

Higher-dimensional analogues of K3 surfaces

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A Kähler manifold X is *hyperkähler* if it is simply connected and carries a holomorphic symplectic form whose cohomology class spans $H^{2,0}(X)$. A hyperkähler manifold of dimension 2 is a K3 surface. In many respects higher-dimensional hyperkähler manifolds behave like K3 surfaces: they are the higher dimensional analogues of K3 surfaces of the title. In each dimension greater than 2 there is more than one deformation class of hyperkähler manifolds. One deformation class of dimension $2n$ is that of the Hilbert scheme $S^{[n]}$ where S is a K3 surface. We will present a program which aims to prove that a numerical K3^[2] is a deformation of K3^[2] — a numerical K3^[2] is a hyperkähler 4-fold X such that there is an isomorphism of abelian groups $H^2(X; \mathbb{Z}) \xrightarrow{\sim} H^2(\text{K3}^{[2]}; \mathbb{Z})$ compatible with the polynomials given by 4-tuple cup-product.

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0. Introduction

K3 surfaces were known classically as complex smooth projective surfaces whose generic hyperplane section is a canonically embedded curve; an example is provided by a smooth quartic surface in \mathbb{P}^3 . One naturally encounters K3's in the Enriques–Kodaira classification of compact complex surfaces: they are defined to be compact Kähler surfaces with trivial canonical bundle and vanishing first Betti number. Here are a few among the wonderful properties of K3's:

- (1) [Kodaira 1964] Any two K3 surfaces are deformation equivalent — thus they are all deformations of a quartic surface.

- (2) The Kähler cone of a K3 surface X is described as follows. Let $\omega \in H_{\mathbb{R}}^{1,1}(X)$ be one Kähler class and \mathcal{N}_X be the set of nodal classes

$$\mathcal{N}_X := \{\alpha \in H_{\mathbb{Z}}^{1,1}(X) \mid \alpha \cdot \alpha = -2, \alpha \cdot \omega > 0\}. \quad (0.0.1)$$

The Kähler cone \mathcal{K}_X is given by

$$\mathcal{K}_X := \{\alpha \in H_{\mathbb{R}}^{1,1}(X) \mid \alpha \cdot \alpha > 0 \text{ and } \alpha \cdot \beta > 0 \text{ for all } \beta \in \mathcal{N}_X\}. \quad (0.0.2)$$

- (3) [Piatetski-Shapiro and Shafarevich 1971; Burns and Rapoport 1975; Looijenga and Peters 1980/81] Weak and strong global Torelli hold. The weak version states that two K3 surfaces X, Y are isomorphic if and only if there exists an integral isomorphism of Hodge structures $f : H^2(X) \xrightarrow{\sim} H^2(Y)$ which is an isometry (with respect to the intersection forms), the strong version states that f is induced by an isomorphism $\phi : Y \xrightarrow{\sim} X$ if and only if it maps effective divisors to effective divisors.¹

The higher-dimensional complex manifolds closest to K3 surfaces are *hyperkähler manifolds* (HK); they are defined to be simply connected Kähler manifolds with $H^{2,0}$ spanned by the class of a holomorphic *symplectic* form. The terminology originates from riemannian geometry: Yau's solution of Calabi's conjecture gives that every Kähler class ω on a HK manifold contains a Kähler metric g with holonomy the compact symplectic group. There is a sphere S^2 (the pure quaternions of norm 1) parametrizing complex structures for which g is a Kähler metric — the *twistor family* associated to g ; it plays a key role in the general theory of HK manifolds.² Notice that a HK manifold has trivial canonical bundle and is of even dimension. An example of Beauville [1983] is the Douady space $S^{[n]}$ parametrizing length- n analytic subsets of a K3 surface S — it has dimension $2n$. (Of course $S^{[n]}$ is a Hilbert scheme if S is projective.) We mention right away two results which suggest that HK manifolds might behave like K3's. Let X be HK:

- (a) By a theorem of Bogomolov [1978] deformations of X are unobstructed;³ that is, the deformation space $\text{Def}(X)$ is smooth of the expected dimension $H^1(T_X)$.
- (b) Since the sheaf map $T_X \rightarrow \Omega_X^1$ given by contraction with a holomorphic symplectic form is an isomorphism it follows that the differential of the

¹Effective divisors have a purely Hodge-theoretic description once we have located one Kähler class.

²Hyperkähler manifolds are also known as *irreducible symplectic*.

³The obstruction space $H^2(T_X)$ might be nonzero — for example, if X is a generalized Kummer. See Section 1.1.

weight-2 period map

$$H^1(T_X) \longrightarrow \text{Hom}(H^{2,0}(X), H^{1,1}(X)) \tag{0.0.3}$$

is injective, i.e., infinitesimal Torelli holds.

Assuming (a) we may prove that the generic deformation of X has $h_{\mathbb{Z}}^{1,1} = 0$ arguing as follows. A given $\alpha \in H^1(\Omega_X^1)$ remains of type $(1, 1)$ to first order in the direction determined by $\kappa \in H^1(T_X)$ if and only if $\text{Tr}(\kappa \cup \alpha) = 0$ (Griffiths). On the other hand if $\alpha \neq 0$ the map

$$H^1(T_X) \rightarrow H^2(\mathbb{O}_X), \quad \kappa \mapsto \text{Tr}(\kappa \cup \alpha) \tag{0.0.4}$$

is surjective by Serre duality; it follows that α does not remain of type $(1, 1)$ on a generic deformation X_t of X (of course what we denote by α is actually the class $\alpha_t \in H^2(X_t)$ obtained from α by Gauss–Manin parallel transport). Item (b) above suggests that the weight-2 Hodge structure of X might capture much of the geometry of X . One is naturally led to ask whether analogues of properties (1)–(3) above hold for higher-dimensional HK manifolds. Let us first discuss (1).

In each (even) dimension greater than 2 we know of two distinct deformation classes of HK manifolds, with one extra deformation class in dimensions 6 and 10. The known examples are distinguished up to deformation by the isomorphism class of their integral weight-2 cohomology group equipped with the top cup-product form — we might name these, together with the dimension, the *basic discrete data* of a HK manifold. Huybrechts [2003b] has shown that the set of deformation classes of HK’s with assigned discrete data is finite. In this paper we will present a program which aims to prove that a HK whose discrete data are isomorphic to those of $\text{K3}^{[2]}$ is in fact a deformation of $\text{K3}^{[2]}$. For the reader’s convenience we spell out the meaning of the previous sentence. A *numerical* $(\text{K3})^{[2]}$ is a HK 4-fold X such that there exists an isomorphism of abelian groups $\psi : H^2(X; \mathbb{Z}) \xrightarrow{\sim} H^2(S^{[2]}; \mathbb{Z})$ (here S is a K3) for which

$$\int_X \alpha^4 = \int_{S^{[2]}} \psi(\alpha)^4 \quad \text{for all } \alpha \in H^2(X; \mathbb{Z}). \tag{0.0.5}$$

Our program aims to prove that a numerical $\text{K3}^{[2]}$ is a deformation of $\text{K3}^{[2]}$. What about analogues of properties (2) and (3) above? We start with (3), global Torelli. On the weight-2 cohomology of a HK there is a natural quadratic form (named after Beauville and Bogomolov) and hence one may formulate a statement — call it *naive Torelli* — analogous to the weak global Torelli statement. The key claim in such a naive Torelli is that if two HK’s have Hodge-isometric H^2 ’s then they are bimeromorphic (one cannot require that they be isomorphic; see [Debarre 1984]). However it has been known for some time [Namikawa 2002; Markman 2010] that naive Torelli is false for HK’s belonging to certain

deformation classes. Recently Verbitsky [2009] proposed a proof of a suitable version of global Torelli valid for arbitrary HK's (see also [Huybrechts 2011]); that result together with Markman's monodromy computations [2010] implies that naive Torelli holds for deformations of $K3^{[p^k+1]}$ where p is a prime. To sum up: an appropriate version of global Torelli holds for any deformation class of HK's. Regarding item (2): Huybrechts [2003c] and Boucksom [2001] have given a description of the Kähler cone in terms of intersections with rational curves (meaning curves with vanishing geometric genus), but that is not a purely Hodge-theoretic description. Hassett and Tschinkel [2001] have formulated a conjectural Hodge-theoretic description of the ample cone of a deformation of $K3^{[2]}$ and they have proved that the divisors satisfying their criterion are indeed ample.

The paper is organized as follows. Following a brief section devoted to the known examples of HK's we introduce basic results on topology and the Kähler cone of a HK in Section 2. After that we will present examples of *explicit* locally complete families of *projective* higher-dimensional HK's. These are analogues of the explicit families of projective K3's such as double covers of \mathbb{P}^2 branched over a sextic curve, quartic surfaces in \mathbb{P}^3 etc. (The list goes on for quite a few values of the degree, thanks to Mukai, but there are theoretical reasons [Gritsenko et al. 2007] why it should stop before degree 80, more or less.) In particular we will introduce double EPW-sextics, these are double covers of special sextic hypersurfaces in \mathbb{P}^5 ; they play a key rôle in our program for proving that a numerical $K3^{[2]}$ is a deformation of $K3^{[2]}$. The last section is devoted to that program: we discuss what has been proved and what is left to be proved.

1. Examples

The surprising topological properties of HK manifolds (see Section 2.1) led Bogomolov [1978] to state erroneously that no higher-dimensional ($\dim > 2$) HK exists. Some time later Fujiki [1983] realized that $K3^{[2]}$ is a higher-dimensional HK manifold.⁴ Beauville [1983] then showed that $K3^{[n]}$ is a HK manifold; moreover by constructing generalized Kummers he exhibited another deformation class of HK manifolds in each even dimension greater than 2. In [O'Grady 1999; 2003] we exhibited two "sporadic" deformation classes, one in dimension 6 the other in dimension 10. No other deformation classes are known other than those mentioned above.

1.1. Beauville. Beauville discovered another class of $2n$ -dimensional HK manifolds besides $(K3)^{[n]}$: generalized Kummers associated to a 2-dimensional

⁴Fujiki described $K3^{[2]}$ not as a Douady space but as the blow-up of the diagonal in the symmetric square of a K3 surface.

compact complex torus. Before defining generalized Kummers we recall that the Douady space $W^{[n]}$ comes with a cycle (Hilbert–Chow) map

$$W^{[n]} \xrightarrow{\kappa_n} W^{(n)}, \quad [Z] \mapsto \sum_{p \in W} \ell(\mathbb{O}_{Z,p})p, \tag{1.1.1}$$

where $W^{(n)}$ is the symmetric product of W . Now suppose that T is a 2-dimensional compact complex torus. We have the summation map $\sigma_n : W^{(n)} \rightarrow W$. Composing the two maps above (with $n+1$ replacing n) we get a locally (in the classical topology) trivial fibration $\sigma_{n+1} \circ \kappa_{n+1} : W^{[n+1]} \rightarrow W$. The $2n$ -dimensional *generalized Kummer* associated to T is

$$K^{[n]}T := (\sigma_{n+1} \circ \kappa_{n+1})^{-1}(0). \tag{1.1.2}$$

The name is justified by the observation that if $n = 1$ then $K^{[1]}T$ is the Kummer surface associated to T (and hence a K3). Beauville [1983] proved that $K^{[n]}(T)$ is a HK manifold. Moreover if $n \geq 2$ then

$$b_2((K3)^{[n]}) = 23, \quad b_2(K^{[n]}T) = 7. \tag{1.1.3}$$

In particular $(K3)^{[n]}$ and $K^{[n]}T$ are not deformation equivalent as soon as $n \geq 2$. The second cohomology of these manifolds is described as follows. Let W be a compact complex surface. There is a “symmetrization map”

$$\mu_n : H^2(W; \mathbb{Z}) \longrightarrow H^2(W^{(n)}; \mathbb{Z}) \tag{1.1.4}$$

characterized by the following property. Let $\rho_n : W^n \rightarrow W^{(n)}$ be the quotient map and $\pi_i : W^n \rightarrow W$ be the i -th projection: then

$$\rho_n^* \circ \mu_n(\alpha) = \sum_{i=1}^n \pi_i^* \alpha, \quad \alpha \in H^2(W; \mathbb{Z}). \tag{1.1.5}$$

Composing with κ_n^* and extending scalars one gets an injection of integral Hodge structures

$$\tilde{\mu}_n := \kappa_n^* \circ \mu_n : H^2(W; \mathbb{C}) \longrightarrow H^2(W^{[n]}; \mathbb{C}). \tag{1.1.6}$$

This map is not surjective unless $n = 1$; we are missing the Poincaré dual of the exceptional set of κ_n , that is,

$$\Delta_n := \{[Z] \in W^{[n]} \mid Z \text{ is nonreduced}\}. \tag{1.1.7}$$

It is known that Δ_n is a prime divisor and that it is divisible⁵ by 2 in $\text{Pic}(W^{[n]})$:

$$\mathbb{O}_{W^{[n]}(\Delta_n)} \cong L_n^{\otimes 2}, \quad L_n \in \text{Pic}(W^{[n]}). \tag{1.1.8}$$

⁵If $n = 2$ Equation (1.1.8) follows from existence of the double cover $Bl_{\text{diag}}(S^2) \rightarrow S^{[2]}$ ramified over Δ_2 .

Let $\xi_n := c_1(L_n)$; one has

$$H^2(W^{[n]}; \mathbb{Z}) = \tilde{\mu}_n H^2(W; \mathbb{Z}) \oplus \mathbb{Z}\xi_n \quad \text{if } H_1(W) = 0. \quad (1.1.9)$$

That describes $H^2((K3)^{[n]})$. Beauville proved that an analogous result holds for generalized Kummer, namely we have an isomorphism

$$H^2(T; \mathbb{Z}) \oplus \mathbb{Z} \xrightarrow{\sim} H^2(K^{[n]}T; \mathbb{Z}), \quad (\alpha, k) \mapsto (\tilde{\mu}_{n+1}(\alpha) + k\xi_{n+1})|_{K^{[n]}T}. \quad (1.1.10)$$

This description of the H^2 gives the following interesting result: if $n \geq 2$ the generic deformation of $S^{[n]}$ where S is a K3 is not isomorphic to $T^{[n]}$ for some other K3 surface T . In fact every deformation of $S^{[n]}$ obtained by deforming S keeps ξ_n of type $(1, 1)$, while, as noticed previously, the generic deformation of a HK manifold has no nontrivial integral $(1, 1)$ -classes. (Notice that if S is a surface of general type then every deformation of $S^{[n]}$ is indeed obtained by deforming S , see [Fantechi 1995].)

1.2. Mukai and beyond. Mukai [1984; 1987b; 1987a] and Tyurin [1987] analyzed moduli spaces of semistable sheaves on projective K3's and abelian surfaces and obtained other examples of HK manifolds. Let S be a projective K3 and \mathcal{M} the moduli space of $\mathcal{O}_S(1)$ -semistable pure sheaves on S with assigned Chern character — by results of Gieseker, Maruyama, and Simpson, \mathcal{M} has a natural structure of projective scheme. A nonzero canonical form on S induces a holomorphic symplectic 2-form on the open $\mathcal{M}^s \subset \mathcal{M}$ parametrizing stable sheaves (notice that \mathcal{M}^s is smooth; see [Mukai 1984]). If $\mathcal{M}^s = \mathcal{M}$ then \mathcal{M} is a HK variety;⁶ in general it is not isomorphic (nor birational) to $(K3)^{[n]}$, but it can be deformed to $(K3)^{[n]}$ (here $2n = \dim \mathcal{M}$). See [Göttsche and Huybrechts 1996; O'Grady 1997; Yoshioka 1999]. Notice that $S^{[n]}$ may be viewed as a particular case of Mukai's construction by identifying it with the moduli space of rank-1 semistable sheaves on S with $c_1 = 0$ and $c_2 = n$. Notice also that these moduli spaces give explicit deformations of $(K3)^{[n]}$ which are not $(K3)^{[n]}$. Similarly one may consider moduli spaces of semistable sheaves on an abelian surface A : in the case when $\mathcal{M} = \mathcal{M}^s$ one gets deformations of the generalized Kummer. To be precise, it is not \mathcal{M} which is a deformation of a generalized Kummer but rather one of its Beauville–Bogomolov factors. Explicitly we consider the map

$$\mathcal{M}(A) \xrightarrow{\mathfrak{a}} A \times \widehat{A}, \quad [F] \mapsto (\text{alb}(c_2(F) - c_2(F_0)), [\det F \otimes (\det F_0)^{-1}]), \quad (1.2.1)$$

where $[F_0] \in \mathcal{M}$ is a “reference” point and $\text{alb} : CH_0^{\text{hom}}(A) \rightarrow A$ is the Albanese map. Then \mathfrak{a} is a locally (classical topology) trivial fibration; Yoshioka [2001] proved that the fibers of \mathfrak{a} are deformations of a generalized Kummer. What can we say about moduli spaces such that $\mathcal{M} \neq \mathcal{M}^s$? The locus $(\mathcal{M} \setminus \mathcal{M}^s)$ parametrizing

⁶A HK variety is a projective HK manifold.

S -equivalence classes of semistable nonstable sheaves is the singular locus of \mathcal{M} except for pathological choices of Chern character which do not give anything particularly interesting; thus we assume that $(\mathcal{M} \setminus \mathcal{M}^s)$ is the singular locus of \mathcal{M} . A natural question is the following: does there exist a crepant desingularization $\tilde{\mathcal{M}} \rightarrow \mathcal{M}$? We constructed such a desingularization in [O’Grady 1999; 2003] (see also [Lehn and Sorger 2006]) for the moduli space $\mathcal{M}_4(S)$ of semistable rank-2 sheaves on a K3 surface S with $c_1 = 0$ and $c_2 = 4$ and for the moduli space $\mathcal{M}_2(A)$ of semistable sheaves on an abelian surface A with $c_1 = 0$ and $c_2 = 2$; the singularities of the moduli spaces are the same in both cases and both moduli spaces have dimension 10. Let M_{10} be our desingularization of $\mathcal{M}_4(S)$ where S is a K3. Since the resolution is crepant Mukai’s holomorphic symplectic form on $(\mathcal{M}(S) \setminus \mathcal{M}(S)^s)$ extends to a holomorphic symplectic form on M_{10} . We proved in [O’Grady 1999] that M_{10} is HK; that is, it is simply connected and $h^{2,0}(M_{10}) = 1$. Moreover M_{10} is not a deformation of one of Beauville’s examples because $b_2(M_{10}) = 24$. (We proved that $b_2(M_{10}) \geq 24$; later Rapagnetta [2008] proved that equality holds.) Next let A be an abelian surface and $\tilde{\mathcal{M}}_2(A) \rightarrow \mathcal{M}_2(A)$ be our desingularization. Composing the map (1.2.1) for $\mathcal{M}(A) = \mathcal{M}_2(A)$ with the desingularization map we get a locally (in the classical topology) trivial fibration $\tilde{\alpha}: \tilde{\mathcal{M}}_2(A) \rightarrow A \times \hat{A}$; let M_6 be any fiber of $\tilde{\alpha}$. We proved in [O’Grady 1999] that M_6 is HK and that $b_2(M_6) = 8$; thus M_6 is not a deformation of one of Beauville’s examples. We point out that while all Betti and Hodge numbers of Beauville’s examples are known [Göttsche 1994] the same is not true of our examples (Rapagnetta [2007] computed the Euler characteristic of M_6). Of course there are examples of moduli spaces \mathcal{M} with $\mathcal{M} \neq \mathcal{M}^s$ in any even dimension; one would like to desingularize them and produce many more deformation classes of HK manifolds. Kaledin, Lehn, and Sorger [Kaledin et al. 2006] have determined exactly when the moduli space has a crepant desingularization. Combining their results with those of [Perego and Rapagnetta 2010] one gets that if there is a crepant desingularization then it is a deformation of M_{10} if the surface is a K3, while in the case of an abelian surface the fibers of map (1.2.1) composed with the desingularization map are deformations of M_6 .⁷ In fact all known examples of HK manifolds are deformations either of Beauville’s examples or of ours.

1.3. Mukai flops. Let X be a HK manifold of dimension $2n$ containing a submanifold Z isomorphic to \mathbb{P}^n . The *Mukai flop* of Z (introduced in [Mukai 1984]) is a bimeromorphic map $X \dashrightarrow X^\vee$ which is an isomorphism away from Z and

⁷To be precise, their result holds if the polarization of the surface is “generic” relative to the chosen Chern character; with this hypothesis the singular locus of \mathcal{M} is, so to speak, as small as possible

replaces Z by the dual plane $Z^\vee := (\mathbb{P}^n)^\vee$. Explicitly let $\tilde{\tau} : \tilde{X} \rightarrow X$ be the blow-up of Z and $E \subset \tilde{X}$ be the exceptional divisor. Since Z is Lagrangian the symplectic form on X defines an isomorphism $N_{Z/X} \cong \Omega_Z = \Omega_{\mathbb{P}^n}$. Thus

$$E \cong \mathbb{P}(N_{Z/X}) = \mathbb{P}(\Omega_{\mathbb{P}^n}) \subset \mathbb{P}^n \times (\mathbb{P}^n)^\vee. \tag{1.3.1}$$

Hence E is a \mathbb{P}^{n-1} -fibration in two different ways: we have $\pi : E \rightarrow \mathbb{P}^n$, i.e., the restriction of τ to E and $\rho : E \rightarrow (\mathbb{P}^n)^\vee$. A straightforward computation shows that the restriction of $N_{E/\tilde{X}}$ to a fiber of ρ is $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$. By the Fujiki–Nakano contractibility criterion there exists a proper map $\tau^\vee : \tilde{X} \rightarrow X^\vee$ to a complex manifold X^\vee which is an isomorphism outside E and which restricts to ρ on E . Clearly $\tau^\vee(E)$ is naturally identified with Z^\vee and we have a bimeromorphic map $X \dashrightarrow X^\vee$ which defines an isomorphism $(X \setminus Z) \xrightarrow{\sim} (X^\vee \setminus Z^\vee)$. Summarizing, we have the commutative diagram

$$\begin{array}{ccc}
 & \widehat{X} & \\
 \tau \swarrow & & \searrow \tau^\vee \\
 X & \dashrightarrow & X^\vee \\
 c \searrow & & \swarrow c^\vee \\
 & W &
 \end{array} \tag{1.3.2}$$

where $c : X \rightarrow W$ and $c^\vee : X^\vee \rightarrow W$ are the contractions of Z and Z^\vee respectively—see the Introduction of [Wierzba and Wiśniewski 2003]. It follows that X^\vee is simply connected and a holomorphic symplectic form on X gives a holomorphic symplectic form on X^\vee spanning $H^0(\Omega_{X^\vee}^2)$; thus X^\vee is HK if it is Kähler. We give an example with X and X^\vee projective. Let $f : S \rightarrow \mathbb{P}^2$ be a double cover branched over a smooth sextic and $\mathcal{O}_S(1) := f^*\mathcal{O}_{\mathbb{P}^2}(1)$; thus S is a K3 of degree 2. Let $X := S^{[2]}$ and \mathcal{M} be the moduli space of pure 1-dimensional $\mathcal{O}_S(1)$ -semistable sheaves on S with typical member $\iota_*\mathcal{L}$ where $\iota : C \hookrightarrow S$ is the inclusion of $C \in |\mathcal{O}_S(1)|$ and \mathcal{L} is a line bundle on C of degree 2. We have a natural rational map

$$\phi : S^{[2]} \dashrightarrow \mathcal{M} \tag{1.3.3}$$

which associates to $[W] \in S^{[2]}$ the sheaf $\iota_*\mathcal{L}$ where C is the unique curve containing W (uniqueness requires W to be generic!) and $\mathcal{L} := \mathcal{O}_C(W)$. If every divisor in $|\mathcal{O}_S(1)|$ is prime (i.e., the branch curve of f has no tritangents) then \mathcal{M} is smooth (projective) and the rational map ϕ is identified with the flop of

$$Z := \{f^{-1}(p) \mid p \in \mathbb{P}^2\}. \tag{1.3.4}$$

Wierzba and Wiśniewsky [2003] have proved that any birational map between HK four-folds is a composition of Mukai flops. In higher dimensions Mukai [1984] defined more general flops in which the indeterminacy locus is a fibration in projective spaces. Markman [2001] constructed *stratified Mukai flops*.

2. General theory

It is fair to state that there are three main ingredients in the general theory of HK manifolds developed by Bogomolov, Beauville, Fujiki, Huybrechts and others:

- (1) Deformations are unobstructed (Bogomolov’s Theorem).
- (2) The canonical Bogomolov–Beauville quadratic form on H^2 of a HK manifold (see the next subsection).
- (3) Existence of the twistor family on a HK manifold equipped with a Kähler class: this is a consequence of Yau’s solution of Calabi’s conjecture.

2.1. Topology. Let X be a HK-manifold of dimension $2n$. Beauville [1983] and Fujiki [1987] proved that there exist an integral indivisible quadratic form

$$q_X : H^2(X) \rightarrow \mathbb{C} \tag{2.1.1}$$

(cohomology is with complex coefficients) and $c_X \in \mathbb{Q}_+$ such that

$$\int_X \alpha^{2n} = c_X \frac{(2n)!}{n!2^n} q_X(\alpha)^n, \quad \alpha \in H^2(X). \tag{2.1.2}$$

This equation determines c_X and q_X with no ambiguity unless n is even. If n is even then q_X is determined up to ± 1 : one singles out one of the two choices by imposing the inequality $q_X(\sigma + \bar{\sigma}) > 0$ for σ a holomorphic symplectic form. The *Beauville–Bogomolov* form and the *Fujiki constant* of X are q_X and c_X respectively. We note that the equation in (2.1.2) is equivalent (by polarization) to

$$\int_X \alpha_1 \wedge \cdots \wedge \alpha_{2n} = c_X \sum_{\sigma \in \mathcal{R}_{2n}} (\alpha_{\sigma(1)}, \alpha_{\sigma(2)})_X \cdot (\alpha_{\sigma(3)}, \alpha_{\sigma(4)})_X \cdots (\alpha_{\sigma(2n-1)}, \alpha_{\sigma(2n)})_X, \tag{2.1.3}$$

where $(\cdot, \cdot)_X$ is the symmetric bilinear form associated to q_X and \mathcal{R}_{2n} is a set of representatives for the left cosets of the subgroup $\mathcal{G}_{2n} < \mathcal{S}_{2n}$ of permutations of $\{1, \dots, 2n\}$ generated by transpositions $(2i - 1, 2i)$ and by products of transpositions $(2i - 1, 2j - 1)(2i, 2j)$ — in other words in the right-hand side of (2.1.3) we avoid repeating addends which are equal.⁸ The existence of q_X, c_X is by no means trivial; we sketch a proof. Let $f : \mathcal{X} \rightarrow T$ be a deformation of X representing $\text{Def}(X)$; more precisely letting $X_t := f^{-1}\{t\}$ for $t \in T$, we are given $0 \in T$, an isomorphism $X_0 \xrightarrow{\sim} X$ and the induced map of germs $(T, 0) \rightarrow \text{Def}(X)$ is an isomorphism. In particular T is smooth in 0 and hence we may assume that it is a polydisk. The Gauss–Manin connection defines an

⁸In defining c_X we have introduced a normalization which is not standard in order to avoid a combinatorial factor in (2.1.3).

integral isomorphism $\phi_t : H^2(X) \xrightarrow{\sim} H^2(X_t)$. The *local period map* of X is given by

$$T \xrightarrow{\pi} \mathbb{P}(H^2(X)), \quad t \mapsto \phi_t^{-1} H^{2,0}(X_t). \tag{2.1.4}$$

By infinitesimal Torelli — see (0.0.3) — $\text{Im } \pi$ is an analytic hypersurface in an open (classical topology) neighborhood of $\pi(0)$ and hence its Zariski closure $V = \overline{\text{Im } \pi}$ is either all of $\mathbb{P}(H^2(X))$ or a hypersurface. One shows that the latter holds by considering the (nonzero) degree- $2n$ homogeneous polynomial

$$H^2(X) \xrightarrow{G} \mathbb{C}, \quad \alpha \mapsto \int_X \alpha^{2n}. \tag{2.1.5}$$

In fact if $\sigma_t \in H^{2,0}(X_t)$ then

$$\int_{X_t} \sigma_t^{2n} = 0 \tag{2.1.6}$$

by type consideration and it follows by Gauss–Manin parallel transport that G vanishes on V . Thus $I(V) = (F)$ where F is an irreducible homogeneous polynomial. By considering the derivative of the period map (0.0.3) one checks easily that V is not a hyperplane and hence $\text{deg } F \geq 2$. On the other hand type consideration gives something stronger than (2.1.6), namely

$$\int_{X_t} \sigma_t^{n+1} \wedge \alpha_1 \cdots \wedge \alpha_{n-1} = 0, \quad \alpha_1, \dots, \alpha_{n-1} \in H^2(X_t). \tag{2.1.7}$$

It follows that all the derivatives of G up to order $(n - 1)$ included vanish on V . Since $\text{deg } G = 2n$ and $\text{deg } F \geq 2$ it follows that $G = c \cdot F^n$ and $\text{deg } F = 2$. By integrality of G there exists $\lambda \in \mathbb{C}^*$ such that $c_X := \lambda c$ is rational positive, $q_X := \lambda \cdot F$ is integral indivisible and (2.1.2) is satisfied.

Of course if X is a K3 then q_X is the intersection form of X (and $c_X = 1$). In general q_X gives $H^2(X; \mathbb{Z})$ a structure of lattice just as in the well-known case of K3 surfaces. Suppose that X and Y are deformation equivalent HK-manifolds: it follows from (2.1.2) that $c_X = c_Y$ and the lattices $H^2(X; \mathbb{Z}), H^2(Y; \mathbb{Z})$ are isometric (see the comment following (2.1.2) if n is even). Consider the case when $X = (\text{K3})^{[n]}$; then $\tilde{\mu}_n$ is an isometry, $\xi_n \perp \text{Im } \tilde{\mu}_n$ and $q_X(\xi_n) = -2(n - 1)$, i.e.,

$$H^2(S^{[n]}; \mathbb{Z}) \cong U^3 \hat{\oplus} E_8 \langle -1 \rangle^2 \hat{\oplus} \langle -2(n - 1) \rangle \tag{2.1.8}$$

where $\hat{\oplus}$ denotes orthogonal direct sum, U is the hyperbolic plane and $E_8 \langle -1 \rangle$ is the unique rank-8 negative definite unimodular even lattice. Moreover the Fujiki constant is

$$c_{S^{[n]}} = 1. \tag{2.1.9}$$

In [Rapagnetta 2008] the reader will find the B-B quadratic form and Fujiki constant of the other known deformation classes of HK manifolds.

Remark 2.1. Let X be a HK manifold of dimension $2n$ and $\omega \in H_{\mathbb{R}}^{1,1}(X)$ be a Kähler class.

(1) Equation (2.1.2) gives that, with respect to $(\cdot, \cdot)_X$

$$H^{p,q}(X) \perp H^{p',q'}(X) \quad \text{unless } (p', q') = (2 - p, 2 - q). \tag{2.1.10}$$

(2) $q_X(\omega) > 0$. In fact let σ be generator of $H^{2,0}(X)$; by (2.1.3) and item (1) above we have

$$0 < \int_X \sigma^{n-1} \wedge \bar{\sigma}^{n-1} \wedge \omega^2 = c_X(n-1)! (\sigma, \bar{\sigma})_X q_X(\omega). \tag{2.1.11}$$

Since $c_X > 0$ and $(\sigma, \bar{\sigma})_X > 0$ we get $q_X(\omega) > 0$ as claimed.

(3) The index of q_X is $(3, b_2(X) - 3)$ (i.e., that is the index of its restriction to $H^2(X; \mathbb{R})$). In fact applying (2.1.3) to $\alpha_1 = \dots = \alpha_{2n-1} = \omega$ and arbitrary α_{2n} we get that ω^\perp is equal to the primitive cohomology $H_{pr}^2(X)$ (primitive with respect to ω). On the other hand (2.1.3) with $\alpha_1 = \dots = \alpha_{2n-2} = \omega$ and $\alpha_{2n-1}, \alpha_{2n} \in \omega^\perp$ gives that a positive multiple of $q_X|_{\omega^\perp}$ is equal to the standard quadratic form on $H_{pr}^2(X)$. By the Hodge index Theorem it follows that the restriction of q_X to $\omega^\perp \cap H^2(X; \mathbb{R})$ has index $(2, b_2(X) - 3)$. Since $q_X(\omega) > 0$ it follows that q_X has index $(3, b_2(X) - 3)$.

(4) Let D be an effective divisor on X ; then $(\omega, D)_X > 0$. (Of course $(\omega, D)_X$ denotes $(\omega, c_1(\mathbb{O}_X(D)))_X$.) In fact the inequality follows from the inequality $\int_D \omega^{2n-1} > 0$ together with (2.1.3) and item (2) above.

(5) Let $f : X \dashrightarrow Y$ be a birational map where Y is a HK manifold. Since X and Y have trivial canonical bundle f defines an isomorphism $U \xrightarrow{\sim} V$ where $U \subset X$ and $V \subset Y$ are open sets with complements of codimension at least 2. It follows that f induces an isomorphism $f^* : H^2(Y; \mathbb{Z}) \xrightarrow{\sim} H^2(X; \mathbb{Z})$; f^* is an isometry of lattices, see Lemma 2.6 of [Huybrechts 1999].

The proof of existence of q_X and c_X may be adapted to prove the following useful generalization of (2.1.2).

Proposition 2.2. *Let X be a HK manifold of dimension $2n$. Let $\mathcal{X} \rightarrow T$ be a representative of the deformation space of X . Suppose that*

$$\gamma \in H_{\mathbb{R}}^{p,p}(X)$$

is a nonzero class which remains of type (p, p) under Gauss–Manin parallel transport (such as the Chern class $c_p(X)$). Then p is even and moreover there exists $c_\gamma \in \mathbb{R}$ such that

$$\int_X \gamma \wedge \alpha^{2n-p} = c_\gamma q_X(\alpha)^{n-p/2}. \tag{2.1.12}$$

Our next topic is Verbitsky's theorem. Let X be a HK-manifold of dimension $2n$. Our sketch proof of (2.1.2) shows that

$$\alpha \in H^2(X) \text{ and } q_X(\alpha) = 0 \implies \alpha^{n+1} = 0 \text{ in } H^{2n+2}(X). \tag{2.1.13}$$

In fact, using the notation in the proof of (2.1.2), we have $0 = \sigma_t^{n+1} \in H^{2n+2}(X_t)$; hence, by Gauss–Manin transport we get $0 = (\psi_t^{-1}\sigma_t)^{n+1} \in H^{2n+2}(X)$. Since the set $\{\psi_t^{-1}\sigma_t \mid t \in T\}$ is Zariski dense in the zero-set $V(q_X) \subset H^2(X)$ we get (2.1.13). Let $I \subset \text{Sym}^\bullet H^2(X)$ be the ideal generated by α^{n+1} where $\alpha \in H^2(X)$ and $q_X(\alpha) = 0$:

$$I := \langle \{\alpha^{n+1} \mid \alpha \in H^2(X), \quad q_X(\alpha) = 0\} \rangle. \tag{2.1.14}$$

By (2.1.13) we have a natural map of \mathbb{C} -algebras

$$\text{Sym}^\bullet H^2(X)/I \longrightarrow H^\bullet(X). \tag{2.1.15}$$

Theorem 2.3 [Verbitsky 1996] (see also [Bogomolov 1996]). *The map (2.1.15) is injective.*

In particular we get that cup-product defines an injection

$$\bigoplus_{q=0}^n \text{Sym}^q H^2(X) \hookrightarrow H^\bullet(X). \tag{2.1.16}$$

S. M. Salamon proved that there is a nontrivial linear constraint on the Betti numbers of a compact Kähler manifold carrying a holomorphic symplectic form (for example a HK manifold); the proof consists in a clever application of the Hirzebruch–Riemann–Roch formula to the sheaves Ω_X^p and the observation that the symplectic form induces an isomorphism $\Omega_X^p \cong \Omega_X^{2n-p}$ where $2n = \dim X$.⁹

Theorem 2.4 [Salamon 1996]. *Let X be a compact Kähler manifold of dimension $2n$ carrying a holomorphic symplectic form. Then*

$$nb_{2n}(X) = 2 \sum_{i=1}^{2n} (-1)^i (3i^2 - n)b_{2n-i}(X). \tag{2.1.17}$$

The following corollary of Verbitsky's and Salamon's results was obtained by Beauville (unpublished) and Guan [2001].

Corollary 2.5 (Beauville and Guan). *Let X be a HK 4-fold. Then $b_2(X) \leq 23$. If equality holds then $b_3(X) = 0$ and moreover the map*

$$\text{Sym}^2 H^2(X; \mathbb{Q}) \longrightarrow H^4(X; \mathbb{Q}) \tag{2.1.18}$$

⁹A nonzero section of the canonical bundle defines an isomorphism $\Omega_X^{2n-p} \cong (\Omega_X^p)^\vee = \wedge^p T_X$ and the symplectic form defines an isomorphism $T_X \cong \Omega_X$ and hence $\wedge^p T_X \cong \Omega_X^p$.

induced by cup-product is an isomorphism.

Proof. Let $b_i := b_i(X)$. Salamon’s equation (2.1.17) for X reads

$$b_4 = 46 + 10b_2 - b_3. \tag{2.1.19}$$

By Verbitsky’s theorem — see (2.1.16) — we have

$$\binom{b_2 + 1}{2} \leq b_4. \tag{2.1.20}$$

Replacing b_4 by the right-hand side of (2.1.19) we get that

$$b_2^2 + b_2 \leq 92 + 20b_2 - 2b_3 \leq 92 + 20b_2. \tag{2.1.21}$$

It follows that $b_2 \leq 23$ and that if equality holds then $b_3 = 0$. Suppose that $b_2 = 23$: then $b_4 = 276$ by (2.1.19) and hence (2.1.18) follows from Verbitsky’s theorem (Theorem 2.3). \square

Guan [2001] has obtained other restrictions on $b_2(X)$ for a HK four-fold X : for example, $8 < b_2(X) < 23$ is “forbidden”.

2.2. The Kähler cone. Let X be a HK manifold of dimension $2n$. The convex cone $\mathcal{K}_X \subset H_{\mathbb{R}}^{1,1}(X)$ of Kähler classes is the *Kähler cone of X* . The inequality in (2.1.2) together with Remark 2.1(3) gives that the restriction of q_X to $H_{\mathbb{R}}^{1,1}(X)$ is nondegenerate of signature $(1, b_2(X) - 3)$; it follows that the cone

$$\{\alpha \in H_{\mathbb{R}}^{1,1}(X) \mid q_X(\alpha) > 0\} \tag{2.2.1}$$

has two connected components. By Remark 2.1(2) \mathcal{K}_X is contained in (2.2.1). Since \mathcal{K}_X is convex it is contained in a single connected component of (2.2.1); that component is the *positive cone* \mathcal{C}_X .

Theorem 2.6 [Huybrechts 2003a]. *Let X be a HK manifold. Let $\mathcal{X} \rightarrow T$ be a representative of $\text{Def}(X)$ with T irreducible. If $t \in T$ is very general (i.e., outside a countable union of proper analytic subsets of T) then*

$$\mathcal{K}_{X_t} = \mathcal{C}_{X_t}. \tag{2.2.2}$$

Proof. Let $0 \in T$ be the point such that $X_0 \cong X$ and the induced map of germs $(T, 0) \rightarrow \text{Def}(X)$ is an isomorphism.¹⁰ By shrinking T around 0 if necessary we may assume that T is simply connected and that $\mathcal{X} \rightarrow T$ represents $\text{Def}(X_t)$ for every $t \in T$. In particular the Gauss–Manin connection gives an isomorphism $P_t : H^\bullet(X; \mathbb{Z}) \xrightarrow{\sim} H^\bullet(X_t; \mathbb{Z})$ for every $t \in T$. Given $\gamma \in H^{2p}(X; \mathbb{Z})$ we let

$$T_\gamma := \{t \in T \mid P_t(\gamma) \text{ is of type } (p, p)\}. \tag{2.2.3}$$

¹⁰The map $(T, 0) \rightarrow \text{Def}(X)$ depends on the choice of an isomorphism $f : X_0 \xrightarrow{\sim} X$ but whether it is an isomorphism or not is independent of f .

Let

$$t \in \left(T \setminus \bigcup_{T_\gamma \neq T} T_\gamma\right) \tag{2.2.4}$$

and $Z \subset X_t$ be a closed analytic subset of codimension p ; we claim that

$$\int_Z \alpha^{2n-p} > 0 \quad \text{if } q_{X_t}(\alpha) > 0. \tag{2.2.5}$$

In fact let $\gamma \in H_{\mathbb{R}}^{p,p}(X_t)$ be the Poincaré dual of Z . By (2.2.4) γ remains of type (p, p) for every deformation of X_t ; by Proposition 2.2 p is even and moreover there exists $c_\gamma \in \mathbb{R}$ such that

$$\int_Z \alpha^{2n-p} = c_\gamma q_X(\alpha)^{n-p/2} \quad \text{for all } \alpha \in H^2(X_t). \tag{2.2.6}$$

Let ω be a Kähler class. Since $0 < \int_Z \omega^{2n-p}$ and $0 < q_X(\omega)$ we get that $c_\gamma > 0$; thus (2.2.5) follows from (2.2.6). Now apply Demailly and Paun's version of the Nakai–Moishezon ampleness criterion [Demailly and Paun 2004]: \mathcal{H}_{X_t} is a connected component of the set $P(X_t) \subset H_{\mathbb{R}}^{1,1}(X_t)$ of classes α such that $\int_Z \alpha^{2n-p} > 0$ for all closed analytic subsets $Z \subset X_t$ (here $p = \text{cod}(Z, X_t)$). Let t be as in (2.2.4). By (2.2.5) $P(X_t) = \mathcal{C}_{X_t} \amalg (-\mathcal{C}_{X_t})$; since $\mathcal{H}_{X_t} \subset \mathcal{C}_{X_t}$ we get the proposition. \square

Theorem 2.6 leads to this *projectivity criterion*:

Theorem 2.7 [Huybrechts 1999]. *A HK manifold X is projective if and only if there exists a (holomorphic) line bundle L on X such that $q_X(c_1(L)) > 0$.*

Boucksom, elaborating on ideas of Huybrechts, gave the following characterization of \mathcal{H}_X for arbitrary X :

Theorem 2.8 [Boucksom 2001]. *Let X be a HK manifold. A class $\alpha \in H_{\mathbb{R}}^{1,1}(X)$ is Kähler if and only if it belongs to the positive cone \mathcal{C}_X and moreover $\int_C \alpha > 0$ for every rational curve C .¹¹*

One would like to have a numerical description of the Kähler (or ample) cone as in the 2-dimensional case. There is this result:

Theorem 2.9 [Hassett and Tschinkel 2009b]. *Let X be a HK variety deformation equivalent to $\mathbb{K}3^{[2]}$ and L_0 an ample line bundle on X . Let L be a line bundle on X such that $c_1(L) \in \mathcal{C}_X$. Suppose that $(c_1(L), \alpha)_X > 0$ for all $\alpha \in H_{\mathbb{Z}}^{1,1}(X)$ such that $(c_1(L_0), \alpha)_X > 0$ and*

- (a) $q_X(\alpha) = -2$ or
- (b) $q_X(\alpha) = -10$ and $(\alpha, H^2(X; \mathbb{Z}))_X = 2\mathbb{Z}$.

Then L is ample.

¹¹A curve is rational if it is irreducible and its normalization is rational.

Hassett and Tschinkel [2001] conjectured that the converse of this theorem holds, in the sense that its conditions are also necessary for L to be ample. We explain the appearance of the conditions in the theorem and why one expects that the converse holds. We start with (a). Let X be a HK manifold deformation equivalent to $K3^{[2]}$ and L a line bundle on X : Hirzebruch–Riemann–Roch for X reads

$$\chi(L) = \frac{1}{8}(q(L) + 4)(q(L) + 6). \tag{2.2.7}$$

(We let $q = q_X$.) It follows that $\chi(L) = 1$ if and only if $q(L) = -2$ or $q(L) = -8$.

Conjecture 2.10 (Folk). Let X be a HK manifold deformation equivalent to $K3^{[2]}$. Let L be a line bundle on X such that $q_X(L) = -2$.

- (1) If $(c_1(L), H^2(X; \mathbb{Z}))_X = \mathbb{Z}$ then either L or L^{-1} has a nonzero section.
- (2) If $(c_1(L), H^2(X; \mathbb{Z}))_X = 2\mathbb{Z}$ then either L^2 or L^{-2} has a nonzero section. (Notice that $q_X(L^{\pm 2}) = -8$.)

If this conjecture holds then given $\alpha \in H_{\mathbb{Z}}^{1,1}(X)$ with $q_X(\alpha) = -2$ we have that either $(\alpha, \cdot)_X$ is strictly positive or strictly negative on \mathcal{K}_X ; in particular the condition corresponding to Theorem 2.9(a) is necessary for a line bundle to be ample. Below are examples of line bundles satisfying the items (1) and (2) in the conjecture.

Example. Let S be a K3 containing a smooth rational curve C and $X = S^{[2]}$. Let

$$D := \{[Z] \in S^{[2]} \mid Z \cap C \neq \emptyset\}. \tag{2.2.8}$$

Let $L := \mathcal{O}_X(D)$; then $c_1(L) = \tilde{\mu}_2(c_1(\mathcal{O}_S(C)))$, where $\tilde{\mu}_2$ is given by (1.1.6). Since $\tilde{\mu}_2$ is an isometry we have $q_X(L) = C \cdot C = -2$; moreover $(c_1(L), H^2(X; \mathbb{Z}))_X = \mathbb{Z}$. For another example see Remark 3.3(5).

Example. Let S be a K3 and $X = S^{[2]}$. Let L_2 be the square root of $\mathcal{O}_X(\Delta_2)$ where $\Delta_2 \subset S^{[2]}$ is the divisor parametrizing nonreduced subschemes — thus $c_1(L_2) = \xi_2$. Then $q(L_2) = -2$ and L_2^2 has “the” nonzero section vanishing on Δ_2 . Notice that neither L_2 nor L_2^{-1} has a nonzero section.

Summarizing: line bundles of square -2 on a HK deformation of $K3^{[2]}$ should be similar to (-2) -classes on a K3. (Recall that if L is a line bundle on a K3 with $c_1(L)^2 = -2$ then by Hirzebruch–Riemann–Roch and Serre duality either L or L^{-1} has a nonzero section.) Next we explain Theorem 2.9(b). Suppose that X is a HK deformation of $K3^{[2]}$ and that $Z \subset X$ is a closed submanifold isomorphic to \mathbb{P}^2 — see Section 1.3. Let $C \subset Z$ be a line. Since $(\cdot, \cdot)_X$ is nondegenerate (but not unimodular!) there exists $\beta \in H^2(X; \mathbb{Q})$ such that

$$\int_C \gamma = (\beta, \gamma)_X \quad \text{for all } \gamma \in H^2(X). \tag{2.2.9}$$

One proves that

$$q_X(\beta) = -\frac{5}{2}. \tag{2.2.10}$$

This follows from the isomorphism (2.1.18) and the good properties of deformations of HK manifolds; see [Hassett and Tschinkel 2009b, Section 4]. Since $(\beta, H^2(X; \mathbb{Z}))_X = \mathbb{Z}$ and the discriminant of $(\cdot, \cdot)_X$ is 2 we have $2\beta \in H^2(X; \mathbb{Z})$; thus $\alpha := 2\beta$ is as in Theorem 2.9(b) and if L is ample then $0 < \int_C c_1(L) = \frac{1}{2}(c_1(L), \alpha)_X$.

Remark 2.11. Hassett and Tschinkel [2009a] stated conjectures that extend Theorem 2.9 and its converse to general HK varieties; in particular they have given a conjectural numerical description of the effective cone of a HK variety. The papers [Boucksom 2004; Druel 2011] contain key results in this circle of ideas. Markman [2009, Section 1.4] formulated a conjecture on HK manifolds deformation equivalent to (K3)^[n] which generalizes Conjecture 2.10 and provided a proof relying on Verbitsky’s global Torelli.

We close the section by stating a beautiful result of Huybrechts [2003c] — the proof is based on results on the Kähler cone and uses in an essential way the existence of the twistor family.

Theorem 2.12. *Let X and Y be bimeromorphic HK manifolds. Then X and Y are deformation equivalent.*

3. Complete families of HK varieties

A pair (X, L) , where X is a HK variety and L is a primitive¹² ample line bundle on X with $q_X(L) = d$, is a *HK variety of degree d* ; an isomorphism $(X, L) \xrightarrow{\sim} (X', L')$ between HK’s of degree d consists of an isomorphism $f : X \xrightarrow{\sim} X'$ such that $f^*L' \cong L$. A family of HK varieties of degree d is a pair

$$(f : \mathcal{X} \rightarrow T, \mathcal{L}) \tag{3.0.1}$$

where $\mathcal{X} \rightarrow T$ is a family of HK varieties deformation equivalent to a fixed HK manifold X and \mathcal{L} is a line bundle such that (X_t, L_t) is a HK variety of degree d for every $t \in T$ (here $X_t := f^{-1}(t)$ and $L_t := \mathcal{L}|_{X_t}$) — we say that it is a family of HK varieties if we are not interested in the value of $q_X(L_t)$. The deformation space of (X, L) is a codimension-1 smooth subgerm $\text{Def}(X, L) \subset \text{Def}(X)$ with tangent space the kernel of the map (0.0.4) with $\alpha = c_1(L)$. The family (3.0.1) is *locally complete* if given any $t_0 \in T$ the map of germs $(T, t_0) \rightarrow \text{Def}(X_{t_0}, L_{t_0})$ is surjective, it is *globally complete* if given any HK variety (Y, L) of degree d with Y deformation equivalent to X there exists $t_0 \in T$ such that $(Y, L) \cong (X_{t_0}, L_{t_0})$. In dimension 2 — that is, for K3 surfaces — one has explicit globally complete

¹²That is, $c_1(L)$ is indivisible in $H^2(X; \mathbb{Z})$.

families of low degree: If $d = 2$ the family of double covers $S \rightarrow \mathbb{P}^2$ branched over a smooth sextic will do,¹³ if $d = 4$ we may consider the family of smooth quartic surfaces $S \subset \mathbb{P}^3$ with the addition of certain “limit” surfaces (double covers of smooth quadrics and certain elliptic K3’s) corresponding to degenerate quartics (double quadrics and the surface swept out by tangents to a rational normal cubic curve respectively). The list goes on for quite a few values of d , see [Mukai 1988; 2006] and then it necessarily stops — at least in this form — because moduli spaces of high-degree K3’s are not unirational [Gritsenko et al. 2007]. We remark that in low degree one shows “by hand” that there exists a globally complete family which is irreducible; the same is true in arbitrary degree but I know of no elementary proof, the most direct argument is via global Torelli. What is the picture in dimensions higher than two? Four distinct (modulo obvious equivalence) locally complete families of higher-dimensional HK varieties have been constructed — they are all deformations of K3^[2]. The families are the following:

- (1) In [O’Grady 2006] we constructed the family of double covers of certain special sextic hypersurfaces in \mathbb{P}^5 that we named EPW-sextics (they had been introduced by Eisenbud, Popescu, and Walter [2001]). The polarization is the pull-back of $\mathcal{O}_{\mathbb{P}^5}(1)$; its degree is 2.
- (2) Let $Z \subset \mathbb{P}^5$ be a smooth cubic hypersurface; Beauville and Donagi [1985] proved that the variety parametrizing lines on Z is a deformation of K3^[2]. The polarization is given by the Plücker embedding: it has degree 6.
- (3) Let σ be a generic 3-form on \mathbb{C}^{10} ; Debarre and Voisin [Debarre and Voisin 2010] proved that the set $Y_\sigma \subset Gr(6, \mathbb{C}^{10})$ parametrizing subspaces on which σ vanishes is a deformation of K3^[2]. The polarization is given by the Plücker embedding: it has degree 22.
- (4) Let $Z \subset \mathbb{P}^5$ be a generic cubic hypersurface; Iliev and Ranestad [2001; 2007] have proved that the variety of sums of powers $VSP(Z, 10)$ ¹⁴ is a deformation of K3^[2]. For the polarization we refer to [Iliev and Ranestad 2007]; the degree is 38 (unpublished computation by Iliev, Ranestad and van Geemen).

For each of these families — more precisely for the family obtained by adding “limits” — one might ask whether it is globally complete for HK varieties of the given degree which are deformations of K3^[2]. As formulated the answer is negative with the possible exception of our family, for a trivial reason: in the

¹³In order to get a global family we must go to a suitable double cover of the parameter space of sextic curves.

¹⁴ $VSP(Z, 10)$ parametrizes 9-dimensional linear spaces of $|\mathcal{O}_{\mathbb{P}^5}(3)|$ which contain Z and are 10-secant to the Veronese $\{[L^3] \mid L \in (H^0(\mathcal{O}_{\mathbb{P}^5}(1)) \setminus \{0\})\}$.

lattice $L := H^2(\mathbf{K}3^{[2]}; \mathbb{Z})$ the orbit of a primitive vector v under the action of $O(L)$ is determined by the value of the B-B form $q(v)$ plus the extra information on whether

$$(v, L) = \begin{cases} \mathbb{Z} & \text{or} \\ 2\mathbb{Z} \end{cases} \tag{3.0.2}$$

In the first case one says that the *divisibility of v* is 1, in the second case that it is 2; if the latter occurs then $q(v) \equiv 6 \pmod{8}$. Thus the divisibility of the polarization in family (1) above equals 1; on the other hand it equals 2 for families (2)–(4). The correct question regarding global completeness is the following. Let X be a HK deformation of $\mathbf{K}3^{[2]}$ with an ample line bundle L such that either $q(L) = 2$ or $q(L) \in \{6, 22, 38\}$ and the divisibility of $c_1(L)$ is equal to 2: does there exist a variety Y parametrized by one of the families above — or a limit of such — and an isomorphism $(X, L) \cong (Y, \mathcal{O}_Y(1))$? Yes, by Verbitsky’s global Torelli and Markmans’ monodromy computations.

None of the families above is as easy to construct as are the families of low-degree K3 surfaces. There is the following Hodge-theoretic explanation. In order to get a locally complete family of varieties one usually constructs complete intersections (or sections of ample vector bundles) in homogeneous varieties: by Lefschetz’s hyperplane theorem such a construction will never produce a higher-dimensional HK. On the other hand the families (1), (2), and (3) are related to complete intersections as follows (I do not know whether one may view the Iliev–Ranestad family from a similar perspective). First if $f : X \rightarrow Y$ is a double EPW-sextic (family (1) above) then f is the quotient map of an involution $X \rightarrow X$ which has one-dimensional (+1)-eigenspace on $H^2(X)$ — in particular it kills $H^{2,0}$ — and “allows” the quotient to be a hypersurface. Regarding family (2): let $Z \subset \mathbb{P}^5$ be a smooth cubic hypersurface and X the variety of lines on Z , the incidence correspondence in $Z \times X$ induces an isomorphism of the primitive Hodge structures $H^4(Z)_{pr} \xrightarrow{\sim} H^2(X)_{pr}$. Thus a Tate twist of $H^2(X)_{pr}$ has become the primitive intermediate cohomology of a hypersurface. A similar comment applies to the Debarre–Voisin family (and there is a similar incidence-type construction of double EPW-sextics given in [Iliev and Manivel 2009]).

In this section we will describe in some detail the family of double EPW-sextics and we will say a few words about analogies with the Beauville–Donagi family.

3.1. Double EPW-sextics, I. We start by giving the definition of EPW-sextic [Eisenbud et al. 2001]. Let V be a 6-dimensional complex vector space. We choose a volume form $vol : \wedge^6 V \xrightarrow{\sim} \mathbb{C}$ and we equip $\wedge^3 V$ with the symplectic form

$$(\alpha, \beta)_V := vol(\alpha \wedge \beta). \tag{3.1.1}$$

Let $\mathbb{L}\mathbb{G}(\wedge^3 V)$ be the symplectic Grassmannian parametrizing Lagrangian subspaces of $\wedge^3 V$ — notice that $\mathbb{L}\mathbb{G}(\wedge^3 V)$ is independent of the chosen volume form vol . Given a nonzero $v \in V$ we let

$$F_v := \{\alpha \in \wedge^3 V \mid v \wedge \alpha = 0\}. \tag{3.1.2}$$

Notice that $(\cdot, \cdot)_V$ is zero on F_v and $\dim(F_v) = 10$, i.e., $F_v \in \mathbb{L}\mathbb{G}(\wedge^3 V)$. Let

$$F \subset \wedge^3 V \otimes \mathbb{O}_{\mathbb{P}(V)} \tag{3.1.3}$$

be the vector subbundle with fiber F_v over $[v] \in \mathbb{P}(V)$. Given $A \in \mathbb{L}\mathbb{G}(\wedge^3 V)$ we let

$$Y_A = \{[v] \in \mathbb{P}(V) \mid F_v \cap A \neq \{0\}\}. \tag{3.1.4}$$

Thus Y_A is the degeneracy locus of the map

$$F \xrightarrow{\lambda_A} (\wedge^3 V/A) \otimes \mathbb{O}_{\mathbb{P}(V)} \tag{3.1.5}$$

where λ_A is given by Inclusion (3.1.3) followed by the quotient map $\wedge^3 V \otimes \mathbb{O}_{\mathbb{P}(V)} \rightarrow (\wedge^3 V/A) \otimes \mathbb{O}_{\mathbb{P}(V)}$. Since the vector bundles appearing in (3.1.5) have equal rank Y_A is the zero-locus of $\det \lambda_A \in H^0(\det F^\vee)$ — in particular it has a natural structure of closed subscheme of $\mathbb{P}(V)$. A straightforward computation gives that $\det F \cong \mathbb{O}_{\mathbb{P}(V)}(-6)$ and hence Y_A is a sextic hypersurface unless it equals $\mathbb{P}(V)$; ¹⁵ if the former holds we say that Y_A is an *EPW-sextic*. What do EPW-sextics look like? The main point is that locally they are the degeneracy locus of a symmetric map of vector bundles (they were introduced by Eisenbud, Popescu and Walter to give examples of a “quadratic sheaf”, namely $coker(\lambda_A)$, which can not be expressed **globally** as the cokernel of a symmetric map of vector bundles on \mathbb{P}^5). More precisely given $B \in \mathbb{L}\mathbb{G}(\wedge^3 V)$ we let $\mathcal{U}_B \subset \mathbb{P}(V)$ be the open subset defined by

$$\mathcal{U}_B := \{[v] \in \mathbb{P}(V) \mid F_v \cap B = \{0\}\}. \tag{3.1.6}$$

Now choose B transversal to A . We have a direct-sum decomposition $\wedge^3 V = A \oplus B$; since A is lagrangian the symplectic form $(\cdot, \cdot)_V$ defines an isomorphism $B \cong A^\vee$. Let $[v] \in \mathcal{U}_B$: since F_v is transversal to B it is the graph of a map

$$\tau_A^B([v]) : A \rightarrow B \cong A^\vee, \quad [v] \in \mathcal{U}_B. \tag{3.1.7}$$

The map $\tau_A^B([v])$ is symmetric because A, B and F_v are lagrangians.

¹⁵Given $[v] \in \mathbb{P}(V)$ there exists $A \in \mathbb{L}\mathbb{G}(\wedge^3 V)$ such that $A \cap F_v = \{0\}$ and hence $[v] \notin Y_A$; thus Y_A is a sextic hypersurface for generic $A \in \mathbb{L}\mathbb{G}(\wedge^3 V)$. On the other hand if $A = F_w$ for some $[w] \in \mathbb{P}(V)$ then $Y_A = \mathbb{P}(V)$.

Remark 3.1. There is one choice of B which produces a “classical” description of Y_A , namely $B = \wedge^3 V_0$ where $V_0 \subset V$ is a codimension-1 subspace.¹⁶ With such a choice of B we have $\mathcal{U}_B = (\mathbb{P}(V) \setminus \mathbb{P}(V_0))$; we identify it with V_0 by choosing $v_0 \in (V \setminus V_0)$ and mapping

$$V_0 \xrightarrow{\sim} \mathbb{P}(V) \setminus \mathbb{P}(V_0), \quad v \mapsto [v_0 + v]. \tag{3.1.8}$$

The direct-sum decomposition $\wedge^3 V = F_{v_0} \oplus \wedge^3 V_0$ and transversality $A \pitchfork \wedge^3 V_0$ allows us to view A as the graph of a (symmetric) map $\tilde{q}_A : F_{v_0} \rightarrow \wedge^3 V_0$. Identifying $\wedge^2 V_0$ with F_{v_0} via the isomorphism

$$\wedge^2 V_0 \xrightarrow{\sim} F_{v_0}, \quad \alpha \mapsto v_0 \wedge \alpha, \tag{3.1.9}$$

we may view \tilde{q}_A as a symmetric map

$$\wedge^2 V_0 \longrightarrow \wedge^3 V_0 = \wedge^2 V_0^\vee. \tag{3.1.10}$$

We let $q_A \in \text{Sym}^2(\wedge^2 V_0^\vee)$ be the quadratic form corresponding to \tilde{q}_A . Given $v \in V_0$ let $q_v \in \text{Sym}^2(\wedge^2 V_0^\vee)$ be the Plücker quadratic form $q_v(\alpha) := \text{vol}(v_0 \wedge v \wedge \alpha \wedge \alpha)$. Modulo the identification (3.1.8) we have

$$Y_A \cap (\mathbb{P}(V) \setminus \mathbb{P}(V_0)) = V(\det(q_A + q_v)). \tag{3.1.11}$$

Equivalently let

$$Z_A := V(q_A) \cap \text{Gr}(2, V_0) \subset \mathbb{P}(\wedge^2 V_0) \cong \mathbb{P}^9. \tag{3.1.12}$$

Then we have an isomorphism

$$\mathbb{P}(V) \xrightarrow{\sim} |\mathcal{I}_{Z_A}(2)|, \quad [\lambda v_0 + \mu v] \mapsto V(\lambda q_A + \mu q_v). \tag{3.1.13}$$

(Here $\lambda, \mu \in \mathbb{C}$ and $v \in V_0$.) Let $D_A \subset |\mathcal{I}_{Z_A}(2)|$ be the discriminant locus; modulo the identification above we have

$$Y_A \cap (\mathbb{P}(V) \setminus \mathbb{P}(V_0)) = D_A \cap (|\mathcal{I}_{Z_A}(2)| \setminus |\mathcal{I}_{\text{Gr}(2, V_0)}(2)|). \tag{3.1.14}$$

Notice that $|\mathcal{I}_{\text{Gr}(2, V_0)}(2)|$ is a hyperplane contained in D_A with multiplicity 4; that explains why $\text{deg } Y_A = 6$ while $\text{deg } D_A = 10$.

We go back to general considerations regarding Y_A . The symmetric map τ_A^B of (3.1.7) allows us to give a structure of scheme to the degeneracy locus

$$Y_A[k] = \{[v] \in \mathbb{P}(V) \mid \dim(A \cap F_v) \geq k\} \tag{3.1.15}$$

by declaring that $Y_A[k] \cap \mathcal{U}_B = V(\wedge^{(11-k)} \tau_A^B)$. By a standard dimension count we expect that the following holds for generic $A \in \mathbb{L}\mathbb{G}(\wedge^3 V)$: $Y_A[3] = \emptyset$,

¹⁶It might happen that there is no V_0 such that $\wedge^3 V_0$ is transversal to A : in that case A is unstable for the natural $PGL(V)$ -action on $\mathbb{L}\mathbb{G}(\wedge^3 V)$ and hence we may forget about it.

$Y_A[2] = \text{sing } Y_A$ and $Y_A[2]$ is a smooth surface (of degree 40 by (6.7) of [Fulton and Pragacz 1998]), in particular Y_A should be a very special sextic hypersurface. This is indeed the case; in order to be less “generic” let

$$\Delta := \{A \in \mathbb{L}\mathbb{G}(\wedge^3 V) \mid Y_A[3] \neq \emptyset\}, \tag{3.1.16}$$

$$\Sigma := \{A \in \mathbb{L}\mathbb{G}(\wedge^3 V) \mid \exists W \in \mathbb{G}r(3, V) \text{ s.t. } \wedge^3 W \subset A\}. \tag{3.1.17}$$

A straightforward computation shows that Σ and Δ are distinct closed irreducible codimension-1 subsets of $\mathbb{L}\mathbb{G}(\wedge^3 V)$. Let

$$\mathbb{L}\mathbb{G}(\wedge^3 V)^0 := \mathbb{L}\mathbb{G}(\wedge^3 V) \setminus \Sigma \setminus \Delta. \tag{3.1.18}$$

Then Y_A has the generic behavior described above if and only if A belongs to $\mathbb{L}\mathbb{G}(\wedge^3 V)^0$. Next let $A \in \mathbb{L}\mathbb{G}(\wedge^3 V)$ and suppose that $Y_A \neq \mathbb{P}(V)$: then Y_A comes equipped with a natural double cover $f_A : X_A \rightarrow Y_A$ defined as follows. Let $i : Y_A \hookrightarrow \mathbb{P}(V)$ be the inclusion map: since $\text{coker}(\lambda_A)$ is annihilated by a local generator of $\det \lambda_A$ we have $\text{coker}(\lambda_A) = i_* \zeta_A$ for a sheaf ζ_A on Y_A . Choose $B \in \mathbb{L}\mathbb{G}(\wedge^3 V)$ transversal to A ; the direct-sum decomposition $\wedge^3 V = A \oplus B$ defines a projection map $\wedge^3 V \rightarrow A$; thus we get a map $\mu_{A,B} : F \rightarrow A \otimes \mathbb{O}_{\mathbb{P}(V)}$. We claim that there is a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \rightarrow & F & \xrightarrow{\lambda_A} & A^\vee \otimes \mathbb{O}_{\mathbb{P}(V)} & \longrightarrow & i_* \zeta_A \longrightarrow 0 \\ & & \downarrow \mu_{A,B} & & \downarrow \mu'_{A,B} & & \downarrow \beta_A \\ 0 & \rightarrow & A \otimes \mathbb{O}_{\mathbb{P}(V)} & \xrightarrow{\lambda'_A} & F^\vee & \longrightarrow & \text{Ext}^1(i_* \zeta_A, \mathbb{O}_{\mathbb{P}(V)}) \longrightarrow 0 \end{array} \tag{3.1.19}$$

(Since A is Lagrangian the symplectic form defines a canonical isomorphism $(\wedge^3 V/A) \cong A^\vee$; that is why we may write λ_A as above.) In fact the second row is obtained by applying the $\text{Hom}(\cdot, \mathbb{O}_{\mathbb{P}(V)})$ -functor to the first row and the equality $\mu'_{A,B} \circ \lambda_A = \lambda'_A \circ \mu_{A,B}$ holds because F is a Lagrangian subbundle of $\wedge^3 V \otimes \mathbb{O}_{\mathbb{P}(V)}$. Lastly β_A is defined to be the unique map making the diagram commutative; as suggested by notation it is independent of B . Next by applying the $\text{Hom}(i_* \zeta_A, \cdot)$ -functor to the exact sequence

$$0 \longrightarrow \mathbb{O}_{\mathbb{P}(V)} \longrightarrow \mathbb{O}_{\mathbb{P}(V)}(6) \longrightarrow \mathbb{O}_{Y_A}(6) \longrightarrow 0 \tag{3.1.20}$$

we get the exact sequence

$$0 \longrightarrow i_* \text{Hom}(\zeta_A, \mathbb{O}_{Y_A}(6)) \xrightarrow{\partial} \text{Ext}^1(i_* \zeta_A, \mathbb{O}_{\mathbb{P}(V)}) \xrightarrow{n} \text{Ext}^1(i_* \zeta_A, \mathbb{O}_{\mathbb{P}(V)}(6)) \tag{3.1.21}$$

where n is locally equal to multiplication by $\det \lambda_A$. Since the second row of (3.1.19) is exact a local generator of $\det \lambda_A$ annihilates $\text{Ext}^1(i_* \zeta_A, \mathbb{O}_{\mathbb{P}(V)})$; thus

$n = 0$ and hence we get a canonical isomorphism

$$\partial^{-1} : \text{Ext}^1(i_*\zeta_A, \mathbb{O}_{\mathbb{P}(V)}) \xrightarrow{\sim} i_* \text{Hom}(\zeta_A, \mathbb{O}_{Y_A}(6)). \tag{3.1.22}$$

Let

$$\zeta_A \times \zeta_A \xrightarrow{\tilde{m}_A} \mathbb{O}_{Y_A}(6), \quad (\sigma_1, \sigma_2) \mapsto (\partial^{-1} \circ \beta_A(\sigma_1))(\sigma_2). \tag{3.1.23}$$

Let $\xi_A := \zeta_A(-3)$; tensoring both sides of (3.1.23) by $\mathbb{O}_{Y_A}(-6)$ we get a multiplication map

$$m_A : \xi_A \times \xi_A \rightarrow \mathbb{O}_{Y_A}. \tag{3.1.24}$$

This multiplication map equips $\mathbb{O}_{Y_A} \oplus \xi_A$ with the structure of a commutative and associative \mathbb{O}_{Y_A} -algebra. We let

$$X_A := \text{Spec}(\mathbb{O}_{Y_A} \oplus \xi_A), \quad f_A : X_A \rightarrow Y_A. \tag{3.1.25}$$

Then X_A is a *double EPW-sextic*. Let \mathcal{U}_B be as in (3.1.6): we may describe $f_A^{-1}(Y_A \cap \mathcal{U}_B)$ as follows. Let M be the symmetric matrix associated to (3.1.7) by a choice of basis of A and M^c be the matrix of cofactors of M . Let $Z = (z_1, \dots, z_{10})^t$ be the coordinates on A associated to the given basis; then $f_A^{-1}(Y_A \cap \mathcal{U}_B) \subset \mathcal{U}_B \times \mathbb{A}_Z^{10}$ and its ideal is generated by the entries of the matrices

$$M \cdot Z, \quad Z \cdot Z^t - M^c. \tag{3.1.26}$$

(The “missing” equation $\det M = 0$ follows by Cramer’s rule.) One may reduce the size of M in a neighborhood of $[v_0] \in \mathcal{U}_B$ as follows. The kernel of the symmetric map $\tau_A^B([v_0])$ equals $A \cap F_{v_0}$; let $J \subset A$ be complementary to $A \cap F_{v_0}$. Diagonalizing the restriction of τ_A^B to J we may assume that

$$M([v]) = \begin{pmatrix} M_0([v]) & 0 \\ 0 & 1_{10-k} \end{pmatrix} \tag{3.1.27}$$

where $k := \dim(A \cap F_{v_0})$ and M_0 is a symmetric $k \times k$ matrix. It follows at once that f_A is étale over $(Y_A \setminus Y_A[2])$. We also get the following description of f_A over a point $[v_0] \in (Y_A[2] \setminus Y_A[3])$ under the hypothesis that there is no $0 \neq v_0 \wedge v_1 \wedge v_2 \in A$. First $f_A^{-1}([v_0])$ is a single point p_0 , secondly X_A is smooth at p_0 and there exists an involution ϕ on (X_A, p_0) with 2-dimensional fixed-point set such that f_A is identified with the quotient map $(X_A, p_0) \rightarrow (X_A, p_0)/\langle \phi \rangle$. It follows that X_A is smooth if $A \in \mathbb{L}\mathbb{G}(\wedge^3 V)^0$. We may fit together all smooth double EPW-sextics by going to a suitable double cover $\rho : \mathbb{L}\mathbb{G}(\wedge^3 V)^* \rightarrow \mathbb{L}\mathbb{G}(\wedge^3 V)^0$; there exist a family of HK four-folds $\mathcal{X} \rightarrow \mathbb{L}\mathbb{G}(\wedge^3 V)^*$ and a relatively ample line bundle \mathcal{L} over \mathcal{X} such that for all $t \in \mathbb{L}\mathbb{G}(\wedge^3 V)^*$ we have $(X_t, L_t) \cong (X_{A_t}, f_{A_t}^* \mathbb{O}_{Y_{A_t}}(1))$ where

$$X_t := \rho^{-1}(t), \quad L_t = \mathcal{L}|_{X_t}, \quad A_t := \rho(t). \tag{3.1.28}$$

Theorem 3.2 [O’Grady 2006]. *Let $A \in \mathbb{L}\mathbb{G}(\wedge^3 V)^0$. Then X_A is a HK four-fold deformation equivalent to $\mathbb{K}3^{[2]}$. Moreover $\mathcal{X} \rightarrow \mathbb{L}\mathbb{G}(\wedge^3 V)^*$ is a locally complete family of HK varieties of degree 2.*

Sketch of proof. The main issue is to prove that X_A is a HK deformation of $\mathbb{K}3^{[2]}$. In fact once this is known the equality

$$\int_{X_A} f_A^* c_1(\mathcal{O}_{Y_A}(1))^4 = 2 \cdot 6 = 12 \tag{3.1.29}$$

together with (2.1.2) gives that $q(f_A^* c_1(\mathcal{O}_{Y_A}(1))) = 2$ and moreover the family $\mathcal{X} \rightarrow \mathbb{L}\mathbb{G}(\wedge^3 V)^*$ is locally complete by the following argument. First Kodaira vanishing and Formula (2.2.7) give that

$$h^0(f_A^* \mathcal{O}_{Y_A}(1)) = \chi(f_A^* \mathcal{O}_{Y_A}(1)) = 6 \tag{3.1.30}$$

and hence the map

$$X_A \xrightarrow{f_A} Y_A \hookrightarrow \mathbb{P}(V) \tag{3.1.31}$$

may be identified with the map $X_A \rightarrow |f_A^* \mathcal{O}_{Y_A}(1)|^\vee$. From this one gets that the natural map $(\mathbb{L}\mathbb{G}(\wedge^3 V)^0 // PGL(V), [A]) \rightarrow \text{Def}(X_A, f_A^* \mathcal{O}_{Y_A}(1))$ is injective. One concludes that $\mathcal{X} \rightarrow \mathbb{L}\mathbb{G}(\wedge^3 V)^*$ is locally complete by a dimension count:

$$\dim(\mathbb{L}\mathbb{G}(\wedge^3 V)^0 // PGL(V)) = 20 = \dim \text{Def}(X_A, f_A^* \mathcal{O}_{Y_A}(1)). \tag{3.1.32}$$

Thus we are left with the task of proving that X_A is a HK deformation of $\mathbb{K}3^{[2]}$ if $A \in \mathbb{L}\mathbb{G}(\wedge^3 V)^0$. We do this by analyzing X_A for

$$A \in (\Delta \setminus \Sigma). \tag{3.1.33}$$

By definition $Y_A[3]$ is nonempty; one shows that it is finite, that $\text{sing } X_A = f_A^{-1} Y_A[3]$ and that $f_A^{-1}[v_i]$ is a single point for each $[v_i] \in Y_A[3]$. There exists a small resolution

$$\pi_A : \widehat{X}_A \longrightarrow X_A, \quad (f_A \circ \pi_A)^{-1}([v_i]) \cong \mathbb{P}^2 \quad \forall [v_i] \in Y_A. \tag{3.1.34}$$

In fact one gets that locally over the points of $\text{sing } X_A$ the above resolution is identified with the contraction c (or c^\vee) appearing in (1.3.2)—in particular \widehat{X}_A is not unique, in fact there are $2^{|Y_A[3]|}$ choices involved in the construction of \widehat{X}_A . The resolution \widehat{X}_A fits into a simultaneous resolution; i.e., given a sufficiently small open (in the classical topology) $A \in U \subset (\mathbb{L}\mathbb{G}(\wedge^3 V) \setminus \Sigma)$ we have proper maps π, ψ

$$\widehat{\mathcal{X}}_U \xrightarrow{\pi} \mathcal{X}_U \xrightarrow{\psi} U \tag{3.1.35}$$

where ψ is a tautological family of double EPW-sextics over U , i.e., $\psi^{-1}A \cong X_A$ and $(\psi \circ \pi)^{-1}A \rightarrow \psi^{-1}A = X_A$ is a small resolution as above if $A \in U \cap \Delta$ while

$\pi^{-1}A \cong X_A$ if $A \in (U \setminus \Delta)$. Thus it suffices to prove that there exist $A \in (\Delta \setminus \Sigma)$ such that \widehat{X}_A is a HK deformation of $K3^{[2]}$. Let $[v_i] \in Y_A[3]$; we define a K3 surface $S_A(v_i)$ as follows. There exists a codimension-1 subspace $V_0 \subset V$ not containing v_i and such that $\wedge^3 V_0$ is transversal to A . Thus Y_A can be described as in Remark 3.1: we adopt notation introduced in that remark, in particular we have the quadric $Q_A := V(q_A) \subset \mathbb{P}(\wedge^2 V_0)$. The singular locus of Q_A is $\mathbb{P}(A \cap F_{v_i})$ — we recall the identification (3.1.9). By hypothesis $\mathbb{P}(A \cap F_{v_i}) \cap \text{Gr}(2, V_0) = \emptyset$; it follows that $\dim \mathbb{P}(A \cap F_{v_i}) = 2$ (by hypothesis $\dim \mathbb{P}(A \cap F_{v_i}) \geq 2$). Let

$$S_A(v_i) := Q_A^\vee \cap \text{Gr}(2, V_0^\vee) \subset \mathbb{P}(\wedge^2 V_0^\vee). \tag{3.1.36}$$

Then $S_A(v_i) \subset \mathbb{P}(\text{Ann}(A \cap F_{v_i})) \cong \mathbb{P}^6$ is the transverse intersection of a smooth quadric and the Fano 3-fold of index 2 and degree 5, i.e., the generic K3 of genus 6. There is a natural degree-2 rational map

$$g_i : S_A(v_i)^{[2]} \dashrightarrow |\mathcal{I}_{S_A(v_i)}(2)|^\vee \tag{3.1.37}$$

which associates to $[Z]$ the set of quadrics in $|\mathcal{I}_{S_A(v_i)}(2)|$ which contain the line spanned by Z — thus g_i is regular if $S_A(v_i)$ contains no lines. One proves that $\text{Im}(g_i)$ may be identified with Y_A ; it follows that there exists a birational map

$$h_i : S_A(v_i)^{[2]} \dashrightarrow \widehat{X}_A \tag{3.1.38}$$

Moreover if $S_A(v_i)$ contains no lines (that is true for generic $A \in (\Delta \setminus \Sigma)$) there is a choice of small resolution \widehat{X}_A such that h_i is regular and hence an isomorphism — in particular \widehat{X}_A is projective.¹⁷ This proves that X_A is a HK deformation of $K3^{[2]}$ for $A \in \mathbb{L}\mathbb{G}(\wedge^3 V)^0$. \square

Remark 3.3. This proof of Theorem 3.2 provides a description of X_A for $A \in (\Delta \setminus \Sigma)$; what about X_A for $A \in \Sigma$? One proves that if $A \in \Sigma$ is generic — in particular there is a unique $W \in \text{Gr}(3, V)$ such that $\wedge^3 W \subset A$ — then the following hold:

- (1) $C_{W,A} := \{[v] \in \mathbb{P}(W) \mid \dim(A \cap F_v) \geq 2\}$ is a smooth sextic curve.
- (2) $\text{sing } X_A = f_A^{-1}\mathbb{P}(W)$ and the restriction of f_A to $\text{sing } X_A$ is the double cover of $\mathbb{P}(W)$ branched over $C_{W,A}$, i.e., a K3 surface of degree 2.
- (3) If $p \in \text{sing } X_A$ the germ (X_A, p) (in the classical topology) is isomorphic to the product of a smooth 2-dimensional germ and an A_1 singularity; thus the blow-up $\widetilde{X}_A \rightarrow X_A$ resolves the singularities of X_A .
- (4) Let $U \subset \mathbb{L}\mathbb{G}(\wedge^3 V)$ be a small open (classical topology) subset containing A . After a base change $\widetilde{U} \rightarrow U$ of order 2 branched over $U \cap \Sigma$ there is a

¹⁷There is no reason a priori why \widehat{X}_A should be Kähler, in fact one should expect it to be non-Kähler for some A and some choice of small resolution.

simultaneous resolution of singularities of the tautological family of double EPW's parametrized by \tilde{U} . It follows that \tilde{X}_A is a HK deformation of $K3^{[2]}$.

- (5) Let E_A be the exceptional divisor of the blow-up $\tilde{X}_A \rightarrow X_A$ and $e_A \in H^2(\tilde{X}_A; \mathbb{Z})$ be its Poincaré dual; then $q(e_A) = -2$ and $(e_A, H^2(\tilde{X}_A; \mathbb{Z})) = \mathbb{Z}$.

3.2. The Beauville–Donagi family. Let $\mathcal{D}, \mathcal{P} \subset |\mathbb{O}_{\mathbb{P}^5}(3)|$ be the prime divisors parametrizing singular cubics and cubics containing a plane respectively. We recall that if $Z \in (|\mathbb{O}_{\mathbb{P}^5}(3)| \setminus \mathcal{D})$ then

$$X = F(Z) := \{L \in \mathbb{G}r(1, \mathbb{P}^5) \mid L \subset X\} \tag{3.2.1}$$

is a HK four-fold deformation equivalent to $K3^{[2]}$. Let H be the Plücker ample divisor on X and $h = c_1(\mathbb{O}_X(H))$; then

$$q(h) = 6, \quad (h, H^2(X; \mathbb{Z}))_X = 2\mathbb{Z}. \tag{3.2.2}$$

These results are proved in [Beauville and Donagi 1985] by considering the codimension-1 locus of Pfaffian cubics; they show that if Z is a generic such Pfaffian cubic then X is isomorphic to $S^{[2]}$ where S is a K3 of genus 8 that one associates to Z , moreover the class h is identified with $2\tilde{\mu}(D) - 5\xi_2$ where D is the class of the (genus 8) hyperplane class of S . Here we will stress the similarities between the HK four-folds parametrized by \mathcal{D}, \mathcal{P} and those parametrized by the loci $\Delta, \Sigma \subset \mathbb{L}\mathbb{G}(\wedge^3 V)$ described in the previous subsection. Let $Z \in \mathcal{D}$ be generic. Then Z has a unique singular point p and it is ordinary quadratic, moreover the set of lines in Z containing p is a K3 surface S of genus 4. The variety $X = F(Z)$ parametrizing lines in Z is birational to $S^{[2]}$; the birational map is given by

$$S^{[2]} \dashrightarrow F(Z), \quad \{L_1, L_2\} \mapsto R, \tag{3.2.3}$$

where $L_1 + L_2 + R = \langle L_1, L_2 \rangle \cdot Z$. Moreover $F(Z)$ is singular with singular locus equal to S . Thus from this point of view \mathcal{D} is similar to Δ . On the other hand let $Z_0 \in (|\mathbb{O}_{\mathbb{P}^5}(3)| \setminus \mathcal{D})$ be “close” to Z ; the monodromy action on $H^2(F(Z_0))$ of a loop in $(|\mathbb{O}_{\mathbb{P}^5}(3)| \setminus \mathcal{D})$ which goes once around \mathcal{D} has order 2 and hence as far as monodromy is concerned \mathcal{D} is similar to Σ . (Let $U \subset |\mathbb{O}_{\mathbb{P}^5}(3)|$ be a small open (classical topology) set containing Z ; it is natural to expect that after a base change $\pi : \tilde{U} \rightarrow U$ of order 2 ramified over \mathcal{D} the family of $F(Z_u)$ for $u \in (\tilde{U} \setminus \pi^{-1}\mathcal{D})$ can be completed over points of $\pi^{-1}\mathcal{D}$ with HK four-folds birational (isomorphic?) to $S^{[2]}$.) Now let $Z \in \mathcal{P}$ be generic, in particular it contains a unique plane P . Let $T \cong \mathbb{P}^2$ parametrize 3-dimensional linear subspaces of \mathbb{P}^5 containing P ; given $t \in T$ and L_t the corresponding 3-space the intersection $L_t \cdot Z$ decomposes as $P + Q_t$ where Q_t is a quadric surface. Let

$E \subset X = F(Z)$ be the set defined by

$$E := \{L \in F(Z) \mid \exists t \in T \text{ such that } L \subset Q_t\}. \tag{3.2.4}$$

For Z generic we have a well-defined map $E \rightarrow T$ obtained by associating to L the unique t such that $L \subset Q_t$; the Stein factorization of $E \rightarrow T$ is $E \rightarrow S \rightarrow T$ where $S \rightarrow T$ is the double cover ramified over the curve $B \subset T$ parametrizing singular quadrics. The locus B is a smooth sextic curve and hence S is a K3 surface of genus 2. The picture is: E is a conic bundle over the K3 surface S and we have

$$q(E) = -2, \quad (e, H^2(X; \mathbb{Z})) = \mathbb{Z}, \quad e := c_1(\mathbb{O}_X(E)). \tag{3.2.5}$$

Thus from this point of view \mathcal{P} is similar to Σ — of course if we look at monodromy the analogy fails.

4. Numerical Hilbert squares

A *numerical Hilbert square* is a HK four-fold X such that c_X is equal to the Fujiki constant of $K3^{[2]}$ and the lattice $H^2(X; \mathbb{Z})$ is isometric to $H^2(K3^{[2]}; \mathbb{Z})$; by (2.1.8), (2.1.9) this holds if and only if

$$H^2(X; \mathbb{Z}) \cong U^3 \hat{\oplus} E_8(-1) \hat{\oplus} \langle -2 \rangle, \quad c_X = 1. \tag{4.0.1}$$

We will present a program which aims to prove that a numerical Hilbert square is a deformation of $K3^{[2]}$, i.e., an analogue of Kodaira’s theorem that any two K3’s are deformation equivalent. First we recall how Kodaira [1964] proved that K3 surfaces form a single deformation class. Let X_0 be a K3. Let $\mathcal{X} \rightarrow T$ be a representative of the deformation space $\text{Def}(X_0)$. The image of the local period map $\pi : T \rightarrow \mathbb{P}(H^2(X_0))$ contains an open (classical topology) subset of the quadric $\mathcal{Q} := V(q_{X_0})$. The set $\mathcal{Q}(\mathbb{Q})$ of rational points of \mathcal{Q} is dense (classical topology) in the set of real points $\mathcal{Q}(\mathbb{R})$; it follows that the image $\pi(T)$ contains a point $[\sigma]$ such that $\sigma^\perp \cap H^2(X_0; \mathbb{Q})$ is generated by a nonzero α such that $q_X(\alpha) = 0$. Let $t \in T$ such that $\pi(t) = [\sigma]$ and set $X := X_t$; by the Lefschetz (1, 1) theorem we have

$$H_{\mathbb{Z}}^{1,1}(X) = \mathbb{Z}c_1(L), \quad q_X(c_1(L)) = 0, \tag{4.0.2}$$

where L is a holomorphic line bundle on X . By Hirzebruch–Riemann–Roch and Serre duality we get that $h^0(L) + h^0(L^{-1}) \geq 2$. Thus we may assume that $h^0(L) \geq 2$. It follows that L is globally generated, $h^0(L) = 2$ and the map $\phi_L : X \rightarrow |L| \cong \mathbb{P}^1$ is an elliptic fibration. Kodaira then proved that any two elliptic K3’s are deformation equivalent. J. Sawon [2003] has launched

a similar program with the goal of classifying deformation classes of higher-dimensional HK manifolds¹⁸ by deforming them to Lagrangian fibrations — we notice that Matsushita [1999; 2001; 2005] has proved quite a few results on HK manifolds which have nontrivial fibrations. The program is quite ambitious; it runs immediately into the problem of proving that if L is a nontrivial line bundle on a HK manifold X with $q_X(c_1(L)) = 0$ then $h^0(L) + h^0(L^{-1}) > 0$.¹⁹ On the other hand Kodaira’s theorem on K3’s can be proved (see [Le Potier 1985]) by deforming X_0 to a K3 surface X such that $H_{\mathbb{Z}}^{1,1}(X) = \mathbb{Z}c_1(L)$ where L is a holomorphic line bundle such that $q_X(L)$ is a small positive integer, say 2. By Hirzebruch–Riemann–Roch and Serre duality $h^0(L) + h^0(L^{-1}) \geq 3$ and hence we may assume that $h^0(L) \geq 3$; it follows easily that L is globally generated, $h^0(L) = 3$ and the map $\phi_L : X \rightarrow |L|^\vee \cong \mathbb{P}^2$ is a double cover ramified over a smooth sextic curve. Thus every K3 is deformation equivalent to a double cover of \mathbb{P}^2 ramified over a sextic; since the parameter space for smooth sextics is connected it follows that any two K3 surfaces are deformation equivalent. Our idea is to adapt this proof to the case of numerical Hilbert squares. In short the plan is as follows. Let X_0 be a numerical Hilbert square. First we deform X_0 to a HK four-fold X such that

$$H_{\mathbb{Z}}^{1,1}(X) = \mathbb{Z}c_1(L), \quad q_X(c_1(L)) = 2 \tag{4.0.3}$$

and the Hodge structure of X is very generic given the constraint (4.0.3), see Section 4.1 for the precise conditions. By Huybrechts’ Projectivity Criterion (Theorem 2.7) we may assume that L is ample and then Hirzebruch–Riemann–Roch together with Kodaira vanishing gives that $h^0(L) = 6$. Thus we must study the map $f : X \dashrightarrow |L|^\vee \cong \mathbb{P}^5$. We prove that either f is the natural double cover of an EPW-sextic or else it is birational onto its image (a hypersurface of degree at most 12). We conjecture that the latter never holds; if the conjecture is true then any numerical Hilbert square is a deformation of a double EPW-sextic and hence is a deformation of K3^[21].

4.1. The deformation. We recall Huybrechts’ theorem on surjectivity of the global period map for HK manifolds. Let X_0 be a HK manifold. Let L be a lattice isomorphic to the lattice $H^2(X_0; \mathbb{Z})$; we denote by $(\cdot, \cdot)_L$ the extension to $L \otimes \mathbb{C}$ of the bilinear symmetric form on L . The period domain $\Omega_L \subset \mathbb{P}(L \otimes \mathbb{C})$ is given by

$$\Omega_L := \{[\sigma] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (\sigma, \sigma)_L = 0, \quad (\sigma, \bar{\sigma})_L > 0\}. \tag{4.1.1}$$

¹⁸One should assume that $b_2 \geq 5$ in order to ensure that the set of rational points in $V(q_X)$ is nonempty (and hence dense in the set of real points).

¹⁹Let $\dim X = 2n$. Hirzebruch–Riemann–Roch gives that $\chi(L) = n + 1$, one would like to show that $h^q(L) = 0$ for $0 < q < 2n$.

A HK manifold X deformation equivalent to X_0 is *marked* if it is equipped with an isometry of lattices $\psi : L \xrightarrow{\sim} H^2(X; \mathbb{Z})$. Two pairs (X, ψ) and (X', ψ') are equivalent if there exists an isomorphism $f : X \rightarrow X'$ such that $H^2(f) \circ \psi' = \pm \psi$. The moduli space \mathcal{M}_{X_0} of marked HK manifolds deformation equivalent to X_0 is the set of equivalence classes of pairs as above. If $t \in \mathcal{M}_{X_0}$ we let (X_t, ψ_t) be a representative of t . Choosing a representative $\mathcal{X} \rightarrow T$ of the deformation space of X_t with T contractible we may put a natural structure of (nonseparated) complex analytic manifold on \mathcal{M}_{X_0} ; see for example Theorem (2.4) of [Looijenga and Peters 1980/81]. The period map is given by

$$\mathcal{M}_{X_0} \xrightarrow{\mathcal{P}} \Omega_L, \quad (X, \psi) \mapsto \psi^{-1} H^{2,0}(X). \tag{4.1.2}$$

(We denote by the same symbol both the isometry $L \xrightarrow{\sim} H^2(X; \mathbb{Z})$ and its linear extension $L \otimes \mathbb{C} \rightarrow H^2(X; \mathbb{C})$.) The map \mathcal{P} is locally an isomorphism by infinitesimal Torelli and local surjectivity of the period map. The following result is proved in [Huybrechts 1999]; the proof is an adaptation of Todorov's proof of surjectivity for K3 surfaces [Todorov 1980].

Theorem 4.1 (Todorov, Huybrechts). *Keep notation as above and let $\mathcal{M}_{X_0}^0$ be a connected component of \mathcal{M}_{X_0} . The restriction of \mathcal{P} to $\mathcal{M}_{X_0}^0$ is surjective.*

Let

$$\Lambda := U^3 \widehat{\oplus} E_8 \langle -1 \rangle^2 \widehat{\oplus} \langle -2 \rangle \tag{4.1.3}$$

be the Hilbert square lattice; see (2.1.8). Thus Ω_Λ is the period space for numerical Hilbert squares. A straightforward computation gives the following result; see Lemma 3.5 of [O'Grady 2008].

Lemma 4.2. *Suppose that $\alpha_1, \alpha_2 \in \Lambda$ satisfy*

$$(\alpha_1, \alpha_1)_\Lambda = (\alpha_2, \alpha_2)_\Lambda = 2, \quad (\alpha_1, \alpha_2)_\Lambda \equiv 1 \pmod{2}. \tag{4.1.4}$$

Let X_0 be a numerical Hilbert square. Let $\mathcal{M}_{X_0}^0$ be a connected component of the moduli space of marked HK four-folds deformation equivalent to X_0 . There exists $1 \leq i \leq 2$ such that for every $t \in \mathcal{M}_{X_0}^0$ the class of $\psi_t(\alpha_i)^2$ in $H^4(X_t; \mathbb{Z})/\text{Tors}$ is indivisible.

Notice that Λ contains (many) pairs α_1, α_2 which satisfy (4.1.4); it follows that there exists $\alpha \in \Lambda$ such that for every $t \in \mathcal{M}_{X_0}^0$ the class of $\psi_t(\alpha)^2$ in $H^4(X_t; \mathbb{Z})/\text{Tors}$ is indivisible. There exists $[\sigma] \in \Omega_\Lambda$ such that

$$\sigma^\perp \cap \Lambda = \mathbb{Z}\alpha. \tag{4.1.5}$$

By Theorem 4.1 there exists $t \in \mathcal{M}_{X_0}$ such that $\mathcal{P}(t) = [\sigma]$. Equality (4.1.5) gives

$$H_{\mathbb{Z}}^{1,1}(X_t) = \mathbb{Z}\alpha. \tag{4.1.6}$$

Since $q(\psi_t(\alpha)) = 2 > 0$ the HK manifold X_t is projective by Theorem 2.7; by (4.1.6) either $\psi_t(\alpha)$ or $\psi_t(-\alpha)$ is ample and hence we may assume that $\psi_t(\alpha)$ is ample. Let $X' := X_t$ and H' be the divisor class such that $c_1(\mathcal{O}_{X'}(H')) = \psi_t(\alpha)$; X' is a first approximation to the deformation of X_0 that we will consider. The reason for requiring that $\psi_t(\alpha)^2$ be indivisible in $H^4(X_t; \mathbb{Z})/\text{Tors}$ will become apparent in the sketch of the proof of Theorem 4.5.

Remark 4.3. If X is a deformation of $K3^{[2]}$ and $\alpha \in H^2(X; \mathbb{Z})$ is an arbitrary class such that $q(\alpha) = 2$ then the class of α^2 in $H^4(X; \mathbb{Z})/\text{Tors}$ is not divisible; see Proposition 3.6 of [O’Grady 2008].

Let $\pi : \mathcal{X} \rightarrow S$ be a representative of the deformation space $\text{Def}(X', H')$. Thus letting $X_s := \pi^{-1}(s)$ there exist $0 \in S$ and a given isomorphism $X_0 \xrightarrow{\sim} X'$ and moreover there is a divisor-class \mathcal{H} on \mathcal{X} which restricts to H' on X_0 ; we let $H_s := \mathcal{H}|_{X_s}$. We will replace (X', H') by (X_s, H_s) for s very general in S in order to ensure that $H^4(X_s)$ has the simplest possible Hodge structure. First we describe the Hodge substructures of $H^4(X_s)$ that are forced by the Beauville–Bogomolov quadratic form and the integral $(1, 1)$ class $\psi_t(\alpha)$. Let X be a HK manifold. The Beauville–Bogomolov quadratic form q_X provides us with a nontrivial class $q_X^\vee \in H_{\mathbb{Q}}^{2,2}(X)$. In fact since q_X is nondegenerate it defines an isomorphism

$$L_X : H^2(X) \xrightarrow{\sim} H^2(X)^\vee. \tag{4.1.7}$$

Viewing q_X as a symmetric tensor in $H^2(X)^\vee \otimes H^2(X)^\vee$ and applying L_X^{-1} we get a class

$$(L_X^{-1} \otimes L_X^{-1})(q_X) \in H^2(X) \otimes H^2(X);$$

applying the cup-product map $H^2(X) \otimes H^2(X) \rightarrow H^4(X)$ to $(L_X^{-1} \otimes L_X^{-1})(q_X)$ we get an element $q_X^\vee \in H^4(X; \mathbb{Q})$ which is of type $(2, 2)$ by (2.1.10). Now we assume that X is a numerical Hilbert square and that H is a divisor class such that $q(H) = 2$. Let $h := c_1(\mathcal{O}_X(H))$. We have an orthogonal (with respect to q_X) direct sum decomposition

$$H^2(X) = \mathbb{C}h \widehat{\oplus} h^\perp \tag{4.1.8}$$

into Hodge substructures of levels 0 and 2 respectively. Since $b_2(X) = 23$ we get by Corollary 2.5 that cup-product defines an isomorphism

$$\text{Sym}^2 H^2(X) \xrightarrow{\sim} H^4(X). \tag{4.1.9}$$

Because of (4.1.9) we will identify $H^4(X)$ with $\text{Sym}^2 H^2(X)$. Thus (4.1.8) gives a direct sum decomposition

$$H^4(X) = \mathbb{C}h^2 \oplus (\mathbb{C}h \otimes h^\perp) \oplus \text{Sym}^2(h^\perp) \tag{4.1.10}$$

into Hodge substructures of levels 0, 2 and 4 respectively. As is easily checked $q_X^\vee \in (\mathbb{C}h^2 \oplus \text{Sym}^2(h^\perp))$. Let

$$W(h) := (q^\vee)^\perp \cap \text{Sym}^2(h^\perp). \tag{4.1.11}$$

(To avoid misunderstandings: the first orthogonality is with respect to the intersection form on $H^4(X)$, the second one is with respect to q_X .) One proves easily (see Claim 3.1 of [O'Grady 2008]) that $W(h)$ is a codimension-1 rational sub Hodge structure of $\text{Sym}^2(h^\perp)$, and that we have a direct sum decomposition

$$\mathbb{C}h^2 \oplus \text{Sym}^2(h^\perp) = \mathbb{C}h^2 \oplus \mathbb{C}q^\vee \oplus W(h). \tag{4.1.12}$$

Thus we have the decomposition

$$H^4(X; \mathbb{C}) = (\mathbb{C}h^2 \oplus \mathbb{C}q^\vee) \oplus (\mathbb{C}h \otimes h^\perp) \oplus W(h) \tag{4.1.13}$$

into sub Hodge structures of levels 0, 2 and 4 respectively.

Claim 4.4 [O'Grady 2008, Proposition 3.2]. *Keep notation as above. Let $s \in S$ be very general, i.e., outside a countable union of proper analytic subsets of S . Then:*

- (1) $H_{\mathbb{Z}}^{1,1}(X_s) = \mathbb{Z}h_s$ where $h_s = c_1(\mathbb{C}_{X_s}(H_s))$.
- (2) Let $\Sigma \in Z_1(X_s)$ be an integral algebraic 1-cycle on X_s and $cl(\Sigma) \in H_{\mathbb{Q}}^{3,3}(X_3)$ be its Poincaré dual. Then $cl(\Sigma) = mh_s^3/6$ for some $m \in \mathbb{Z}$.
- (3) If $V \subset H^4(X_s)$ is a rational sub Hodge structure then $V = V_1 \oplus V_2 \oplus V_3$ where $V_1 \subset (\mathbb{C}h_s^2 \oplus \mathbb{C}q_{X_s}^\vee)$, V_2 is either 0 or equal to $\mathbb{C}h_s \otimes h_s^\perp$, and V_3 is either 0 or equal to $W(h_s)$.
- (4) The image of h_s^2 in $H^4(X_s; \mathbb{Z})/\text{Tors}$ is indivisible.
- (5) $H_{\mathbb{Z}}^{2,2}(X_s)/\text{Tors} \subset \mathbb{Z}(h_s^2/2) \oplus \mathbb{Z}(q_{X_s}^\vee/5)$.

Let $s \in S$ be such that the five conclusions of Claim 4.4 hold. Let $X := X_s$, $H := H_s$ and $h := c_1(\mathbb{C}_X(H))$. Since H is in the positive cone and h generates $H_{\mathbb{Z}}^{1,1}(X)$ we get that H is ample. By construction X is a deformation of our given numerical Hilbert square. The goal is to analyze the linear system $|H|$. First we compute its dimension. A computation (see pp. 564-565 of [O'Grady 2008]) gives that $c_2(X) = 6q_X^\vee/5$; it follows that (2.2.7) holds for numerical Hilbert squares. Thus $\chi(\mathbb{C}_X(H)) = 6$. By Kodaira vanishing we get $h^0(\mathbb{C}_X(H)) = 6$. Thus we have the map

$$f : X \dashrightarrow |H|^\vee \cong \mathbb{P}^5. \tag{4.1.14}$$

Theorem 4.5 [O’Grady 2008]. *Let (X, H) be as above. One of the following holds:*

- (a) *The line bundle $\mathbb{C}_X(H)$ is globally generated and there exist an antisymplectic involution $\phi : X \rightarrow X$ and an inclusion $X/\langle\phi\rangle \hookrightarrow |H|^\vee$ such that the map f of (4.1.14) is identified with the composition*

$$X \xrightarrow{\rho} X/\langle\phi\rangle \hookrightarrow |H|^\vee \tag{4.1.15}$$

where ρ is the quotient map.

- (b) *The map f of (4.1.14) is birational onto its image (a hypersurface of degree between 6 and 12).*

Sketch of proof. We use the following result, which follows from conclusions (4) and (5) of Claim 4.4 plus a straightforward computation; see Proposition 4.1 of [O’Grady 2008].

Claim 4.6. *If $D_1, D_2 \in |H|$ are distinct then $D_1 \cap D_2$ is a reduced irreducible surface.*

In fact we chose h such that h^2 is not divisible in $H^4(X; \mathbb{Z})/\text{Tors}$ precisely to ensure that this claim holds. Let $Y \subset \mathbb{P}^5$ be the image of f (to be precise the closure of the image by f of its regular points). Thus (abusing notation) we have $f : X \dashrightarrow Y$. Of course $\dim Y \leq 4$. Suppose that $\dim Y = 4$ and that $\deg f = 2$. Then there exists a nontrivial rational involution $\phi : X \dashrightarrow X$ commuting with f . Since $\text{Pic}(X) = \mathbb{Z}[H]$ we get that $\phi^*H \sim H$; since $K_X \sim 0$ it follows that ϕ is regular; it follows easily that (a) holds. Thus it suffices to reach a contradiction assuming that $\dim Y < 4$ or $\dim Y = 4$ and $\deg f > 2$. One goes through a (painful) case-by-case analysis. In each case, with the exception of Y a quartic 4-fold, one invokes either Claim 4.6 or Claim 4.4(3). We give two “baby” cases. First suppose that Y is a quadric 4-fold. Let Y_0 be an open dense subset containing the image by f of its regular points. There exists a 3-dimensional linear space $L \subset \mathbb{P}^5$ such that $L \cap Y_0$ is a reducible surface. Now L corresponds to the intersection of two distinct $D_1, D_2 \in |H|$ and since $L \cap Y_0$ is reducible so is $D_1 \cap D_2$; this contradicts Claim 4.6. As a second example we suppose that Y is a smooth cubic 4-fold and f is regular. Notice that

$$H \cdot H \cdot H \cdot H = 12 \tag{4.1.16}$$

by (2.1.3) and hence $\deg f = 4$. Let $H^4(Y)_{pr} \subset H^4(Y)$ be the primitive cohomology. By Claim 4.4(3) we must have $f^*H^4(Y)_{pr} \subset \mathbb{C}h \otimes h^\perp$. The restriction to $f^*H^4(Y; \mathbb{Q})_{pr}$ of the intersection form on $H^4(X)$ equals the intersection form on $H^4(Y; \mathbb{Q})_{pr}$ multiplied by 4 because $\deg f = 4$; one gets a contradiction by comparing discriminants. □

Conjecture 4.7. Item (b) of Theorem 4.5 does not occur.

As we will explain in the next subsection, Conjecture 4.7 implies that a numerical Hilbert square is in fact a deformation of $K3^{[2]}$. The following question arose in connection with the proof of Theorem 4.5.

Question 4.8. Let X be a HK 4-fold and H an ample divisor on X . Is $\mathcal{O}_X(2H)$ globally generated?

The analogous question in dimension 2 has a positive answer; see for example [Mayer 1972]. We notice that if X is a 4-fold with trivial canonical bundle and H is ample on X then $\mathcal{O}_X(5H)$ is globally generated, by [Kawamata 1997]. The relation between Question 4.8 and Theorem 4.5 is the following.

Claim 4.9. *Suppose that the answer to Question 4.8 is positive. Let X be a numerical Hilbert square equipped with an ample divisor H such that $q_X(H) = 2$. Let $Y \subset |H|^\vee$ be the closure of the image of the set of regular points of the rational map $X \dashrightarrow |H|^\vee$. Then one of the following holds:*

- (1) $\mathcal{O}_X(H)$ is globally generated.
- (2) Y is contained in a quadric.

Proof. Suppose that alternative (2) does not hold. Then multiplication of sections defines an injection $\text{Sym}^2 H^0(\mathcal{O}_X(H)) \hookrightarrow H^0(\mathcal{O}_X(2H))$; on the other hand we have

$$\dim \text{Sym}^2 H^0(\mathcal{O}_X(H)) = 21 = \dim H^0(\mathcal{O}_X(2H)). \tag{4.1.17}$$

(The last equation holds by (2.2.7), which is valid for numerical Hilbert squares as noticed above.) Since $\mathcal{O}_X(2H)$ is globally generated it follows that $\mathcal{O}_X(H)$ is globally generated as well, and alternative (1) holds. \square

We remark that alternatives (1) and (2) of the claim are not mutually exclusive. In fact let $S \subset \mathbb{P}^3$ be a smooth quartic surface (a $K3$) not containing lines. We have a finite map

$$S^{[2]} \xrightarrow{f} \text{Gr}(1, \mathbb{P}^3) \subset \mathbb{P}^5, \quad [Z] \mapsto \langle Z \rangle \tag{4.1.18}$$

with image the Plücker quadric in \mathbb{P}^5 . Let $H := f^* \mathcal{O}_{\mathbb{P}^5}(1)$; since f is finite H is ample. Moreover $q(H) = 2$ because $H \cdot H \cdot H \cdot H = 12$; thus (4.1.18) may be identified with the map associated to the complete linear system $|H|$.

4.2. Double EPW-sextics, II. Let (X, H) be as in Theorem 4.5(a). In [O'Grady 2006] we proved that there exists $A \in \mathbb{L}G(\wedge^3 \mathbb{C}^6)^0$ such that $Y_A = f(X)$ and the double cover $X \rightarrow f(X)$ may be identified with the canonical double cover $X_A \rightarrow Y_A$. Since X_A is a deformation of $K3^{[2]}$ it follows that if Conjecture 4.7 holds then numerical Hilbert squares are deformations of $K3^{[2]}$. The precise result is this:

Theorem 4.10 [O’Grady 2006]. *Let X be a numerical Hilbert square. Suppose that H is an ample divisor class on X such that the following hold:*

- (1) $q_X(H) = 2$ (and hence $\dim |H| = 5$).
- (2) $\mathcal{O}_X(H)$ is globally generated.
- (3) *There exist an antisymplectic involution $\phi : X \rightarrow X$ and an inclusion $X/\langle\phi\rangle \hookrightarrow |H|^\vee$ such that the map $X \rightarrow |H|^\vee$ is identified with the composition*

$$X \xrightarrow{\rho} X/\langle\phi\rangle \hookrightarrow |H|^\vee \tag{4.2.1}$$

where ρ is the quotient map.

Then there exists $A \in \mathbb{L}G(\wedge^3 \mathbb{C}^6)^0$ such that $Y_A = Y$ and the double cover $X \rightarrow f(X)$ may be identified with the canonical double cover $X_A \rightarrow Y_A$.

Proof. Step I. Let $Y := f(X)$; abusing notation we let $f : X \rightarrow Y$ be the double cover which is identified with the quotient map for the action of $\langle\phi\rangle$. We have the decomposition $f_*\mathcal{O}_X = \mathcal{O}_Y \oplus \eta$ where η is the (-1) -eigensheaf for the action of ϕ on \mathcal{O}_X . One proves that $\zeta := \eta \otimes \mathcal{O}_Y(3)$ is globally generated — an intermediate step is the proof that $3H$ is very ample. Thus we have an exact sequence

$$0 \rightarrow G \rightarrow H^0(\zeta) \otimes \mathcal{O}_{|H|^\vee} \rightarrow i_*\zeta \rightarrow 0. \tag{4.2.2}$$

where $i : Y \hookrightarrow |H|^\vee$ is inclusion.

Step II. One computes $h^0(\zeta)$ as follows. First $H^0(\zeta)$ is equal to $H^0(\mathcal{O}_X(3H))^-$, the space of ϕ -anti-invariant sections of $\mathcal{O}_X(3H)$. Using Equation (2.2.7) one gets that $h^0(\zeta) = 10$. A local computation shows that G is locally free. By invoking Beilinson’s spectral sequence for vector bundles on projective spaces one gets that $G \cong \Omega^3_{|H|^\vee}(3)$. On the other hand one checks easily (Euler sequence) that the vector bundle F of (3.1.3) is isomorphic to $\Omega^3_{\mathbb{P}(V)}(3)$. Hence if we identify $\mathbb{P}(V)$ with $|H|^\vee$ then F is isomorphic to the sheaf G appearing in (4.2.2). In other words (4.2.2) starts looking like the top horizontal sequence of (3.1.19).

Step III. The multiplication map $\eta \otimes \eta \rightarrow \mathcal{O}_Y$ defines an isomorphism $\beta : i_*\zeta \xrightarrow{\sim} \text{Ext}^1(i_*\zeta, \mathcal{O}_{|H|^\vee})$. Applying general results of Eisenbud, Popescu, and Walter [Eisenbud et al. 2001] (alternatively see the proof of Claim (2.1) of [Casnati and Catanese 1997]) one gets that β fits into a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega^3_{|H|^\vee}(3) & \xrightarrow{\kappa} & H^0(\theta) \otimes \mathcal{O}_{|H|^\vee} & \longrightarrow & i_*\zeta & \longrightarrow & 0 \\ & & \downarrow s^t & & \downarrow s & & \downarrow \beta & & \\ 0 & \longrightarrow & H^0(\theta)^\vee \otimes \mathcal{O}_{|H|^\vee} & \xrightarrow{\kappa^t} & \Theta^3_{|H|^\vee}(-3) & \xrightarrow{\partial} & \text{Ext}^1(i_*\zeta, \mathcal{O}_{|H|^\vee}) & \longrightarrow & 0 \end{array} \tag{4.2.3}$$

where the second row is obtained from the first one by applying $\text{Hom}(\cdot, \mathcal{O}_{|H|^\vee})$.

Step IV. One checks that

$$\Omega_{|H|^\vee}^3 \xrightarrow{(\kappa, s^t)} (H^0(\zeta) \oplus H^0(\zeta)^\vee) \otimes \mathcal{O}_{|H|^\vee} \quad (4.2.4)$$

is an injection of vector bundles. The transpose of the map above induces an isomorphism $(H^0(\zeta)^\vee \oplus H^0(\zeta)) \xrightarrow{\sim} H^0(\Omega_{|H|^\vee}^3)^\vee$. The same argument shows that the transpose of (3.1.3) induces an isomorphism $\wedge^3 V^\vee \xrightarrow{\sim} H^0(F^\vee)$. Since F is isomorphic to $\Omega_{|H|^\vee}^3$ we get an isomorphism $\rho: H^0(\zeta) \oplus H^0(\zeta)^\vee \xrightarrow{\sim} \wedge^3 V$ such that (abusing notation) $\rho(\Omega_{|H|^\vee}^3) = F$. Lastly one checks that the standard symplectic form on $(H^0(\zeta) \oplus H^0(\zeta)^\vee)$ is identified (up to a multiple) via ρ with the symplectic form $(\cdot, \cdot)_V$ of (3.1.1). Now let $A = \rho(H^0(\zeta)^\vee)$; then (4.2.3) is identified with (3.1.19). This ends the proof of Theorem 4.10. \square

References

- [Beauville 1983] A. Beauville, “Variétés Kähleriennes dont la première classe de Chern est nulle”, *J. Differential Geom.* **18**:4 (1983), 755–782.
- [Beauville and Donagi 1985] A. Beauville and R. Donagi, “La variété des droites d’une hypersurface cubique de dimension 4”, *C. R. Acad. Sci. Paris Sér. I Math.* **301**:14 (1985), 703–706.
- [Bogomolov 1978] F. A. Bogomolov, “Hamiltonian Kählerian manifolds”, *Dokl. Akad. Nauk SSSR* **243**:5 (1978), 1101–1104. In Russian; translated in *Soviet Math. Dokl.* **19** (1978), 1462–1465.
- [Bogomolov 1996] F. A. Bogomolov, “On the cohomology ring of a simple hyper-Kähler manifold (on the results of Verbitsky)”, *Geom. Funct. Anal.* **6**:4 (1996), 612–618.
- [Boucksom 2001] S. Boucksom, “Le cône kählérien d’une variété hyperkählérienne”, *C. R. Acad. Sci. Paris Sér. I Math.* **333**:10 (2001), 935–938.
- [Boucksom 2004] S. Boucksom, “Divisorial Zariski decompositions on compact complex manifolds”, *Ann. Sci. École Norm. Sup. (4)* **37**:1 (2004), 45–76.
- [Burns and Rapoport 1975] D. Burns, Jr. and M. Rapoport, “On the Torelli problem for kählerian K-3 surfaces”, *Ann. Sci. École Norm. Sup. (4)* **8**:2 (1975), 235–273.
- [Casnati and Catanese 1997] G. Casnati and F. Catanese, “Even sets of nodes are bundle symmetric”, *J. Differential Geom.* **47**:2 (1997), 237–256.
- [Debarre 1984] O. Debarre, “Un contre-exemple au théorème de Torelli pour les variétés symplectiques irréductibles”, *C. R. Acad. Sci. Paris Sér. I Math.* **299**:14 (1984), 681–684.
- [Debarre and Voisin 2010] O. Debarre and C. Voisin, “Hyper-Kähler fourfolds and Grassmann geometry”, *J. Reine Angew. Math.* **649** (2010), 63–87.
- [Demailly and Paun 2004] J.-P. Demailly and M. Paun, “Numerical characterization of the Kähler cone of a compact Kähler manifold”, *Ann. of Math. (2)* **159**:3 (2004), 1247–1274.
- [Druel 2011] S. Druel, “Quelques remarques sur la décomposition de Zariski divisorielle sur les variétés dont la première classe de Chern est nulle”, *Math. Z.* **267**:1-2 (2011), 413–423.
- [Eisenbud et al. 2001] D. Eisenbud, S. Popescu, and C. Walter, “Lagrangian subbundles and codimension 3 subcanonical subschemes”, *Duke Math. J.* **107**:3 (2001), 427–467.
- [Fantechi 1995] B. Fantechi, “Deformation of Hilbert schemes of points on a surface”, *Compositio Math.* **98**:2 (1995), 205–217.

- [Fujiki 1983] A. Fujiki, “On primitively symplectic compact Kähler V -manifolds of dimension four”, pp. 71–250 in *Classification of algebraic and analytic manifolds* (Katata, 1982), Progr. Math. **39**, Birkhäuser, Boston, 1983.
- [Fujiki 1987] A. Fujiki, “On the de Rham cohomology group of a compact Kähler symplectic manifold”, pp. 105–165 in *Algebraic geometry* (Sendai, 1985), Adv. Stud. Pure Math. **10**, North-Holland, Amsterdam, 1987.
- [Fulton and Pragacz 1998] W. Fulton and P. Pragacz, *Schubert varieties and degeneracy loci*, Lecture Notes in Mathematics **1689**, Springer, Berlin, 1998.
- [Göttsche 1994] L. Göttsche, *Hilbert schemes of zero-dimensional subschemes of smooth varieties*, Lecture Notes in Mathematics **1572**, Springer, Berlin, 1994.
- [Göttsche and Huybrechts 1996] L. Göttsche and D. Huybrechts, “Hodge numbers of moduli spaces of stable bundles on K3 surfaces”, *Internat. J. Math.* **7**:3 (1996), 359–372.
- [Gritsenko et al. 2007] V. A. Gritsenko, K. Hulek, and G. K. Sankaran, “The Kodaira dimension of the moduli of K3 surfaces”, *Invent. Math.* **169**:3 (2007), 519–567.
- [Guan 2001] D. Guan, “On the Betti numbers of irreducible compact hyperkähler manifolds of complex dimension four”, *Math. Res. Lett.* **8**:5-6 (2001), 663–669.
- [Hassett and Tschinkel 2001] B. Hassett and Y. Tschinkel, “Rational curves on holomorphic symplectic fourfolds”, *Geom. Funct. Anal.* **11**:6 (2001), 1201–1228.
- [Hassett and Tschinkel 2009a] B. Hassett and Y. Tschinkel, “Intersection numbers of extremal rays on holomorphic symplectic four-folds”, preprint, 2009. arXiv 0909.4745
- [Hassett and Tschinkel 2009b] B. Hassett and Y. Tschinkel, “Moving and ample cones of holomorphic symplectic fourfolds”, *Geom. Funct. Anal.* **19**:4 (2009), 1065–1080.
- [Huybrechts 1999] D. Huybrechts, “Compact hyper-Kähler manifolds: basic results”, *Invent. Math.* **135**:1 (1999), 63–113. Erratum in [Huybrechts 2003a].
- [Huybrechts 2003a] D. Huybrechts, “Erratum to ‘Compact hyper-Kähler manifolds: basic results’”, *Invent. Math.* **152**:1 (2003), 209–212.
- [Huybrechts 2003b] D. Huybrechts, “Finiteness results for compact hyperkähler manifolds”, *J. Reine Angew. Math.* **558** (2003), 15–22.
- [Huybrechts 2003c] D. Huybrechts, “The Kähler cone of a compact hyperkähler manifold”, *Math. Ann.* **326**:3 (2003), 499–513.
- [Huybrechts 2011] D. Huybrechts, “A global Torelli theorem for hyperkaehler manifolds (after Verbitsky)”, preprint, 2011. arXiv 1106.5573
- [Iliev and Manivel 2009] A. Iliev and L. Manivel, “Fano manifolds of degree ten and EPW sextics”, preprint, 2009. arXiv 0907.2781
- [Iliev and Ranestad 2001] A. Iliev and K. Ranestad, “K3 surfaces of genus 8 and varieties of sums of powers of cubic fourfolds”, *Trans. Amer. Math. Soc.* **353**:4 (2001), 1455–1468.
- [Iliev and Ranestad 2007] A. Iliev and K. Ranestad, “Addendum to ‘K3 surfaces of genus 8 and varieties of sums of powers of cubic fourfolds’ [Trans. Amer. Math. Soc. **353** (2001), no. 4, 1455–1468; MR1806733]”, *C. R. Acad. Bulgare Sci.* **60**:12 (2007), 1265–1270.
- [Kaledin et al. 2006] D. Kaledin, M. Lehn, and C. Sorger, “Singular symplectic moduli spaces”, *Invent. Math.* **164**:3 (2006), 591–614.
- [Kawamata 1997] Y. Kawamata, “On Fujita’s freeness conjecture for 3-folds and 4-folds”, *Math. Ann.* **308**:3 (1997), 491–505.
- [Kodaira 1964] K. Kodaira, “On the structure of compact complex analytic surfaces, I”, *Amer. J. Math.* **86** (1964), 751–798.

- [Le Potier 1985] J. Le Potier, “Simple connexité des surfaces K3”, pp. 79–89 in *Geometry of K3 surfaces: moduli and periods* (Palaiseau, 1981/1982), Astérisque **126**, Soc. Math. de France, 1985.
- [Lehn and Sorger 2006] M. Lehn and C. Sorger, “La singularité de O’Grady”, *J. Algebraic Geom.* **15**:4 (2006), 753–770.
- [Looijenga and Peters 1980/81] E. Looijenga and C. Peters, “Torelli theorems for Kähler K3 surfaces”, *Compositio Math.* **42**:2 (1980/81), 145–186.
- [Markman 2001] E. Markman, “Brill-Noether duality for moduli spaces of sheaves on K3 surfaces”, *J. Algebraic Geom.* **10**:4 (2001), 623–694.
- [Markman 2009] E. Markman, “Prime exceptional divisors on holomorphic symplectic varieties and modromy-reflections”, preprint, 2009. arXiv 0912.4981
- [Markman 2010] E. Markman, “Integral constraints on the monodromy group of the hyperKähler resolution of a symmetric product of a K3 surface”, *Internat. J. Math.* **21**:2 (2010), 169–223.
- [Matsushita 1999] D. Matsushita, “On fibre space structures of a projective irreducible symplectic manifold”, *Topology* **38**:1 (1999), 79–83.
- [Matsushita 2001] D. Matsushita, “Addendum: “On fibre space structures of a projective irreducible symplectic manifold” [Topology **38** (1999), no. 1, 79–83; MR1644091 (99f:14054)]”, *Topology* **40**:2 (2001), 431–432.
- [Matsushita 2005] D. Matsushita, “Higher direct images of dualizing sheaves of Lagrangian fibrations”, *Amer. J. Math.* **127**:2 (2005), 243–259.
- [Mayer 1972] A. L. Mayer, “Families of $K - 3$ surfaces”, *Nagoya Math. J.* **48** (1972), 1–17.
- [Mukai 1984] S. Mukai, “Symplectic structure of the moduli space of sheaves on an abelian or K3 surface”, *Invent. Math.* **77**:1 (1984), 101–116.
- [Mukai 1987a] S. Mukai, “Moduli of vector bundles on K3 surfaces and symplectic manifolds”, *Sūgaku* **39**:3 (1987), 216–235. Reprinted in *Sugaku Expositions* **1**:2 (1988), 139–174.
- [Mukai 1987b] S. Mukai, “On the moduli space of bundles on K3 surfaces, I”, pp. 341–413 in *Vector bundles on algebraic varieties*, Tata Inst. Fund. Res. Stud. Math. **11**, Tata Inst. Fund. Res., Bombay, 1987.
- [Mukai 1988] S. Mukai, “Curves, K3 surfaces and Fano 3-folds of genus ≤ 10 ”, pp. 357–377 in *Algebraic geometry and commutative algebra*, vol. I, Kinokuniya, Tokyo, 1988.
- [Mukai 2006] S. Mukai, “Polarized K3 surfaces of genus thirteen”, pp. 315–326 in *Moduli spaces and arithmetic geometry*, Adv. Stud. Pure Math. **45**, Math. Soc. Japan, Tokyo, 2006.
- [Namikawa 2002] Y. Namikawa, “Counter-example to global Torelli problem for irreducible symplectic manifolds”, *Math. Ann.* **324**:4 (2002), 841–845.
- [O’Grady 1997] K. G. O’Grady, “The weight-two Hodge structure of moduli spaces of sheaves on a K3 surface”, *J. Algebraic Geom.* **6**:4 (1997), 599–644.
- [O’Grady 1999] K. G. O’Grady, “Desingularized moduli spaces of sheaves on a K3”, *J. Reine Angew. Math.* **512** (1999), 49–117.
- [O’Grady 2003] K. G. O’Grady, “A new six-dimensional irreducible symplectic variety”, *J. Algebraic Geom.* **12**:3 (2003), 435–505.
- [O’Grady 2006] K. G. O’Grady, “Irreducible symplectic 4-folds and Eisenbud-Popescu-Walter sextics”, *Duke Math. J.* **134**:1 (2006), 99–137.
- [O’Grady 2008] K. G. O’Grady, “Irreducible symplectic 4-folds numerically equivalent to $(K3)^{[2]}$ ”, *Commun. Contemp. Math.* **10**:4 (2008), 553–608.

- [Perego and Rapagnetta 2010] A. Perego and A. Rapagnetta, “Deformation of the O’Grady moduli spaces”, preprint, 2010. arXiv 1008.0190
- [Piatetski-Shapiro and Shafarevich 1971] I. Piatetski-Shapiro and I. R. Shafarevich, “A Torelli theorem for algebraic surfaces of type K3”, *Math. USSR Izvestija* **5** (1971), 547–588.
- [Rapagnetta 2007] A. Rapagnetta, “Topological invariants of O’Grady’s six dimensional irreducible symplectic variety”, *Math. Z.* **256**:1 (2007), 1–34.
- [Rapagnetta 2008] A. Rapagnetta, “On the Beauville form of the known irreducible symplectic varieties”, *Math. Ann.* **340**:1 (2008), 77–95.
- [Salamon 1996] S. M. Salamon, “On the cohomology of Kähler and hyper-Kähler manifolds”, *Topology* **35**:1 (1996), 137–155.
- [Sawon 2003] J. Sawon, “Abelian fibred holomorphic symplectic manifolds”, *Turkish J. Math.* **27**:1 (2003), 197–230.
- [Todorov 1980] A. N. Todorov, “Applications of the Kähler–Einstein–Calabi–Yau metric to moduli of K3 surfaces”, *Invent. Math.* **61**:3 (1980), 251–265.
- [Tyurin 1987] A. N. Tyurin, “Cycles, curves and vector bundles on an algebraic surface”, *Duke Math. J.* **54**:1 (1987), 1–26.
- [Verbitsky 1996] M. Verbitsky, “Cohomology of compact hyper-Kähler manifolds and its applications”, *Geom. Funct. Anal.* **6**:4 (1996), 601–611.
- [Verbitsky 2009] M. Verbitsky, “Mapping class group and a global Torelli theorem for hyper-Kähler manifolds”, preprint, 2009. arXiv 0908.4121v5
- [Wierzba and Wiśniewski 2003] J. Wierzba and J. A. Wiśniewski, “Small contractions of symplectic 4-folds”, *Duke Math. J.* **120**:1 (2003), 65–95.
- [Yoshioka 1999] K. Yoshioka, “Some examples of Mukai’s reflections on K3 surfaces”, *J. Reine Angew. Math.* **515** (1999), 97–123.
- [Yoshioka 2001] K. Yoshioka, “Moduli spaces of stable sheaves on abelian surfaces”, *Math. Ann.* **321**:4 (2001), 817–884.

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