Current Developments in Algebraic Geometry
MSRI Publications
Volume **59**, 2011

# Geometry of varieties of minimal rational tangents

JUN-MUK HWANG

We present the theory of varieties of minimal rational tangents (VMRT), with an emphasis on its own structural aspect, rather than applications to concrete problems in algebraic geometry. Our point of view is based on differential geometry, in particular, Cartan's method of equivalence. We explain various aspects of the theory, starting with the relevant basic concepts in differential geometry and then relating them to VMRT. Several open problems are proposed, which are natural from the view point of understanding the geometry of VMRT itself.

## 1. Introduction

The concept of varieties of minimal rational tangents (VMRT) on uniruled projective manifolds first appeared as a tool to study the deformation of Hermitian symmetric spaces [Hwang and Mok 1998]. For many classical examples of uniruled manifolds, VMRT is a very natural geometric object associated to low degree rational curves, and as such, it had been studied and used long before its formal definition appeared in that reference. At a more conceptual level, namely, as a tool to investigate unknown varieties, it had been already used in [Mok 1988] for manifolds with nonnegative curvature. However, in the context of that work, its very special relation with the curvature property of the Kähler metric somewhat overshadowed its role as an algebro-geometric object, so it had not been considered for general uniruled manifolds. Thus it is fair to say that the concept as an independent geometric object defined on uniruled projective manifolds really originated from [Hwang and Mok 1998]. Shortly after this formal debut, numerous examples of its applications to classical problems of algebraic geometry were discovered. In the early MSRI survey [Hwang and Mok 1999], written only a couple of years after the first discovery, one can already find a substantial list of problems in a wide range of topics, which can be solved by the help of VMRT.

Supported by National Researcher Program 2010-0020413 of NRF and MEST..
*Keywords:* varieties of minimal rational tangents, equivalence problem, cone structure, $G$-structure, distribution, foliation.

Since the beginning VMRT has been studied exclusively in relation with some classical problems, namely, problems which do not involve VMRT itself explicitly. In particular, this is the case for most of my collaboration with N. Mok. In other words, VMRT has mostly served as a *tool* to study uniruled manifolds. However, after more than a decade's service, I believe it is time to give due recognition and it is not unreasonable to start to regard VMRT itself as a central object of research. The purpose of this exposition is to introduce and advertise this new viewpoint. In fact, the title of the current article, as opposed to that of my old survey [Hwang 2001], is deliberately chosen to emphasize this shift of perspective.

As a result, in this article, I intentionally avoided talking about applications. Also, only a minimal number of examples are given. This omission is happily justifiable by the appearance of the excellent survey paper [Mok 2008a], which covers many recent applications. The old surveys [Hwang and Mok 1999] and [Hwang 2001] as well as [Mok 2008a] all emphasize the applications to concrete geometric problems. The reader is encouraged to look at these surveys for explicit examples and applications.

There is another reason that I believe it is justifiable to give such an emphasis on the theoretical aspect of the theory. After seeing many applications of the techniques developed so far, it seems to me that we need a considerable advancement of the structural theory of VMRT itself, to enhance the applicability of the theory to a wider class of geometric problems. With this motivation in mind, I will propose several open problems of this sort throughout the article, which I believe are not only natural, but will be useful in applications.

Most of these open problems are likely to be of less interest unless one believes that VMRT itself is an interesting object. In this regard, part of my aim is to advertise VMRT, trying to convince the reader that the subject is exciting and amusing. In other words, by presenting these open problems, I hope to transfer to the reader the kind of perspective I have about this subject. The reader is encouraged to try to think about the meaning of the open problems and why they are interesting, to understand the underlying philosophy.

The basic framework of my presentation is essentially differential geometric, belonging to Cartanian geometry. Since this is an article for algebraic geometers, very little knowledge of differential geometry will be assumed. Essentially all differential geometric concepts are explained from the very definition. This differential geometric framework has been in the background of most of my joint work with N. Mok, but has not been explicitly explained in publications so far. The basic idea is that VMRT is a special kind of cone structure and one of the key issue is to understand what is special about it. In this article, we will mostly concentrate on the existence of a characteristic connection among the special properties. Schematically, we may put it as

$$\{ \text{ cone strucures } \} \supset \{ \text{ characteristic connections } \} \supset \{ \text{ VMRT } \}.$$

Some of the discussion below works for cone structures, some for characteristic connections and some for VMRT.

For me, the most amusing aspect of the study of VMRT is the interaction, or rather the fusion, of algebraic geometry and differential geometry. I hope that this expository article helps algebraic geometers to become more familiar with the concepts and methods originating from differential geometry. Most of the sections start with an introduction of certain differential geometric concepts and then mix them with the algebraic geometry of rational curves.

In another direction, although it is written for algebraic geometers, I hope this article will attract differential geometers, especially those working on Cartanian geometry, to problems arising from the algebraic geometry of rational curves. Many of the problems I propose have differential geometric components. Moreover, I think the theory of VMRT provides a lot of new examples of geometric structures which are highly interesting from a differential geometric point of view.

## 2. Preliminaries on distributions

Throughout the paper, we will work over the complex numbers. All differential geometric objects are holomorphic. In this section, we collect some terms and facts about distributions. These will be used throughout the paper.

A *distribution* $D$ on a complex manifold $U$ means a vector subbundle $D \subset T(U)$ of the tangent bundle. In particular, $T(U)/D$ is locally free.

The *Frobenius tensor* of the distribution $D$ is the homomorphism of vector bundles $\beta : \bigwedge^2 D \to T(U)/D$ defined by

$$\beta(v, w) = [\tilde{v}, \tilde{w}] \mod D,$$

where for $x \in U$ and two vectors $v, w \in D_x$, $\tilde{v}$ and $\tilde{w}$ are local sections of $D$ extending $v$ and $w$ in a neighborhood of $x$, and $[\tilde{v}, \tilde{w}]$ denotes the bracket of $\tilde{v}$ and $\tilde{w}$ as holomorphic vector fields. It is easy to see that $\beta(v, w)$ does not depend on the choice of the extensions $\tilde{v}, \tilde{w}$. By the Frobenius theorem, if the Frobenius tensor is identically zero then the distribution comes from a foliation, i.e., a partition of $U$ into complex submanifolds whose tangent spaces correspond to $D$. In this case, we say that the distribution is *integrable*.

For each $x \in U$ define

$$Ch(D)_x := \{v \in D_x,\ \beta(v, w) = 0 \text{ for all } w \in D_x.\}.$$

In a Zariski open subset $U' \subset U$,

$$\{Ch(D)_x, x \in U'\}$$

defines a distribution, called the *Cauchy characteristic* of $D$ and denoted by $Ch(D)$. This distribution is always integrable.

Given a distribution $D$ on $U$, its *first derived system*, denoted by $\partial D$, is the distribution defined on a Zariski open subset of $U$ whose associated sheaf corresponds to $\mathcal{O}(D) + [\mathcal{O}(D), \mathcal{O}(D)]$. Define successively

$$\partial^1 D := \partial D, \quad \partial^k D := \partial(\partial^{k-1} D).$$

There exists some $\ell$ such that $\partial^\ell D = \partial^{\ell+1} D$ so that the Frobenius tensor of $\partial^\ell D$ is zero. The foliation on a Zariski open subset of $U$ determined by $\partial^\ell D$ is called the *foliation generated by* $D$. We say that the distribution is *bracket-generating* if $\partial^\ell D = T(U')$ on some Zariski open subset $U'$ and $\ell > 0$.

Let $f : M \to B$ be a holomorphic submersion between two complex manifolds, i.e., $df : T(M) \to f^*T(B)$ is a surjective bundle homomorphism. The distribution $\mathrm{Ker}\, df$ on $M$ is integrable and the corresponding foliation of $M$ has the fibers of $f$ as leaves. Given a distribution $D$ on $B$, we have a distribution $f^{-1}D$ on $M$, called the *inverse-image of the distribution* $D$, given by the subbundle $df^{-1}(f^*D)$ of $T(M)$ where $f^*D \subset f^*T(B)$ is the pull-back of $D \subset T(B)$. It is clear that $\mathrm{Ker}\, df \subset Ch(f^{-1}D)$.

## 3. Equivalence of cone structures

A well-known philosophy, going back to Klein's Erlangen program, is that the fundamental problem in any area of geometry is the study of invariant properties under equivalence relations. Algebraic geometry is no exception. In classical projective geometry, the most fundamental equivalence relation is the equivalence of two subvarieties of projective space under a projective transformation, or more generally, the equivalence of two families of subvarieties under a family of projective transformations. One possible formulation of this equivalence relation is the following.

**Definition 3.1.** Let $U$ and $U'$ be a (connected) complex manifold. Let $\mathcal{V}$ and $\mathcal{V}'$ be vector bundles on $U$ and $U'$, respectively, and let $\mathbb{P}\mathcal{V}$ and $\mathbb{P}\mathcal{V}'$ be their projectivizations, as sets of 1-dimensional subspaces in the fibers. Given (not necessarily irreducible) subvarieties $\mathcal{C} \subset \mathbb{P}\mathcal{V}$ and $\mathcal{C}' \subset \mathbb{P}\mathcal{V}'$ of pure dimension, surjective over $U$ and $U'$ respectively, we say that $\mathcal{C}$ and $\mathcal{C}'$ are *equivalent as families of projective subvarieties* if there exist a biholomorphic map $\varphi : U \to U'$

and a projective bundle isomorphism $\psi : \mathbb{P}\mathcal{V} \to \mathbb{P}\mathcal{V}'$ with a commuting diagram

$$\begin{array}{ccc} \mathbb{P}\mathcal{V} & \xrightarrow{\psi} & \mathbb{P}\mathcal{V}' \\ \downarrow & & \downarrow \\ U & \xrightarrow{\phi} & U' \end{array}$$

such that $\psi(\mathcal{C}) = \mathcal{C}'$. For a point $x \in U$ and a point $x' \in U'$, we say that the family $\mathcal{C}$ at $x$ is *locally equivalent* to the family $\mathcal{C}'$ at $x'$, if there exist a neighborhood $W \subset U$ of $x$ and a neighborhood $W' \subset U'$ of $x'$ such that the restriction $\mathcal{C}|_W \subset \mathbb{P}\mathcal{V}|_W$ is equivalent to the restriction $\mathcal{C}'|_{W'} \subset \mathbb{P}\mathcal{V}'|_{W'}$.

Suppose that $\mathcal{V}$ and $\mathcal{V}'$ in Definition 3.1 are the tangent bundles $T(U)$ and $T(U')$. Then we have the following finer equivalence relation.

**Definition 3.2.** For a complex manifold $U$, a subvariety of pure dimension $\mathcal{C} \subset \mathbb{P}T(U)$ which is surjective over the base $U$ will be called a *cone structure* on $U$. Here, we do not assume that $\mathcal{C}$ is irreducible. The fiber dimension of the projection $\mathcal{C} \to U$, i.e., $\dim \mathcal{C} - \dim U$, will be called the *projective rank* of the cone structure. The *rank* of the cone structure is the projective rank of the cone structure plus one. A cone structure $\mathcal{C} \subset \mathbb{P}T(U)$ on $U$ and a cone structure $\mathcal{C}' \subset \mathbb{P}T(U')$ on $U'$ are *equivalent as cone structures* if there exists a biholomorphic map $\varphi : U \to U'$ such that the projective bundle isomorphism $\psi : \mathbb{P}T(U) \to \mathbb{P}T(U')$ induced by the differential $d\varphi : T(U) \to T(U')$ of $\varphi$

$$\begin{array}{ccc} \mathbb{P}T(U) & \xrightarrow{\psi = d\varphi} & \mathbb{P}T(U') \\ \downarrow & & \downarrow \\ U & \xrightarrow{\phi} & U' \end{array}$$

satisfies $\psi(\mathcal{C}) = \mathcal{C}'$. For a point $x \in U$ and a point $x' \in U'$, we say that the cone structure $\mathcal{C}$ at $x$ is *locally equivalent* to the cone structure $\mathcal{C}'$ at $x'$, if there exist a neighborhood $W \subset U$ of $x$ and a neighborhood $W' \subset U'$ such that the restriction $\mathcal{C}|_W \subset \mathbb{P}T(W)$ is equivalent as cone structures to the restriction $\mathcal{C}'|_{W'} \subset \mathbb{P}T(W')$.

Notice the essential difference between Definitions 3.1 and 3.2: the projective bundle isomorphism $\psi$ is arbitrary in the former as long as it is compatible with the map $\varphi$ while $\psi$ is completely determined by $\varphi$ in the latter. Since $\psi$ comes from the derivative of $\varphi$ in Definition 3.2, the equivalence of cone structures has features of differential geometry as well as algebraic geometry. Let us look at two classical examples.

**Example 3.3.** A cone structure $\mathcal{C} \subset \mathbb{P}T(U)$ where each fiber $\mathcal{C}_x$, $x \in U$, is a linear subspace of $\mathbb{P}T_x(U)$ is equivalent to a distribution on $U$. The rank of the distribution (as a subbundle of $T(U)$) is equal to the rank of the cone structure.

Two such cone structures $\mathscr{C} \subset \mathbb{P}T(U)$ and $\mathscr{C}' \subset \mathbb{P}T(U')$ of the same projective rank on complex manifolds $U, U'$ of the same dimension are always locally equivalent as families of projective subvarieties. Their local equivalence as cone structures is much more subtle. For example, an integrable distribution cannot be locally equivalent to a non-integrable one.

**Example 3.4.** A cone structure $\mathscr{C} \subset \mathbb{P}T(U)$ where each fiber $\mathscr{C}_x$, $x \in U$, is a nonsingular quadric hypersurface of $\mathbb{P}T_x(U)$ is called a *conformal structure* on $U$. Locally, a conformal structure is determined by a nondegenerate holomorphic symmetric bilinear form $g : S^2T(U) \to \mathscr{O}_U$, i.e., a holomorphic Riemannian metric, up to multiplication by nowhere-zero holomorphic functions. Two conformal structures $\mathscr{C} \subset \mathbb{P}T(U)$ and $\mathscr{C}' \subset \mathbb{P}T(U')$ on complex manifolds $U$ and $U'$ of the same dimension are always locally equivalent as families of projective subvarieties. They are locally equivalent as cone structures if the associated holomorphic Riemannian metrics are conformally isometric. The study of this equivalence relation is the subject of conformal geometry, an active area of research in differential geometry.

A more “modern” version of Definition 3.1 is the equivalence of families of polarized projective varieties. Let us assume that the family is smooth for simplicity. One possible formulation is as follows.

**Definition 3.5.** A *polarized family* $f : M \to U$ is just a smooth projective morphism between two complex manifolds $M$ and $U$ with a line bundle $L$ on $M$ which is $f$-ample. Here we assume that $U$ is connected, $M$ is of pure dimension, but not necessarily connected. The line bundle $L$ is called a *polarization*. Two polarized families $f : M \to U$ with a polarization $L$ and $f' : M' \to U'$ with a polarization $L'$ are *equivalent as polarized families* if there exist a biholomorphism $\varphi : U \to U'$ and a biholomorphism $\psi : M \to M'$ satisfying $\varphi \circ f = f' \circ \psi$ and $L \cong \psi^* L'$.

In Definition 3.5, taking a sufficiently high power $L^{\otimes m}$ to make it $f$-very-ample, we get an embedding $M \to \mathbb{P}(f_* L^{\otimes m})^*$ whose image $\mathscr{C}$ is a family of projective subvarieties. The equivalence in Definition 3.5 implies the equivalence in the sense of Definition 3.1 for this family $\mathscr{C}$ of projective subvarieties. The more intrinsic formulation of Definition 3.5 is often more convenient than the classical version in Definition 3.1. Analogously, sometimes it is convenient to have a more intrinsic formulation of Definition 3.2 as follows.

**Definition 3.6.** Given a polarized family $f : M \to U$ with a polarization $L$, a distribution $\mathscr{J} \subset T(M)$ on $M$ is called a *precone* structure if $\operatorname{Ker} df \subset \mathscr{J}$ and the quotient bundle $\mathscr{J}/\operatorname{Ker} df$ is a line bundle isomorphic to the dual line bundle $L^*$ of $L$. Two precone structures $(f : M \to U, L, \mathscr{J})$ and $(f' : M' \to U', L', \mathscr{J}')$ are equivalent if they are equivalent as polarized families in the sense of Definition 3.5 such that the differential $d\psi : T(M) \to T(M')$ sends the distribution $\mathcal{J}$ to $\mathcal{J}'$.

The relation between Definition 3.2 and Definition 3.6 is given by the following proposition, which is essentially [Yamaguchi 1982, Lemma 1.5], attributed to N. Tanaka.

**Proposition 3.7.** *Given a precone structure, $(f : M \to U, L, \mathcal{J})$, define a morphism $\tau : M \to \mathbb{P}T(U)$ by*

$$\textit{for each } \alpha \in M,\ \tau(\alpha) := df(\mathcal{J}_\alpha),$$

*called the tangent morphism. The image of $\tau$ determines a cone structure $\mathscr{C} := \tau(M) \subset \mathbb{P}T(U)$. Moreover, when $\mathbb{O}(1)$ is the relative hyperplane bundle on $\mathbb{P}T(U)$, the polarization $L$ is isomorphic to $\tau^*\mathbb{O}(1)$, implying that $\tau$ is a finite morphism over its image. The rank of this cone structure is equal to the rank of the distribution $\mathcal{J}$.*

Note that although $M \to U$ in Definition 3.6 is assumed to be a smooth morphism, the induced cone structure $\mathscr{C} \to U$ by the tangent morphism is not necessarily smooth. This is one advantage of Definition 3.6, in the sense that a cone structure coming from a precone structure has a hidden regularity. The VMRT structure in the next section is such an example. It is easy to see that when a cone structure $\mathscr{C} \to U$ is smooth, it comes from a precone structure:

**Proposition 3.8.** *Given a cone structure $\mathscr{C} \subset \mathbb{P}T(U)$ such that the projection $f : \mathscr{C} \to U$ is a smooth morphism, the distribution $\mathcal{J}$ on $\mathscr{C}$ defined by*

$$\textit{for each } \alpha \in \mathscr{C},\ \mathcal{J}_\alpha := df_\alpha^{-1}(\hat{\alpha})$$

*where $\hat{\alpha} \subset T_x(U)$, $x = f(\alpha)$, is the 1-dimensional subspace corresponding to $\alpha$, is a precone structure on $\mathscr{C}$. The cone structure induced by this precone structure via Proposition 3.7 agrees with the original cone structure $\mathscr{C} \subset \mathbb{P}T(U)$.*

## 4. Varieties of minimal rational tangents

Now we define the cone structure which is our main interest.

**Definition 4.1.** A rational curve $C \subset X$ on a projective manifold $X$ is *free* if under the normalization $\nu : \mathbb{P}_1 \to C$, the vector bundle $\nu^*T(X)$ is nef. The *normalized space of free rational curves* on $X$, to be denoted by $\mathrm{FRC}(X)$, is a smooth scheme with countably many components, by [Kollár 1996, II.3]. We have the universal family $\mathrm{Univ}(X)$ with a $\mathbb{P}_1$-bundle structure $\mathrm{Univ}(X) \to \mathrm{FRC}(X)$ and the evaluation morphism $\mathrm{Univ}(X) \to X$.

Note that $\mathrm{FRC}(X) \neq \varnothing$ if and only if $X$ is a uniruled projective manifold.

**Definition 4.2.** Let $X$ be a uniruled projective manifold. An irreducible component $\mathcal{K}$ of FRC$(X)$ is called a *minimal component* if for the universal family $\rho : \mathcal{U} \to \mathcal{K}$ and $\mu : \mathcal{U} \to X$ obtained by restricting Univ$(X)$ to $\mathcal{K}$, the morphism $\mu$ is generically projective, i.e., the fiber $\mathcal{K}_x := \mu^{-1}(x)$ over a general point $x \in X$ is projective. A member of $\mathcal{K}$ is called a *minimal free rational curve*.

**Proposition 4.3.** *For a minimal component $\mathcal{K}$, there exists a Zariski open subset $X_o \subset X$ such that*

(i) *each fiber $\mu^{-1}(x)$, $x \in X_o$, is smooth*;

(ii) $\operatorname{Ker} d\mu \cap \operatorname{Ker} d\rho = 0$ *at every point of $\mu^{-1}(X_o)$; and*

(iii) *the dual bundle $L$ of the line bundle* $\operatorname{Ker} d\rho \subset T(\mathcal{U})$ *is $\mu$-ample on $\mu^{-1}(X_o)$.*

*In particular, the distribution $\mathcal{J} :=$* $\operatorname{Ker} d\mu + \operatorname{Ker} d\rho$ *defines a precone structure on the family $\mu|_{\mu^{-1}(X_o)} : \mu^{-1}(X_o) \to X_o$ of smooth projective varieties with the polarization $L$.*

*Proof.* Part (i) is [Kollár 1996, Corollary II.3.11.5]. Parts (ii) and (iii) follow from [Kebekus 2002, Theorem 3.4]. □

**Definition 4.4.** The cone structure $\mathscr{C} \subset \mathbb{P}T(X_o)$ associated to the precone structure of Proposition 4.3 via Proposition 3.7 is called the *family of varieties of minimal rational tangents* (in short, VMRT) of the minimal component $\mathcal{K}$. Its fiber $\mathscr{C}_x$ at $x \in X_o$ is called the *variety of minimal rational tangents at $x$*. For each $x \in X_o$, the restriction $\tau_x : \mu^{-1}(x) \to \mathscr{C}_x$ of the tangent morphism $\tau : \mu^{-1}(X_o) \to \mathscr{C}$ defined in Proposition 3.7 is called the *tangent morphism at $x$*.

This cone structure, VMRT, is our main interest. Before going into the study of VMRT in detail, let us give at least one reason why it is interesting to consider the equivalence problem for such a cone structure. The following is the main result of [Hwang and Mok 2001].

**Theorem 4.5.** *Let $X$ and $X'$ be two Fano manifolds of Picard number* 1. *Let $\mathcal{K}$ and $\mathcal{K}'$ be minimal components on $X$ and $X'$, respectively, with associated VMRT $\mathscr{C}$ and $\mathscr{C}'$). Assume that the VMRT $\mathscr{C}_x \subset \mathbb{P}T_x(X)$ at a general point $x \in X$ is not a finite union of linear subspaces. Suppose $\mathscr{C}$ at some point $x \in X_o$ is locally equivalent as cone structures to $\mathscr{C}'$ at some point $x' \in X'_o$. Then $X$ and $X'$ are biregular.*

To be precise, in [Hwang and Mok 2001], Theorem 4.5 is proved under the stronger assumption that $\mathscr{C}_x \subset \mathbb{P}T_x(X)$ has generically finite Gauss map. However, one can extend the argument to the above form by using results from [Hwang and Mok 2004]. See [Mok 2008a, Theorem 9] for a discussion of this extension.

Theorem 4.5 is not just of theoretical interest. It is used to identify certain Fano manifolds of Picard number 1 in a number of classical problems. See [Hwang and Mok 1999], [Hwang 2001] and [Mok 2008a] for concrete examples.

The VMRT is an algebraic object defined as a quasi-projective variety in $\mathbb{P}T(X)$. It is convenient to introduce a local version of this definition:

**Definition 4.6.** A cone structure $\mathscr{C}' \subset \mathbb{P}T(U)$ on a complex manifold $U$ is a *VMRT structure* if there exists a VMRT $\mathscr{C} \subset \mathbb{P}T(X_o)$ as in Definition 4.4 such that $\mathscr{C}' \to U$ is locally equivalent as cone structures to $\mathscr{C} \to X_o$ at every point of $U$.

This is not a truly local definition. It is introduced merely for linguistic convenience. A truly local definition of VMRT structure as a cone structure with certain distinguished differential geometric properties is still lacking. One special property is obvious: from the very definition, it is provided with a connection in the following sense.

**Definition 4.7.** Let $\mathscr{J} \subset T(M)$ be a precone structure on a polarized family $(f : M \to U, L)$. A *connection* of the precone structure is a line subbundle $F \subset \mathscr{J}$ with an isomorphism $F \cong L^*$ splitting the exact sequence

$$0 \longrightarrow \operatorname{Ker} df \longrightarrow \mathscr{J} \longrightarrow L^* \longrightarrow 0.$$

By abuse of terminology, we will also say that $F$ is a connection for the cone structure $\mathscr{C} \subset \mathbb{P}T(U)$ induced by the precone structure. When $\mathscr{C} = \mathbb{P}T(U)$, a connection is called a *projective connection* on $U$.

From the fact that the set of splittings of the exact sequence in Definition 4.7 is $H^0(M, \operatorname{Ker} df \otimes L)$, we have

**Proposition 4.8.** *Given a precone structure on $M \to U$, if the fiber $M_x$ at some $x \in U$ satisfies $H^0(M_x, T(M_x) \otimes L) = 0$, then a connection is unique if it exists.*

A VMRT structure is naturally equipped with a connection given by $\operatorname{Ker} d\rho$ of Proposition 4.3. This connection has a distinguished property.

**Definition 4.9.** A connection $F \subset \mathscr{J}$ in Definition 4.7 is a *characteristic connection* if $F \subset Ch(\partial\mathscr{J})$ on an open subset of $M$.

The following result is in [Hwang and Mok 2004, Proposition 8].

**Proposition 4.10.** *In Proposition 4.3, the distribution $\partial\mathscr{J}$ on $\mu^{-1}(X_o)$ is of the form $\rho^{-1}D$ for some distribution $D$ on $\mathscr{K}$. In particular, the line subbundle* $\operatorname{Ker} d\rho$ *is a characteristic connection of the precone structure.*

The existence of the characteristic connection is a key property of VMRT structure. Most of the algebraic geometric applications of the local differential geometry of VMRT come from this property. There are other local differential geometric properties of VMRT. For example, the admissibility condition of [Bernstein and Gindikin 2003] holds for minimal free rational curves, which can be interpreted as a property of the cone structure. However, I feel that the investigation of these additional properties is not yet mature enough to be discussed here.

An important property of a characteristic connection is the relation with the projective differential geometry of the fibers of the cone structure. Let us start the discussion by recalling some definitions from projective geometry.

**Definition 4.11.** For each point $v \in \mathbb{P}V$, let $\hat{v} \subset V$ be the 1-dimensional subspace corresponding to $v$. Let $Z \subset \mathbb{P}V$ be a projective subvariety and let $\hat{Z} \subset V$ be the affine cone of $Z$. Denote by $Sm(Z)$ the smooth locus of $Z$. For each $z \in Sm(Z)$, let $\hat{T}_z(Z) \subset V$ be the *affine tangent space* to $Z$ at $z$, i.e., the affine cone of the projective tangent space to $Z$ at $z$:

$$\hat{T}_z(Z) = T_{z'}(\hat{Z}) \text{ for any } z' \in \hat{z} \setminus \{0\}.$$

Let $\mathbf{Gr}(p, V)$ be the Grassmannian of $p$-dimensional subspaces of $V$. The *Gauss map* of $Z$ is the morphism $\gamma : Sm(Z) \to \mathbf{Gr}(\dim \hat{Z}, V)$, defined by $\gamma(z) := \hat{T}_z(Z)$. The *second fundamental form* at $z \in Sm(Z)$ is the derivative of the Gauss map at $z$ defined as the homomorphism

$$II_z(Z) : S^2 T_z(Z) \to T_z(\mathbb{P}V)/T_z(Z).$$

The following is an immediate property of having a connection. It follows essentially from [Hwang and Mok 2004, Proposition 1].

**Proposition 4.12.** *Let $(f : M \to U, \mathcal{J})$ be a precone structure with a connection. For a point $\alpha \in M$ where the tangent morphism $\tau : M \to \mathscr{C} \subset \mathbb{P}T(U)$ in Proposition 3.7 is immersive,*

$$df_\alpha(\partial \mathcal{J}) = \hat{T}_{\tau(\alpha)}(\mathscr{C}_x)$$

*with $x = f(\alpha)$.*

The following proposition is proved in [Hwang and Mok 2004, Proposition 2], where a precise meaning of “describes the second fundamental form” is given.

**Proposition 4.13.** *Let $\mathcal{J}$ be a precone structure on $f : M \to U$ with characteristic connection $F$. Then the Frobenius tensor of $\partial \mathcal{J}$ at a point $\alpha \in M$ describes the second fundamental form of the projective variety $\mathscr{C}_x \subset \mathbb{P}T_x(U)$, $x = f(\alpha)$ at the point $\tau(\alpha)$, via Proposition 4.12. In particular, the second fundamental form remains unchanged along a leaf of $F \subset Ch(\partial \mathcal{J})$.*

Proposition 4.13 gives a necessary condition for a polarized family to admit a precone structure with characteristic connection.

One reason the characteristic connection is important is its uniqueness under a mild assumption. The following is in [Hwang and Mok 2004, Proposition 3].

**Proposition 4.14.** *Let $\mathscr{C} \subset \mathbb{P}T(U)$ be a cone structure associated to a precone structure $(M, \mathscr{J})$ such that a general fiber $\mathscr{C}_x \subset \mathbb{P}T_x(U)$ has generically finite Gauss map. If $(M, \mathscr{J})$ has a characteristic connection $F$, then $F = Ch(\partial\mathscr{J})$ on an open subset in $M$. In particular, a characteristic connection is unique if it exists.*

The condition that $\mathscr{C}_x \subset \mathbb{P}T_x(U)$ has generically finite Gauss map holds as long as $\mathscr{C}_x$ is smooth and its components are not linear subspaces. The smoothness of $\mathscr{C}_x$ holds in most natural examples, as discussed below. The non-linearity condition will be discussed in Section 7. One can say that the uniqueness of characteristic connection holds in all essential cases.

Regarding the smoothness of VMRT, the following has been one of the most tantalizing questions.

**Problem 4.15.** In Definition 4.4, is the tangent morphism $\tau_x$ at a general point $x$ an immersion? Is it an embedding?

The immersiveness of $\tau_x$ at a point $\alpha \in \mathscr{U}$ can be interpreted as a geometric property of the rational curve $\rho(\alpha)$.

**Definition 4.16.** A free rational curve $C \subset X$ is *standard* if under the normalization $\nu : \mathbb{P}_1 \to C$,

$$\nu^* T(X) \cong \mathscr{O}(2) \oplus \mathscr{O}(1)^p \oplus \mathscr{O}^{\dim X - p - 1}$$

where $p$ is the nonnegative integer satisfying $(-K_X) \cdot C = p + 2$.

The following is a consequence of Mori's bend-and-break argument and basic deformation theory of rational curves (cf. [Hwang 2001, Proposition 1.4]).

**Proposition 4.17.** *Given a minimal component $\mathscr{K}$, a general member of $\mathscr{K}$ is a standard rational curve. For such a standard rational curve, the integer $p$ in Definition 4.16 is the dimension of $\mathscr{K}_x$ and is equal to the projective rank of the VMRT. For a point $x \in X_o$, the tangent morphism $\tau_x$ is immersive at $\alpha \in \mathscr{K}_x$ if and only if $\rho(\alpha) \in \mathscr{K}$ corresponds to a standard rational curve.*

Thus checking the immersiveness of $\tau_x$ is equivalent to showing that all members of $\mathscr{K}_x$ are standard. Although many people believe that this is true for a general point $x$, no plausible approach has been suggested up to now.

Toward the injectivity of $\tau_x$, the best result so far is the following result of [Hwang and Mok 2004].

**Theorem 4.18.** *The tangent morphism $\tau : \mu^{-1}(X_o) \to \mathscr{C}$ is birational. Consequently, $\tau_x$ is birational for a general point $x$.*

Another result on the injectivity is Proposition 7.7 discussed below, in the particular case when the components of $\mathscr{C}_x$ are linear subspaces. There is also a study of the injectivity of $\tau_x$ in [Kebekus and Kovács 2004], relating the problem to the existence of certain singular rational curves.

As these discussions show, Problem 4.15 is quite difficult and its solution will be very important in this subject. On the other hand, since it holds in all concrete examples, sometimes it is OK to work under the assumption that it is true. More precisely, it is meaningful to work with projective manifolds and minimal components whose VMRT is smooth.

The uniqueness of the characteristic connection in Proposition 4.14 suggests the following stronger uniqueness question.

**Problem 4.19.** Can a polarized family $(f : \mathscr{C} \to U, L)$ have two distinct precone structures inducing non-equivalent VMRT structures on $U$?

We will see below an example where this is not unique, i.e., two VMRT's with the same underlying polarized family (Example 5.9). However, this example is very special. It is likely that there are many examples of polarized families for which uniqueness holds. Some cases will be discussed in Theorem 5.11 and Theorem 5.12. One can also ask the following weaker question.

**Problem 4.20.** For a polarized family $(f : \mathscr{C} \to U, L)$, can there exist a positive dimensional family of precone structures inducing locally non-equivalent VMRT structures on U?

Problem 4.20 is closely related to the deformation of Fano manifolds of Picard number 1 via Theorem 4.5.

## 5. Isotrivial VMRT

In this section, we will discuss a special class of cone structures, for which there exists a good differential geometric tool to study the equivalence problem. Let us start by recalling the relevant notion in differential geometry. Chapter VII of [Sternberg 1983] is a good reference.

**Definition 5.1.** Fix a vector space $V$. For a complex manifold $U$ of dimension equal to $\dim V$, its *frame bundle* $\mathbf{Fr}(U)$ is a $\mathbf{GL}(V)$-principal fiber bundle with the fiber at $x \in U$ defined by

$$\mathbf{Fr}_x(U) := \mathrm{Isom}(V, T_x(U))$$

the set of isomorphisms from $V$ to $T_x(U)$. Given an algebraic subgroup $G \subset \mathbf{GL}(V)$, a *$G$-structure* on $U$ means a $G$-principal subbundle $\mathscr{G} \subset \mathbf{Fr}(U)$. Two $G$-structures $\mathscr{G} \subset \mathbf{Fr}(U)$ and $\mathscr{G}' \subset \mathbf{Fr}(U')$ are *equivalent* if there is a biholomorphic map $\varphi : U \to U'$ whose differential $\varphi_* : \mathbf{Fr}(U) \to \mathbf{Fr}(U')$ sends $\mathcal{G}$ to $\mathcal{G}'$. The local equivalence of $G$-structures is defined similarly.

As a trivial example:

**Example 5.2.** The tangent bundle $T(V)$ of a vector space $V$ is naturally isomorphic to $V \times V$. The frame bundle $\mathbf{Fr}(V)$ is naturally isomorphic to $\mathbf{GL}(V) \times V$. For any $G \subset \mathbf{GL}(V)$, we have a natural $G$-structure

$$G \times V \subset \mathbf{GL}(V) \times V = \mathbf{Fr}(V)$$

on the manifold $V$. This is called the *flat* $G$-structure on $V$. A $G$-structure $\mathcal{G}$ on a manifold is said to be *locally flat* if it is locally equivalent to the flat $G$-structure.

Many classical geometric structures in differential geometry are $G$-structures for various choices of $G$. For this reason, the equivalence problem for $G$-structures has been studied extensively. For the following special class of cone structures, the equivalence problem can be reduced to that of certain $G$-structures.

**Definition 5.3.** Let $Z \subset \mathbb{P}V$ be a projective subvariety. A cone structure $\mathcal{C} \subset \mathbb{P}T(U)$ is *$Z$-isotrivial* if the fiber $\mathcal{C}_x \subset \mathbb{P}T_x(U)$ at each $x \in U$ is isomorphic to $Z \subset \mathbb{P}V$.

The simplest example is the following analog of Example 5.2.

**Example 5.4.** The projectivized tangent bundle $\mathbb{P}T(V)$ of a vector space $V$ is naturally isomorphic to $\mathbb{P}V \times V$. For a given subvariety $Z \subset \mathbb{P}V$, we have a natural cone structure

$$Z \times V \subset \mathbb{P}V \times V = \mathbb{P}T(V)$$

on the manifold $V$. This will be called the $Z$-isotrivial *flat* cone structure on $V$.

The equivalence problem for isotrivial cone structures can be reduced to that of G-structures.

**Definition 5.5.** When $\mathcal{C} \subset \mathbb{P}T(U)$ is a $Z$-isotrivial cone structure, the subbundle $\mathcal{G} \subset \mathbf{Fr}(U)$ with the fiber at $x$ defined by

$$\mathcal{G}_x := \{h \in \mathrm{Isom}(V, T_x(U)),\ h(\hat{Z}) = \hat{\mathcal{C}}_x\}$$

is a $G$-structure with $G = \mathrm{Aut}(\hat{Z}) \subset \mathbf{GL}(V)$, the group of linear automorphisms of $\hat{Z} \subset V$. This is called the *$G$-structure induced by the isotrivial cone structure*.

For example, Example 5.2 is the G-structure induced by Example 5.4. It is easy to see that two $Z$-isotrivial cone structures are locally equivalent as cone structures if and only if the G-structures induced by them are locally equivalent. Thus we can use the the theory of G-structures to study isotrivial cone structures. However, this does not mean that the theory of isotrivial cone structures can be completely reduced to the theory of G-structures: it is a highly non-trivial problem to translate conditions on an isotrivial cone structure into the language of G-structures. The problem gets more serious when we consider isotrivial VMRT structures.

Let us say that a VMRT for a uniruled manifold is $Z$-isotrivial if it is a $Z$-isotrivial cone structure at a general point. The first question one can ask is whether there is any restriction on the subvariety $Z \subset \mathbb{P}V$ for the existence of a $Z$-isotrivial VMRT on a uniruled projective manifold. Problem 4.15 suggests that $Z \subset \mathbb{P}V$ should be nonsingular. The next example shows that this is the only necessary condition, if $Z$ is irreducible.

**Example 5.6.** Let $Z \subset \mathbb{P}_{n-1} \subset \mathbb{P}_n$ be a nonsingular irreducible projective variety contained in a hyperplane. Let $X_Z \to \mathbb{P}_n$ be the blow-up of $\mathbb{P}_n$ with center $Z$. Let $\mathcal{K}_Z$ be the family of curves on $X_Z$ which are proper transforms of lines in $\mathbb{P}^n$ intersecting $Z$. Then $\mathcal{K}_Z$ determines a minimal component of $X_Z$ with $Z$-isotrivial VMRT. In fact, the VMRT at a general point is locally equivalent to Example 5.4.

The situation is quite different when $Z$ is reducible. The construction of Example 5.6 does not work, because $\mathcal{K}_Z$ there would be reducible. In fact, the following problem has not been studied.

**Problem 5.7.** Given a nonsingular variety $Z \subset \mathbb{P}V$ with more than one irreducible component, does there exist a uniruled projective manifold $X$ with $Z$-isotrivial VMRT?

Going back to the irreducible case, the most basic question one can ask about isotrivial VMRT is the following.

**Problem 5.8.** Let $Z \subset \mathbb{P}V$ be an irreducible nonsingular variety. Let $X$ be an $n$-dimensional uniruled projective manifold with $Z$-isotrivial VMRT. Is the VMRT at a general point of $X$ locally equivalent to that of Example 5.6?

Recall that $Z \subset \mathbb{P}V$ is degenerate if it is contained in a hyperplane of $\mathbb{P}V$ and nondegenerate otherwise. When $Z \subset \mathbb{P}V$ is degenerate, there are many examples where the answer is negative, as will be seen in Section 6. Even for a nondegenerate $Z \subset \mathbb{P}V$, the answer to Problem 5.8 is not always affirmative.

**Example 5.9.** Let $W$ be a $2\ell$-dimensional complex vector space with a symplectic form. Fix an integer $k$, $1 < k < \ell$ and let $S$ be the variety of all $k$-dimensional isotropic subspaces of $W$. $S$ is a uniruled homogeneous projective manifold. There is a unique minimal component consisting of all lines on $S$ under the Plücker embedding. The VMRT is $Z$-isotrivial where $Z$ is the projectivization of the vector bundle $\mathcal{O}(-1)^{2\ell-2k} \oplus \mathcal{O}(-2)$ on $\mathbb{P}_{k-1}$ embedded by the dual tautological bundle of the projective bundle (cf. Proposition 3.2.1 of [Hwang and Mok 2005]).

Let us denote it by $Z \subset \mathbb{P}V$. There is a distinguished hypersurface $R \subset Z$ corresponding to $\mathbb{P}\mathbb{O}(-1)^{2\ell-2k}$. Let $D$ be the linear span of $R$ in $V$. This $D$ defines a distribution on $S$ which is not integrable (cf. Section 4 of [Hwang and Mok 2005]). However, the corresponding distribution on $X_Z$ of Example 5.6 is integrable. Thus VMRT of $S$ cannot be locally equivalent to that of Example 5.6.

For $Z$ in Example 5.9 or degenerate $Z \subset \mathbb{P}(V)$, the group $\mathrm{Aut}(\hat{Z}) \subset \mathfrak{gl}(V)$ is not reductive. Thus it is reasonable to refine Problem 5.8 to

**Problem 5.10.** Let $Z \subset \mathbb{P}V$ be an irreducible nonsingular subvariety such that $\mathrm{Aut}(\hat{Z}) \subset \mathbf{GL}(V)$ is reductive. Let $X$ be an $n$-dimensional projective manifold with $Z$-isotrivial VMRT. Is the VMRT locally equivalent to that of Example 5.6?

What is nice about Problem 5.10 is that we have a classical differential geometric tool to check local flatness. In fact, given a $Z$-isotrivial cone structure $\mathscr{C} \subset \mathbb{P}T(U)$, we get an induced $G$-structure where $G = \mathrm{Aut}(\hat{Z})$. The flatness of a $G$-structure for a reductive group $G$ can be checked by the vanishing of certain curvature tensors (cf. [Hwang and Mok 1997]). Thus Problem 5.10 is reduced to checking the vanishing of the curvature tensors using properties of VMRT structures.

There are two classes of examples for which Problem 5.10 has been answered in the affirmative. The first one are those covered by the next theorem of Mok [2008b].

**Theorem 5.11.** *Let $S$ be an $n$-dimensional irreducible Hermitian symmetric space of compact type with a base point $o \in S$. There exists a unique minimal component on $S$. Let $\mathscr{C}_o \subset \mathbb{P}T_o(S)$ be the VMRT at $o$. If the projective variety $Z \subset \mathbb{P}V$ is isomorphic to $\mathscr{C}_o \subset \mathbb{P}T_o(S)$, then Problem 5.10 has an affirmative answer.*

For example when $S$ is the $n$-dimensional quadric hypersurface, $Z \subset \mathbb{P}V$ is just an $(n-2)$-dimensional non-singular quadric hypersurface. Then $\mathscr{C}_x \subset \mathbb{P}T_x(X)$ in Problem 5.10 defines a conformal structure at the general point of $X$. In this case, Theorem 5.11 says that this conformal structure is locally flat.

Let us recall Mok's strategy for the proof of Theorem 5.11. The main point is to show that the G-structure which is defined at the general point of $X$ can be extended to a G-structure in a *neighborhood* of a standard rational curve, by exploiting Proposition 4.13 and the fact that $Z \subset \mathbb{P}V$ in Theorem 5.11 is determined by the second fundamental form. Once this extension is obtained, one can deduce the flatness by applying [Hwang and Mok 1997], which shows the vanishing of the curvature tensor from *global* information of the tangent bundle of $X$ on the standard rational curve.

It is very difficult to apply Mok's approach to other $Z \subset \mathbb{P}V$. Since the projective variety $Z \subset \mathbb{P}V$ treated in Theorem 5.11 is the highest weight variety associated to an irreducible representation, one would hope that a similar approach holds for the highest weight variety $Z \subset \mathbb{P}V$ associated to other irreducible representation. However, this is not possible. In fact, [Hwang and Mok 1997] shows that the only irreducible G-structure which can be extended to a neighborhood of a standard rational curve is the one covered by Theorem 5.11.

The other class of examples for which an affirmative answer to Problem 5.10 is known belong to the opposite case when the projective automorphism group of $Z \subset \mathbb{P}V$ is 0-dimensional, i.e. when $\mathrm{Aut}_o(\hat{Z}) = \mathbb{C}^*$. In this case, we cannot use Mok's approach, i.e., the $G$-structure with $G = \mathbb{C}^*$ *cannot* be extended to a neighborhood of a standard rational curve. One can see this as follows. Suppose it is possible to extend the $G$-structure to a neighborhood $U$ of a standard rational curve. For simplicity, let us assume that the automorphism group of $Z \subset \mathbb{P}V$ is trivial. In $\mathbb{P}T(U)$ we have a submanifold $\mathscr{C} \subset \mathbb{P}T(U)$ with each fiber $\mathscr{C}_x \subset \mathbb{P}T_x(U)$ isomorphic to $Z \subset \mathbb{P}_{n-1}$ and since the automorphism group is trivial, we get a unique trivialization of the projective bundle $\mathbb{P}T(U)$. But on a standard rational curve, $T(U)$ splits into $\mathbb{O}(2) \oplus \mathbb{O}(1)^p \oplus \mathbb{O}^{n-1-p}$ for some $p \geq 0$, a contradiction.

In this case, the flatness of the G-structure, or the vanishing of the corresponding curvature tensors, must be proved only at general points. In other words, it must come from the flatness of cone structures satisfying certain differential geometric condition. In this direction, we have the following result from [Hwang 2010, Theorem 1.11].

**Theorem 5.12.** *Assume that $Z \subset \mathbb{P}V$ satisfies the following conditions.*

(1) *$Z$ is nonsingular and linearly normal, i.e.,* $H^0(Z, \mathbb{O}(1)) = V^*$.

(2) *The variety of tangent lines to $Z$, defined as a subvariety of* $\mathbf{Gr}(2, V) \subset \mathbb{P}(\bigwedge^2 V)$, *is nondegenerate in* $\mathbb{P}(\bigwedge^2 V)$.

(3) $H^0(Z, T(Z) \otimes \mathbb{O}(1)) = H^0(Z, \mathrm{ad}(T(Z)) \otimes \mathbb{O}(1)) = 0$ *where* $\mathrm{ad}(T(Z))$ *denotes the bundle of traceless endomorphisms of the tangent bundle of $Z$.*

*Then a $Z$-isotrivial cone structure with characteristic connection is locally equivalent to that of Example 5.6.*

Note that nondegeneracy and $H^0(Z, T(Z) \otimes \mathbb{O}(1)) = 0$ imply that the projective automorphism group of $Z \subset \mathbb{P}V$ is 0-dimensional.

Let us recall the strategy of the proof of Theorem 5.12 in [Hwang 2010]. Given a $Z$-isotrivial cone structure $\mathscr{C} \subset \mathbb{P}T(U)$ with $\mathrm{Aut}_o(\hat{Z}) = \mathbb{C}^*$, we have a uniquely determined trivialization $\bar{\theta} : \mathbb{P}T(U) \cong \mathbb{P}V \times U$ with $\bar{\theta}(\mathscr{C}) = Z \times U$. Up to multiplication by a scalar function, this means a trivialization $\theta : T(U) \cong V \times U$, $\theta(\hat{\mathscr{C}}) = \hat{Z} \times U$. Such a trivialization $\theta$ determines an affine connection on $U$

and consequently a projective connection on $\mathbb{P}T(U)$. This projective connection is tangent to $\mathscr{C} \subset \mathbb{P}T(U)$ from the way $\theta$ is chosen. In particular, it induces a natural connection on the cone structure $\mathscr{C}$. One difficulty is that there is no reason why this natural connection on $\mathscr{C}$ agrees with the characteristic connection on $\mathscr{C}$ whose existence is assumed in Theorem 5.12. This difficulty is avoided by the first condition in Theorem 5.12 (3) via Proposition 4.8. In particular, the connection on $\mathscr{C}$ induced by $\theta$ is a characteristic connection. This enables us to relate the projective geometry of $\mathscr{C}_x$ to the curvature tensors of the G-structure. The rest of the conditions in Theorem 5.12 are used to derive the vanishing of the curvature tensors via this relation.

The conditions of Theorem 5.12 are rather restrictive, although some examples of $Z \subset \mathbb{P}V$ satisfying them are given in [Hwang 2010]. The main remaining problem is how to weaken these conditions, by using more properties of VMRT-structures, other than the existence of a characteristic connection.

Another interesting problem, in view of Theorem 5.12, is to prove a local version of Theorem 5.11, namely, the local flatness of a $Z$-isotrivial cone structure, for the same $Z$ as in Theorem 5.11, with some additional differential geometric conditions on the cone structure, which always holds for VMRT structures.

## 6. Distribution spanned by VMRT

When the fibers of a cone structure are degenerate, the cone structure defines a non-trivial distribution as follows.

**Definition 6.1.** Let $\mathscr{C} \subset \mathbb{P}T(U)$ be a cone structure. For each $x \in U$, let $D_x \subset T_x(U)$ be the vector space spanned by the cone $\hat{\mathscr{C}}_x$. Let $U_o \subset U$ be the open subset where the dimension of $D_x$ is constant. Then $\{D_x, x \in U_o\}$ determine a distribution on $U_o$, denoted by $\mathrm{Dist}(\mathscr{C})$ and called the *distribution spanned by* $\mathscr{C}$.

When $\mathscr{C}$ admits a characteristic connection or a VMRT structure, $\mathrm{Dist}(\mathscr{C})$ has a special feature. There is an intricate relation between the projective geometry of the fibers $\mathscr{C}_x \subset \mathbb{P}T_x(U)$ and the Frobenius tensor of $\mathrm{Dist}(\mathscr{C})$. The following was first proved in [Hwang and Mok 1998, Proposition 10], when $\mathscr{C}$ is a VMRT structure, by a more geometric argument using a family of standard rational curves.

**Theorem 6.2.** *Let $M \to U$ be a precone structure with a characteristic connection and $\tau : M \to \mathbb{P}T(U)$ be the tangent morphism in Proposition 3.7. Let $D := \mathrm{Dist}(\mathscr{C})$, $\mathscr{C} = \tau(M)$, and let $\beta : \bigwedge^2 D \to T(U)/D$ be the Frobenius tensor of $D$. For a general point $x \in U$, any point $\alpha \in M_x$ and a tangent vector $v \in T_\alpha(M_x)$, let $a \in D_x$ be a vector belonging to $\tau(\alpha) \subset \mathbb{P}T_x(U)$ and $b \in D_x$ be a vector proportional to the image of $d\tau(v) \in T(\mathscr{C}_x)$ in $T_x(U)$. Then the Frobenius tensor satisfies $\beta(a \wedge b) = 0$.*

*Proof.* Let $\pi : M \to U$ be the natural projection and $\mathcal{J}$ be the distribution of the precone structure. The distribution $\pi^{-1}D$ on $M$ contains the distribution $\partial\mathcal{J}$ by Proposition 4.12. The vertical distribution $\mathrm{Ker}\, d\pi$ is in $Ch(\pi^{-1}(D))$. Let

$$\delta : \bigwedge^2(\pi^{-1}D) \to T(M)/\pi^{-1}D$$

be the Frobenius tensor of $\pi^{-1}D$. Then for any $w_1, w_2 \in (\pi^{-1}D)_\alpha \subset T_\alpha(M)$,

$$\beta(d\pi(w_1), d\pi(w_2)) = d\pi(\delta(w_1, w_2))$$

where $d\pi$ on the right-hand side refers to the natural map

$$T(M)/\pi^{-1}D \to T(U)/D$$

induced by $d\pi : T(M) \to T(U)$. For any $a \in \hat{\alpha} \in T_x(U)$ and $b \in \hat{T}_\alpha(\mathscr{C}_x) \subset T_x(U)$, we have their lifts $w_1, w_2 \in (\pi^{-1}D)_\alpha$ with $d\pi(w_1) = a, d\pi(w_2) = b$ such that $w_1 \in F_\alpha$ where $F$ is the characteristic connection and $w_2 \in (\partial\mathcal{J})_\alpha$ by Proposition 4.12. By the definition of a characteristic connection, if $\lambda : \bigwedge^2(\partial\mathcal{J}) \to T(\mathscr{C})/(\partial\mathcal{J})$ is the Frobenius tensor of $\partial\mathcal{J}$, then $\lambda(w_1, w_2) = 0$. Since $\partial\mathcal{J}$ is a sub-distribution of $\pi^{-1}D$, we have $\delta|_{\wedge^2(\partial\mathcal{J})} = \lambda \mod \pi^{-1}D$. It follows that $\delta(w_1, w_2) = 0$ and consequently, $\beta(a, b) = 0$. □

In other words, at a general point $x \in X$, if $H \subset \bigwedge^2 D_x$ denotes the linear span of the points in $\mathbb{P}\bigwedge^2 D_x$ corresponding to the bivectors given by the tangent lines of $\mathscr{C}_x \subset \mathbb{P}T_x(U)$, then $H$ is in the kernel of the Frobenius tensor of $D = \mathrm{Dist}(\mathscr{C})$. This gives non-trivial information about the Frobenius tensor of the distribution spanned by a VMRT structure. Are there more restrictions on the Frobenius tensor enforced by a VMRT structure? This is a very interesting question to study. More specifically, one can ask the following.

**Problem 6.3.** Given a nondegenerate nonsingular projective variety $Z \subset \mathbb{P}V$, let $H \subset \bigwedge^2 V$ be the linear span of the variety of tangent lines to $Z$. Then does there exist a uniruled manifold $X$ with VMRT $\mathscr{C} \subset \mathbb{P}T(X_o)$ such that for a general $x \in X_o$, $\mathscr{C}_x \subset \mathbb{P}\mathrm{Dist}(\mathscr{C})_x$ is isomorphic to $Z \subset \mathbb{P}V$ and the kernel of the Frobenius tensor of $\mathrm{Dist}(\mathscr{C})$ is precisely $H$?

Problem 6.3 is trivial if $H = \bigwedge^2 V$. As was noticed in [Hwang and Mok 1999, Proposition 1.3.2], this is the case if $\dim Z > \frac{1}{2}\dim V - 1$. Thus we may assume that $\dim Z \le \frac{1}{2}\dim V - 1$ in Problem 6.3. A special case of Problem 6.3 is when $Z \subset \mathbb{P}V$ is the highest weight variety associated to an irreducible representation. Even in this case, the answer is unknown in general. A known example is when $Z$ comes from the VMRT of the rational homogenous space $G/P$ associated with a long simple root, as explained in [Hwang and Mok 2002, Proposition 5]. In this case, the condition that the Frobenius tensor is determined by $H$ has to do with the finiteness condition in Serre's presentation of simple Lie algebras.

Now let us turn our attention from Dist($\mathscr{C}$) to the foliation it generates, in the sense explained in Section 2. When the cone structure is a VMRT on a uniruled projective manifold, the leaves of this foliation have a strong algebraic property. To explain this, we will give a general construction of a foliation of a uniruled projective manifold by members of a component of FRC($X$), the space of free rational curves on $X$. Firstly, recall the following basic fact.

**Proposition 6.4.** *Let $X^o$ be an irreducible quasi-projective variety. Suppose that for each irreducible subvariety $W \subset X^o$, we have associated a subvariety $C_W \subset X^o$ with finitely many components such that each irreducible component of $C_W$ contains $W$ and if $W \subset W'$, then $C_W \subset C_{W'}$. We say that an irreducible subvariety $W$ is saturated if $C_W = W$. For each $x \in X^o$, define*

$$Z_x := \text{ the intersection of all saturated subvarieties through } x.$$

*Then the followings hold.*

(i) *$Z_x$ is irreducible and saturated.*

(ii) *Let $Z^0 = \{x\}$ and let $Z^{i+1}$ be a component of $C_{Z^i}$ for $i \geq 0$. Then $Z^n = Z_x$ where $n = \dim X$.*

(iii) *There exists a Zariski open subset $X^* \subset X^o$ and a foliation on $X^*$ whose leaves are algebraic such that the leaf through a very general point $x \in X^*$ is $Z_x \cap X^*$.*

*Proof.* For (i), it suffices to show that each component $Y$ of $Z_x$ containing $x$ is saturated. Suppose that $W$ is a saturated subvariety through $x$. Then $Y \subset W$, hence $C_Y \subset C_W = W$. It follows that $C_Y$ is contained in any saturated variety through $x$, implying that $Y \subset C_Y \subset Z_x$. But each component of $C_Y$ contains $Y$. Consequently, $C_Y = Y$ and $Y$ is saturated, implying $Z_x = Y$. For (ii), note that if $W$ is not saturated, then every component of $C_W$ has dimension strictly bigger than $\dim W$. Thus $Z^n$ must be a saturated variety containing $x$, implying $Z_x \subset Z^n$. On the other hand, if $W$ is any saturated variety through $x$, then $Z^1 \subset C_x \subset C_W = W$. Thus we get inductively $Z^i \subset C_{Z^i} \subset C_W = W$. It follows that $Z^n \subset W$ for any saturated subvariety $W$ through $x$, showing $Z^n \subset Z_x$. For (iii), since the collection of subvarieties $Z_x$ cover $X^o$, there must be a flat family of subvarieties whose very general member is of the form $Z_x$ for some $x \in X^o$ and the members of the family cover a Zariski dense open subset in $X^o$. To show (iii), it suffices to see that for two $Z_x$ and $Z_y$ in this family, if $y \in Z_x$ then $Z_y = Z_x$. From the definition of $Z_y$, we get $Z_y \subset Z_x$. Then we get equality by flatness of the family. □

**Definition 6.5.** Let $\mathscr{K}$ be a component of FRC($X$). Let $X^o \subset X$ be a Zariski open subset in the union of members of $\mathscr{K}$. Given an irreducible subvariety

$W \subset X$ with $W \cap X^o \neq \varnothing$, define

$$C_W := \text{ closure of } \bigcup_{C \in \mathcal{K}, C \cap W \neq \varnothing} C.$$

Since the evaluation morphism for $\mathcal{K}$ is smooth, each component of $C_W$ contains $W$. If $W'$ is another irreducible subvariety of $X$ with $W \subset W'$, then $C_W \subset C_{W'}$. Thus we can apply Proposition 6.4 to get a foliation $\mathcal{F}$ on a Zariski open subset $X^* \subset X$ such that for a very general $x \in X^*$, the leaf through $x$ is $Z_x \cap X^*$. In fact, we can choose $X^*$ such that $\mathrm{codim}(X \setminus X^*) \geq 2$. This foliation $F$ is called the *foliation generated by* $\mathcal{K}$.

The construction of the foliation generated by a minimal component $\mathcal{K}$ is purely algebro-geometric. We want to relate it to a differential geometric construction.

**Proposition 6.6.** *Let $X$ be a uniruled projective manifold and $\mathcal{C} \subset \mathbb{P}T(X)$ the VMRT associated to a minimal component $\mathcal{K}$. Then the foliation generated by* $\mathrm{Dist}(\mathcal{C})$ *in the sense of Section* 2 *coincides, on a dense open subset of $X$, with the foliation generated by $\mathcal{K}$ in the sense of Definition 6.5. In particular, for the normalization of a member $\nu : \mathbb{P}_1 \to X$ of $\mathcal{K}$ through a very general point $x \in X$, if $f^*T(X) = P \oplus N$ is the decomposition into an ample vector bundle $P$ and a trivial vector bundle $N$ on $\mathbb{P}_1$, then the fiber $P_x$ lies in the tangent space $T_x(Z_x)$.*

*Proof.* Let $\mathcal{F}$ be the foliation generated by $\mathcal{K}$ and let $\mathcal{F}'$ be the foliation generated by $\mathrm{Dist}(\mathcal{C})$. For a very general point $x \in X$, the leaf of $\mathcal{F}$ corresponds to an open subset in $Z_x$. Let $\mathcal{L}$ be the leaf of $\mathcal{F}'$ through $x$. From the construction of $Z_x$ in Proposition 6.4 (iii), the germ of $Z_x$ must be contained in $\mathcal{L}$. Thus $\mathcal{F}$ is a foliation contained in $\mathcal{F}'$. On the other hand, the tangent space to $Z_x$ at a general point must contain all vectors tangent to members of $\mathcal{K}_x$. This implies that $\mathcal{F}'$ is a foliation contained in $\mathcal{F}$. We conclude that the two foliations coincide. The second statement follows because $P_x$ must lie in $\mathrm{Dist}(\mathcal{C})_x$. □

To my knowledge, the next theorem has not appeared in print.

**Theorem 6.7.** *Let $\mathcal{F}$ be the foliation defined in Proposition 6.6, extended to a foliation on a maximal open subset of $X$ and let $\mathcal{L}$ be a general leaf of $\mathcal{F}$. Then for a general point $x \in \mathcal{L}$, all members of $\mathcal{K}_x$ lie in $\mathcal{L}$. In particular, if $\tilde{\mathcal{L}}$ is a desingularization of $\bar{\mathcal{L}}$, then $\tilde{\mathcal{L}}$ is a uniruled projective manifold and there exists a minimal component $\tilde{\mathcal{K}}$ with a natural identification $\mathcal{K}_x = \tilde{\mathcal{K}}_x$ for a general point $x \in \mathcal{L}$. Consequently, the VMRT-structure $\mathcal{C}$ of $X$ restricted $\mathcal{L}$, i.e., $\mathcal{C} \cap \mathbb{P}T(\mathcal{L})$ is equivalent to a VMRT-structure of the manifold $\tilde{\mathcal{L}}$.*

*Proof.* From the construction of the foliation generated by $\mathcal{K}$ in Definition 6.5, there exists a rational map $\eta : X \to B$ surjective over a projective manifold $B$ such that the fiber of $\eta$ through a very general point $x \in X$ corresponds to $Z_x$.

Let $\tilde{X} \subset X \times B$ be the graph of $\eta$ with the birational morphism $p_1 : \tilde{X} \to X$ and the morphism $p_2 : \tilde{X} \to B$ which is an elimination of the indeterminacy locus of $\eta$.

We claim that the proper transforms of members of $\mathscr{K}$ to $\tilde{X}$, which intersect the exceptional divisors of $p_1$ do not cover $\tilde{X}$. Suppose not. An exceptional divisor $E$ of $p_1$ is covered by curves which are contracted by $p_1$ but not contracted by $p_2$. Thus we get a 1-dimensional family $\{C_t \subset X, t \in \Delta\}$ of members of $\mathscr{K}$ passing through general points $x_t \in C_t$ with a common point $y \in \cap C_t$ such that $C_t \subset Z_{x_t}$ with $Z_{x_t} \neq Z_{x_s}$ for $t \neq s \in \Delta$. Let $f_t : \mathbb{P}_1 \to C_t$ be the normalization with $f_t(o) = y$ and $f_t(\infty) = x_t$ for two fixed points $o, \infty \in \mathbb{P}_1$. If $\sigma_t \in H^0(\mathbb{P}_1, f_t^* T(X))$ is the infinitesimal deformation of $f_t$, then $\sigma_t(o) = 0$. By Proposition 6.6, we see that $\sigma(\infty) \in T_{x_t}(Z_{x_t})$ for all $t \in \Delta$. But this is contradiction, because the deformation $C_t$ moves out of $Z_{x_t}$. This verifies the claim.

Since a general fiber of $p_2$ is smooth, the complement of the exceptional divisors of $p_1$ in the general fiber of $p_2$ is sent into a general leaf $\mathscr{L}$ in $X$. By the claim, all members of $\mathscr{K}_x$ for a general point $x \in X$ lie on the leaf $\mathscr{L}$ through $x$, completing the proof. □

The following is essentially [Hwang and Mok 1998, Proposition 13].

**Proposition 6.8.** *Let $X$ be a uniruled projective manifold of Picard number* 1. *Then the distribution spanned by the VMRT of a minimal component is bracket-generating.*

*Proof.* Suppose it is not bracket-generating. Then in the proof of Theorem 6.7, $\dim B > 0$. Choose a hypersurface $H \subset B$ to get the hypersurface $p_1(p_2^{-1}(H)) \subset X$. Let $C$ be a general member of $\mathscr{K}$ whose proper transform in $\tilde{X}$ is disjoint from $p_2^{-1}(H)$. By the proof of Theorem 6.7, the proper transform of $C$ is disjoint from the exceptional divisors of $p_1$. It follows that $C$ is disjoint from the divisor $p_1(p_2^{-1}(H))$ in $X$, a contradiction to the fact that all effective divisors on $X$ are ample. □

Even when the Picard number of $X$ is bigger than 1, Theorem 6.7 implies that to study the VMRT-structure, it is necessary to study the VMRT-structure of the desingularized leaf closure $\tilde{\mathscr{L}}$. This justifies that when studying VMRT, it makes sense to assume that the distribution spanned by the cone is bracket-generating.

## 7. Linear VMRT

In this section, we will look at the case when the fiber $\mathscr{C}_x$ of a cone structure $\mathscr{C} \subset \mathbb{P}T(U)$ is a union of linear subspaces. To start with, we need some differential geometric concepts.

**Definition 7.1.** Recall that a connection on the full cone structure $\mathscr{C} = \mathbb{P}T(U)$ is called a *projective connection* on $U$. It is automatically a characteristic connection. We say that a projective connection is *locally flat* if it is locally equivalent to the one on $\mathbb{P}_n$ induced by the family $\mathscr{K}$ of lines in $\mathbb{P}_n$ where $n = \dim U$.

**Definition 7.2.** A *web* (of rank $m$) on a complex manifold $U$ is a finite collection $\{F_1, \ldots, F_\ell\}$ of foliations (of rank $m$) on $U$. A cone structure $\mathscr{C} \subset \mathbb{P}T(U)$ is a *linear cone structure* if each fiber $\mathscr{C}_x$ is a union of linear subspaces. A web defines a linear cone structure $\mathscr{C} := \mathbb{P}F_1 \cup \cdots \cup \mathbb{P}F_\ell$.

**Proposition 7.3.** *Let $\mathscr{C} \subset \mathbb{P}T(U)$ be a linear cone structure. Shrinking $U$, assume that $\mathscr{C} = \mathscr{C}_1 \cup \cdots \cup \mathscr{C}_\ell$ such that each $\mathscr{C}_i$ is just a distribution. If $\mathscr{C}$ has a characteristic connection, then each $\mathscr{C}_i$ is integrable and $\mathscr{C}$ defines a web structure. Moreover, each leaf of the web has a projective connection. Conversely, a web with a projective connection on each leaf gives rise to a cone structure with characteristic connection whose fibers are union of linear subspaces.*

The only non-trivial part in Proposition 7.3 is the integrability of each distribution $\mathscr{C}_i$ when there is a characteristic connection. This is a direct consequence of Theorem 6.2.

Now when the cone structure is a VMRT structure, something special happens, namely, the projective connection becomes locally flat. In fact, the following general structure theorem is proved in [Araujo 2006, Theorem 3.1] and [Hwang 2007, Proposition 1].

**Proposition 7.4.** *Let $X$ be a uniruled manifold and $\mathscr{C} \subset \mathbb{P}T(X_o)$ be the VMRT of a minimal component $\mathscr{K}$. Assume that $\mathscr{C}$ is a linear cone structure. Then there exists a normal variety $\tilde{X}$ with a finite holomorphic map $\eta : \tilde{X} \to X$ and a dense open subset $\tilde{X}_o$ of $\tilde{X}$ equipped with a proper holomorphic map $\varphi : \tilde{X}_o \to T$ such that each fiber of $\varphi$ is biregular to $\mathbb{P}_k$ where $k$ is the rank of the VMRT and each member of $\mathscr{K}_x$ for a general $x \in X$ is the image of a line in some fibers of $\varphi$. Moreover, for each $t \in T$, let $P_t := \eta(\varphi^{-1}(t))$ be a subvariety in $X$. Then $P_t$ is an immersed submanifold with trivial normal bundle in $X$, $\rho|_{\varphi^{-1}(t)}$ is its normalization, and for two distinct points $t_1 \neq t_2 \in T$, the two subvarieties $P_{t_1}$ and $P_{t_2}$ are distinct.*

One may say that a linear VMRT defines a web structure whose "leaves" are immersed projective spaces with trivial normal bundles. Proposition 6.8 has the following consequence.

**Proposition 7.5.** *In the situation of Proposition 7.4, assume that $X$ has Picard number* 1. *Then the morphism $\eta$ has degree* $> 1$.

The most interesting case of a linear VMRT is one with projective rank 0, i.e., when the VMRT at a general point is finite. In all known examples of linear

VMRT on uniruled manifolds of Picard number 1, the VMRT turns out to be of projective rank 0. This leads to the following question.

**Problem 7.6.** In the situation of Proposition 7.4, suppose the degree of $\mu$ is bigger than 1. Is $\dim \mathscr{C}_x = 0$ at a general point $x$?

Regarding Problem 7.6, there is at least a restriction on $\dim \mathscr{C}_x$. The following was in [Hwang 2007, Proposition 2].

**Proposition 7.7.** *In the case of Proposition 7.4, $\mathscr{C}_x$ is smooth. In other words, components of $\mathscr{C}_x$ are disjoint from each other. In particular, if the degree of $\mu$ is bigger than* 1, *then* $2 \dim \mathscr{C}_x \leq \dim X - 2$.

The geometric idea behind this result is as follows. From Proposition 7.4, in an unramified cover of a neighborhood of $\mathbb{P}_k$, we have a foliation with leaves isomorphic to $\mathbb{P}_k$. If two different components of $\mathscr{C}_x$ intersect, then one of the foliations defines in the leaf $\mathbb{P}_k$ of the other foliation a positive-dimensional subvariety with trivial normal bundle, a contradiction to the ampleness of the tangent bundle of $\mathbb{P}_k$.

When $X$ is embedded in projective space such that members of $\mathscr{K}$ are lines, one can go one step further from Proposition 7.7: Theorem 1.1 of [Novelli and Occhetta 2011] excludes the case when $2 \dim \mathscr{C}_x = \dim X - 2$ under this assumption. Their argument seems difficult to generalize to arbitrary uniruled projective manifolds.

In Theorem 4.5, the case of linear VMRT was excluded. When VMRT is linear, a counterexample can be constructed.

**Example 7.8.** Note that when $f : X \to Y$ is a finite morphism between two Fano manifolds and $Y$ has a minimal component $\mathscr{K}_Y$ with VMRT of projective rank 0, i.e., the VMRT at a general point is finite, then the inverse images of members of $\mathscr{K}_Y$ under $f$ form a minimal component $\mathscr{K}_X$ on $X$ with VMRT of projective rank 0. See, for example, [Hwang and Mok 2003, Proposition 6] for a proof. Thus to get a counterexample to an analogue of Theorem 4.5 for VMRT of projective rank 0, it suffices to provide a finite morphism $f : X \to Y$ between two non-isomorphic Fano manifolds such that $Y$ has VMRT of projective rank 0. Such an example is given by [Schuhmann 1999, Example 1.1]. More precisely, let $Y \subset \mathbb{P}_4$ be a cubic threefold. Let $X_1 \subset \mathbb{P}_5$ be the cone over $Y$ and $X_2 \subset \mathbb{P}_5$ be a quadric hypersurface such that the intersection $X = X_1 \cap X_2$ is a smooth threefold in $\mathbb{P}_5$. Then $X$ is a Fano threefold of index 1 and projection from the vertex of the cone $X_1$ onto $Y$ induces a finite morphism from $X$ to $Y$.

It is natural to ask what partial result toward Theorem 4.5 holds when VMRT is linear. We hope that the following has an affirmative answer.

**Problem 7.9.** Let $X$ be a uniruled projective manifold of Picard number 1 with linear VMRT. Let $\varphi : U \to U'$ be a biholomorphic map between two connected open subsets in $X$ such that $d\varphi$ sends $\mathscr{C}|_U$ to $\mathscr{C}|_{U'}$. Does $\varphi$ extend to a biregular automorphism $\tilde{\varphi} : X \to X$?

We point out that $U$ and $U'$ in Problem 7.9 are open subsets in the classical topology. In fact, if $U$ and $U'$ are Zariski open and $\varphi$ is birational, it is easy to show that $\varphi$ extends to a biregular morphism $\tilde{\varphi}$, as explained in [Hwang and Mok 2001, Proposition 4.4].

Problem 7.9 can be viewed as a generalization of the Liouville theorem in conformal geometry (e.g., [Dubrovin et al. 1984, 15.2]) which says that for the flat conformal model of dimension $\geq 3$ a local conformal transformation comes from a global conformal transformation. There are only a few examples where the answer to Problem 7.9 is known.

When $X \subset \mathbb{P}_{n+1}$ is a smooth hypersurface of degree $n$ and $\mathscr{K}$ is a family of lines covering $X$, the VMRT has projective rank 0 (cf. [Hwang 2001, 1.4.2]). In fact, the ideal defining these finite points in $\mathbb{P}T_x(X)$ is given by the complete intersection of homogeneous polynomials of degree $2, 3, \dots, n$. The quadric polynomial corresponds to the second fundamental form of the hypersurface $X$ at $x$ and the cubic polynomial corresponds to the Fubini cubic form in the language of [Jensen and Musso 1994]. In particular, the second fundamental form and the Fubini cubic form are determined by VMRT. Thus by the result of [Jensen and Musso 1994] or [Sasaki 1988], we have an affirmative answer for Problem 7.9 when $X \subset \mathbb{P}_{n+1}$ is a hypersurface of degree $n$.

One can also ask the same question for the hypersurface $X \subset \mathbb{P}_{n+1}$ of degree $n+1$. For $\mathscr{K}$ consisting of conics covering $X$, then the VMRT has projective rank 0. However, for this example, it is still unknown whether Problem 7.9 has an affirmative answer.

Another example of Problem 7.9 is Mukai–Umemura threefolds in [Mukai and Umemura 1983]. Recall that these are Fano threefolds of Picard number 1, which are quasi-homogeneous under the three-dimensional Lie group $SL(3, \mathbb{C})$. In fact, they are equivariant compactifications of $SL(3, \mathbb{C})/\mathbf{O}$ and $SL(3, \mathbb{C})/\mathbf{I}$ where $\mathbf{O}$ and $\mathbf{I}$ denote the octahedral and icosahedral groups, respectively. The choice of a Cartan subgroup of $SL(3, \mathbb{C})$ determines a rational curve on $X$, whose orbits under $SL(3, \mathbb{C})$ give a minimal component $\mathscr{K}$. The VMRT at a base point $x \in X$ in the open orbit is given by the orbit of the Cartan subalgebra by the action of $\mathbf{O}$ or $\mathbf{I}$. Using this one can explicitly describe the web structure in a neighborhood of $x$, from which one can check that Problem 7.9 has an affirmative answer.

There are many examples with VMRT of projective rank 0; see [Hwang and Mok 2003]. For example, all Fano threefolds of Picard number 1, other than $\mathbb{P}_3$ and the quadric threefold in $\mathbb{P}_4$, provide such examples. For most of these examples, Problem 7.9 is still open.

On the other hand, one may wonder whether a counterexample to Problem 7.9 can be constructed in a way analogous to Example 7.8. This is not the case. It is related to the following well-known problem.

**Problem 7.10.** Let $X$ be a Fano manifold of Picard number 1 different from projective space. If $f : X \to X$ is a finite self-morphism, should $f$ be bijective?

An affirmative answer is known for Problem 7.10 when $X$ has linear VMRT. This was proved when the projective rank is 0, by [Hwang and Mok 2003, Corollary 3]. The proof for any projective rank, which also gives a simpler and different proof for projective rank 0 case, is given in [Hwang and Nakayama 2011, Theorem 1.3].

## 8. Symmetries of cone structures

An important component of any equivalence problem is its symmetries, i.e., the self-equivalence, or the automorphisms, of the geometric structure. In the study of continuous symmetries, the investigation of the Lie algebra of the symmetry group is an efficient method. In this section, we present the theory of the local symmetries of the cone structure. More precisely, for a given cone structure $\mathscr{C} \subset \mathbb{P}T(U)$, we want to understand the Lie algebra of germs of holomorphic vector fields at a point $x \in U$ which preserve the cone structure in the following sense.

**Definition 8.1.** Given a G-structure $\mathscr{G} \subset \mathbf{Fr}(U)$ (resp. a cone structure $\mathscr{C} \subset \mathbb{P}T(U)$), a holomorphic vector field $\sigma$ on $U$ *preserves* the G-structure (resp. cone structure) if the induced vector field $\tilde{\sigma}$ on $\mathbf{Fr}(U)$ (resp. on $\mathbb{P}T(U)$) is tangent to the subvariety $\mathscr{G}$ (resp. $\mathscr{C}$).

A convenient notion in studying symmetries of G-structures is the following.

**Definition 8.2.** Let $V$ be a vector space. Let $\mathfrak{gl}(V)$ be the Lie algebra of endomorphisms of $V$. Given a Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}(V)$, its $m$-th *prolongation* is the subspace $\mathfrak{g}^{(m)} \subset \mathrm{Hom}(S^{m+1}V, V)$ consisting of multi-linear homomorphisms $\sigma : S^{m+1}V \to V$ such that for any fixed $v_1, \dots, v_m \in V$, the endomorphism

$$v \in V \mapsto \sigma(v, v_1, \dots, v_m) \in V$$

belongs to $\mathfrak{g}$.

**Lemma 8.3.** *The following properties are immediate.*

(i) $\mathfrak{g}^{(0)} = \mathfrak{g}$.

(ii) *If* $\mathfrak{g}^{(m)} = 0$ *for some* $m \geq 0$, *then* $\mathfrak{g}^{(m+1)} = 0$.

(iii) *If* $\mathfrak{h} \subset \mathfrak{g} \subset \mathfrak{gl}(V)$ *is a Lie subalgebra, then* $\mathfrak{h}^{(m)} \subset \mathfrak{g}^{(m)}$ *for each* $m \geq 0$.

This is related to the symmetries of G-structures as follows, which is explained well in [Yamaguchi 1993, Section 2.1].

**Proposition 8.4.** *Let $G \subset \mathbf{GL}(V)$ be a connected algebraic subgroup and $\mathfrak{g} \subset \mathfrak{gl}(V)$ be its Lie algebra. Let $\mathcal{G} \subset \mathbf{Fr}(U)$ be a $G$-structure. For a point $x \in U$, let $\mathfrak{f}$ be the Lie algebra of all germs of holomorphic vector fields at $x$ which preserve $\mathcal{G}$. Let $\mathfrak{f}^k$ be the Lie subalgebra of $\mathfrak{f}$ consisting of vector fields which vanish at $x$ to order $\geq k+1$ for some integer $k \geq -1$. For each $k \geq 0$, regard the quotient space $\mathfrak{f}^k/\mathfrak{f}^{k+1}$ as a subspace of* $\mathrm{Hom}(S^{k+1}V, V)$ *by taking the leading coefficients of the Taylor expansion of the vector fields at $x$. Then*

$$\mathfrak{f}^k/\mathfrak{f}^{k+1} \subseteq \mathfrak{g}^{(k)}$$

*and equality holds for a locally flat $G$-structure.*

In other words, the prolongations of $\mathfrak{g}$ are the graded pieces of the Lie algebra of infinitesimal symmetries of the $G$-structure. The prolongations of subalgebras of $\mathfrak{gl}(V)$ have been much studied in differential geometry. When $\mathfrak{g}$ is reductive, the theory of Lie algebras and their representations is particularly powerful and one can use it to get a good understanding of the prolongations. A fundamental result is the following result stated by E. Cartan [1909], with modern proofs in [Kobayashi and Nagano 1965] and [Singer and Sternberg 1965].

**Theorem 8.5.** *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be an irreducible representation of a Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}^{(2)} = 0$ unless $\mathfrak{g} = \mathfrak{gl}(V)$, $\mathfrak{sl}(V)$, $\mathfrak{csp}(V)$, or $\mathfrak{sp}(V)$, where in the last two cases $V$ is an even-dimensional vector space provided with a symplectic form.*

Now we want to modify these notions on G-structures to cone structures.

**Definition 8.6.** Let $Z \subset \mathbb{P}V$ be a projective subvariety and let $\mathrm{aut}(\hat{Z}) \subset \mathfrak{gl}(V)$ be the Lie algebra of $\mathrm{Aut}(\hat{Z})$. In other words,

$$\mathrm{aut}(\hat{Z}) = \{A \in \mathfrak{gl}(V), \text{ for each smooth point } z \in Z, A(\hat{z}) \subset \hat{T}_z(Z).\}.$$

Let $\mathcal{C}$ be a $Z$-isotrivial cone structure. It is easy to see that a germ of a vector field at a point of $U$ preserves the cone structure if and only if it preserves the induced G-structure. Thus to understand the infinitesimal symmetries of a $Z$-isotrivial cone structure it suffices to understand the prolongations of $\mathrm{aut}(\hat{Z})$. This is true even for non-isotrivial cone structures:

**Proposition 8.7.** *Let $\mathcal{C} \subset \mathbb{P}T(U)$ be a smooth cone structure. For a point $x \in U$, let $\mathfrak{f}$ be the Lie algebra of all germs of holomorphic vector fields at $x$ which preserve $\mathcal{C}$. Let $\mathfrak{f}^k$ be the Lie subalgebra of $\mathfrak{f}$ consisting of vector fields which vanish at $x$ to order $\geq k+1$ for some integer $k \geq -1$. Let $V = T_x(U)$ and $Z \subset \mathbb{P}V$ be the fiber of $\mathcal{C}$ at $x$. For each $k \geq 0$, regard the quotient space $\mathfrak{f}^k/\mathfrak{f}^{k+1}$*

*as a subspace of* $\mathrm{Hom}(S^{k+1}V, V)$ *by taking the leading coefficients of the Taylor expansion of the vector fields at* $x$. *Then*

$$\mathfrak{f}^k/\mathfrak{f}^{k+1} \subset \mathrm{aut}(\hat{Z})^{(k)}.$$

This is checked in [Hwang and Mok 2005, Proposition 1.2.1]. The argument is analogous to the case of G-structures, i.e., Proposition 8.4.

This gives the hope that the theory of G-structures provides an effective tool to study the symmetries of cone structures. However, the classical theory of G-structures is not very powerful in dealing with non-reductive groups and it is more efficient to work with the cone structure directly in many cases. Technically, one has to replace the use of Lie theory by projective geometry of the fibers of the cone structure to investigate the symmetry. One example of this approach is in [HM05] Section 1. In particular, the following generalization of Theorem 8.5 is proved there.

**Theorem 8.8.** *Let* $Z \subset \mathbb{P}V$ *be an irreducible nonsingular nondegenerate subvariety. Then* $\mathrm{aut}(\hat{Z})^{(2)} = 0$ *unless* $Z = \mathbb{P}V$.

The assumption of irreducibility and nondegeneracy in Theorem 8.8 is necessary: just consider a linear subspace $Z \subset \mathbb{P}V$. It is easy to check that $\mathrm{aut}(\hat{Z})^{(m)} \neq 0$ for all $m \geq 0$. Since nonsingularity is also a reasonable condition in view of Problem 4.15, Theorem 8.8 is a fairly satisfactory result.

Theorem 8.8 implies Theorem 8.5. In fact, let $Z \subset \mathbb{P}V$ be the highest weight variety of $\mathfrak{g} \subset \mathfrak{gl}(V)$ in Theorem 8.5 so that $\mathfrak{g} \subset \mathrm{aut}(\hat{Z})$. If $\mathfrak{g}^{(2)} \neq 0$, then $\mathrm{aut}(\hat{Z})^{(2)} \neq 0$ by Lemma 8.3. Since $Z$ is nonsingular and nondegenerate, $Z = \mathbb{P}V$. But it is well-known that an irreducible Lie subalgebra of $\mathfrak{gl}(V)$ whose highest weight variety is $\mathbb{P}V$ is one of the four listed in Theorem 8.5.

The proof of Theorem 8.8 is quite different from the old proofs of Theorem 8.5. Since $\mathrm{aut}(\hat{Z})$ is a priori not reductive, Lie theory is not so helpful in the proof. One has to replace Lie theory by projective geometry of $Z \subset \mathbb{P}V$. The proof in [Hwang and Mok 2005] involves a complicated induction, using the theory of VMRT.

Theorem 8.8 implies, by Proposition 8.4, that the symmetry group of a cone structure is finite dimensional if a fiber of $\mathscr{C} \to U$ is irreducible, nonsingular and nondegenerate with rank $< \dim U$. In fact, the dimension of the group must be bounded by

$$\dim V + \dim \mathfrak{gl}(V) + \dim \mathfrak{gl}(V)^{(1)}.$$

It is natural to ask for the following extension of Theorem 8.8.

**Problem 8.9.** Classify all nonsingular linearly normal subvarieties $Z \subset \mathbb{P}V$ with $\mathrm{aut}(\hat{Z})^{(1)} \neq 0$.

The additional assumption of linear normality, i.e., $H^0(Z, \mathbb{O}(1)) = V^*$, is added to simplify the problem. Under this condition, Theorem 1.1.3 of [Hwang and Mok 2005] says that

$$\dim \mathrm{aut}(Z)^{(1)} \leq \dim V$$

and $Z$ must be a quasi-homogeneous variety. When $Z$ is a homogeneous variety, we have the following classification result of [Kobayashi and Nagano 1964].

**Theorem 8.10.** *Let $Z \subset \mathbb{P}V$ be the highest weight variety of an irreducible representation. Then* $\mathrm{aut}(\hat{Z})^{(1)} \neq 0$ *if and only if $Z$ is the highest weight variety of the isotropy representation of an irreducible Hermitian symmetric space of compact type*, *i.e.*, *$Z \subset \mathbb{P}V$ in Theorem 5.11.*

There are non-homogeneous examples of $Z \subset \mathbb{P}V$ with $\mathrm{aut}(\hat{Z})^{(1)} \neq 0$. See, e.g., [Hwang and Mok 2005, Propositions 4.2.3 and 7.2.3]. Their automorphism groups are not reductive, making it hard to approach Problem 8.9 by Lie theory. It seems to require a good amount of classical projective algebraic geometry.

By Proposition 8.7, we can reformulate Theorem 8.8 in terms of symmetries of cone structures:

**Theorem 8.11.** *Let $\mathscr{C} \subset \mathbb{P}T(U)$ be a cone structure with irreducible nonsingular and nondegenerate fibers. Suppose there exists a nonzero element of $\mathfrak{f}^2$ in the notation of Proposition 8.7 which preserves the cone structure. Then $\mathscr{C} = \mathbb{P}T(U)$.*

When stated this way, further questions arise. It is very natural to replace the nondegeneracy assumption of Theorem 8.11 by a bracket-generating condition. For example one can raise the following question, as a generalization of Theorem 8.11.

**Problem 8.12.** Let $\mathscr{C} \subset \mathbb{P}T(U)$ be a cone structure with irreducible nonsingular fibers such that the distribution $\mathrm{Dist}(\mathscr{C})$ spanned by $\mathscr{C}$ is bracket-generating. Suppose there exists a nonzero element of $\mathfrak{f}^2$ in the notation of Proposition 8.7. What are the possible fibers of $\mathscr{C}$?

There are serious difficulties in generalizing the method of the proof of Theorem 8.11 in the direction of Problem 8.12. The differential geometric problem of Theorem 8.11 has been reduced to purely projective geometric problem of Theorem 8.8 by Proposition 8.7. So far, the differential geometric theory needed to make such a reduction for Problem 8.12 has not been fully developed.

One may wonder why in this section we consider a general cone structure. Maybe by restricting to the more special case of cone structures with characteristic connection or VMRT-structures we can get better results? It is likely that such a restriction gives a non-trivial improvement. This direction has not been pursued so far. I hope that a more refined theory can be developed by such an approach, leading to a better formulation of Problems 8.9 and 8.12.

## References

[Araujo 2006] C. Araujo, “Rational curves of minimal degree and characterizations of projective spaces”, *Math. Ann.* **335**:4 (2006), 937–951.

[Bernstein and Gindikin 2003] J. Bernstein and S. Gindikin, “Notes on integral geometry for manifolds of curves”, pp. 57–80 in *Lie groups and symmetric spaces*, Amer. Math. Soc. Transl. Ser. 2 **210**, Amer. Math. Soc., Providence, RI, 2003.

[Cartan 1909] E. Cartan, “Les groupes de transformations continus, infinis, simples”, *Ann. Sci. École Norm. Sup.* (3) **26** (1909), 93–161.

[Dubrovin et al. 1984] B. A. Dubrovin, A. T. Fomenko, and S. P. Novikov, *Modern Geometry I*, Grad. Texts in Math. **93**, Springer, Berlin, 1984.

[Hwang 2001] J.-M. Hwang, “Geometry of minimal rational curves on Fano manifolds”, pp. 335–393 in *School on Vanishing Theorems and Effective Results in Algebraic Geometry* (Trieste, 2000), ICTP Lect. Notes **6**, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2001.

[Hwang 2007] J.-M. Hwang, “Deformation of holomorphic maps onto Fano manifolds of second and fourth Betti numbers 1”, *Ann. Inst. Fourier* (*Grenoble*) **57**:3 (2007), 815–823.

[Hwang 2010] J.-M. Hwang, “Equivalence problem for minimal rational curves with isotrivial varieties of minimal rational tangents”, *Ann. Sci. Éc. Norm. Supér.* (4) **43**:4 (2010), 607–620.

[Hwang and Mok 1997] J.-M. Hwang and N. Mok, “Uniruled projective manifolds with irreducible reductive $G$-structures”, *J. Reine Angew. Math.* **490** (1997), 55–64.

[Hwang and Mok 1998] J.-M. Hwang and N. Mok, “Rigidity of irreducible Hermitian symmetric spaces of the compact type under Kähler deformation”, *Invent. Math.* **131**:2 (1998), 393–418.

[Hwang and Mok 1999] J.-M. Hwang and N. Mok, “Varieties of minimal rational tangents on uniruled projective manifolds”, pp. 351–389 in *Several complex variables* (Berkeley, 1995–1996), Math. Sci. Res. Inst. Publ. **37**, Cambridge Univ. Press, Cambridge, 1999.

[Hwang and Mok 2001] J.-M. Hwang and N. Mok, “Cartan–Fubini type extension of holomorphic maps for Fano manifolds of Picard number 1”, *J. Math. Pures Appl.* (9) **80**:6 (2001), 563–575.

[Hwang and Mok 2002] J.-M. Hwang and N. Mok, “Deformation rigidity of the rational homogeneous space associated to a long simple root”, *Ann. Sci. École Norm. Sup.* (4) **35**:2 (2002), 173–184.

[Hwang and Mok 2003] J.-M. Hwang and N. Mok, “Finite morphisms onto Fano manifolds of Picard number 1 which have rational curves with trivial normal bundles”, *J. Algebraic Geom.* **12**:4 (2003), 627–651.

[Hwang and Mok 2004] J.-M. Hwang and N. Mok, “Birationality of the tangent map for minimal rational curves”, *Asian J. Math.* **8**:1 (2004), 51–63.

[Hwang and Mok 2005] J.-M. Hwang and N. Mok, “Prolongations of infinitesimal linear automorphisms of projective varieties and rigidity of rational homogeneous spaces of Picard number 1 under Kähler deformation”, *Invent. Math.* **160**:3 (2005), 591–645.

[Hwang and Nakayama 2011] J.-M. Hwang and N. Nakayama, “On endomorphisms of Fano manifolds of Picard number 1”, *Pure Appl. Math. Q.* **7**:4 (2011), 1407–1426.

[Jensen and Musso 1994] G. R. Jensen and E. Musso, “Rigidity of hypersurfaces in complex projective space”, *Ann. Sci. École Norm. Sup.* (4) **27**:2 (1994), 227–248.

[Kebekus 2002] S. Kebekus, "Families of singular rational curves", *J. Algebraic Geom.* **11**:2 (2002), 245–256.

[Kebekus and Kovács 2004] S. Kebekus and S. J. Kovács, "Are rational curves determined by tangent vectors?", *Ann. Inst. Fourier* (*Grenoble*) **54**:1 (2004), 53–79.

[Kobayashi and Nagano 1964] S. Kobayashi and T. Nagano, "On filtered Lie algebras and geometric structures, I", *J. Math. Mech.* **13** (1964), 875–907.

[Kobayashi and Nagano 1965] S. Kobayashi and T. Nagano, "On filtered Lie algebras and geometric structures, III", *J. Math. Mech.* **14** (1965), 679–706.

[Kollár 1996] J. Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **32**, Springer, Berlin, 1996.

[Mok 1988] N. Mok, "The uniformization theorem for compact Kähler manifolds of nonnegative holomorphic bisectional curvature", *J. Differential Geom.* **27**:2 (1988), 179–214.

[Mok 2008a] N. Mok, "Geometric structures on uniruled projective manifolds defined by their varieties of minimal rational tangents", pp. 151–205 in *Géométrie différentielle, physique mathématique, mathématiques et société*, vol. 2, Astérisque **322**, Soc. Math. de France, Paris, 2008.

[Mok 2008b] N. Mok, "Recognizing certain rational homogeneous manifolds of Picard number 1 from their varieties of minimal rational tangents", pp. 41–61 in *Third International Congress of Chinese Mathematicians*, vol. 2, AMS/IP Stud. Adv. Math. **42,** pt. 1, Amer. Math. Soc., Providence, RI, 2008.

[Mukai and Umemura 1983] S. Mukai and H. Umemura, "Minimal rational threefolds", pp. 490–518 in *Algebraic geometry* (Tokyo/Kyoto, 1982), Lecture Notes in Math. **1016**, Springer, Berlin, 1983.

[Novelli and Occhetta 2011] C. Novelli and G. Occhetta, "Projective manifolds containing a large linear subspace with nef normal bundle", *Mich. Math. J.* **60**:2 (2011), 441–462.

[Sasaki 1988] T. Sasaki, "On the projective geometry of hypersurfaces", pp. 115–161 in *Équations différentielles dans le champ complexe* (Strasbourg, 1985), vol. III, Univ. Louis Pasteur, Strasbourg, 1988.

[Schuhmann 1999] C. Schuhmann, "Morphisms between Fano threefolds", *J. Algebraic Geom.* **8**:2 (1999), 221–244.

[Singer and Sternberg 1965] I. M. Singer and S. Sternberg, "The infinite groups of Lie and Cartan, I: The transitive groups", *J. Analyse Math.* **15** (1965), 1–114.

[Sternberg 1983] S. Sternberg, *Lectures on differential geometry*, 2nd ed., Chelsea, New York, 1983.

[Yamaguchi 1982] K. Yamaguchi, "Contact geometry of higher order", *Japan. J. Math.* (*N.S.*) **8**:1 (1982), 109–176.

[Yamaguchi 1993] K. Yamaguchi, "Differential systems associated with simple graded Lie algebras", pp. 413–494 in *Progress in differential geometry*, Adv. Stud. Pure Math. **22**, Math. Soc. Japan, Tokyo, 1993.

jmhwang@kias.re.kr *Korea Institute for Advanced Study, Hoegiro 87, Seoul, 130-722, Korea*