extension $\bar{\nu}$ then the image of $\bar{\nu}(s)$ in G_s is equal to $\sigma_{\mathbb{Z},s}(\nu)$ for all $s \in \bar{S}$. If $\sigma_{\mathbb{Z},s}(\nu) = 0$ for all $s \in \bar{S}$ then $\bar{\nu}$ restricts to a holomorphic section of $J_{\bar{S}}(\mathcal{H})^0$ over each strata.*

REMARK 98. More generally, this is true under the following hypothesis:

(1) S is a complex manifold and $j : S \to \bar{S}$ is a partial compactification of S as an analytic space;

(2) $\mathcal{H}$ is a variation of Hodge structure on S of negative weight, which need not have unipotent monodromy.

To construct the identity component $J_{\bar{S}}(\mathcal{H})^0$, let $\nu : S \to J(\mathcal{H})$ be an admissible normal function which is represented by an extension

$$0 \to \mathcal{H} \to \mathcal{V} \to \mathbb{Z}(0) \to 0 \tag{5-3}$$

and $j : S \to \bar{S}$ denote the inclusion map. Also, given $s \in \bar{S}$ let $i_s : \{s\} \to \bar{S}$ denote the inclusion map. Then, the short exact sequence (5-3) induces an exact sequence of mixed Hodge structures

$$0 \to H_s \to H^0(i_s^* R j_* \mathcal{V}) \to \mathbb{Z}(0) \to H^1(i_s^* R j_* \mathcal{H}), \tag{5-4}$$

where the arrow $\mathbb{Z}(0) \to H^1(i_s^* R j_* \mathcal{H})$ is given by $1 \mapsto \sigma_{\mathbb{Z},s}(\nu)$. Accordingly, if $\sigma_{\mathbb{Z},s}(\nu) = 0$ then (5-4) determines a point $\bar{\nu}(s) \in J(H_s)$. Therefore, as a set, we define

$$J_{\bar{S}}(\mathcal{H})^0 = \coprod_{s \in \bar{S}} J(H_s)$$

and topologize by identifying it with a subspace of the Zucker extension $J^Z_{\bar{S}}(\mathcal{H})$.

Now, by condition (b) of Theorem 97 and the theory of mixed Hodge modules[S4], it follows that if $i_\alpha : S_\alpha \to \bar{S}$ are the inclusion maps then $H^k(i_\alpha^* R j_* \mathcal{H})$ are admissible variations of mixed Hodge structure over each stratum S_α. In particular, the restriction of $J_{\bar{S}}(\mathcal{H})^0$ to S_α is a complex Lie group.

Suppose now that $\nu : S \to J(\mathcal{H})$ is an admissible normal function with extension $\bar{\nu} : \bar{S} \to J_{\bar{S}}(\mathcal{H})$ such that $\sigma_{\mathbb{Z},s}(\nu) = 0$ for each $s \in \bar{S}$. Then, in order to prove that $\bar{\nu}$ is a continuous section of $J_{\bar{S}}(\mathcal{H})^0$ which restricts to a holomorphic section over each stratum, it is sufficient to prove that $\bar{\nu}$ coincides with the section of the Zucker extension (cf. [S1, Proposition 2.3]). For this, it is in turn sufficient to consider the curve case by restriction to the diagonal curve $\Delta \to \Delta^r$ by $t \mapsto (t, \ldots, t)$; see [BPS, § 1.4].

It remains now to construct $J_{\bar{S}}(\mathcal{H})$ via the following gluing procedure: Let U be an open subset of $\bar{S}$ and $\nu : U \to J(\mathcal{H})$ be an admissible normal function with cohomological invariant

$$\sigma_{\mathbb{Z},U}(\nu) = \partial(1) \in H^1(U, \mathbb{H}_\mathbb{Z})$$

defined by the map

$$\partial : H^0(U, \mathbb{Z}(0)) \to H^1(U, \mathbb{H}_\mathbb{Z})$$

induced by the short exact sequence (5-3) over U. Then, we declare $J_U(\mathcal{H}_{U\cap S})^\nu$ to be the component of $J_{\bar S}(\mathcal{H})$ over U, and equip $J_U(\mathcal{H}_{U\cap S})^\nu$ with a canonical morphism

$$J_U(\mathcal{H}_{U\cap S})^\nu \to J_U(\mathcal{H}_{U\cap S})^0$$

which sends ν to the zero section. If μ is another admissible normal function over U with $\sigma_{\mathbb{Z},U}(\nu) = \sigma_{\mathbb{Z},U}(\mu)$ then there is a canonical isomorphism

$$J_U(\mathcal{H}_{U\cap S})^\nu \cong J_U(\mathcal{H}_{U\cap S})^\mu$$

which corresponds to the section $\nu - \mu$ of $J_U(\mathcal{H}_{U\cap S})^0$ over U.

Addendum to § 5.2. Since the submission of this article, there have been several important developments in the theory of Néron models for admissible normal functions on which we would like to report here. To this end, let us suppose that $\mathcal{H}$ is a variation of Hodge structure of level 1 over a smooth curve $S \subset \bar S$. Let $\mathcal{A}_S$ denote the corresponding abelian scheme with Néron model $\mathcal{A}_{\bar S}$ over $\bar S$. Then, we have a canonical morphism

$$\mathcal{A}_{\bar S} \to J_{\bar S}(\mathcal{H})$$

which is an isomorphism over S. However, unless $\mathcal{H}$ has unipotent local monodromy about each point $s \in \bar S - S$, this morphism is not an isomorphism [BPS]. Recently however, building upon his work on local duality and mixed Hodge modules [Sl2], Christian Schnell has found an alternative construction of the identity component of a Néron model which contains the construction of [BPS] in the case of unipotent local monodromy and agrees [SS] with the classical Néron model for VHS of level 1 in the case of nonunipotent monodromy. In the paragraphs below, we reproduce a summary of this construction which has been generously provided by Schnell for inclusion in this article.

The genesis of the new construction is in unpublished work of Clemens on normal functions associated to primitive Hodge classes. When Y is a smooth hyperplane section of a smooth projective variety X of dimension $2n$, and $H_\mathbb{Z} = H^{2n-1}(Y,\mathbb{Z})_{\text{van}}$ its vanishing cohomology modulo torsion, the intermediate Jacobian $J(Y)$ can be embedded into a bigger object, $K(Y)$ in Clemens's notation, defined as

$$K(Y) = \frac{\left(H^0\left(X, \Omega_X^{2n}(nY)\right)\right)^\vee}{H^{2n-1}(Y,\mathbb{Z})_{\text{van}}}.$$

The point is that the vanishing cohomology of Y is generated by residues of meromorphic $2n$-forms on X, with the Hodge filtration determined by the order of the pole (provided that $\mathcal{O}_X(Y)$ is sufficiently ample). Clemens introduced $K(Y)$ with the hope of obtaining a weak, topological form of Jacobi inversion for its points, and because of the observation that the numerator in its definition

makes sense *even when Y becomes singular*. In his Ph.D. thesis [Sl3], Schnell proved that residues and the pole order filtration actually give a filtered holonomic $\mathcal{D}$-module on the projective space parametrizing hyperplane sections of X; and that this $\mathcal{D}$-module underlies the polarized Hodge module corresponding to the vanishing cohomology by Saito's theory. At least in the geometric case, therefore, there is a close connection between the question of extending intermediate Jacobians, and filtered $\mathcal{D}$-modules (with the residue calculus providing the link).

The basic idea behind Schnell's construction is to generalize from the geometric setting above to arbitrary bundles of intermediate Jacobians. As before, let $\mathcal{H}$ be a variation of polarized Hodge structure of weight -1 on a complex manifold S, and M its extension to a polarized Hodge module on $\bar{S}$. Let $(\mathcal{M}, F)$ be its underlying filtered left $\mathcal{D}$-module: $\mathcal{M}$ is a regular holonomic $\mathcal{D}$-module, and $F = F_\bullet\mathcal{M}$ a good filtration by coherent subsheaves. In particular, $F_0\mathcal{M}$ is a coherent sheaf on $\bar{S}$ that naturally extends the Hodge bundle $F^0\mathcal{H}_{\mathcal{O}}$. Now consider the analytic space over $\bar{S}$, given by

$$T = T(F_0\mathcal{M}) = \mathrm{Spec}_{\bar{S}}\big(\mathrm{Sym}_{\mathcal{O}_{\bar{S}}}(F_0\mathcal{M})\big),$$

whose sheaf of sections is $(F_0\mathcal{M})^\vee$. (Over S, it is nothing but the vector bundle corresponding to $(F^0\mathcal{H}_{\mathcal{O}})^\vee$.) It naturally contains a copy $T_{\mathbb{Z}}$ of the étale space of the sheaf $j_*\mathbb{H}_{\mathbb{Z}}$; indeed, every point of that space corresponds to a local section of $\mathbb{H}_{\mathbb{Z}}$, and it can be shown that every such section defines a map of $\mathcal{D}$-modules $\mathcal{M} \to \mathcal{O}_{\bar{S}}$ via the polarization.

Schnell proves that $T_{\mathbb{Z}} \subseteq T$ is a closed analytic subset, discrete on fibers of $T \to \bar{S}$. This makes the fiberwise quotient space $\bar{J} = T/T_{\mathbb{Z}}$ into an analytic space, naturally extending the bundle of intermediate Jacobians for H. He also shows that admissible normal functions with no singularities extend uniquely to holomorphic sections of $\bar{J} \to \bar{S}$. To motivate the extension process, note that the intermediate Jacobian of a polarized Hodge structure of weight -1 has two models,

$$\frac{H_{\mathbb{C}}}{F^0H_{\mathbb{C}} + H_{\mathbb{Z}}} \simeq \frac{(F^0H_{\mathbb{C}})^\vee}{H_{\mathbb{Z}}},$$

with the isomorphism coming from the polarization. An extension of mixed Hodge structure of the form

$$0 \to H \to V \to \mathbb{Z}(0) \to 0 \tag{5-5}$$

gives a point in the second model in the following manner.

Let $H^* = \mathrm{Hom}(H, \mathbb{Z}(0))$ be the dual Hodge structure, isomorphic to $H(-1)$ via the polarization. After dualizing, we have

$$0 \to \mathbb{Z}(0) \to V^* \to H^* \to 0,$$

and thus an isomorphism $F^1V^*_{\mathbb{C}} \simeq F^1H^*_{\mathbb{C}} \simeq F^0H_{\mathbb{C}}$. Therefore, any $v \in V_{\mathbb{Z}}$ lifting $1 \in \mathbb{Z}$ gives a linear map $F^0H_{\mathbb{C}} \to \mathbb{C}$, well-defined up to elements of $H_{\mathbb{Z}}$; this is the point in the second model of $J(H)$ that corresponds to the extension in (5-5).

It so happens that this second construction is the one that extends to all of $\bar{S}$. Given a normal function ν on S, let

$$0 \to \mathbb{H}_{\mathbb{Z}} \to \mathbb{V}_{\mathbb{Z}} \to \mathbb{Z}_S \to 0$$

be the corresponding extension of local systems. By applying j_*, it gives an exact sequence

$$0 \to j_*\mathbb{H}_{\mathbb{Z}} \to j_*\mathbb{V}_{\mathbb{Z}} \to \mathbb{Z}_{\bar{S}} \to R^1j_*\mathbb{H}_{\mathbb{Z}},$$

and when ν has no singularities, an extension of sheaves

$$0 \to j_*\mathbb{H}_{\mathbb{Z}} \to j_*\mathbb{V}_{\mathbb{Z}} \to \mathbb{Z}_{\bar{S}} \to 0.$$

Using duality for filtered $\mathcal{D}$-modules, one obtains local sections of $(F_0\mathcal{M})^\vee$ from local sections of $j_*\mathbb{V}_{\mathbb{Z}}$, just as above, and thus a well-defined holomorphic section of $\bar{J} \to \bar{S}$ that extends ν.

As in the one-variable case, where the observation is due to Green, Griffiths, and Kerr, horizontality constrains such extended normal functions to a certain subset of $\bar{J}$; Schnell proves that this subset is precisely the identity component of the Néron model constructed by Brosnan, Pearlstein, and Saito. With the induced topology, the latter is therefore a Hausdorff space, as expected. This provides an additional proof for the algebraicity of the zero locus of an admissible normal function, similar in spirit to the one-variable result in Saito's paper, in the case when the normal function has no singularities.

The other advance, is the recent construction [KNU2] of log intermediate Jacobians by Kato, Nakayama and Usui. Although a proper exposition of this topic would take us deep into logarithmic Hodge theory [KU], the basic idea is as follows: Let $\mathcal{H} \to \Delta^*$ be a variation of Hodge structure of weight -1 with unipotent monodromy. Then, we have a commutative diagram

$$\begin{array}{ccc} J(\mathcal{H}) & \xrightarrow{\tilde{\varphi}} & \tilde{\Gamma} \setminus \mathcal{M} \\ \downarrow & & \downarrow \mathrm{Gr}^W_{-1} \\ \Delta^* & \xrightarrow{\varphi} & \Gamma \setminus \mathcal{D} \end{array} \tag{5-6}$$

where $\tilde{\varphi}$ and φ are the respective period maps. In [KU], Kato and Usui explained how to translate the bottom row of this diagram into logarithmic Hodge theory. More generally, building on the ideas of [KU] and the several variable SL_2-orbit theorem [KNU1], Kato, Nakayama and Usui are able to construct a theory of

logarithmic mixed Hodge structures which they can then apply to the top row of the previous diagram. In this way, they obtain a log intermediate Jacobian which serves the role of a Néron model and allows them to give an alternate proof of Theorem 57 [KNU3].

5.3. Singularities of normal functions overlying nilpotent orbits. We now consider the group of components G_s of $J_{\bar{S}}(\mathcal{H})$ at $s \in \bar{S}$. For simplicity, we first consider the case where $\mathcal{H}$ is a nilpotent orbit $\mathcal{H}_{\text{nilp}}$ over $(\Delta^*)^r$. To this end, we recall that in the case of a variation of Hodge structure $\mathcal{H}$ over $(\Delta^*)^r$ with unipotent monodromy, the intersection cohomology of $\mathbb{H}_{\mathbb{Q}}$ is computed by the cohomology of a complex $(B^\bullet(N_1,\dots,N_r), d)$ (cf. § 3.4). Furthermore, the short exact sequence of sheaves

$$0 \to \mathbb{H}_{\mathbb{Q}} \to \mathbb{V}_{\mathbb{Q}} \to \mathbb{Q}(0) \to 0$$

associated to an admissible normal function $\nu : (\Delta^*)^r \to J(\mathcal{H})$ with unipotent monodromy gives a connecting homomorphism

$$\partial : IH^0(\mathbb{Q}(0)) \to IH^1(\mathbb{H}_{\mathbb{Q}})$$

such that

$$\partial(1) = [(N_1(e_o^{\mathbb{Q}}), \dots, N_r(e_o^{\mathbb{Q}})] = \text{sing}_0(\nu),$$

where $e_o^{\mathbb{Q}}$ is an element in the reference fiber $V_{\mathbb{Q}}$ of $\mathbb{V}_{\mathbb{Q}}$ over $s_o \in (\Delta^*)^r$ which maps to $1 \in \mathbb{Q}(0)$. After passage to complex coefficients, the admissibility of $\mathcal{V}$ allows us to pick an alternate lift $e_o \in V_{\mathbb{C}}$ to be of type $(0,0)$ with respect to the limit MHS of $\mathcal{V}$. It also forces $h_j = N_j(e_o)$ to equal $N_j(f_j)$ for some element $f_j \in H_{\mathbb{C}}$ of type $(0,0)$ with respect to the limit MHS of $\mathcal{H}$. Moreover, $e_0^{\mathbb{Q}} - e_0 =: h$ maps to $0 \in \text{Gr}_0^W$, hence lies in $H_{\mathbb{C}}$, so

$$(N_1(e_0^{\mathbb{Q}}), \dots, N_r(e_0^{\mathbb{Q}})) \equiv (N_1(e_0), \dots, N_r(e_0)) \text{ modulo } d(B^0) = \text{im} \bigoplus_{j=1}^{r} N_j$$

(i.e., up to $(N_1(h), \dots, N_r(h))$).

COROLLARY 99. *$\text{sing}_0(\nu)$ is a rational class of type $(0,0)$ in $IH^1(\mathbb{H}_{\mathbb{Q}})$.*

SKETCH OF PROOF. This follows from the previous paragraph together with the explicit description of the mixed Hodge structure on the cohomology of $B^\bullet(N_1,\dots,N_r)$ given in [CKS2]. □

Conversely, we have:

LEMMA 100. *Let $\mathcal{H}_{\text{nilp}} = e^{\sum_j z_j N_j}.F_\infty$ be a nilpotent orbit of weight -1 over Δ^{*r} with rational structure $\mathbb{H}_{\mathbb{Q}}$. Then, any class β of type $(0,0)$ in $IH^1(\mathbb{H}_{\mathbb{Q}})$ is representable by a $\mathbb{Q}$-normal function ν which is an extension of $\mathbb{Q}(0)$ by $\mathcal{H}_{\text{nilp}}$ such that $\text{sing}_0(\nu) = \beta$.*

PROOF. By the remarks above, β corresponds to a collection of elements $h_j \in N_j(H_{\mathbb{C}})$ such that

(a) $h_1, \ldots, h_r$ are of type $(-1,-1)$ with respect to the limit mixed Hodge structure of $\mathcal{H}_{\text{nilp}}$,

(b) $d(h_1, \ldots, h_r) = 0$, i.e., $N_j(h_k) - N_k(h_j) = 0$, and

(c) There exists $h \in H_{\mathbb{C}}$ such that $N_j(h) + h_j \in H_{\mathbb{Q}}$ for each j, i.e., the class of $(h_1, \ldots, h_r)$ in $IH^1(\mathbb{H}_{\mathbb{C}})$ belongs to the image $IH^1(\mathbb{H}_{\mathbb{Q}}) \to IH^1(\mathbb{H}_{\mathbb{C}})$.

We now define the desired nilpotent orbit by formally setting $V_{\mathbb{C}} = \mathbb{C}e_o \oplus H_{\mathbb{C}}$, where e_o is of type $(0,0)$ with respect to the limit mixed Hodge structure and letting $V_{\mathbb{Q}} = \mathbb{Q}(e_o + h) \oplus H_{\mathbb{Q}}$. We define $N_j(e_o) = h_j$. Then, following Kashiwara [Ka]:

(a) The resulting nilpotent orbit $\mathcal{V}_{\text{nilp}}$ is pre-admissible.

(b) The relative weight filtration of

$$W_{-2} = 0, \qquad W_{-1} = H_{\mathbb{Q}}, \qquad W_0 = V_{\mathbb{Q}}$$

with respect to each N_j exists.

Consequently $\mathcal{V}_{\text{nilp}}$ is admissible, and the associated normal function ν has singularity β at 0. □

5.4. Obstructions to the existence of normal functions with prescribed singularity class. Thus, in the case of a nilpotent orbit, we have a complete description of the group of components of the Néron model $\otimes\mathbb{Q}$. In analogy with nilpotent orbits, one might expect that given a variation of Hodge structure $\mathcal{H}$ of weight -1 over $(\Delta^*)^r$ with unipotent monodromy, the group of components of the Néron model $\otimes\mathbb{Q}$ to equal the classes of type $(0,0)$ in $IH^1(\mathbb{H}_{\mathbb{Q}})$. However, Saito [S6] has managed to construct examples of variations of Hodge structure over $(\Delta^*)^r$ which do not admit any admissible normal functions with nontorsion singularities. We now want to describe Saito's class of examples. We begin with a discussion of the deformations of an admissible nilpotent orbit into an admissible variation of mixed Hodge structure over $(\Delta^*)^r$.

Let $\varphi : (\Delta^*)^r \to \Gamma\backslash\mathcal{D}$ be the period map of a variation of pure Hodge structure with unipotent monodromy. Then, after lifting the period map of $\mathcal{H}$ to the product of upper half-planes U^r, the work of Cattani, Kaplan and Schmid on degenerations of Hodge structure gives us a local normal form of the period map

$$F(z_1, \ldots, z_r) = e^{\sum_j z_j N_j} e^{\Gamma(s)} . F_\infty .$$

Here, $(s_1, \ldots, s_r)$ are the coordinates on Δ^r, $(z_1, \ldots, z_r)$ are the coordinates on U^r relative to which the covering map $U^r \to (\Delta^*)^r$ is given by $s_j = e^{2\pi i z_j}$;

$$\Gamma : \Delta^r \to \mathfrak{g}_{\mathbb{C}}$$

is a holomorphic function which vanishes at the origin and takes values in the subalgebra

$$\mathfrak{q} = \bigoplus_{p<0} \mathfrak{g}^{p,q};$$

and $\bigoplus_{p,q} \mathfrak{g}^{p,q}$ denotes the bigrading of the MHS induced on $\mathfrak{g}_{\mathbb{C}}$ (cf. §4.2) by the limit MHS $(F_\infty, W(N_1 + \cdots N_r)[1])$ of $\mathcal{H}$. The subalgebra $\mathfrak{q}$ is graded nilpotent

$$\mathfrak{q} = \bigoplus_{a<0} \mathfrak{q}_a, \qquad \mathfrak{q}_a = \bigoplus_b \mathfrak{g}^{a,b},$$

with $N_1, \ldots, N_r \in \mathfrak{q}_{-1}$. Therefore,

$$e^{\sum_j z_j N_j} e^{\Gamma(s)} = e^{X(z_1,\ldots,z_r)},$$

where X takes values in $\mathfrak{q}$, and hence the horizontality of the period map becomes

$$e^{-X} \partial e^X = \partial X_{-1},$$

where $X = X_{-1} + X_{-2} + \cdots$ relative to the grading of $\mathfrak{q}$. Equality of mixed partial derivatives then forces

$$\partial X_{-1} \wedge \partial X_{-1} = 0$$

Equivalently,

$$\left[N_j + 2\pi i s_j \frac{\partial \Gamma_{-1}}{\partial s_j},\ N_k + 2\pi i s_k \frac{\partial \Gamma_{-1}}{\partial s_k} \right] = 0. \tag{5-7}$$

REMARK 101. The function Γ and the local normal form of the period map appear in [CK].

In his letter to Morrison [De4], Deligne showed that for VHS over $(\Delta^*)^r$ with maximal unipotent boundary points, one could reconstruct the VHS from data equivalent to the nilpotent orbit and the function Γ_{-1}. More generally, one can reconstruct the function Γ starting from ∂X_{-1} using the equation

$$\partial e^X = e^X \partial X_{-1}$$

subject to the integrability condition $\partial X_{-1} \wedge \partial X_{-1} = 0$. This is shown by Cattani and Javier Fernandez in [CF].

The above analysis applies to VMHS over $(\Delta^*)^r$ as well: As discussed in the previous section, a VMHS is given by a period map from the parameter space into the quotient of an appropriate classifying space of graded-polarized mixed Hodge structure $\mathcal{M}$. As in the pure case, we have a Lie group G which acts on $\mathcal{M}$ by biholomorphisms and a complex Lie group $G_{\mathbb{C}}$ which acts on the "compact dual" $\check{\mathcal{M}}$.

As in the pure case (and also discussed in §4), an admissible VMHS with nilpotent orbit $(e^{\sum_j z_j N_j}.F_\infty, W)$ will have a local normal form

$$F(z_1,\dots,z_r) = e^{\sum_j z_j N_j} e^{\Gamma(s)}.F_\infty,$$

where $\Gamma : \Delta^r \to \mathfrak{g}_{\mathbb{C}}$ takes values in the subalgebra

$$\mathfrak{q} = \bigoplus_{p<0} \mathfrak{g}^{p,q}.$$

Conversely (given an admissible nilpotent orbit), subject to the integrability condition (5-7) above, any function Γ_{-1} determines a corresponding admissible VMHS; see [Pe1, Theorem 6.16].

Returning to Saito's examples (which for simplicity we only consider in the two-dimensional case), let $\mathcal{H}$ be a variation of Hodge structure of weight -1 over Δ^* with local normal form $F(z) = e^{zN}e^{\Gamma(s)}.F_\infty$. Let $\pi : \Delta^2 \to \Delta$ by $\pi(s_1,s_2) = s_1s_2$. Then, for $\pi^*(\mathcal{H})$, we have

$$\Gamma_{-1}(s_1,s_2) = \Gamma_{-1}(s_1s_2).$$

In order to construct a normal function, we need to extend $\Gamma_{-1}(s_1,s_2)$ and $N_1 = N_2 = N$ on the reference fiber $H_{\mathbb{C}}$ of $\mathcal{H}$ to include a new class u_0 of type $(0,0)$ which projects to 1 in $\mathbb{Z}(0)$. Set

$$N_1(u_0) = h_1, \qquad N_2(u_0) = h_2, \qquad \Gamma_{-1}(s_1,s_2)u_0 = \alpha(s_1,s_2).$$

Note that (h_1,h_2) determines the cohomology class of the normal function so constructed, and that $h_2 - h_1$ depends only on the cohomology class, and not the particular choice of representative (h_1,h_2).

In order to construct a normal function in this way, we need to check horizontality. This amounts to checking the equation

$$N\left(s_2\frac{\partial\alpha}{\partial s_2} - s_1\frac{\partial\alpha}{\partial s_1}\right) + s_1s_2\Gamma'_{-1}(s_1s_2)(h_2-h_1) + 2\pi i\, s_1s_2\Gamma'_{-1}(s_1s_2)\left(s_2\frac{\partial\alpha}{\partial s_2} - s_1\frac{\partial\alpha}{\partial s_1}\right) = 0.$$

Computation shows that the coefficient of $(s_1s_2)^m$ on the left side is

$$\frac{1}{(m-1)!}\Gamma^{(m)}_{-1}(0)(h_2-h_1). \tag{5-8}$$

Hence, a necessary condition for the cohomology class represented by (h_1,h_2) to arise from an admissible normal function is for h_2-h_1 to belong to the kernel of $\Gamma_{-1}(t)$. This condition is also sufficient since, under this hypothesis, one can simply set $\alpha = 0$.

EXAMPLE 102. Let $\mathcal{X} \xrightarrow{\rho} \Delta$ be a family of Calabi–Yau 3-folds (smooth over Δ^*, smooth total space) with Hodge numbers $h^{3,0} = h^{2,1} = h^{1,2} = h^{0,3} = 1$ and central singular fiber having an ODP. Setting $\mathcal{H} := \mathcal{H}^3_{\mathcal{X}^*/\Delta^*}(2)$, the LMHS has as its nonzero $I^{p,q}$'s $I^{-2,1}$, $I^{-1,-1}$, $I^{0,0}$, and $I^{1,-2}$. Assume that the Yukawa coupling $(\nabla_{\delta_s})^3 \in \mathrm{Hom}_{\mathcal{O}_\Delta}(\mathcal{H}_e^{3,0}, \mathcal{H}_e^{0,3})$ is nonzero ($\delta_s = s\,d/ds$), and thus the restriction of $\Gamma_{-1}(s)$ to $\mathrm{Hom}_{\mathcal{O}_\Delta}(I^{-1,-1}, I^{-2,1})$, does not vanish identically. Then, for any putative singularity class, $0 \neq h_2 - h_1 \in (I^{-1.-1})_{\mathbb{Q}} \cong \ker(N)^{(-1,-1)}_{\mathbb{Q}}$ (this being isomorphic to (2-10) in this case, which is just one-dimensional) for admissible normal functions overlying $\pi^*\mathcal{H}$, nonvanishing of $\Gamma_{-1}(s)(h_2 - h_1)$ on Δ implies that (5-8) cannot be zero for every m.

5.5. Implications for the Griffiths–Green conjecture. Returning now to the work of Griffiths and Green on the Hodge conjecture via singularities of normal functions, it follows using the work of Richard Thomas that for a sufficiently high power of L, the Hodge conjecture implies that one can force ν_ξ to have a singularity at a point $p \in \hat{X}$ such that $\pi^{-1}(p)$ has only ODP singularities. In general, on a neighborhood of such a point $\hat{X}$ need not be a normal crossing divisor. However, the image of the monodromy representation is nevertheless abelian. Using a result of Steenbrink and Némethi [NS], it then follows from the properties of the monodromy cone of a nilpotent orbit of pure Hodge structure that $\mathrm{sing}_p(\nu_\xi)$ persists under blowup. Therefore, it is sufficient to study ODP degenerations in the normal crossing case (cf. [BFNP, sec. 7]). What we will find below is that the "infinitely many" conditions above (vanishing of (5-8) for all m) are replaced by surjectivity of a single logarithmic Kodaira–Spencer map at each boundary component. Consequently, as suggested in the introduction, it appears that M. Saito's examples are not a complete show-stopper for existence of singularities for Griffiths–Green normal functions.

The resulting limit mixed Hodge structure is of the form

$$\begin{array}{ccccccccc} & & & & I^{0,0} & & & & \\ \cdots & & I^{-2,1} & & I^{-1,0} & & I^{0,-1} & & I^{1,-2} & \cdots \\ & & & & I^{-1,-1} & & & & \end{array}$$

and $N^2 = 0$ for every element of the monodromy cone $\mathcal{C}$. The weight filtration is given by

$$M_{-2}(N) = \sum_j N_j(H_{\mathbb{C}}), \quad M_{-1}(N) = \bigcap_j \ker(N_j), \quad M_0(N) = H_{\mathbb{C}}.$$

For simplicity of notation, let us restrict to a two parameter version of such a degeneration, and consider the obstruction to constructing an admissible normal function with cohomology class represented by (h_1, h_2) as above. As in Saito's

example, we need to add a class u_o of type $(0,0)$ such that $N_j(u_o) = h_j$ and construct $\alpha = \Gamma_{-1}(u_o)$. Then, the integrability condition $\partial X_{-1} \wedge \partial X_{-1} = 0$ becomes

$$-(2\pi i s_2)\frac{\partial \Gamma_{-1}}{\partial s_2}(h_1) + (2\pi i s_1)\frac{\partial \Gamma_{-1}}{\partial s_1}(h_2) + (2\pi i s_1)(2\pi i s_2)\left(\frac{\partial \Gamma_{-1}}{\partial s_1}\frac{\partial \alpha}{\partial s_2} - \frac{\partial \Gamma_{-1}}{\partial s_2}\frac{\partial \alpha}{\partial s_1}\right) = 0, \quad (5\text{-}9)$$

since $\alpha = \Gamma_{-1}(u_o)$ takes values in $M_{-1}(N)$.

Write $\alpha = \sum_{j,k} s_1^j s_2^k \alpha_{jk}$ and $\Gamma_{-1} = \sum_{p,q} s_1^p s_2^q \gamma_{pq}$ on $H_{\mathbb{C}}$. Then, for ab nonzero, the coefficient of $s_1^a s_2^b$ on the left side of equation (5-9) is

$$-2\pi i b \gamma_{ab}(h_2) + 2\pi i a \gamma_{ab}(h_1) + (2\pi i)^2 \sum_{\substack{p+j=a \\ q+k=b}} (pk - qj)\gamma_{pq}(\alpha_{jk}).$$

Define

$$\zeta_{ab} = 2\pi i b \gamma_{ab}(h_2) - 2\pi i a \gamma_{ab}(h_1) - (2\pi i)^2 \sum_{\substack{p+j=a \\ q+k=b \\ pq\neq 0}} (pk - qj)\gamma_{pq}(\alpha_{jk}).$$

Then, equation (5-9) is equivalent to

$$(2\pi i)^2 b \gamma_{10}(\alpha_{(a-1)b}) - (2\pi i)^2 a \gamma_{01}(\alpha_{a(b-1)}) = \zeta_{ab},$$

where α_{jk} occurs in ζ_{ab} only in total degree $j + k < a + b - 1$. Therefore, *provided* that

$$\gamma_{10}, \gamma_{01} : F_\infty^{-1}/F_\infty^0 \to F_\infty^{-2}/F_\infty^{-1}$$

are surjective, we can always solve (nonuniquely!) for the coefficients α_{jk}, and hence formally (i.e., modulo checking convergence of the resulting series) construct the required admissible normal function with given cohomology class.

REMARK 103. (i) Of course, it is not necessary to have only ODP singularities for the analysis above to apply. It is sufficient merely that the limit mixed Hodge structure have the stated form. In particular, this is always true for degenerations of level 1. Furthermore, in this case $\mathrm{Gr}_{F_\infty}^{-2} = 0$, and hence, after tensoring with $\mathbb{Q}$, the group of components of the Néron model surjects onto the Tate classes of type $(0,0)$ in $IH^1(\mathbb{H}_{\mathbb{Q}})$.

(ii) In Saito's examples from §5.4, even if $\Gamma'_{-1}(0) \neq 0$, we will have $\gamma_{01} = 0 = \gamma_{10}$, since the condition of being a pullback via $(s_1, s_2) \mapsto s_1 s_2$ means $\Gamma_{-1}(s_1, s_2) = \sum_{p,q} s_1^p s_2^q \gamma_{pq} = \sum_r s_1^r s_2^r \gamma_{rr}$.

EXAMPLE 104. In the case of a degeneration of Calabi–Yau threefolds with limit mixed Hodge structure on the middle cohomology (shifted to weight -1)

$$\begin{array}{ccccc} & & I^{0,0} & & \\ I^{-2,1} & & I^{-1,0} & I^{0,-1} & I^{1,-2} \\ & & I^{-1,-1} & & \end{array}$$

the surjectivity of the partial derivatives of Γ_{-1} are related to the Yukawa coupling as follows: Let

$$F(z) = e^{\sum_j z_j N_j} e^{\Gamma(s)} . F_\infty$$

be the local normal form of the period map as above. Then, a global nonvanishing holomorphic section of the canonical extension of $\mathcal{F}^1$ (i.e., of $\mathcal{F}^3$ before we shift to weight -1) is of the form

$$\Omega = e^{\sum_j z_j N_j} e^{\Gamma(s)} \sigma_\infty(s),$$

where $\sigma_\infty : \Delta^r \to I^{1,-2}$ is holomorphic and nonvanishing. Then, the Yukawa coupling of Ω is given by

$$Q(\Omega, D_j D_k D_\ell \Omega), \qquad D_a = \frac{\partial}{\partial z_a}.$$

In keeping with the notation above, let $e^X = e^{\sum_j z_j N_j} e^{\Gamma(s)}$ and $A_j = D_j X_{-1}$. Using the first Hodge–Riemann bilinear relation and the fact that e^X is an automorphism of Q, it follows that

$$Q(\Omega, D_j D_k D_\ell \Omega) = Q(\sigma_\infty(s), A_j A_k A_\ell \sigma_\infty(s)).$$

Moreover (cf. [CK; Pe1]), the horizontality of the period map implies that

$$\left[\Gamma_{-1}|_{s_k=0} , N_k \right] = 0$$

Using this relation, it then follows that

$$\lim_{s\to 0} \frac{Q(\Omega, D_j D_k D_\ell \Omega)}{(2\pi i s_j)(2\pi i s_k)(2\pi i s_\ell)} = Q(\sigma_\infty(0), G_j G_k G_\ell \sigma_\infty(0))$$

for $j \neq k$, where

$$G_a = \frac{\partial \Gamma_{-1}}{\partial s_a}(0).$$

In particular, if for each index j there exist indices k and ℓ with $k \neq \ell$ such that the left-hand side of the previous equation is nonzero then $G_j : (F_\infty^{-1}/F_\infty^0) \to (F_\infty^{-2}/F_\infty^{-1})$ is surjective.

6. Global considerations: monodromy of normal functions

Returning to a normal function $\mathcal{V} \in \mathrm{NF}^1(S, \mathcal{H})^{\mathrm{ad}}_{\bar{S}}$ over a *complete* base, we want to speculate a bit about how one might "force" singularities to exist. The (inconclusive) line of reasoning we shall pursue rests on two basic principles:

(i) maximality of the geometric (global) monodromy group of $\mathbb{V}$ may be deduced from hypotheses on the torsion locus of $\mathcal{V}$; and

(ii) singularities of $\mathcal{V}$ can be interpreted in terms of the local monodromy of $\mathbb{V}$ being sufficiently large.

While it is unclear what hypotheses (if any) would allow one to pass from global to local monodromy-largeness, the proof of the first principle is itself of interest as a first application of algebraic groups (the algebraic variety analog of Lie groups, originally introduced by Picard) to normal functions.

6.1. Background. Mumford–Tate groups of Hodge structures were introduced by Mumford [Mu] for pure HS and by André [An] in the mixed setting. Their power and breadth of applicability is not well-known, so we will first attempt a brief summary. They were first brought to bear on $H^1(A)$ for A an abelian variety, which has led to spectacular results:

- Deligne's theorem [De2] that $\mathbb{Q}$-Bettiness of a class in $F^p H^{2p}_{dR}(A_k)$ for k algebraically closed is independent of the embedding of k into $\mathbb{C}$ ("Hodge implies absolute Hodge");
- the proofs by Hazama [Ha] and Murty [Mr] of the HC for A "nondegenerate" (MT of $H^1(A)$ is maximal in a sense to be defined below); and
- the density of special (Shimura) subvarieties in Shimura varieties and the partial resolution of the André–Oort Conjecture by Klingler and Yafaev [KY].

More recently, MT groups have been studied for higher weight HS's; one can still use them to define special $\bar{\mathbb{Q}}$-subvarieties of (non-Hermitian-symmetric) period domains D, which classify polarized HS's with fixed Hodge numbers (and polarization). In particular, the 0-dimensional subdomains — still dense in D — correspond to HS with CM (complex multiplication); that is, with abelian MT group. One understands these HS well: their irreducible subHS may be constructed directly from primitive CM types (and have endomorphism algebra equal to the underlying CM field), which leads to a complete classification; and their Weil and Griffiths intermediate Jacobians are CM abelian varieties [Bo]. Some further applications of MT groups include:

- Polarizable CM-HS are motivic [Ab]; when they come from a CY variety, the latter often has good modularity properties;

- Given H^* of a smooth projective variety, the level of the MT Lie algebra furnishes an obstruction to the variety being dominated by a product of curves [Sc];
- Transcendence degree of the space of periods of a VHS (over a base S), viewed as a field extension of $\mathbb{C}(S)$ [An];

and specifically in the mixed case:

- the recent proof [AK] of a key case of the Beilinson–Hodge Conjecture for semiabelian varieties and products of smooth curves.

The latter paper, together with [An] and [De2], are the best references for the definitions and properties we now summarize.

To this end, recall that an algebraic group G over a field k is an algebraic variety over k together with k-morphisms of varieties $1_G : \mathrm{Spec}(k) \to G$, "multiplication" $\mu_G : G \times G \to G$, and "inversion" $\iota_G : G \to G$ satisfying obvious compatibility conditions. The latter ensure that for any extension K/k, the K-points $G(K)$ form a group.

DEFINITION 105. (i) A (k-)closed algebraic subgroup $M \leq G$ is one whose underlying variety is (k-)Zariski closed.

(ii) Given a subgroup $\mathcal{M} \leq G(K)$, the k-closure of $\mathcal{M}$ is the smallest k-closed algebraic subgroup M of G with K-points $M(K) \geq \mathcal{M}$.

If $\mathcal{M} := M(K)$ for an algebraic k-subgroup $M \leq G$, then the k-closure of $\mathcal{M}$ is just the k-Zariski closure of $\mathcal{M}$ (i.e., the algebraic variety closure).

But in general, this is not true: instead, M may be obtained as the k-Zariski (algebraic variety) closure of the group generated by the k-spread of $\mathcal{M}$.

We refer the reader to [Sp] (especially Chapter 6) for the definitions of reductive, semisimple, unipotent, etc. in this context (which are less crucial for the sequel). We will write $DG := [G, G]\,(\trianglelefteq G)$ for the derived group.

6.2. Mumford–Tate and Hodge groups. Let V be a (graded-polarizable) mixed Hodge structure with dual $V^\vee$ and tensor spaces

$$T^{m,n}V := V^{\otimes m} \otimes (V^\vee)^{\otimes n}$$

($n, m \in \mathbb{Z}_{\geq 0}$). These carry natural MHS, and any $g \in \mathrm{GL}(V)$ acts naturally on $T^{m,n}V$.

DEFINITION 106. (i) A *Hodge* (p, p)*-tensor* is any $\tau \in (T^{m,n}V)^{(p,p)}_{\mathbb{Q}}$.

(ii) The *MT group* M_V of V is the (largest) $\mathbb{Q}$-algebraic subgroup of $\mathrm{GL}(V)$ fixing[12] the Hodge $(0, 0)$-tensors for all m, n. The weight filtration $W_\bullet$ on V is preserved by M_V.

[12] "Fixing" means fixing pointwise; the term for "fixing as a set" is "stabilizing".

Similarly, the *Hodge group* M_V° of V is the $\mathbb{Q}$-algebraic subgroup of $\mathrm{GL}(V)$ fixing the Hodge (p,p)-tensors for all m,n,p. (In an unfortunate coincidence of terminology, these are completely different objects from — though not unrelated to — the finitely generated abelian groups $\mathrm{Hg}^m(H)$ discussed in § 1.)

(iii) The weight filtration on V induces one on MT/Hodge:

$$W_{-i}M_V^{(\circ)} := \left\{ g \in M_V^{(\circ)} \;\middle|\; (g-\mathrm{id})W_\bullet V \subset W_{\bullet-i}V \right\} \trianglelefteq M_V^{(\circ)}.$$

One has: $W_0 M_V^{(\circ)} = M_V^{(\circ)}$; $W_{-1}M_V^{(\circ)}$ is unipotent; and $\mathrm{Gr}_0^W M_V^{(\circ)} \cong M_{V^{\mathrm{split}}}^{(\circ)}$ ($V^{\mathrm{split}} := \bigoplus_{\ell\in\mathbb{Z}} \mathrm{Gr}_\ell^W V$), cf. [An].

Clearly $M_V^\circ \trianglelefteq M_V$; and unless V is pure of weight 0, we have $M_V/M_V^\circ \cong \mathbb{G}_m$. If V has polarization $Q \in Hom_{\mathrm{MHS}}(V\otimes V, \mathbb{Q}(-k))$ for $k\in\mathbb{Z}\backslash\{0\}$, then M_V° is of finite index in $M_V \cap \mathrm{GL}(V,Q)$ (where $g\in\mathrm{GL}(V,Q)$ means $Q(gv,gw) = Q(v,w)$), and if in addition $V(=H)$ is pure (or at least split) then both are reductive. One has in general that $W_{-1}M_V \subseteq DM_V \subseteq M_V^\circ \subseteq M_V$.

DEFINITION 107. (i) If M_V is abelian ($\Longleftrightarrow$ $M_V(\mathbb{C}) \cong (\mathbb{C}^*)^{\times a}$), V is called a *CM-MHS*. (A subMHS of a CM-MHS is obviously CM.)

(ii) The endomorphisms $\mathrm{End}_{\mathrm{MHS}}(V)$ can be interpreted as the $\mathbb{Q}$-points of the algebra $(\mathrm{End}(V))^{M_V} =: E_V$. One always has $M_V \subset \mathrm{GL}(V,E_V)$ (=centralizer of E_V); if this is an equality, then V is said to be *nondegenerate*.

Neither notion implies the other; however: any CM or nondegenerate MHS is ($\mathbb{Q}$-)*split*, i.e., V ($= V^{\mathrm{split}}$) is a direct sum of pure HS in different weights.

REMARK 108. (a) We point out why CM-MHS are split. If M_V is abelian, then $M_V \subset E_V$ and so $M_V(\mathbb{Q})$ consists of morphisms of MHS. But then any $g\in W_{-1}M_V(\mathbb{Q})$, hence $g-\mathrm{id}$, is a morphism of MHS with $(g-\mathrm{id})W_\bullet \subset W_{\bullet-1}$; so $g=\mathrm{id}$, and $M_V = M_{V^{\mathrm{split}}}$, which implies $V = V^{\mathrm{split}}$.

(b) For an arbitrary MHS V, the subquotient tensor representations of M_V killing DM_V (i.e., factoring through the abelianization) are CM-MHS. By (a), they are split, so that $W_{-1}M_V$ acts trivially; this gives $W_{-1}M_V \subseteq DM_V$.

Now we turn to the representation-theoretic point of view on MHS. Define the algebraic $\mathbb{Q}$-subgroups $U \subset S \subset \mathrm{GL}_2$ via their complex points:

$$\begin{array}{ccccc}
S(\mathbb{C}): & = & \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \;\middle|\; \begin{array}{c} \alpha,\beta\in\mathbb{C} \\ (\alpha,\beta)\neq(0,0) \end{array} \right\} & \xrightarrow[\text{eigenvalues}]{\cong} & \mathbb{C}^*\times\mathbb{C}^* \qquad \left(z,\frac{1}{z}\right) \\
\uparrow & & \uparrow & & \uparrow \\
U(\mathbb{C}): & = & \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \;\middle|\; \begin{array}{c} \alpha,\beta\in\mathbb{C} \\ \alpha^2+\beta^2=1 \end{array} \right\} & \xrightarrow{\cong} & \mathbb{C}^* \qquad\quad z
\end{array} \tag{6-1}$$

where the top map sends $\left(\begin{smallmatrix}\alpha & \beta\\ -\beta & \alpha\end{smallmatrix}\right) \mapsto (\alpha+i\beta, \alpha-i\beta) =: (z,w)$. (Points in $S(\mathbb{C})$ will be represented by the "eigenvalues" (z,w).) Let

$$\varphi:\ S(\mathbb{C}) \to \mathrm{GL}(V_{\mathbb{C}})$$

be given by

$$\varphi(z,w)|_{I^{p,q}(H)} := \text{multiplication by } z^p w^q \quad (\forall\, p,q).$$

Note that this map is in general only defined over $\mathbb{C}$, though in the pure case it is defined over $\mathbb{R}$ (and as $S(\mathbb{R}) \subset S(\mathbb{C})$ consists of tuples $(z,\bar{z})$, one tends not to see precisely the approach above in the literature). The following useful result[13] allows one to compute MT groups in some cases.

PROPOSITION 109. *M_V is the $\mathbb{Q}$-closure of $\varphi(S(\mathbb{C}))$ in* $\mathrm{GL}(V)$.

REMARK 110. In the pure ($V = H$) case, this condition can be replaced by $M_H(\mathbb{R}) \supset \varphi(S(\mathbb{R}))$, and M_H° defined similarly as the $\mathbb{Q}$-closure of $\varphi(U(\mathbb{R}))$; unfortunately, for V a non-$\mathbb{Q}$-split MHS the $\mathbb{Q}$-closure of $\varphi(U(\mathbb{C}))$ is smaller than M_H°.

Now let H be a pure polarizable HS with Hodge numbers $h^{p,q}$, and take D (with compact dual $\check{D}$) to be the classifying space for such. We may view $\check{D}$ as a quasiprojective variety over $\mathbb{Q}$ in a suitable flag variety. Consider the subgroup $M_{H,\varphi}^\circ \subset M_H^\circ$ with real points $M_{H,\varphi}^\circ(\mathbb{R}) := (M_H^\circ(\mathbb{R}))^{\varphi(S(\mathbb{R}))}$. If we view M_H° as acting on a Hodge flag of $H_{\mathbb{C}}$ with respect to a (fixed) basis of $H_{\mathbb{Q}}$,then $M_{H,\varphi}^\circ$ is the stabilizer of the Hodge flag. This leads to a Noether–Lefschetz-type substratum in D:

PROPOSITION 111. *The MT domain*

$$D_H := \frac{M_H^\circ(\mathbb{R})}{M_{H,\varphi}^\circ(\mathbb{R})}\left(\subset \frac{M_H^\circ(\mathbb{C})}{M_{H,\varphi}^\circ(\mathbb{C})} =: \check{D}_H\right)$$

classifies HS with Hodge group contained in M_H, or equivalently with Hodge-tensor set containing that of H. The action of M_H° upon H embeds $\check{D}_H \hookrightarrow \check{D}$ as a quasiprojective subvariety, defined over an algebraic extension of $\mathbb{Q}$. The $\mathrm{GL}(H_{\mathbb{Q}}, Q)$*-translates of $\check{D}_H$ give isomorphic subdomains* (*with conjugate MT groups*) *dense in $\check{D}$.*

A similar definition works for certain kinds of MHS. The trouble with applying this in the variational setting (which is our main concern here), is that the "tautological VHS" (or VMHS) over such domains (outside of a few classical cases in low weight or level) violate Griffiths transversality hence are not actually VHS.

[13] Proof of this, and of Proposition 111 below, will appear in a work of the first author with P. Griffiths and M. Green.

Still, it can happen that MT domains in non-Hermitian symmetric period domains are themselves Hermitian symmetric. For instance, taking Sym^3 of HS's embeds the classifying space ($\cong \mathfrak{H}$) of (polarized) weight-1 Hodge structures with Hodge numbers $(1,1)$ into that of weight-3 Hodge structures with Hodge numbers $(1,1,1,1)$.

6.3. MT groups in the variational setting. Let S be a smooth quasiprojective variety with good compactification $\bar{S}$, and $\mathcal{V} \in \mathrm{VMHS}(S)^{\mathrm{ad}}_{\bar{S}}$; assume $\mathcal{V}$ is graded-polarized, which means we have

$$Q \in \bigoplus_i \mathrm{Hom}_{\mathrm{VMHS}(S)}\big((\mathrm{Gr}_i^W \mathcal{V})^{\otimes 2}, \mathbb{Q}(-i)\big)$$

satisfying the usual positivity conditions. The Hodge flag embeds the universal cover $\hat{S}(\twoheadrightarrow S)$ in a flag variety; let the *image-point* of $\hat{s}_0(\mapsto s_0)$ be of maximal transcendence degree. (One might say $s_0 \in S(\mathbb{C})$ is a "very general point in the sense of Hodge"; we are *not* saying s_0 is of maximal transcendence degree.) Parallel translation along the local system $\mathbb{V}$ gives rise to the monodromy representation $\rho : \pi_1(S, s_0) \to \mathrm{GL}(V_{s_0,\mathbb{Q}}, W_\bullet, Q)$. Moreover, taking as basis for $V_{s,\mathbb{Q}}$ the parallel translate of one for $V_{s_0,\mathbb{Q}}$, M_{V_s} is constant on paths (from s_0) avoiding a countable union T of proper analytic subvarieties of S, where in fact $S^\circ := S \backslash T$ is pathwise connected. (At points $t \in T$, $M_{V_t} \subset M_{V_s}$; and even the MT group of the LMHS $\psi_{\underline{s}}\mathcal{V}$ at $x \in \bar{S}\backslash S$ naturally includes in M_{V_s}.)

DEFINITION 112. (i) We call $M_{V_{s_0}} =: M_{\mathcal{V}}$ the *MT group*, and $M^\circ_{V_{s_0}} =: M^\circ_{\mathcal{V}}$ the *Hodge group*, of $\mathcal{V}$. One has $\mathrm{End}_{\mathrm{MHS}}(V_{s_0}) \cong \mathrm{End}_{\mathrm{VMHS}(S)}(\mathcal{V})$; see [PS2].

(ii) The identity connected component $\Pi_{\mathcal{V}}$ of the $\mathbb{Q}$-closure of $\rho(\pi_1(S, s_0))$ is the geometric monodromy group of $\mathcal{V}$; it is invariant under finite covers $\tilde{S} \twoheadrightarrow S$ (and semisimple in the split case).

PROPOSITION 113. *(André)* $\Pi_{\mathcal{V}} \trianglelefteq DM_{\mathcal{V}}$.

SKETCH OF PROOF. By a theorem of Chevalley, any closed $\mathbb{Q}$-algebraic subgroup of $\mathrm{GL}(V_{s_0})$ is the stabilizer, for some multitensor $\mathfrak{t} \in \bigoplus_i T^{m_i,n_i}(V_{s_0,\mathbb{Q}})$ of $\mathbb{Q}\langle \mathfrak{t} \rangle$. For $M_{\mathcal{V}}$, we can arrange for this $\mathfrak{t}_{\mathcal{V}}$ to be *itself* fixed and to lie in $\bigoplus_i \big(T^{m_i,n_i}(V_{s_0})\big)^{(0,0)}_{\mathbb{Q}}$. By genericity of s_0, $\mathbb{Q}\langle \mathfrak{t}_{\mathcal{V}} \rangle$ extends to a subVMHS with (again by $\exists$ of Q) finite monodromy group, and so $\mathfrak{t}_{\mathcal{V}}$ is fixed by $\Pi_{\mathcal{V}}$. This proves $\Pi_{\mathcal{V}} \subset M_{\mathcal{V}}$ (in fact, $\subset M^\circ_{\mathcal{V}}$ since monodromy preserves Q). Normality of this inclusion then follows from the Theorem of the Fixed Part: the largest constant sublocal system of any $T^{m,n}(\mathbb{V})$ (stuff fixed by $\Pi_{\mathcal{V}}$) is a subVMHS, hence subMHS at s_0 and stable under $M_{\mathcal{V}}$.

Now let

$$M^{\mathrm{ab}}_{\mathcal{V}} := \frac{M_{\mathcal{V}}}{DM_{\mathcal{V}}}, \qquad \Pi^{\mathrm{ab}}_{\mathcal{V}} := \frac{\Pi_{\mathcal{V}}}{\Pi_{\mathcal{V}} \cap DM_{\mathcal{V}}} \subset M^{\mathrm{ab}}_{\mathcal{V}},$$

(which is a connected component of the $\mathbb{Q}$-closure of some $\pi^{\mathrm{ab}} \subset M_{\mathcal{V}}^{\mathrm{ab},\circ}(\mathbb{Z})$), and (taking a more exotic route than André) V^{ab} be the (CM)MHS corresponding to a faithful representation of $M_{\mathcal{V}}^{\mathrm{ab}}$. For each irreducible $H \subset V^{\mathrm{ab}}$, the image $\overline{M_{\mathcal{V}}^{\mathrm{ab}}}$ has integer points $\cong \mathcal{O}_L^*$ for some CM field L, and $\overline{M_{\mathcal{V}}^{\mathrm{ab},\circ}}(\mathbb{Q}) \subset L$ consists of elements of norm 1 under any embedding. The latter generate L (a well-known fact for CM fields) but, by a theorem of Kronecker, have finite intersection with $\mathcal{O}_L^*$: the roots of unity. It easily follows from this that $\overline{\Pi_{\mathcal{V}}^{\mathrm{ab}}}$, hence $\Pi_{\mathcal{V}}^{\mathrm{ab}}$, is trivial. □

DEFINITION 114. Let $x \in \bar{S}$ with neighborhood $(\Delta^*)^k \times \Delta^{n-k}$ in S and local (commuting) monodromy logarithms $\{N_i\}$;[14] define the weight monodromy filtration $M_\bullet^x := M(N, W)_\bullet$ where $N := \sum_{i=1}^k N_i$. In the following we assume a choice of path from s_0 to x:

(a) Write $\pi_{\mathcal{V}}^x$ for the *local monodromy group* in $\mathrm{GL}(V_{s_0,\mathbb{Z}}, W_\bullet, Q)$ generated by the $T_i = (T_i)_{ss} e^{N_i}$, and ρ^x for the corresponding representation.

(b) We say that $\mathcal{V}$ is *nonsingular at* x if $V_{s_0} \cong \bigoplus_j \mathrm{Gr}_j^W V_{s_0}$ as ρ^x-modules. In this case, the condition that $\psi_{\underline{s}}\mathcal{V} \cong \bigoplus_j \psi_{\underline{s}} \mathrm{Gr}_j^W \mathcal{V}$ is independent of the choice of local coordinates $(s_1, \ldots, s_n)$ at x, and $\mathcal{V}$ is called *semisplit (nonsingular)* at x when this is satisfied.

(c) The $\mathrm{Gr}_i^{M^x} \psi_{\underline{s}}\mathcal{V}$ are always independent of $\underline{s}$. We say that $\mathcal{V}$ is *totally degenerate (TD)* at x if these Gr_i^M are (pure) Tate and *strongly degenerate* (SD) at x if they are CM-HS. Note that the SD condition is interesting already for the nonboundary points ($x \in S$, $k = 0$).

We can now generalize results of André [An] and Mustafin [Ms].

THEOREM 115. *If $\mathcal{V}$ is semisplit TD* (*resp. SD*) *at a point $x \in \bar{S}$, then $\Pi_{\mathcal{V}} = M_{\mathcal{V}}^\circ$* (*resp. $DM_{\mathcal{V}}^\circ$*).

REMARK 116. Note that semisplit SD at $x \in S$ simply means that V_x is a CM-MHS (this case is done in [An]). Also, if $\Pi_{\mathcal{V}} = M_{\mathcal{V}}^\circ$ then in fact $\Pi_{\mathcal{V}} = DM_{\mathcal{V}}^\circ = M_{\mathcal{V}}^\circ$.

PROOF. Passing to a finite cover to identify $\Pi_{\mathcal{V}}$ and $\overline{\rho(\pi_1)}$, if we can show that any invariant tensor $\mathfrak{t} \in \left(T^{m,n} V_{s_0,\mathbb{Q}}\right)^{\Pi_{\mathcal{V}}}$ is also fixed by $M_{\mathcal{V}}^\circ$ (resp. $DM_{\mathcal{V}}^\circ$), we are done by Chevalley. Now the span of $M_{\mathcal{V}}^\circ \mathfrak{t}$ is (since $\Pi_{\mathcal{V}} \trianglelefteq M_{\mathcal{V}}^\circ$) fixed by $\rho(\pi_1)$, and (using the Theorem of the Fixed Part) extends to a constant subVMHS $\mathcal{U} \subset T^{m,n}\mathcal{V} =: \mathcal{T}$. Now the hypotheses on $\mathcal{V}$ carry over to $\mathcal{T}$ and taking LMHS at x, $\mathcal{U} = \psi_{\underline{s}}\mathcal{U} = \bigoplus_i \psi_{\underline{s}} \mathrm{Gr}_i^W \mathcal{U} = \bigoplus_i \mathrm{Gr}_i^W \mathcal{U}$, we see that $\mathcal{U}$ splits (as VMHS). As $\mathcal{T}$ is TD (resp. SD) at x, $\mathcal{U}$ is split Hodge–Tate (resp. CM-MHS).

[14] Though this has been suppressed so far throughout this paper, one has $\{N_i\}$ and LMHS even in the general case where the local monodromies T_i are only quasi-unipotent, by writing $T_i =: (T_i)_{ss}(T_i)_u$ uniquely as a product of semisimple and unipotent parts (Jordan decomposition) and setting $N_i := \log((T_i)_u)$.

If $\mathcal{U}$ is H-T then it consists of Hodge tensors; so $M_{\mathcal{V}}^{\circ}$ acts trivially on $\mathcal{U}$ hence on $\mathfrak{t}$.

If $\mathcal{U}$ is CM then $M_{\mathcal{V}}^{\circ}|_{\mathcal{U}} = M_{\mathcal{U}}^{\circ}$ is abelian; and so the action of $M_{\mathcal{V}}^{\circ}$ on $\mathcal{U}$ factors through $M_{\mathcal{V}}^{\circ}/DM_{\mathcal{V}}^{\circ}$, so that $DM_{\mathcal{V}}^{\circ}$ fixes $\mathfrak{t}$. □

A reason why one would want this "maximality" result $\Pi_{\mathcal{V}} = M_{\mathcal{V}}^{\circ}$ is to satisfy the hypothesis of the following interpretation of Theorem 91 (which was a partial generalization of results of [Vo1] and [Ch]). Recall that a VMHS $\mathcal{V}/S$ is k-motivated if there is a family $\mathcal{X} \to S$ of quasiprojective varieties defined over k with $V_s =$ the canonical (Deligne) MHS on $H^r(X_s)$ for each $s \in S$.

PROPOSITION 117. *Suppose $\mathcal{V}$ is motivated over k with trivial fixed part, and let $T_0 \subset S$ be a connected component of the locus where $M_{V_s}^{\circ}$ fixes some vector (in V_s). If T_0 is algebraic (over $\mathbb{C}$), $M_{\mathcal{V}_{T_0}}^{\circ}$ has* only *one fixed line, and $\Pi_{\mathcal{V}_{T_0}} = M_{\mathcal{V}_{T_0}}^{\circ}$, then T_0 is defined over $\bar{k}$.*

Of course, to be able to use this one also needs a result on algebraicity of T_0, i.e., a generalization of the theorems of [CDK] and [BP3] to arbitrary VMHS. One *now has this* by work of Brosnan, Schnell, and the second author:

THEOREM 118. *Given any integral, graded-polarized $\mathcal{V} \in \mathrm{VMHS}(S)^{\mathrm{ad}}_{\bar{S}}$, the components of the Hodge locus of any $\alpha \in V_s$ yield complex algebraic subvarieties of S.*

6.4. MT groups of (higher) normal functions. We now specialize to the case where $\mathcal{V} \in \mathrm{NF}^r(S, \mathcal{H})^{\mathrm{ad}}_{\bar{S}}$, with $\mathcal{H} \to S$ the underlying VHS of weight $-r$. $M_{\mathcal{V}}^{\circ}$ is then an extension of $M_{\mathcal{H}}^{\circ} \cong M_{\mathcal{V}^{\mathrm{split}}(=\mathcal{H}\oplus\mathbb{Q}_S(0))}^{\circ}$ by (and a semidirect product with) an additive (unipotent) group

$$U := W_{-r} M_{\mathcal{V}}^{\circ} \cong \mathbb{G}_a^{\times\mu},$$

with $\mu \leq \mathrm{rank}\mathbb{H}$. Since $M_{\mathcal{V}}^{\circ}$ respects weights, there is a natural map $\eta : M_{\mathcal{V}}^{\circ} \twoheadrightarrow M_{\mathcal{H}}^{\circ}$ and one might ask when this is an isomorphism.

PROPOSITION 119. *$\mu = 0 \iff \mathcal{V}$ is torsion.*

PROOF. First we note that $\mathcal{V}$ is torsion if and only if, for some finite cover $\tilde{S} \twoheadrightarrow S$, we have

$$\begin{aligned}\{0\} \neq \mathrm{Hom}_{\mathrm{VMHS}(\tilde{S})}(\mathbb{Q}_S(0), \mathcal{V}) &= \mathrm{End}_{\mathrm{VMHS}(\tilde{S})}(\mathcal{V}) \cap \mathrm{ann}(\mathcal{H}) \\ &= \mathrm{End}_{\mathrm{MHS}}(V_{s_0}) \cap \mathrm{ann}(H_{s_0}) = \left(\mathrm{Hom}_{\mathbb{Q}}((V_{s_0}/H_{s_0}), V_{s_0})\right)^{M_{\mathcal{V}}^{\circ}}.\end{aligned}$$

The last expression can be interpreted as consisting of vectors $\underline{w} \in H_{s_0,\mathbb{Q}}$ that satisfy $(\mathrm{id} - M)\underline{w} = \underline{u}$ whenever $\left(\begin{smallmatrix} 1 & 0 \\ \underline{u} & M \end{smallmatrix}\right) \in M_{\mathcal{V}}^{\circ}$. This is possible only if there is one $\underline{u}$ for each M, i.e., if $\eta : M_{\mathcal{V}}^{\circ} \to M_{\mathcal{H}}^{\circ}$ is an isomorphism. Conversely,

assuming this, write $\underline{u} = \eta^{-1}(M)$ [noting

$$\tilde{\eta}^{-1}(M_1 M_2) = \tilde{\eta}^{-1}(M_1) + M_1 \tilde{\eta}^{-1}(M_2)] \qquad (*)$$

and set $\underline{w} := \tilde{\eta}^{-1}(0)$. Taking $M_2 = 0$ and $M_1 = M$ in $(*)$, we get $(\mathrm{id} - M)\underline{w} = (\mathrm{id} - M)\eta^{-1}(0) = \tilde{\eta}^{-1}(M) = \underline{u}$ for all $M \in M^\circ_{\mathcal{H}}$. □

We can now address the problem which lies at the heart of this section: what can one say about the monodromy of the normal function above and beyond that of the underlying VHS — for example, about the kernel of the natural map $\Theta : \Pi_{\mathcal{V}} \twoheadrightarrow \Pi_{\mathcal{H}}$? One can make some headway simply by translating Definition 114 and Theorem 115 into the language of normal functions; all vanishing conditions are $\otimes\mathbb{Q}$.

PROPOSITION 120. *Let $\mathcal{V}$ be an admissible higher normal function over S, and let $x \in \bar{S}$ with local coordinate system $\underline{s}$.*

(i) *$\mathcal{V}$ is nonsingular (as AVMHS) at x if and only if* $\mathrm{sing}_x(\mathcal{V}) = 0$. *Assuming this, $\mathcal{V}$ is semi-simple at x if and only if* $\lim_x(\mathcal{V}) = 0$. (*In case $x \in S$,* $\mathrm{sing}_x(\mathcal{V}) = 0$ *is automatic and* $\lim_x(\mathcal{V}) = 0$ *if and only if x is in the torsion locus of $\mathcal{V}$.*)

(ii) *$\mathcal{V}$ is TD (resp. SD) at x if and only if the underlying VHS $\mathcal{H}$ is.* (*For $x \in S$, this just means that H_x is CM.*)

(iii) *If* $\mathrm{sing}_x(\mathcal{V})$, $\lim_x(\mathcal{V})$ *vanish and $\psi_{\underline{s}}\mathcal{H}$ is graded CM, then $\Pi_{\mathcal{V}} = DM_{\mathcal{V}}$.* (*For $x \in S$, we are just hypothesizing that the torsion locus of $\mathcal{V}$ contains a CM point of $\mathcal{H}$.*)

(iv) *Let $x \in \bar{S} \setminus S$. If* $\mathrm{sing}_x(\mathcal{V})$, $\lim_x(\mathcal{V})$ *vanish and $\psi_{\underline{s}}\mathcal{H}$ is Hodge–Tate, then $\Pi_{\mathcal{V}} = M^\circ_{\mathcal{V}}$.*

(v) *Under the hypotheses of* (iii) *and* (iv), $\dim(\ker(\Theta)) = \mu$. (*In general one has* $\leq$.)

PROOF. All parts are self-evident except for (v), which follows from observing (in both cases (iii) and (iv)) via the diagram

$$\begin{array}{ccccc} \mathbb{G}_a^{\times\mu} \cong W_{-1}M_{\mathcal{V}}^{(\circ)} = \ker(\eta) \subseteq DM_{\mathcal{V}} & = & \Pi_{\mathcal{V}} & \hookrightarrow & M_{\mathcal{V}}^{(\circ)} \\ & & \downarrow \Theta & & \downarrow \eta \\ & & \Pi_{\mathcal{H}} & \hookrightarrow & M_{\mathcal{H}}^{(\circ)} \end{array} \qquad (6\text{-}2)$$

that $\ker(\eta) = \ker(\Theta)$. □

EXAMPLE 121. The Morrison–Walcher normal function from §1.7 (Example 13) lives "over" the VHS $\mathcal{H}$ arising from $R^3\pi_*\mathbb{Z}(2)$ for a family of "mirror quintic" CY 3-folds, and vanishes at $z = \infty$. (One should take a suitable, e.g., order 2 or 10 pullback so that $\mathcal{V}$ is well-defined.) The underlying HS H at this point is of CM type (the fiber is the usual $(\mathbb{Z}/5\mathbb{Z})^3$ quotient of $\{\sum_{i=0}^4 Z_i^5 = 0\} \subset \mathbb{P}^4$),

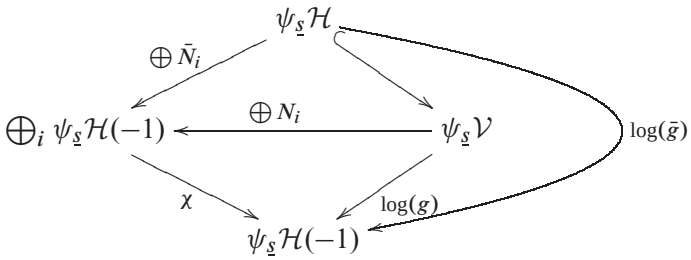
with $M_H(\mathbb{Q}) \cong \mathbb{Q}(\zeta_5)$. So $\mathcal{V}$ would satisfy the conditions of Proposition 120(iii). It should be interesting to work out the consequences of the resulting equality $\Pi_{\mathcal{V}} = DM_{\mathcal{V}}$.

There is a different aspect to the relationship between local and global behavior of $\mathcal{V}$. Assuming for simplicity that the local monodromies at x are unipotent, let $\kappa_x := \ker(\pi_{\mathcal{V}}^x \twoheadrightarrow \pi_{\mathcal{H}}^x)$ denote the local monodromy kernel, and μ_x the dimensions of its $\mathbb{Q}$-closure $\overline{\kappa_x}$. This is an additive (torsion-free) subgroup of $\ker(\Theta)$, and so $\dim(\ker(\Theta)) \geq \mu_x$ $(\forall x \in \bar{S}\backslash S)$. Writing $\{N_i\}$ for the local monodromy logarithms at x, we have the

PROPOSITION 122. (*i*) $\mu_x > 0$ *implies* $\mathrm{sing}_x(\mathcal{V}) \neq 0$ (*nontorsion singularity*)

(*ii*) *The converse holds assuming* $r = 1$ *and* $\mathrm{rank}(N_i) = 1$ $(\forall i)$.

PROOF. Let $g \in \pi_{\mathcal{V}}^x$, and define $\underline{m} \in \mathbb{Q}^{\oplus k}$ by $\log(g) =: \sum_{i=1}^k m_i N_i$. Writing $\bar{g}$, $\bar{N}_i$ for $g|_{\mathbb{H}}$, $N_i|_H$, consider the (commuting) diagram of morphisms of MHS



(6-3)

where $\chi(\underline{w}_1, \ldots, \underline{w}_k) = \sum_{i=1}^k m_i \underline{w}_i$ and $\log(g) = \sum_{i=1}^k m_i \bar{N}_i$. Then $\mathrm{sing}_x(\mathcal{V})$ is nonzero if and only if $\left(\bigoplus N_i\right)\nu_{\mathbb{Q}}$ does not lie in $\mathrm{im}(\oplus \bar{N}_i)$, where $\nu_{\mathbb{Q}}$ (see Definition 2(b)) generates $\psi_{\underline{s}}\mathcal{V}/\psi_{\underline{s}}\mathcal{H}$.

(i) Suppose $g \in \kappa_x\backslash\{1\}$. Then $0 = \log(\bar{g})$ implies $0 = \chi(\mathrm{im}(\oplus \bar{N}_i))$ while $0 \neq \log g$ implies $0 \neq (\log(g))\nu_{\mathbb{Q}} = \chi((\oplus N_i)\nu_{\mathbb{Q}})$. So χ "detects" a singularity.

(ii) If $r = 1$ we may replace $\bigoplus_{i=1}^k \psi_{\underline{s}}\mathcal{H}(-1)$ in the diagram by the subspace $\bigoplus_{i=1}^k (N_i(\psi_{\underline{s}}\mathcal{H}))$. Since each summand is of dimension 1, and

$$\left(\bigoplus N_i\right)\nu_{\mathbb{Q}} \notin \mathrm{im}\left(\bigoplus \bar{N}_i\right)$$

(by assumption), we can choose $\underline{m} = \{m_i\}$ in order that χ kill $\mathrm{im}\left(\bigoplus \bar{N}_i\right)$ but not $\left(\bigoplus N_i\right)\nu_{\mathbb{Q}}$. Using the diagram, $\log(\bar{g}) = 0 \neq \log(g)$ implies $g \in \kappa_x\backslash\{1\}$. □

REMARK 123. (a) The existence of a singularity *always* implies that $\mathcal{V}$ is nontorsion, hence $\mu > 0$.

(b) In the situation of [GG], we have $r = 1$ and rank 1 local monodromy logarithms; hence, by Proposition 122(ii), the existence of a singularity implies $\dim(\ker(\Theta)) > 0$, consistent with (a).

(c) By Proposition 122(i), in the normal function case ($r = 1$), $\mu_x = 0$ along codimension-1 boundary components.

(d) In the "maximal geometric monodromy" situation of Proposition 120(v), $\mu \geq \mu_x \ \forall x \in \bar{S}\backslash S$.

Obviously, for the purpose of forcing singularities to exist, the inequality in (d) points in the wrong direction. One wonders if some sort of cone or spread on a VMHS might be used to translate global into local monodromy kernel, but this seems unlikely to be helpful.

We conclude with an amusing application of differential Galois theory related to a result of André [An]:

PROPOSITION 124. *Consider a normal function $\mathcal{V}$ of geometric origin together with an $\mathcal{O}_S$-basis $\{\omega_i\}$ of holomorphic sections of $\mathcal{F}^0\mathcal{H}$. (That is, V_s is the extension of MHS corresponding to $AJ(Z_s) \in J^p(X_s)$ for some flat family of cycles on a family of smooth projective varieties over S.) Let K denote the extension of $\mathbb{C}(S)$ by the (multivalued) periods of the $\{\omega_i\}$; and L denote the further extension of K via the (multivalued) Poincaré normal functions given by pairing the ω_i with an integral lift of $1 \in \mathbb{Q}_S(0)$ (i.e., the membrane integrals $\int_{\Gamma_s} \omega_i(s)$ where $\partial\Gamma_s = Z_s$). Then* $\operatorname{trdeg}(L/K) = \dim(\ker(\Theta))$.

The proof rests on a result of N. Katz [Ka, Corollary 2.3.1.1] relating transcendence degrees and dimensions of differential Galois groups, together with the fact that the $\{\int_{\Gamma_s} \omega_i\}$ (for each i) satisfy a homogeneous linear ODE with regular singular points [Gr1]. (This fact implies equality of differential Galois and geometric monodromy groups, since monodromy invariant solutions of such an ODE belong to $\mathbb{C}(S)$ which is the fixed field of the Galois group.) In the event that $\mathcal{H}$ has no fixed part (so that L can introduce no new constants and one has a "Picard–Vessiot field extension") and the normal function is motivated over $k = \bar{k}$, one can probably replace $\mathbb{C}$ by k in the statement.

References

[Ab] S. Abdulali, *Hodge structures of CM type*, J. Ramanujan Math. Soc. 20 (2005), no. 2, 155–162.

[An] Y. Andre, *Mumford–Tate groups of mixed Hodge structures and the theorem of the fixed part*, Compositio Math. 82 (1992), no. 1, 1–24.

[Ar] D. Arapura, *The Leray spectral sequence is motivic*, Invent. Math. 160 (2005), no. 3, 567–589.

[AK] D. Arapura and M. Kumar, *Beilinson–Hodge cycles on semiabelian varieties*, Math. Res. Lett. 16 (2009), no. 4, 557–562.

[BBD] A. Beilinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, in "Analysis and topology of singular spaces, I" (Luminy, 1981), pp. 5–171, Astérisque 100, Soc. Math. France, Paris, 1982.

[Be] A. A. Beilinson, *Height pairing between algebraic cycles*, in "K-theory, arithmetic, and geometry (Moscow, 1984–6)", pp. 1–25, LNM 1289, Springer, Berlin, 1987.

[Bl] S. Bloch, *Algebraic cycles and higher K-theory*, Adv. Math. 61 (1986), no. 3, 267–304.

[Bo] C. Borcea, *Calabi–Yau threefolds and complex multiplication*, in "Essays on mirror manifolds", 489–502, Intl. Press, Hong Kong, 1992.

[BP1] P. Brosnan and G. Pearlstein, *The zero-locus of an admissible normal function*, Ann. Math. 170 (2009), 883–897.

[BP2] ———, *Zero loci of normal functions with torsion singularities*, Duke Math. J. 150 (2009), no. 1, 77–100.

[BP3] ———, *On the algebraicity of the zero-locus of an admissible normal function*, arXiv:0910.0628v1.

[BFNP] P. Brosnan, H. Fang, Z. Nie, and G. Pearlstein, *Singularities of admissible normal functions*, Inventiones Math. 117 (2009), 599–629.

[BPS] P. Brosnan, G. Pearlstein and M. Saito, *A generalization of the Néron models of Griffiths, Green and Kerr*, arXiv:0809.5185, preprint.

[BPSc] P. Brosnan, G. Pearlstein and C. Schnell, *The locus of Hodge classes in an admissible variation of mixed Hodge structure*, C. R. Acad. Sci. Paris, Ser. I 348 (2010), 657–660.

[BZ] J.-L. Brylinski and S. Zucker, *An overview of recent advances in Hodge theory*, in "Several complex variables, VI", 39–142, Encyclopedia Math. Sci. 69, Springer, Berlin, 1990.

[Ca] J. Carlson, *Extensions of mixed Hodge structures*, in "Journées de Géometrie Algébrique d'Angers" (1979), pp. 107–127, Sijthoff & Noordhoff, Alphen aan def Rijn—Germantown, Md., 1980.

[CDK] E. Cattani, P. Deligne, and A Kaplan, *On the locus of Hodge classes*, JAMS 8 (1995), 483–506.

[CF] E. Cattani and J. Fernandez, *Asymptotic Hodge theory and quantum products*, math.AG/001113, in "Advances in algebraic geometry motivated my physics" (Lowell, MA), pp. 115–136, Contemp. Math. 276, AMS, Providence, RI, 2001.

[CK] E. Cattani and A. Kaplan, *Degenerating variations of Hodge structure*, in "Actes du Colloque de Théorie de Hodge" (Luminy), Astérisque 179–180 (1989), 9, 67–96.

[CKS1] E. Cattani, A. Kaplan, and W. Schmid, *Degeneration of Hodge structures*, Ann. of Math. 123 (1986), 457–535.

[CKS2] ———, *L^2 and intersection cohomologies for a polarizable variation of Hodge structure*, Invent. Math. 87 (1987), 217–252.

[CMP] J. Carlson, S. Müller-Stach, and C. Peters, "Period mappings and period domains", Cambridge Stud. Adv. Math. 85, Cambridge University Press, Cambridge, 2003.

[Ch] F. Charles, *On the zero locus of normal functions and the étale Abel–Jacobi map*, IMRN vol. 2010, no. 12, 2283–2304.

[Cl1] H. Clemens, *Degeneration of Kähler manifolds*, Duke Math. J. 44 (1977), 212–290.

[Cl2] ———, The Néron model for families of intermediate Jacobians acquiring "algebraic" singularities, Publ. IHES 58 (1983), 5–18.

[Co] A. Collino, *Griffiths's infinitesimal invariant and higher K-theory on hyperelliptic Jacobians*, J. Alg. Geom. 6 (1997), no.3, 393–415.

[dCM] M. A. de Cataldo and L. Migliorini, *On singularities of primitive cohomology classes*, Proc. Amer. Math. Soc. 137 (2009), 3593–3600.

[DM1] P. del Angel and S. Müller-Stach, *Differential equations associated to families of algebraic cycles*, Ann. Inst. Fourier (Grenoble) 58 (2008), no. 6, 2075–2085.

[DM2] ———, *Picard–Fuchs equations, integrable systems, and higher algebraic K-theory*, in "Calabi–Yau Varieties and Mirror Symmetry" (Yui and Lewis, editors), pp. 43–56, AMS, 2003.

[De1] P. Deligne, *Théorie de Hodge II, III* , Inst. Hautes Etudes Sci. Publ. Math. 40 (1971), 5–57; 44 (1974), 5–77.

[De2] P. Deligne, *Hodge cycles on abelian varieties* (notes by J. Milne), in "Hodge cycles, motives, and Shimura varieties", LNM 900, Springer, New York, 1982.

[De3] ———, letter to Cattani and Kaplan (unpublished).

[De4] ———, *Local behavior of Hodge structures at infinity*, AMS/IP Stud. Adv. Math. 1 (1997), AMS.

[DK] C. Doran and M. Kerr, *Algebraic K-theory of toric hypersurfaces*, preprint, arXiv:0809.5279.

[EZ] F. El Zein and S. Zucker, *Extendability of normal functions associated to algebraic cycles*, in "Topics in transcendental algebraic geometry", Ann. Math. Stud. 106, Princeton Univ. Press, Princeton, NJ, 1984.

[GGM] A. Galligo, M. Granger, P. Maisonobe, *$\mathcal{D}$-modules at faisceaux pervers dont le support singulier est un croisement normal*, Ann. Inst. Fourier 35 (1985), no. 1, 1–48.

[GMV] M. Green, J. Murre, and C. Voisin, "Algebraic cycles and Hodge theory" (Albano and Bardelli, editors), LNM 1594, Springer, 1994.

[Gr1] P. Griffiths, *A theorem concerning the differential equations satisfied by normal functions attached to algebraic cycles*, Amer. J. Math. 101 (1979), no. 1, 94–131.

[Gr2] ———, *Poincaré and algebraic geometry*, Bull. AMS 6 (1982), no. 2, 147–159.

[Gr3] ———, *On the periods of certain rational integrals II*, Ann. of Math. 90 (1969), no. 2,496–541.

[Gr4] ———, *Singularities of admissible normal functions*, to appear in "Cycles, motives, and Shimura varieties" (Bombay, 2008).

[GC] P. Griffiths and H. Clemens, *The intermediate Jacobian of the cubic threefold*, Ann. of Math. (2) 95 (1972), 281–356.

[GG] P. Griffiths and M. Green, *Algebraic cycles and singularities of normal functions*, in "Algebraic cycles and motives", pp. 206–263, LMS Lect. Not. Ser. 343, Cambridge Univ. Press, Cambridge, 2007.

[GGK] P. Griffiths, M. Green and M. Kerr, *Néron models and limits of Abel–Jacobi mappings*, Compositio Math. 146 (2010), 288–366.

[GH] P. Griffiths and J. Harris, "Principles of algebraic geometry", Wiley, New York, 1978.

[GS] P. Griffiths and W. Schmid, *Locally homogeneous complex manifolds*, Acta Math. 123 (1969), 253–302.

[Ha] F. Hazama, *Hodge cycles on certain abelian varieties and powers of special surfaces*, J. Fac. Sci. Univ. Tokyo, Sect. 1a, 31 (1984), 487–520

[K] M. Kashiwara, *A study of variation of mixed Hodge structure*, Publ. Res. Inst. Math. Sci. 22 (1986), no. 5, 991–1024.

[KNU1] K. Kato, C. Nakayama and S. Usui, SL(2)*-orbit theorem for degeneration of mixed Hodge structure*, J. Algebraic Geom. 17 (2008), 401–479.

[KNU2] ———, *Log intermediate Jacobians*, Proc. Japan Acad. Ser. A Math. Sci. Volume 86, Number 4 (2010), 73–78.

[KNU3] ———, private communication.

[KU] K. Kato and S. Usui, "Classifying spaces of degenerating polarized Hodge structures", Ann. Math. Stud. 169, 2009.

[Ka] N. Katz, "Exponential sums and differential equations", Ann. of Math. Study 124, Princeton University Press, Princeton, NJ, 1990.

[Ki] J. King, *Log complexes of currents and functorial properties of the Abel–Jacobi map*, Duke Math. J. 50 (1983), no. 1, 1–53.

[KL] M. Kerr and J. Lewis, The Abel–Jacobi map for higher Chow groups, II, Invent. Math 170 (2007), no. 2, 355–420.

[KLM] M. Kerr, J. Lewis and S. Muller-Stach, *The Abel–Jacobi map for higher Chow groups*, Compositio Math. 142 (2006), no. 2, 374–396.

[KP] M. Kerr and G. Pearlstein, Normal functions and the GHC, preprint, available at http://www.math.wustl.edu/˜matkerr/KPGHC.pdf.

[KY] B. Klingler and A. Yafaev, *The André–Oort Conjecture*, preprint, available at http://www.math.jussieu.fr/˜klingler/papiers/KY12.pdf.

[L] S. Lefschetz, "L'Analysis situs et la géometrie algébrique", Gauthier-Villars, Paris, 1924.

[Le1] J. Lewis, "A survey of the Hodge Conjecture" (2nd ed.), CRM Monograph Ser. 10, AMS, Providence, RI, 1999.

[Le2] ———, *A filtration on the Chow groups of a complex projective variety*, Compositio Math. 128 (2001), no. 3, 299–322.

[Li] D. Lieberman, *On the module of intermediate Jacobians*, Amer. J. Math. 91 (1969), 671–682.

[Ma] Y. I. Manin, *Rational points on algebraic curves over function fields*, Izv. Akad. Nauk SSSR Ser. Mat. 27 (1963), 1395–1440.

[Mo] D. Morrison, *The Clemens–Schmid exact sequence and applications*, in "Topics in transcendental algebraic geometry", pp. 101–119, Ann. Math. Studies, Princeton Univ. Press, Princeton, NJ, 1984.

[MW] D. Morrison and J. Walcher, *D-branes and normal functions*, Adv. Theor. Math. Phys. 13 (2009), no. 2, 553–598.

[Mu] D. Mumford, *Families of abelian varieties*, in "Algebraic groups and discontinuous subgroups" (Boulder, CO, 1965), pp. 347–351, AMS, Providence, RI, 1966.

[Mr] V. K. Murty, *Exceptional Hodge classes on certain abelian varieties*, Math. Ann. 268 (1984), 197–206.

[Ms] G. A. Mustafin, *Families of algebraic varieties and invariant cycles*, Izv. Akad. Mauk. SSSR Ser. Mat. 49 (1985), no. 5, pp. 948–978.

[Na] I. Nakamura, *Relative compactification of the Néron model and its application*, in "Complex analysis and algebraic geometry", pp. 207–225, Iwanami Shoten, Tokyo, 1977.

[NS] A. Némethi and J. Steenbrink, *Extending Hodge bundles for abelian variations*, Annals of Math. 142 (1996), 1–18.

[Pe1] G. Pearlstein,*Variations of mixed Hodge structure, Higgs fields and quantum cohomology*, Manuscripta Math. 102 (2000), 269–310.

[Pe2] ———, *SL_2-orbits and degenerations of mixed Hodge structure*, J. Differential Geom. 74 (2006), no. 1, 1–67.

[Pe3] ———, *Hodge conjecture and the period map at infinity*, preprint.

[PS1] C. Peters and J. Steenbrink, "Mixed Hodge structures", Ergebnisse der Mathematik ser. 3 vol. 52, Springer, Berlin, 2008.

[PS2] ———, *Monodromy of variations of Hodge structure*, Acta Appl. Math. 75 (2003), no. 1–3, 183–194.

[P1] H. Poincaré, *Sur les courbes tracées sur les surfaces algébriques*, Ann. Sci. de l'École Norm. Sup. 27 (1910), 55–108.

[P2] ———, *Sur les courbes tracées sur une surface algébrique*, Sitz. der Berliner Math. Gesellschaft 10 (1911), 28–55.

[RS] A. Rosenschon and M. Saito, *Cycle map for strictly decomposable cycles*, Amer. J. Math. 125 (2003), 773–790.

[S1] M. Saito, *Admissible normal functions*, J. Alg. Geom. 5 (1996), no. 2, 235–276.

[S2] ———, *Direct image of logarithmic complexes and infinitesimal invariants of cycles*, in "Algebraic cycles and motives", vol. 2, pp. 304–318, LMS Lect. Not. Ser. 344, Cambridge Univ. Press, Cambridge, 2007.

[S3] ———, *Cohomology classes of admissible normal functions*, arXiv:0904.1593, preprint.

[S4] ———, *Mixed Hodge modules*, Publ. Res. Inst. Math. Sci. 26 (1990), no. 2, 221–333.

[S5] ———, *Hausdorff property of the Zucker extension at the monodromy invariant subspace*, arXiv:0803.2771, preprint.

[S6] ———, *Variations of Hodge structures having no cohomologically nontrivial admissible normal functions*, preprint.

[S7] ———, *On the theory of mixed Hodge modules*, AMS Translations (Series 2) 160 (1994), 47–61. Translated from Sūgaku.

[S8] ———, *Modules de Hodge polarisables*, Publ. Res. Inst. Math. Sci. 24 (1988), no. 6, 849–995.

[SS] M. Saito and C. Schnell, *A variant of Néron models over curves*, to appear in Manuscripta Math.

[Sa] S. Saito, *Motives and filtrations on Chow groups*, Invent. Math. 125 (1996), 149–196.

[Sc] W. Schmid, *Variation of Hodge structure: the singularities of the period mapping*, Invent. Math. 22 (1973), 211–319.

[Sl1] C. Schnell, *Two observations about normal functions*, Clay Math. Proc. 9 (2010), 75–79.

[Sl2] ———, *Local duality and polarized Hodge modules*, arXiv:0904.3480, to appear in Publ. Math. Res. Inst. Sci.

[Sl3] ———, "The boundary behavior of cohomology classes and singularities of normal functions," Ph.D. Thesis, Ohio State University, 2008. Available at `OhioLink ETD`.

[Sl4] ———, *An alternative construction of the Néron model*, preprint, 2009.

[Sn] C. Schoen, *Varieties dominated by product varieties*, Internat. J. Math. 7 (1996), no. 4, 541–571.

[Sh] T. Shioda, *A note on a theorem of Griffiths on the Abel–Jacobi map*, Invent. Math. 82 (1985), no. 3, 461–465.

[Sp] T. A. Springer, "Linear algebraic groups" (2nd ed.), Birkhäuser, Boston, 1998.

[St] J. Steenbrink, *Limits of Hodge structures*, Inventiones Math. 31 (1976), 229–257.

[SZ] J. Steenbrink and S. Zucker, *Variation of mixed Hodge structure, I*, Invent. Math. 80 (1985), no. 3, 489–542.

[Th] R. Thomas, *Nodes and the Hodge conjecture*, J. Algebraic Geom. 14 (2005), no. 1, 177–185.

[Vo1] C. Voisin, *Hodge loci and absolute Hodge classes*, Compos. Math. 143 (2007), no. 4, 945–958.

[Vo2] ———, "Hodge theory and complex algebraic geometry" (2 vol.), Cambridge Stud. Adv. Math. 76, 77, Cambridge Univ. Press, Cambridge, 2007.

[Yo] A. Young, "Complex analytic Néron models for abelian varieties over higher dimensional parameter spaces", Princeton Univ. Ph.D. thesis.

[Zu1] S. Zucker, *The Hodge conjecture for cubic fourfolds*, Compositio Math. 34 (1977), no. 2, 199–209.

[Zu2] ———, *Generalized intermediate Jacobians and the theorem on normal functions*, Inventiones Math. 33 (1976), no. 2, 185–222.

MATT KERR
DEPARTMENT OF MATHEMATICS
WASHINGTON UNIVERSITY IN ST. LOUIS
CUPPLES I HALL
ONE BROOKINGS DRIVE
ST. LOUIS, MO 63130
UNITED STATES
matkerr@math.wustl.edu

GREGORY PEARLSTEIN
DEPARTMENT OF MATHEMATICS
MICHIGAN STATE UNIVERSITY
EAST LANSING, MI 48824
UNITED STATES
gpearl@math.msu.edu