

Interpolation

JOE HARRIS

This is an overview of interpolation problems: when, and how, do zero-dimensional schemes in projective space fail to impose independent conditions on hypersurfaces?

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1. The interpolation problem

We give an overview of the exciting class of problems in algebraic geometry known as interpolation problems: basically, when points (or more generally zero-dimensional schemes) in projective space may fail to impose independent conditions on polynomials of a given degree, and by how much.

We work over an arbitrary field K . Our starting point is this elementary theorem:

Theorem 1.1. *Given any $z_1, \dots, z_{d+1} \in K$ and $a_1, \dots, a_{d+1} \in K$, there is a unique $f \in K[z]$ of degree at most d such that*

$$f(z_i) = a_i, \quad i = 1, \dots, d + 1.$$

More generally:

Theorem 1.2. *Given any $z_1, \dots, z_k \in K$, natural numbers $m_1, \dots, m_k \in \mathbb{N}$ with $\sum m_i = d + 1$, and*

$$a_{i,j} \in K, \quad 1 \leq i \leq k; \quad 0 \leq j \leq m_i - 1,$$

there is a unique $f \in K[z]$ of degree at most d such that

$$f^{(j)}(z_i) = a_{i,j} \quad \text{for all } i, j.$$

The problem we'll address here is simple: *What can we say along the same lines for polynomials in several variables?*

First, introduce some language/notation. The “starting point” statement Theorem 1.1 says that the evaluation map

$$H^0(\mathcal{O}_{\mathbb{P}^1}(d)) \rightarrow \bigoplus K_{p_i}$$

is surjective; or, equivalently,

$$h^1(\mathcal{I}_{\{p_1, \dots, p_e\}}(d)) = 0$$

for any distinct points $p_1, \dots, p_e \in \mathbb{P}^1$ whenever $e \leq d + 1$. More generally, Theorem 1.2 says that

$$h^1(\mathcal{I}_{p_1}^{m_1} \dots \mathcal{I}_{p_k}^{m_k}(d)) = 0$$

when $\sum m_i \leq d + 1$. To generalize this, let $\Gamma \subset \mathbb{P}^r$ be an subscheme of dimension 0 and degree n . We say that Γ *imposes independent conditions* on hypersurfaces of degree d if the evaluation map

$$\rho : H^0(\mathcal{O}_{\mathbb{P}^r}(d)) \rightarrow H^0(\mathcal{O}_{\Gamma}(d))$$

is surjective, that is, if

$$h^1(\mathcal{I}_{\Gamma}(d)) = 0;$$

we'll say it *imposes maximal conditions* if ρ has maximal rank—that is, is either injective or surjective, or equivalently if $h^0(\mathcal{I}_{\Gamma}(d))h^1(\mathcal{I}_{\Gamma}(d)) = 0$. Note that the rank of ρ is just the value of the Hilbert function of Γ at d :

$$\text{rank}(\rho) = h_{\Gamma}(d);$$

and we'll denote it in this way in the future.

In these terms, the starting point statement is that *any subscheme of \mathbb{P}^1 imposes maximal conditions on polynomials of any degree*. Accordingly, we ask in general when a zero-dimensional subscheme $\Gamma \subset \mathbb{P}^r$ may fail to impose maximal conditions, and by how much: that is, we want to

- characterize geometrically subschemes that fail to impose independent conditions; and
- say by how much they may fail: that is, how large $h^1(\mathcal{I}_{\Gamma}(d))$ may be (equivalently, how small $h_{\Gamma}(d)$ may be).

We will focus primarily on two cases: when Γ is reduced; and when Γ is a union of “fat points”—that is, the scheme

$$\Gamma = V(\mathcal{I}_{p_1}^{m_1} \dots \mathcal{I}_{p_k}^{m_k})$$

defined by a product of powers of maximal ideals of points. Other cases have been studied, such as curvilinear schemes (zero-dimensional schemes having tangent spaces of dimension at most 1; see [Ciliberto and Miranda 1998b]), but

we'll focus on these two here. (It's unreasonable to ask about arbitrary zero-dimensional subschemes $\Gamma \subset \mathbb{P}^r$, since we have no idea what they look like.)

As we'll see, these two cases give rise to very different questions and answers, but there is a common thread to both, and it is this that we hope to bring out in the course of this note.

2. Reduced schemes

In this case, the first observation is that *general points always impose maximal conditions*. So, we ask when special points may fail to impose maximal conditions, and by how much—that is, how small $h_\Gamma(d)$ can be.

In the absence of further conditions, this is trivial: $h_\Gamma(d)$ is minimal for Γ contained in a line. It's still trivial if we require Γ to be nondegenerate: the minimum then is to put $n - r + 1$ points on a line. So we typically impose a “uniformity” condition, such as linear general position—that is, we require that any $r + 1$ or fewer of the points of Γ are linearly independent. In this case, we have the fundamental

Theorem 2.1 (Castelnuovo). *If $\Gamma \subset \mathbb{P}^r$ is a collection of n points in linear general position, then*

$$h_\Gamma(d) \geq \min\{rd + 1, n\}.$$

The proof is elementary: when $n \geq rd + 1$, we exhibit hypersurfaces of degree d containing rd points of Γ and no others by the union of d hyperplanes, each spanned by r of the points of Γ . What is striking, given the apparent crudeness of the argument, is that in fact *this inequality is sharp*: configurations Γ lying on a rational normal curve $C \subset \mathbb{P}^r$ have exactly this Hilbert function.

Even more striking, though, is the converse:

Theorem 2.2 (Castelnuovo). *If $\Gamma \subset \mathbb{P}^r$ is a collection of $n \geq 2r + 3$ points in linear general position, and*

$$h_\Gamma(2) = 2r + 1,$$

then Γ is contained in a rational normal curve.

Thus we have a complete characterization of at least the extremal examples of failure to impose independent conditions. The question is, can we extend this? We believe we can: we have the

Conjecture 2.3. *For $\alpha = 1, 2, \dots, r - 1$, if $\Gamma \subset \mathbb{P}^r$ is a collection of $n \geq 2r + 2\alpha + 1$ points in uniform position, and*

$$h_\Gamma(2) \leq 2r + \alpha,$$

then Γ is contained in a curve $C \subset \mathbb{P}^r$ of degree at most $r - 1 + \alpha$.

“Uniform position” means that, if $\Gamma', \Gamma'' \subset \Gamma$ are subsets of the same cardinality, then $h_{\Gamma'} = h_{\Gamma''}$. This is in some sense a strong form of linear general position: given that the points of Γ span \mathbb{P}^r , linear general position is tantamount to the statement that $h_{\Gamma'}(1) = h_{\Gamma''}(1)$ for subsets $\Gamma', \Gamma'' \subset \Gamma$ of the same cardinality. It is not very restrictive; for example, if $C \subset \mathbb{P}^{r+1}$ is any irreducible curve, the points of a general hyperplane section of C have this property [Arbarello et al. 1985].

There are a number of remarks to make about this conjecture. The first is that it is known in cases $\alpha = 2$ (Fano; see [Harris 1982]) and $\alpha = 3$ [Petrakiev 2008]. A second is that it can't be extended as stated beyond $\alpha = r - 1$: for example, configurations $\Gamma \subset \mathbb{P}^r$ contained in a rational normal surface scroll satisfy $h_{\Gamma}(2) \leq 3r$, but need not lie on a curve of small degree.

A third remark is that we know how to classify irreducible, nondegenerate subvarieties $X \subset \mathbb{P}^r$ with Hilbert function $h_X(2) = 2r + \alpha$. Thus all we have to do to prove the conjecture is to show that the intersection of the quadrics containing Γ is positive-dimensional.

Finally, and perhaps most importantly, a proof of the conjecture would yield a complete answer to the classical problem: *for which triples (n, d, g) does there exist a smooth, irreducible, nondegenerate curve $C \subset \mathbb{P}^n$ of degree d and genus g ?*

We will take a moment out to describe this connection, since it's the original motivation for much of the study of Hilbert functions of points. Let $n = r + 1$, and let $C \subset \mathbb{P}^n$ be an irreducible, nondegenerate curve of degree d and genus g ; let $\Gamma \subset \mathbb{P}^r$ be a general hyperplane section of C . Briefly, Castelnuovo observed that for large m ,

$$g = dm - h_C(m) + 1;$$

and using the inequality

$$h_C(m) - h_C(m - 1) \leq h_{\Gamma}(m)$$

we arrive at the bound

$$g \leq \sum_{m=1}^{\infty} (d - h_{\Gamma}(m)) = \sum_{m=1}^{\infty} h^1(\mathcal{G}_{\Gamma}(m)).$$

Applying the bound in Theorem 2.1, Castelnuovo then arrives at his bound on the genus

$$g \leq \pi(d, n) = \binom{m_0}{2} (n - 1) + m_0 \epsilon,$$

where $m_0 = \lfloor \frac{d-1}{n-1} \rfloor$ and $\epsilon = d - 1 - m_0(n - 1)$.

Now suppose we have a curve C as above that achieves this maximal genus. Assuming $d > 2n$, then, we can apply the converse Theorem 2.2 to conclude that C must lie on a surface $S \subset \mathbb{P}^n$ of minimal degree $n - 1$; and indeed when we look on such surfaces we find curves of this maximal genus, showing that the bound is in fact sharp.

But this is just the beginning of the story. Assuming Conjecture 2.3, we can bound from below the Hilbert function of a configuration of points in uniform position not lying on a rational normal curve, and conclude that any curve $C \subset \mathbb{P}^n$ that does *not* lie on a surface of minimal degree must satisfy a stronger bound

$$g \leq \pi_1(d, n) \sim \frac{d^2}{2n}.$$

In other words, any curve as above with genus $g > \pi_1(d, n)$ must lie on a surface of minimal degree. Now, we know what those surfaces look like (they are either rational normal scrolls or the quadratic Veronese surface), and we know correspondingly exactly what the arithmetic genus of a curve of given degree d on such a surface may be; thus we can say exactly which g in the range $\pi_1(d, n) < g \leq \pi(d, n)$ occur as the genus of an irreducible, nondegenerate curve of degree d in \mathbb{P}^n .

Similarly, if we assume the conjecture in general, we can define a series of functions

$$\pi_\alpha(d, n) \sim \frac{d^2}{2(n + \alpha - 1)}, \quad \alpha = 1, 2, \dots, n - 1$$

such that any curve C of genus $g > \pi_\alpha(d, n)$ must lie on a surface of degree at most $n + \alpha - 2$. Again, we know what all such surfaces look like, and what may be the genera of curves on them (we're in the range $\alpha \leq n - 1$, so all such surfaces are rational or ruled), and so we can say exactly which $g > \pi_{n-1}(d, n)$ occur as the genus of an irreducible, nondegenerate curve of degree d in \mathbb{P}^n . Finally, I think it's the case that every genus $g \leq \pi_{n-1}(d, n)$ occurs, and in fact occurs on a K3 surface $S \subset \mathbb{P}^n$ of degree $2n - 2$.

Returning to the original question of Hilbert functions of collections of points in \mathbb{P}^r , we can express the bottom line as follows: Configurations $\Gamma \subset \mathbb{P}^r$ of points having small Hilbert function do so because they lie on small subvarieties $X \subset \mathbb{P}^r$ —meaning, subvarieties with small Hilbert function. In this case, for small d the hypersurfaces of degree d containing Γ will just be the hypersurfaces containing X ; in particular, X will be the intersection of the quadrics containing Γ .

Usually, to prove results along these lines it's enough to show the base locus $|\mathcal{I}_\Gamma(d)|$ is positive-dimensional.

3. Fat points

We now take up the second case of our general question: we let $p_1, \dots, p_k \in \mathbb{P}^r$ be points, $m_1, \dots, m_k \in \mathbb{N}$, and let

$$\Gamma = V(\mathcal{F}_{p_1}^{m_1} \cdots \mathcal{F}_{p_k}^{m_k}).$$

Right off the bat, we see a fundamental difference from the case of reduced points: it is *not* always the case that for $p_1, \dots, p_k \in \mathbb{P}^r$ general, Γ imposes maximal conditions on hypersurfaces of degree d ! So the first question is: assuming the points p_i are general, *for what values of the integers r, k, m_1, \dots, m_k and d does Γ fail to impose maximal conditions?*

This is a very different flavor of question, if only because the answer is numerical rather than geometric. The fact is, we don't even have a conjectured answer in general! One case where we do know the answer is the case of double points—that is, where all $m_i = 2$. Here we have:

Theorem 3.1 [Alexander and Hirschowitz 1995]. *For $p_i \in \mathbb{P}^r$ general,*

$$\Gamma = V(\mathcal{F}_{p_1}^2 \cdots \mathcal{F}_{p_k}^2)$$

imposes maximal conditions on hypersurfaces of degree d , with four exceptions:

- (1) $k \geq 2, d = 2$;
- (2) $r = 2, k = 5, d = 4$;
- (3) $r = 3, k = 9, d = 4$;
- (4) $r = 4, k = 7, d = 3$;

It's straightforward to see that the first three cases are counterexamples to the general statement. For example, it's three conditions for a polynomial on \mathbb{P}^2 to vanish to order 2, and the vector space of quadratic polynomials is six-dimensional, so we might expect that there is no conic double at each of two assigned points $p, q \in \mathbb{P}^2$, but there is: the double of the line \overline{pq} . Another way to say this is that if we require a quadratic polynomial to vanish to order 2 at a point $p \in \mathbb{P}^r$ and simply to vanish at another point q , it must vanish identically along the line $L = \overline{pq}$; the condition that its directional derivative at q in the direction of L also vanish is thus redundant.

Similarly, we don't expect that there should be a quartic curve in \mathbb{P}^2 double at five assigned points, but there is: if a quartic in \mathbb{P}^2 is double at four points p_1, \dots, p_4 and passes through a fifth p_5 , it necessarily contains the conic through all five, so one of the two additional conditions to be double at p_5 is dependent. The third example is likewise clear: since the space of quartic polynomials on \mathbb{P}^3 has dimension 35, and it's four conditions to vanish to order two at a point,

there shouldn't be a quartic double at nine points; but the double of the quadric containing them is one such.

The last example is trickier. As in the last two, we don't expect that there will be a cubic hypersurface in \mathbb{P}^4 double at seven general points, but there is: the secant variety of the (unique) rational normal quartic curve passing through the seven points.

For general multiplicities m_i and general r , as we said we don't even have a conjectured answer. For $r = 2$, though, we do. To express it, we introduce some more notation:

Let $p_1, \dots, p_k \in \mathbb{P}^2$ be general, and let

$$S = \text{Bl}_{\{p_1, \dots, p_k\}} \mathbb{P}^2$$

be the blow-up of the plane at the p_i . Let H be the divisor class of the preimage of a line in \mathbb{P}^2 , and E_i the exceptional divisor over the point p_i . Let L be the line bundle

$$\mathcal{O}_S(dH - \sum m_i E_i)$$

on S . Then

$$h^i(L) = h^i(\mathcal{I}_{p_1}^{m_1} \dots \mathcal{I}_{p_k}^{m_k}(d)).$$

In particular, the "expected dimension" of $h^0(L)$ is

$$\frac{(d+1)(d+2)}{2} - \sum \frac{m_i(m_i+1)}{2}$$

and this is exceeded exactly when the scheme Γ fails to impose independent conditions in degree d .

In these terms, we can interpret the basic example of conics in \mathbb{P}^2 double at two points p, q as saying that, since the restriction to the line \overline{pq} of the line bundle $\mathcal{O}_{\mathbb{P}^2}(2)$ has degree 2, the requirement that the restriction of a section vanish four times along \overline{pq} is necessarily redundant. Equivalently, if we let S be the blow-up of \mathbb{P}^2 at p and q , and $D \subset S$ the proper transform of the line \overline{pq} , and set

$$L = \mathcal{O}_S(2H - E_p - E_q),$$

then $L|_D$ has degree -2 , and from an examination of the exact sequence

$$0 \rightarrow L(-D) \rightarrow L \rightarrow L|_D \rightarrow 0$$

we see that $h^1(L|_D) \neq 0$ implies that $h^1(L) \neq 0$. A similar interpretation can be given for the example of quartics double at 5 points (the line bundle $L = \mathcal{O}_S(4H - E_1 - \dots - E_5)$ has degree -2 on the proper transform of the conic through the five); and in general we have the

Conjecture 3.2 (Harbourne–Hirschowitz). *Let S be the blow-up of \mathbb{P}^2 at k general points, L any line bundle on S . Then $h^1(L) \neq 0$ if and only there is a (-1) -curve $E \subset S$ such that*

$$\deg(L|_E) \leq -2.$$

An equivalent formulation is that if $h^1(L) \neq 0$, then the base locus of the linear system $|L|$ contains a multiple (-1) -curve.

There are a number of remarks to be made here. The first is that if true, the Harbourne–Hirschowitz conjecture gives a complete answer to our question for $r = 2$: conjecturally (more about this in a moment), we know where the (-1) -curves on S are, and can check the condition $\deg(L|_E) \leq -2$.

Two cases where the conjecture is known are for $k \leq 9$ (in this case S has an effective anticanonical divisor), and when $\max\{m_i\} \leq 7$ (Stephanie Yang; [Yang 2007])

To explicate the conjecture, and the fact that it does answer our question, we should make a small digression to discuss our abysmal ignorance about curves of negative self-intersection on surfaces. To start, let X be any smooth, projective surface, and consider the self-intersections of curves of X ; that is, set

$$\Sigma = \{(C \cdot C) : C \subset S \text{ integral}\} \subset \mathbb{Z}.$$

The first question we might ask is: *is Σ bounded below?*

The answer isn't known in characteristic 0, though János Kollár points out that there are examples in characteristic p of surfaces with integral curves of arbitrarily negative self-intersection: take B a smooth curve of genus $g \geq 2$, $S = B \times B$ and $C_n \subset S$ the graph of the n^{th} power of Frobenius. In characteristic zero, we don't know the answer even for $X = S$ a blow-up of the plane!

We can, however, make a strong conjecture in this case. Consider an arbitrary line bundle $L = \mathcal{O}_S(dH - \sum m_i E_i)$ on a general blow-up S . The expected dimension of $h^0(L)$ is

$$\frac{(d+1)(d+2)}{2} - \sum \frac{m_i(m_i+1)}{2};$$

and the genus of a curve $C \in |L|$ is

$$\frac{(d-1)(d-2)}{2} - \sum \frac{m_i(m_i-1)}{2}.$$

If we assume the first is positive and the second nonnegative, it follows that the self-intersection of C is

$$(C \cdot C) = d^2 - \sum m_i^2 \geq -1.$$

Thus we may make the following conjecture:

Conjecture 3.3. *Let S be a general blow-up of the plane, $C \subset S$ any integral curve. Then*

$$(C \cdot C) \geq -1,$$

and if equality holds then C is a smooth rational curve.

If we believe this, the Harbourne–Hirschowitz conjecture should be equivalent to the weaker version:

Conjecture 3.4 (Harbourne–Hirschowitz; weak form). *Let S be the blow-up of \mathbb{P}^2 at general points, L any line bundle on S . If the linear system $|L|$ contains an integral curve, then $h^1(L) = 0$.*

If we believe the weak Harbourne–Hirschowitz, then by the calculation above Conjecture 3.3 on self-intersections of curves on S follows, and we can in turn deduce strong Harbourne–Hirschowitz. Thus, it’s possible to prove that the two versions of Harbourne–Hirschowitz are equivalent. Note moreover that if we believe any version, it’s possible to locate all the (-1) -curves on S , as primitive solutions of the system of equations

$$\frac{(d-1)(d-2)}{2} - \sum \frac{m_i(m_i-1)}{2} = 0 \quad \text{and} \quad d^2 - \sum m_i^2 \geq -1.$$

Thus the condition that the line bundle L have degree -2 on a (-1) -curve $E \subset S$ is algorithmically checkable.

It’s worth taking a moment to describe some approaches to Harbourne–Hirschowitz. Briefly, all approaches taken to Harbourne–Hirschowitz (in case $k > 9$) involve specialization—Ciliberto and Miranda ([Ciliberto and Miranda 2000], [Ciliberto and Miranda 1998a]) specialize a subset of the points p_i onto a line $L \subset \mathbb{P}^2$; Yang specializes the points onto a line one at a time. Either approach involves an “apparent” loss of conditions; the goal is to understand what conditions the limit of the linear series $|\mathcal{F}_\Gamma(d)|$ will satisfy beyond the obvious multiplicity ones. These questions are fascinating in their own right.

As an example: suppose $d = 4$, $k = 5$ and $(m_1, \dots, m_5) = (1, 1, 1, 1, 3)$; suppose that p_1, \dots, p_4 already lie on a line L and we specialize p_5 onto L . The limits of the curves passing through p_1, \dots, p_4 and triple at p_5 will be of the form $L + C$, with C a cubic double at p_5 . But there are too many of these: cubics double at p_5 form a 6-dimensional linear system, while the system of quartics passing through p_1, \dots, p_4 and triple at the general p_5 is only 4-dimensional. So the question is: which cubics actually appear in the limiting curves? The answer, somewhat unexpectedly, is: cubics with a cusp at p_5 , with tangent line L there.

It would be wonderful to understand better this limiting behavior. For example, does something like this occur when we specialize similarly defined linear systems on more general surfaces?

Before moving on, we should summarize one common thread running through our discussions of Castelnuovo theory and the Harbourne–Hirschowitz conjecture.

The content of the Harbourne–Hirschowitz conjectures may be thought of as this: *if general multiple points in \mathbb{P}^2 fail to impose maximal conditions, they do so because they lie on a “small” curve—in this case, a curve of negative self-intersection.*

It’s hard to say how this might generalize to higher-dimensional space—as we said, we don’t even have a conjectured answer to the question of when general multiple points impose independent conditions in general. Based on our experience in \mathbb{P}^2 , though, we might be led to make a qualitative conjecture:

Conjecture 3.5. *Let $p_1, \dots, p_k \in \mathbb{P}^r$ be general. If*

$$h^1(\mathcal{I}_{p_1}^{m_1} \cdots \mathcal{I}_{p_k}^{m_k}(d)) \neq 0$$

then the base locus of the linear series $|\mathcal{I}_{p_1}^{m_1} \cdots \mathcal{I}_{p_k}^{m_k}(d)|$ must have positive dimension.

4. Recasting the problem

There is a common theme to our results and conjectures so far: we believe in many cases that when a subscheme $\Gamma \subset \mathbb{P}^r$ fails to impose independent conditions on hypersurfaces of degree d —that is, has small Hilbert function $h_\Gamma(d)$ —it’s because it’s contained in a small positive-dimensional subscheme $X \subset \mathbb{P}^r$; and moreover, in this case X will appear as the intersection of the hypersurfaces of degree d containing Γ .

So let’s recast the problem: let’s drop all the conditions we’ve put on Γ at various points above, and instead make just one assumption: that the intersection of the hypersurfaces of degree d containing Γ is zero-dimensional; in other words, Γ is a subscheme of a complete intersection of r hypersurfaces of degree d . We ask: what bounds can we give on $h^1(\mathcal{I}_\Gamma(d))$ (or $h_\Gamma(d)$, or $h^0(\mathcal{I}_\Gamma(d))$) under this hypothesis?

One further wrinkle: instead of specifying the degree n of Γ and asking for estimates on the size of $h^0(\mathcal{I}_\Gamma(d))$, let’s turn it around: let’s specify the dimension $h^0(\mathcal{I}_\Gamma(d))$, and ask for a bound on the degree of Γ . Thus, the question is:

• *Let $V \subset H^0(\mathbb{C}_{\mathbb{P}^r}(d))$ be an N -dimensional linear system of hypersurfaces of degree d , with finite intersection Γ . How large can the degree of Γ be?*

As a first example, let’s try $d = 2$ and $N = r + 1$. The question is, in effect:

- How many common zeroes can $r + 1$ quadrics in \mathbb{P}^r have, if they have only finitely many common zeroes?
- Let $\{p_1, \dots, p_{2r}\} \subset \mathbb{P}^r$ be a complete intersection of quadrics in \mathbb{P}^r . How many of the points p_i can a quadric Q contain without containing them all?

The first few cases can be worked out ad hoc: for example, in case $r = 2$, the answer is visibly 3. In case $r = 3$, the Cayley–Bacharach theorem [Eisenbud et al. 1996] says that any quadric containing 7 of the 8 points of a complete intersection of quadrics in \mathbb{P}^3 contains the eighth as well; the answer is 6. And in case $r = 4$, let $C = Q_1 \cap Q_2 \cap Q_3$. If two more quadrics had 13 common zeroes on C , they would cut out a g_3^1 on C . But C is not trigonal; thus the answer is 12.

All this leads us to the

Conjecture 4.1 (Green, Eisenbud, Harris). *If $Q_1, \dots, Q_{r+1} \subset \mathbb{P}^r$ are linearly independent quadrics and $\Gamma = Q_1 \cap \dots \cap Q_{r+1}$ their zero-dimensional intersection, then*

$$\deg(\Gamma) \leq 3 \cdot 2^{r-2}.$$

In fact, this is just the first case of a general conjecture about linear systems of quadrics, and of higher-degree hypersurfaces; the full statement can be found in [Eisenbud et al. 1996]. And this particular case is in fact no longer a conjecture; it’s been proved by Rob Lazarsfeld, under the mild extra hypothesis that Γ is reduced.

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harris@math.harvard.edu

*Mathematics Department, Harvard University,
1 Oxford Street, Cambridge, MA 02138, United States*