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Compactifications of moduli of abelian varieties: an introduction

MARTIN OLSSON

We survey the various approaches to compactifying moduli stacks of polarized abelian varieties. To motivate the different approaches to compactifying, we first discuss three different points of view of the moduli stacks themselves. Then we explain how each point of view leads to a different compactification. Throughout we emphasize maximal degenerations which capture much of the essence of the theory without many of the technicalities.

1. Introduction

A central theme in modern algebraic geometry is to study the degenerations of algebraic varieties, and its relationship with compactifications of moduli stacks. The standard example considered in this context is the moduli stack $\mathcal{M}_g$ of genus g curves (where $g \geq 2$) and the Deligne–Mumford compactification $\mathcal{M}_g \subset \overline{\mathcal{M}}_g$ [Deligne and Mumford 1969]. The stack $\overline{\mathcal{M}}_g$ has many wonderful properties:

(1) It has a moduli interpretation as the moduli stack of stable genus g curves.
(2) The stack $\overline{\mathcal{M}}_g$ is smooth.
(3) The inclusion $\mathcal{M}_g \hookrightarrow \overline{\mathcal{M}}_g$ is a dense open immersion and $\overline{\mathcal{M}}_g \backslash \mathcal{M}_g$ is a divisor with normal crossings in $\overline{\mathcal{M}}_g$.

Unfortunately the story of the compactification $\mathcal{M}_g \subset \overline{\mathcal{M}}_g$ is not reflective of the general situation. There are very few known instances where one has a moduli stack $\mathcal{M}$ classifying some kind of algebraic varieties and a compactification $\mathcal{M} \subset \overline{\mathcal{M}}$ with the three properties above.

After studying moduli of curves, perhaps to next natural example to consider is the moduli stack $\mathcal{A}_g$ of principally polarized abelian varieties of a fixed dimension g. Already here the story becomes much more complicated, though work of several people has led to a compactification $\mathcal{A}_g \subset \overline{\mathcal{A}}_g$ which enjoys the following properties:

(1) The stack $\overline{\mathcal{A}}_g$ is the solution to a natural moduli problem.
(2′) The stack $\overline{\mathcal{A}}_g$ has only toric singularities.

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(3′) The inclusion $\mathscr{A}_g \hookrightarrow \overline{\mathscr{A}}_g$ is a dense open immersion, and the complement $\overline{\mathscr{A}}_g \backslash \mathscr{A}_g$ defines an fs-log structure $M_{\overline{\mathscr{A}}_g}$ (in the sense of Fontaine and Illusie [Kato 1989]) on $\overline{\mathscr{A}}_g$ such that $(\overline{\mathscr{A}}_g, M_{\overline{\mathscr{A}}_g})$ is log smooth over Spec($\mathbb{Z}$).

Our aim in this paper is to give an overview of the various approaches to compactifying $\mathscr{A}_g$, and to outline the story of the canonical compactification $\mathscr{A}_g \hookrightarrow \overline{\mathscr{A}}_g$. In addition, we also consider higher degree polarizations.

What one considers a 'natural' compactification of $\mathscr{A}_g$ depends to a large extent on one's view of $\mathscr{A}_g$ itself. There are three basic points of view of this moduli stack (which of course are all closely related):

(*The standard approach*). Here one views $\mathscr{A}_g$ as classifying pairs (A, λ), where A is an abelian variety of dimension g and $\lambda : A \to A^t$ is an isomorphism between A and its dual (a *principal polarization*), such that λ is equal to the map defined by an ample line bundle, but one does not fix such a line bundle. This point of view is the algebraic approach most closely tied to Hodge theory.

(*Moduli of pairs approach*). This is the point of view taken in Alexeev's work [2002]. Here one encodes the ambiguity of the choice of line bundle defining λ into a torsor under A. So $\mathscr{A}_g$ is viewed as classifying collections of data (A, P, L, θ), where A is an abelian variety of dimension g, P is an A-torsor, L is an ample line bundle on P defining a principal polarization on A (see 2.2.3), and $\theta \in \Gamma(P, L)$ is a nonzero global section.

(*Theta group approach*). This point of view comes out of Mumford's theory [1966; 1967] of the theta group, combined with Alexeev's approach via torsors. Here one considers triples (A, P, L), where A is an abelian variety of dimension g, P is an A-torsor, and L is an ample line bundle on P defining a principal polarization on A (but one does not fix a section of L). This gives a stack $\mathscr{T}_g$ which is a gerbe over $\mathscr{A}_g$ bound by $\mathbb{G}_m$. Using a standard stack-theoretic construction called *rigidification* one can then construct $\mathscr{A}_g$ from $\mathscr{T}_g$, but in the theta group approach the stack $\mathscr{T}_g$ is the more basic object.

In Section 2 we discuss each of these three points of view of the moduli of principally polarized abelian varieties (and moduli of abelian varieties with higher degree polarization). Then in sections 3 and 4 we discuss how each of these three approaches leads to different compactifications (toroidal, Alexeev, and $\overline{\mathscr{A}}_g$ respectively). We discuss in some detail in the maximally degenerate case the relationship between degenerating abelian varieties and quadratic forms. This relationship is at the heart of all of the different approaches to compactification. We do not discuss the case of partial degenerations where one has to introduce the theory of biextensions (for this the reader should consult [Faltings and Chai 1990]), since most of the main ideas can already be seen in the maximally degenerate case.

Finally in Section 5 we give an overview of how the canonical compactification can be used to compactify moduli stacks for abelian varieties with level structure and higher degree polarizations, using the theta group approach.

Our aim here is not to give a complete treatment, but rather to give the reader an indication of some of the basic ideas involved. Much of our focus is on the local structure of these moduli stacks at points of maximal degeneration in the boundary of the various compactifications (i.e., points where the degeneration of the abelian scheme is a torus). This is because the local structure of the moduli stacks can be seen more clearly here, and because the case of partial degeneration introduces many more technicalities (in particular, in this paper we do not discuss the theory of biextensions). We hardly touch upon the issues involved in going from the local study to the global. The interested reader should consult the original sources [Alexeev 2002; Faltings and Chai 1990; Olsson 2008].

Perhaps preceding the entire discussion of this paper is the theory of the Sataka/Baily–Borel/minimal compactification of $\mathscr{A}_g$, and the connection with modular forms. We should also remark that a beautiful modular interpretation of the toroidal compactifications using log abelian varieties has been developed by Kajiwara, Kato, and Nakayama [Kajiwara et al. 2008a; 2008b]. We do not, however, discuss either of these topics here.

Acknowledgements. The aim of this article is to give a survey of known results, and there are no new theorems. The results discuss here are the fruits of work of many people. We won't try to make an exhaustive list, but let us at least mention two basic sources: [Faltings and Chai 1990] and [Alexeev 2002], from which we learned the bulk of the material on toroidal compactifications and Alexeev's compactification, respectively. We thank the referee for helpful comments on the first version of the paper.

Prerequisites and conventions. We assume that the reader is familiar with the basic theory of abelian varieties as developed for example in [Mumford 1970]. We also assume the reader is familiar with stacks at the level of [Laumon and Moret-Bailly 2000]. Finally knowledge of logarithmic geometry in the sense of Fontaine and Illusie [Kato 1989] will be assumed for sections 4.5 and 5.

Our conventions about algebraic stacks are those of [Laumon and Moret-Bailly 2000].

2. Three perspectives on $\mathscr{A}_g$

2.1. *The standard definition.*

2.1.1. Let k be an algebraically closed field, and let A/k be an abelian variety. Let A^t denote the dual abelian variety of A (see [Mumford 1970, Chapter III,

§13]). Recall that A^t is the connected component of the identity in the Picard variety $\underline{\mathrm{Pic}}_{A/k}$ of A. If L is a line bundle on A, then we obtain a map

$$\lambda_L : A \to A^t, \quad x \mapsto [t_x^* L \otimes L^{-1}],$$

where $t_x : A \to A$ denotes translation by the point x. If L is ample then λ_L is finite and the kernel is a finite group scheme over k (by [Mumford 1970, Application 1 on p. 60]) whose rank is a square by [Mumford 1970, Riemann–Roch theorem, p. 150]. The *degree* of an ample line bundle L is defined to be the positive integer d for which the rank of $\mathrm{Ker}(\lambda_L)$ is d^2. The degree d can also be characterized as the dimension of the k-space $\Gamma(A, L)$ (loc. cit.).

Definition 2.1.2. Let $d \geq 1$ be an integer. A *polarization of degree* d on an abelian variety A/k is a morphism $\lambda : A \to A^t$ of degree d^2, which is equal to λ_L for some ample line bundle L on A. A *principal polarization* is a polarization of degree 1.

Remark 2.1.3. If L and L' are two ample line bundles on an abelian variety A/k, then $\lambda_L = \lambda_{L'}$ if and only if $L' \simeq t_x^* L$ for some point $x \in A(k)$. Indeed $\lambda_L = \lambda_{L'}$ if and only if

$$\lambda_{L' \otimes L^{-1}} = \{e\} \quad \text{(constant map)},$$

which by the definition of the dual abelian variety (see for example [Mumford 1970, p. 125]) is equivalent to the statement that the line bundle $L' \otimes L^{-1}$ defines a point of A^t. Since λ_L is surjective, this in turn is equivalent to the statement that there exists a point $x \in A(k)$ such that

$$t_x^* L \otimes L^{-1} \simeq L' \otimes L^{-1},$$

or equivalently that $t_x^* L \simeq L'$. The same argument shows that if L and L' are line bundles such that $\lambda_L = \lambda_{L'}$ then L is ample if and only if L' is ample.

2.1.4. These definitions extend naturally to families. Recall [Mumford 1965, Definition 6.1] that if S is a scheme then an *abelian scheme over* S is a smooth proper group scheme A/S with geometrically connected fibers. As in the case of abelian varieties, the group scheme structure on A is determined by the zero section [Mumford 1965, Corollary 6.6].

For an abelian scheme A/S, one can define the dual abelian scheme A^t/S as a certain subgroup scheme of the relative Picard scheme $\underline{\mathrm{Pic}}_{A/S}$ (see [Mumford 1965, Corollary 6.8] for more details). As in the case of a field, any line bundle L on A defines a homomorphism

$$\lambda_L : A \to A^t.$$

If L is relatively ample then λ_L is finite and flat, and the kernel $\mathrm{Ker}(\lambda_L)$ has rank d^2 for some locally constant positive integer-valued function d on S. If $\pi : A \to S$ denotes the structure morphism, then we have

$$R^i \pi_* L = 0, \quad i > 0,$$

and $\pi_* L$ is a locally free sheaf of rank d on S whose formation commutes with arbitrary base change $S' \to S$ (this follows from the vanishing theorem for higher cohomology over fields [Mumford 1970, p. 150] and cohomology and base change).

Definition 2.1.5. Let $d \geq 1$ be an integer. A *polarization of degree d* on an abelian scheme A/S is a homomorphism $\lambda : A \to A^t$ such that for every geometric point $\bar{s} \to S$ the map on geometric fibers $A_{\bar{s}} \to A^t_{\bar{s}}$ is a polarization of degree d in the sense of 2.1.2.

Remark 2.1.6. By a similar argument as in 2.1.3, if A/S is an abelian scheme over a base S, and if L and L' are two relatively ample line bundles on A, then $\lambda_L = \lambda_{L'}$ if and only if there exists a point $x \in A(S)$ such that L' and $t_x^* L$ differ by the pullback of a line bundle on S.

2.1.7. If (A, λ) and (A', λ') are two abelian schemes over a scheme S with polarizations of degree d, then an isomorphism $(A, \lambda) \to (A', \lambda')$ is an isomorphism of abelian schemes

$$f : A \to A'$$

such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\lambda'} \\ A^t & \xleftarrow[f^t]{} & A'^t \end{array}$$

commutes, where f^t denotes the isomorphism of dual abelian schemes induced by f.

Lemma 2.1.8. *Let A/S be an abelian scheme and $\lambda : A \to A^t$ a homomorphism. Suppose $s \in S$ is a point such that the restriction $\lambda_s : A_s \to A^t_s$ of λ to the fiber at s is equal to λ_{L_s} for some ample line bundle L_s on A_s. Then after replacing S by an étale neighborhood of s, there exists a relatively ample line bundle L on A such that $\lambda = \lambda_L$.*

Proof. By a standard limit argument, it suffices to consider the case when S is of finite type over an excellent Dedekind ring. By the Artin approximation theorem [1969, 2.2] applied to the functor

$$F : (S\text{-schemes})^{\mathrm{op}} \to \mathrm{Sets}$$

sending an S-scheme T to the set of isomorphism classes of line bundles L on A_T such that $\lambda = \lambda_L$, it suffices to consider the case when $S = \mathrm{Spec}(R)$ is the spectrum of a complete noetherian local ring. In this case it follows from [Oort 1971, 2.3.2 and its proof] that there exists a line bundle L on A whose fiber over the closed point s is isomorphic to L_s. Now note that the two maps

$$\lambda_L, \lambda : A \to A^t$$

are equal by [Mumford 1965, Chapter 6, Corollary 6.2]. □

Lemma 2.1.9. *Let A/S be an abelian scheme over a scheme S, and let $\lambda : A \to A^t$ be a polarization. Then fppf-locally on S there exists a relatively ample line bundle L on A such that $\lambda = \lambda_L$. If 2 is invertible on S, then there exists such a line bundle étale locally on S.*

Proof. Consider first the case when $S = \mathrm{Spec}(k)$, for some field k. In this case, there exists by [Mumford 1965, Chapter 6, Proposition 6.10] a line bundle M on A such that $\lambda_M = 2\lambda$. Let Z denote the fiber product of the diagram

$$\begin{array}{ccc} & & \mathrm{Spec}(k) \\ & & \downarrow{\scriptstyle [M]} \\ \underline{\mathrm{Pic}}_{A/k} & \xrightarrow{\cdot 2} & \underline{\mathrm{Pic}}_{A/k}. \end{array}$$

The scheme Z represents the fppf-sheaf associated to the presheaf which to any k-scheme T associates the set of isomorphism classes of line bundles L for which $L^{\otimes 2} \simeq M$.

By assumption, there exists a field extension $k \to K$ and a line bundle L on A_K such that $\lambda|_{A_K} = \lambda_L$. Then

$$\lambda_{L^{\otimes 2}} = 2\lambda = \lambda_M,$$

so by 2.1.3 there exists, after possibly replacing K by an even bigger field extension, a point $x \in A(K)$ such that $t_x^*(L^{\otimes 2}) \simeq M$. It follows that $t_x^* L$ defines a point of $Z(K)$. Note also that if L is a line bundle on A such that $L^{\otimes 2} \simeq M$ then for any other line bundle R on A the product $L \otimes R$ defines a point of Z if and only if the class of the line bundle R is a point of $A^t[2]$. From this we conclude that Z is a torsor under $A^t[2]$. In particular, Z is étale if 2 is invertible in k, whence in this case there exists étale locally a section of Z.

To conclude the proof in the case of a field, note that if L is a line bundle on A with $L^{\otimes 2} \simeq M$, then

$$\lambda_L - \lambda : A \to A^t$$

has image in $A^t[2]$ since $2\lambda_L = 2\lambda$, and since $A^t[2]$ is affine the map $\lambda_L - \lambda$ must be the trivial homomorphism.

For the general case, let $s \in S$ be a point. Then we can find a finite field extension $k(s) \to K$ and a line bundle L on A_K such that $\lambda_L = \lambda|_{A_K}$. By the above we can further assume $k(s) \to K$ is separable if 2 is invertible in S. Now by [EGA 1961, chapitre 0, proposition 10.3.1, p. 20] there exists a quasifinite flat morphism $S' \to S$ and a point $s' \in S'$ such that the induced extension

$$k(s) \to k(s')$$

is isomorphic to $k(s) \to K$. If $k(s) \to K$ is separable then we can even choose $S' \to S$ to be étale. Now we obtain the result from 2.1.8 applied to $s' \in S'$. □

2.1.10. For integers $d, g \geq 1$, let $\mathscr{A}_{g,d}$ denote the fibered category over the category of schemes, whose fiber over a scheme S is the groupoid of pairs $(A/S, \lambda)$, where A is an abelian scheme of dimension g and $\lambda : A \to A^t$ is a polarization of degree d. We denote $\mathscr{A}_{g,1}$ simply by $\mathscr{A}_g$.

The basic result on the fibered category $\mathscr{A}_{g,d}$ is the following:

Theorem 2.1.11. *The fibered category $\mathscr{A}_{g,d}$ is a Deligne–Mumford stack over $\mathbb{Z}$, with quasiprojective coarse moduli space $A_{g,d}$. Over $\mathbb{Z}[1/d]$ the stack $\mathscr{A}_{g,d}$ is smooth.*

Proof. For the convenience of the reader we indicate how to obtain this theorem from the results of [Mumford 1965], which does not use the language of stacks.

Recall that if S is a scheme and A/S is an abelian scheme, then for any integer n invertible on S the kernel of multiplication by n on A

$$A[n] := \mathrm{Ker}(\cdot n : A \to A)$$

is a finite étale group scheme over S of rank n^{2g}, étale locally isomorphic to $(\mathbb{Z}/(n))^{2g}$. Define a *full level-n-structure on A/S* to be an isomorphism

$$\sigma : (\mathbb{Z}/(n))^{2g} \simeq A[n],$$

and let $\mathscr{A}_{g,d,n}$ be the fibered category over $\mathbb{Z}[1/n]$ whose fiber over a $\mathbb{Z}[1/n]$-scheme S is the groupoid of triples (A, λ, σ), where $(A, \lambda) \in \mathscr{A}_{g,d}(S)$ and σ is a full level-n-structure on A. Here an isomorphism between two objects

$$(A, \lambda, \sigma), \quad (A', \lambda', \sigma') \in \mathscr{A}_{g,d,n}(S)$$

is an isomorphism $f : (A, \lambda) \to (A', \lambda')$ in $\mathscr{A}_{g,d}(S)$ such that the diagram

$$\begin{array}{ccc} & (\mathbb{Z}/(n))^{2g} & \\ {}^{\sigma}\swarrow & & \searrow^{\sigma'} \\ A[n] & \xrightarrow{\quad f \quad} & A'[n] \end{array}$$

commutes. By [Mumford 1965, Chapter 7, Theorem 7.9 and remark following its proof], if $n \geq 3$ then $\mathscr{A}_{g,d,n}$ is equivalent to the functor represented by a quasiprojective $\mathbb{Z}[1/n]$-scheme. Let us also write $\mathscr{A}_{g,d,n}$ for this scheme. There is a natural action of $\mathrm{GL}_{2g}(\mathbb{Z}/(n))$ on $\mathscr{A}_{g,d,n}$ for which $g \in \mathrm{GL}_{2g}(\mathbb{Z}/(n))$ sends (A, λ, σ) to $(A, \lambda, \sigma \circ g)$. Furthermore, we have an isomorphism

$$\mathscr{A}_{g,d}|_{\mathbb{Z}[1/n]} \simeq [\mathscr{A}_{g,d,n}/\,\mathrm{GL}_{2g}(\mathbb{Z}/(n))].$$

Now choose two integer $n, n' \geq 3$ such that n and n' are relatively prime. We then get a covering

$$\mathscr{A}_{g,d} \simeq [\mathscr{A}_{g,d,n}/\,\mathrm{GL}_{2g}(\mathbb{Z}/(n))] \cup [\mathscr{A}_{g,d,n'}/\,\mathrm{GL}_{2g}(\mathbb{Z}/(n'))]$$

of $\mathscr{A}_{g,d}$ by open substacks which are Deligne–Mumford stacks, whence $\mathscr{A}_{g,d}$ is also a Deligne–Mumford stack.

By [Keel and Mori 1997, 1.3] the stack $\mathscr{A}_{g,d}$ has a coarse moduli space, which we denote by $A_{g,d}$. A priori $A_{g,d}$ is an algebraic space, but we show that $A_{g,d}$ is a quasiprojective scheme as follows.

Recall from [Mumford 1965, Chapter 6, Propositon 6.10], that to any object $(A, \lambda) \in \mathscr{A}_{g,d}(S)$ over some scheme S, there is a canonically associated relatively ample line bundle M on A which is rigidified at the zero section of A and such that $\lambda_M = 2\lambda$. By [Mumford 1970, theorem on p. 163] and cohomology and base change, the sheaf $M^{\otimes 3}$ is relatively very ample on A/S, and if $f : A \to S$ denotes the structure morphism then $f_*(M^{\otimes 3})$ is a locally free sheaf on S whose formation commutes with arbitrary base change $S' \to S$ and whose rank N is independent of (A, λ).

Let

$$f : \mathscr{X} \to \mathscr{A}_{g,d}$$

denote the universal abelian scheme, and let $\mathscr{M}$ denote the invertible sheaf on $\mathscr{X}$ given by the association

$$(A, \lambda, \sigma) \mapsto M.$$

For $r \geq 1$, let $\mathscr{E}_r$ denote the vector bundle on $\mathscr{A}_{g,d}$ given by $f_*(\mathscr{M}^{\otimes 3r})$, and let $\mathscr{L}_r$ denote the top exterior power of $\mathscr{E}_r$. We claim that for suitable choices of r and s the line bundle $\mathscr{L}_{mr}^{\otimes ms}$ descends to an ample line bundle on $A_{g,d}$ for any $m \geq 1$. Note that if this is the case, then the descended line bundle is unique up to unique isomorphism, for if R is any line bundle on $A_{g,d}$ then the adjunction map

$$R \to \pi_*\pi^*R$$

is an isomorphism, where $\pi : \mathscr{A}_{g,d} \to A_{g,d}$ is the projection. To verify this claim it suffices to verify it after restricting to $\mathbb{Z}[1/p]$, where p is a prime. In this case the claim follows from the proof of [Mumford 1965, Chapter 7, Theorem 7.10].

Finally the statement that $\mathscr{A}_{g,d}$ is smooth over $\mathbb{Z}[1/d]$ follows from [Oort 1971, 2.4.1]. □

2.2. ***Moduli of pairs.***

2.2.1. In [Alexeev 2002], Alexeev introduced a different perspective on $\mathscr{A}_g$. The key point is to encode into a torsor the ambiguity in the choice of line bundle for a given polarization. To make this precise let us first introduce some basic results about torsors under abelian varieties.

2.2.2. Let S be a scheme and A/S an abelian scheme. An A-*torsor* is a smooth scheme $f : P \to S$ with an action of A on P over S such that the graph of the action map

$$A \times_S P \to P \times_S P, \quad (a, p) \mapsto (p, a * p)$$

is an isomorphism. This implies that if we have a section $s : S \to P$ of f then the induced map

$$A \to P, \quad a \mapsto a * s$$

is an isomorphism of schemes compatible with the A-action, where A acts on itself by left translation. In particular, f is a proper morphism.

2.2.3. If A/S is an abelian scheme, and P/S is an A-torsor, then any line bundle L on P defines a homomorphism

$$\lambda_L : A \to A^t.$$

Namely, since $P \to S$ is smooth, there exists étale locally a section $s : S \to P$ which defines an isomorphism $\iota_s : A \to P$. In this situation we define λ_L to be the map

$$\lambda_{\iota_s^* L} : A \to A^t, \quad a \mapsto t_a^*(\iota_s^* L) \otimes \iota_s^* L^{-1}.$$

We claim that this is independent of the choice of section s. To see this let $s' : S \to P$ be another section. Since P is an A-torsor there exists a unique point $b \in A(S)$ such that $s' = b * s$. It follows that $\iota_{s'}^* L \simeq t_b^* \iota_s^* L$, so the claim follows from [Alexeev 2002, 4.1.12]. It follows that even when there is no section of P/S, we can define the map λ_L by descent theory using local sections.

2.2.4. With notation as in the preceding paragraph, suppose L is an ample line bundle on P, and let $f : P \to S$ be the structure morphism. Then:

(1) $f_* L$ is a locally free sheaf of finite rank on S whose formation commutes with arbitrary base change on S.

(2) If d denotes the rank of $f_* L$, then the kernel of $\lambda_L : A \to A^t$ is a finite flat group scheme over S of rank d^2.

Indeed both these assertions are local on S in the étale topology, so to prove them it suffices to consider the case when P admits a section, in which case they follow from the corresponding statements for ample line bundles on abelian schemes.

2.2.5. The most important example of torsors for this paper is the following. Let S be a scheme and let A/S be an abelian scheme with a principal polarization $\lambda : A \to A^t$. Consider the functor

$$P : (S\text{-schemes}) \to \text{Sets}$$

which to any S-scheme T associates the set of isomorphism classes of pairs (L, ϵ), where L is a line bundle on A_T such that $\lambda_L = \lambda|_{A_T}$ and $\epsilon : \mathcal{O}_T \to e^*L$ is an isomorphism of $\mathcal{O}_T$-modules. Note that two objects (L, ϵ) and (L', ϵ') are isomorphic if and only if the line bundles L and L' are isomorphic, in which case there exists a unique isomorphism $\sigma : L \to L'$ such that the induced diagram

$$\begin{array}{ccc} & & e^*L \\ & \overset{\epsilon}{\nearrow} & \\ \mathcal{O}_T & & \big\downarrow {\scriptstyle e^*\sigma} \\ & \underset{\epsilon}{\searrow} & \\ & & e^*L' \end{array}$$

commutes.

There is an action of A on P defined as follows. Given an S-scheme T, a T-valued point $x \in A(T)$, and an element $(L, \epsilon) \in P(T)$, define $x * (L, \epsilon)$ to be the line bundle

$$t_x^* L \otimes_{\mathcal{O}_S} x^* L^{-1} \otimes_{\mathcal{O}_S} e^* L$$

on A_T, where $t_x : A_T \to A_T$ is the translation, and let $x * \epsilon$ be the isomorphism obtained from ϵ and the canonical isomorphism

$$e^*(t_x^* L \otimes_{\mathcal{O}_S} x^* L^{-1} \otimes_{\mathcal{O}_S} e^* L) \simeq x^* L \otimes x^* L^{-1} \otimes e^* L \simeq e^* L.$$

Then the functor P is representable, and the action of A makes P an A-torsor. Note that we can also think of P is the sheaf (with respect to the étale topology) associated to the presheaf which to any S-scheme T associates the set of isomorphism classes of line bundles L on A_T such that $\lambda_L = \lambda|_T$.

On P there is a tautological line bundle $\mathcal{L}$ together with a global section $\theta \in \Gamma(P, \mathcal{L})$. Indeed giving such a line bundle and section is equivalent to giving for every scheme-valued point $p \in P(T)$ a line bundle $\mathcal{L}_p$ on T together with a section $\theta_p \in \Gamma(T, \mathcal{L}_p)$. We obtain such a pair by noting that since P is a torsor, the point p corresponds to a pair (L_p, ϵ_p) on A_T, and we define $\mathcal{L}_p$ to be $e^* L_p$

with the section θ_p being the image of 1 under the map

$$\epsilon_p : \mathbb{O}_T \to e^* L_p.$$

2.2.6. Define $\mathcal{A}_g^{\text{Alex}}$ to be the fibered category over the category of schemes, whose fiber over a scheme S is the groupoid of quadruples (A, P, L, θ) as follows:

(1) A/S is an abelian scheme of relative dimension g.

(2) P is an A-torsor. Let $f : P \to S$ denote the structure morphism.

(3) L is an ample line bundle on P such that $\lambda_L : A \to A^t$ is an isomorphism.

(4) $\theta : \mathbb{O}_S \to f_* L$ is an isomorphism of line bundles on S.

Note that for any $(A, P, L, \theta) \in \mathcal{A}_g^{\text{Alex}}(S)$ the pair (A, λ_L) is an object of $\mathcal{A}_g(S)$. We therefore get a morphism of fibered categories

$$F : \mathcal{A}_g^{\text{Alex}} \to \mathcal{A}_g \tag{1}$$

Proposition 2.2.7. *The morphism* (1) *is an equivalence.*

Proof. The construction in 2.2.5 defines another functor

$$G : \mathcal{A}_g \to \mathcal{A}_g^{\text{Alex}}$$

which we claim is a quasi-inverse to F.

For this note that given a quadruple $(A, P, L, \theta) \in \mathcal{A}_g^{\text{Alex}}(S)$ over some scheme S, and if (A, P', L', θ') denote the object obtained by applying $G \circ F$, then there is a natural map of A-torsors

$$\rho : P \to P'$$

obtained by associating to any S-scheme-valued point $p \in P(T)$ the class of the line bundle $\iota_p^* L$, where

$$\iota_p : A \to P$$

is the A-equivariant isomorphism obtained by sending $e \in A$ to p (here we think of P' as the sheaf associated to the presheaf of isomorphism classes of line bundles on A defining λ). By construction the isomorphism ρ has the property that $\rho^* L'$ and L are locally on S isomorphic. Since the automorphism group scheme of any line bundle on P is isomorphic to $\mathbb{G}_m$, we see that there exists a unique isomorphism

$$\tilde{\rho} : \rho^* L' \to L$$

sending θ' to θ. We therefore obtain a natural isomorphism

$$(A, P, L, \theta) \simeq (A, P', L', \theta')$$

in $\mathcal{A}_g^{\text{Alex}}$. This construction defines an isomorphism of functors $\text{id} \to G \circ F$.

To construct an isomorphism $\mathrm{id} \to F \circ G$, it suffices to show that if (A, λ) belongs to $\mathscr{A}_g(S)$ for some scheme S, and if (A, P, L, θ) denotes $G(A, P, L, \theta)$, then $\lambda_L = \lambda$, which is immediate from the construction in 2.2.5. □

Remark 2.2.8. In what follows we will usually not use the notation $\mathscr{A}_g^{\mathrm{Alex}}$.

Remark 2.2.9. While we find the language of line bundles with sections most convenient, note that giving the pair (L, θ) is equivalent to giving the corresponding Cartier divisor $D \hookrightarrow P$.

2.3. ***Approach via theta group.*** The third approach to the moduli stacks $\mathscr{A}_{g,d}$ is through a study of theta groups of line bundles. Before explaining this we first need a general stack theoretic construction that will be needed. The notion of *rigidification* we describe below has been discussed in various level of generality in many papers (see for example [Abramovich et al. 2003, Theorem 5.1.5]).

2.3.1. Let $\mathscr{X}$ be an algebraic stack, and let $\mathscr{I}_{\mathscr{X}} \to \mathscr{X}$ be its inertia stack. By definition, the stack $\mathscr{I}_{\mathscr{X}}$ has fiber over a scheme S the groupoid of pairs (x, α), where $x \in \mathscr{X}(S)$ and $\alpha : x \to x$ is an automorphism of x. In particular, $\mathscr{I}_{\mathscr{X}}$ is a relative group space over $\mathscr{X}$. The stack $\mathscr{I}_{\mathscr{X}}$ can also be described as the fiber product of the diagram

$$\begin{array}{ccc} & & \mathscr{X} \\ & & \downarrow{\scriptstyle \Delta} \\ \mathscr{X} & \xrightarrow{\Delta} & \mathscr{X} \times \mathscr{X}. \end{array}$$

Suppose further given a closed substack $\mathscr{G} \subset \mathscr{I}_{\mathscr{X}}$ such that the following hold:

(i) For every $x : S \to \mathscr{X}$ with S a scheme, the base change $\mathscr{G}_S \hookrightarrow \mathscr{I}_S$ is a normal subgroup space of the group space $\mathscr{G}_S$.

(ii) The structure map $\mathscr{G} \to \mathscr{X}$ is flat.

Then one can construct a new stack $\overline{\mathscr{X}}$, called the *rigidification of* $\mathscr{X}$ *with respect to* $\mathscr{G}$, together with a map

$$\pi : \mathscr{X} \to \overline{\mathscr{X}}$$

such that the following hold:

(i) The morphism on inertia stacks

$$\mathscr{I}_{\mathscr{X}} \to \mathscr{I}_{\overline{\mathscr{X}}}$$

sends $\mathscr{G}$ to the identity in $\mathscr{I}_{\overline{\mathscr{X}}}$.

(ii) The morphism π is universal with respect to this property: If $\mathscr{Y}$ is any algebraic stack, then

$$\pi^* : \mathrm{HOM}(\overline{\mathscr{X}}, \mathscr{Y}) \to \mathrm{HOM}(\mathscr{X}, \mathscr{Y})$$

identifies the category $\mathrm{HOM}(\overline{\mathscr{X}}, \mathscr{Y})$ with the full subcategory of $\mathrm{HOM}(\mathscr{X}, \mathscr{Y})$ of morphisms $f : \mathscr{X} \to \mathscr{Y}$ for which the induced morphism of inertia stacks

$$\mathscr{I}_{\mathscr{X}} \to \mathscr{I}_{\mathscr{Y}}$$

sends $\mathscr{G}$ to the identity.

(iii) The map π is faithfully flat, and $\mathscr{X}$ is a gerbe over $\overline{\mathscr{X}}$.

2.3.2. The stack $\overline{\mathscr{X}}$ is obtained as the stack associated to the prestack $\overline{\mathscr{X}}^{\mathrm{ps}}$ whose objects are the same as those of $\mathscr{X}$ but whose morphisms between two objects $x, x' \in \mathscr{X}(S)$ over a scheme S is given by the quotient of $\mathrm{Hom}_{\mathscr{X}(S)}(x, x')$ by the natural action of $\mathscr{G}(S, x)$ (a subgroup scheme of the scheme of automorphisms of x). One checks (see for example [Olsson 2008, §1.5]) that the composition law for morphisms in $\mathscr{X}$ descends to a composition law for morphisms modulo the action of $\mathscr{G}$.

Remark 2.3.3. The faithful flatness of the map π implies that one can frequently descend objects from $\mathscr{X}$ to $\overline{\mathscr{X}}$. Let us explain this in the case of quasicoherent sheaves, but the same argument applies in many other contexts (in particular to finite flat group schemes and logarithmic structures, which will be considered later). For an object $x \in \mathscr{X}(S)$ over a scheme S, let $\mathscr{G}_x$ denote the pullback of $\mathscr{G}$, so $\mathscr{G}_x$ is a flat group scheme over S. If $\mathscr{F}$ is a quasicoherent sheaf on $\mathscr{X}$ then pullback by x also defines a quasicoherent sheaf $\mathscr{F}_x$ on S, and there is an action of the group $\mathscr{G}_x(S)$ on $\mathscr{F}_x$. It is immediate that if $\mathscr{F}$ is of the form $\pi^*\overline{\mathscr{F}}$ for some quasicoherent sheaf $\overline{\mathscr{F}}$ on $\overline{\mathscr{X}}$, then these actions of $\mathscr{G}_x(S)$ on the $\mathscr{F}_x$ are trivial. An exercise in descent theory, which we leave to the reader, shows that in fact π^* induces an equivalence of categories between quasicoherent sheaves on $\overline{\mathscr{X}}$ and the category of quasicoherent sheaves $\mathscr{F}$ on $\mathscr{X}$ such that for every object $x \in \mathscr{X}(S)$ the action of $\mathscr{G}_x(S)$ on $\mathscr{F}_x$ is trivial.

2.3.4. We will apply this rigidification construction to get another view on $\mathscr{A}_{g,d}$.

Consider first the case of $\mathscr{A}_g$. Let $\mathscr{T}_g$ denote the fibered category over the category of schemes whose fiber over a scheme S is the groupoid of triples (A, P, L), where A/S is an abelian scheme of relative dimension g, P is an A-torsor, and L is a relatively ample line bundle on P such that the induced map

$$\lambda_L : A \to A^t$$

is an isomorphism.

Note that for any such triple, there is a natural inclusion

$$\mathbb{G}_m \hookrightarrow \underline{\mathrm{Aut}}_{\mathscr{T}_g}(A, P, L) \tag{2}$$

given by sending $u \in \mathbb{G}_m$ to the automorphism which is the identity on A and P and multiplication by u on L.

Proposition 2.3.5. *The stack* $\mathcal{T}_g$ *is algebraic, and the map*

$$\mathcal{T}_g \to \mathcal{A}_g, \quad (A, P, L) \mapsto (A, \lambda_L) \tag{3}$$

identifies $\mathcal{A}_g$ *with the rigidification of* $\mathcal{T}_g$ *with respect to the subgroup space* $\mathcal{G} \hookrightarrow \mathcal{I}_{\mathcal{T}_g}$ *defined by the inclusions* (2).

Proof. Since any object of $\mathcal{A}_g$ is locally in the image of (3), it suffices to show that for any scheme S and two objects (A, P, L) and (A', P', L') in $\mathcal{T}_g(S)$, the map sheaves on S-schemes (with the étale topology)

$$\underline{\mathrm{Hom}}_{\mathcal{T}_g}((A, P, L), (A', P', L')) \to \underline{\mathrm{Hom}}_{\mathcal{A}_g}((A, \lambda_L), (A', \lambda_{L'}))$$

provides an identification between $\underline{\mathrm{Hom}}_{\mathcal{A}_g}((A, \lambda_L), (A', \lambda_{L'}))$ and the sheaf quotient of $\underline{\mathrm{Hom}}_{\mathcal{T}_g}((A, P, L), (A', P', L'))$ by the natural action of $\mathbb{G}_m$. To verify this we may work étale locally on S, and hence may assume that P and P' are trivial torsors. Fix trivializations of these torsors, and view L and L' as line bundles on A and A' respectively.

In this case we need to show that for any isomorphism $\sigma : A \to A'$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\sigma} & A' \\ \downarrow{\scriptstyle \lambda_L} & & \downarrow{\scriptstyle \lambda_{L'}} \\ A^t & \xleftarrow{\sigma^*} & A'^t \end{array}$$

commutes, there exists a unique point $a \in A(S)$ such that the two line bundles

$$L, \quad t_a^* \sigma^* L'$$

are locally on S isomorphic. This follows from 2.1.6 applied to the two line bundles L and $\sigma^* L'$ which define the same principal polarization on A. □

2.3.6. For any object $(A, P, L) \in \mathcal{T}_g(S)$ over a scheme S, we have a line bundle $\mathcal{W}_{(A,P,L)}$ on S given by $f_* L$, where $f : P \to S$ is the structure morphisms, and the formation of this line bundle commutes with arbitrary base change $S' \to S$. It follows that we get a line bundle $\mathcal{W}$ on the stack $\mathcal{T}_g$. Let

$$\mathcal{V} \to \mathcal{T}_g$$

denote the $\mathbb{G}_m$-torsor corresponding to $\mathcal{W}$. As a stack, $\mathcal{V}$ classifies quadruples (A, P, L, θ), where $(A, P, L) \in \mathcal{T}_g$ and $\theta \in \mathcal{W}_{(A,P,L)}$ is a nowhere vanishing section. From this and 2.2.7 we conclude that the composite map

$$\mathcal{V} \to \mathcal{T}_g \to \mathcal{A}_g$$

is an isomorphism, and therefore defines a section

$$s : \mathscr{A}_g \to \mathscr{T}_g.$$

Since $\mathscr{T}_g$ is a $\mathbb{G}_m$-gerbe over $\mathscr{A}_g$, we conclude that in fact

$$\mathscr{T}_g \simeq \mathscr{A}_g \times B\mathbb{G}_m.$$

2.3.7. The description of $\mathscr{A}_g$ in 2.3.5 can be generalized to higher degree polarizations as follows.

Let S be a scheme and consider a triple (A, P, L), where A/S is an abelian scheme, P is an A-torsor, and L is a line bundle on P. Define the *theta group* of (A, P, L), denoted $\mathscr{G}_{(A,P,L)}$ to be the functor on S-schemes which to any S'/S associates the group of pairs (α, ι), where $\alpha : P_{S'} \to P_{S'}$ is a morphism of $A_{S'}$-torsors, and $\iota : \alpha^* L_{S'} \to L_{S'}$ is an isomorphism of line bundles. Here $P_{S'}$, $A_{S'}$, and $L_{S'}$ denote the base changes to S'. Note that α is equal to translation by a, for a unique point $a \in A(S')$.

It follows that there is a natural map

$$\mathscr{G}_{(A,P,L)} \to A. \tag{4}$$

Its image consists of scheme-valued points $b \in A$ for which $t_b^* L$ and L are locally isomorphic. This is precisely the kernel of λ_L. Note also that there is a natural central inclusion

$$\mathbb{G}_m \hookrightarrow \mathscr{G}_{(A,P,L)}$$

given by sending a unit u to (id_P, u). This is in fact the kernel of (4) so we have an exact sequence of functors

$$1 \to \mathbb{G}_m \to \mathscr{G}_{(A,P,L)} \to K_{(A,P,L)} \to 1,$$

where

$$K_{(A,P,L)} := \mathrm{Ker}(\lambda_L).$$

In particular, if L is ample then $K_{(A,P,L)}$ is a finite flat group scheme over S, which also implies that $\mathscr{G}_{(A,P,L)}$ is a group scheme flat over S.

2.3.8. Suppose now that L is relatively ample on P, so that $K_{(A,P,L)}$ is a finite flat group scheme over S. We then get a skew-symmetric pairing

$$e : K_{(A,P,L)} \times K_{(A,P,L)} \to \mathbb{G}_m,$$

defined by sending sections $x, y \in K_{(A,P,L)}$ to the commutator

$$e(x, y) := \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1},$$

where $\tilde{x}, \tilde{y} \in \mathscr{G}_{(A,P,L)}$ are local liftings of x and y respectively. Note that this is well-defined (in particular independent of the choices of liftings) since $\mathbb{G}_m$ is central in $\mathscr{G}_{(A,P,L)}$.

The pairing e is called the *Weil pairing* and is nondegenerate. Indeed, this can be verified étale locally on S, so it suffices to consider the case when P is a trivial torsor in which case the result is [Mumford 1970, Corollary 2, p. 234].

2.3.9. Fix integers $g, d \geq 1$, and let $\mathcal{T}_{g,d}$ be the stack over the category of schemes whose fiber over a scheme S is the groupoid of triples (A, P, L), where A is an abelian scheme of relative dimension g over S, P is an A-torsor, and L is a relatively ample line bundle on P of degree d.

Proposition 2.3.10. *The stack $\mathcal{T}_{g,d}$ is an algebraic stack. If $\mathcal{G} \subset \mathcal{I}_{\mathcal{T}_{g,d}}$ denotes the subgroup of the inertia stack defined by the theta groups, then $\mathcal{G}$ is flat over $\mathcal{T}_{g,d}$ and the rigidification of $\mathcal{T}_{g,d}$ with respect to $\mathcal{G}$ is canonically isomorphic to $\mathcal{A}_{g,d}$.*

Proof. This follows from an argument similar to the proof of 2.3.5, which we leave to the reader. □

2.3.11. The stacks $\mathcal{A}_{g,d}$ arise naturally when considering level structures, even if one is only interested in principally polarized abelian varieties. Namely, suppose $d' = d \cdot k$ is a second integer. Then there is a natural map

$$\mathcal{A}_{g,d} \to \mathcal{A}_{g,d'}, \quad (A, \lambda) \mapsto (A, k \cdot \lambda). \tag{5}$$

This map is obtained by passing to rigidifications from the map

$$\mathcal{T}_{g,d} \to \mathcal{T}_{g,d'}, \quad (A, P, L) \mapsto (A, P, L^{\otimes k}).$$

Proposition 2.3.12. *Over $\mathbb{Z}[1/d]$, the map* (5) *is an open and closed immersion.*

Proof. See [Olsson 2008, 6.2.3]. □

2.3.13. As we discuss in Section 5 below, this result can be used to study moduli of principally polarized abelian varieties with level structure using moduli stacks for abelian varieties with higher degree polarizations.

3. Degenerations

3.1. *Semiabelian schemes.*

3.1.1. By a *torus* over a scheme S, we mean a commutative group scheme T/S which étale locally on S is isomorphic to $\mathbb{G}_m^r$, for some integer $r \geq 0$. For such a group scheme T, let

$$X_T := \underline{\mathrm{Hom}}(T, \mathbb{G}_m)$$

be the sheaf on the big étale site of S classifying homomorphisms $T \to \mathbb{G}_m$. Then X_T is a locally constant sheaf of free finitely generated abelian groups (indeed this can be verified étale locally where it follows from the fact that $\mathrm{Hom}(\mathbb{G}_m^r, \mathbb{G}_m) \simeq \mathbb{Z}^r$), and the natural map

$$T \to \underline{\mathrm{Hom}}(X_T, \mathbb{G}_m), \quad u \mapsto (\chi \mapsto \chi(u))$$

is an isomorphism of group schemes (again to verify this it suffices to consider the case when $T = \mathbb{G}_m^r$). The sheaf X_T is called the *sheaf of characters* of T.

One can also consider the *sheaf of cocharacters* of T defined to be the sheaf

$$Y_T := \underline{\mathrm{Hom}}(\mathbb{G}_m, T)$$

of homomorphisms $\mathbb{G}_m \to T$. Again this is a locally constant sheaf of finitely generated free abelian groups and the natural map

$$X_T \times Y_T \to \underline{\mathrm{Hom}}(\mathbb{G}_m, \mathbb{G}_m) \simeq \mathbb{Z}, \quad (\chi, \rho) \mapsto \chi \circ \rho$$

identifies Y_T with $\underline{\mathrm{Hom}}(X_T, \mathbb{Z})$. Furthermore, the natural map

$$Y_T \otimes_{\mathbb{Z}} \mathbb{G}_m \to T, \quad \rho \otimes u \mapsto \rho(u)$$

is an isomorphism (where both sides are viewed as sheaves on the big étale site of S).

3.1.2. A *semiabelian variety* over a field k is a commutative group scheme G/k which fits into an exact sequence

$$1 \to T \to G \to A \to 1,$$

where T is a torus and A is an abelian variety over k.

Lemma 3.1.3. *For any scheme S and abelian scheme A/S there are no nonconstant homomorphisms*

$$\mathbb{G}_{m,S} \to A$$

over S.

Proof. Consider first the case when $S = \mathrm{Spec}(k)$ is the spectrum of a field k. If $f : \mathbb{G}_m \to A$ is a homomorphism, then since A is proper f extends to a $\mathbb{G}_m$-equivariant morphism

$$\mathbb{P}^1 \to A,$$

where $\mathbb{G}_m$ acts on A through f. Since $0, \infty \in \mathbb{P}^1(k)$ are fixed points for the $\mathbb{G}_m$-action, their images in A must also be fixed points of the $\mathbb{G}_m$-action, which implies that f is constant.

For the general case, note first that by a standard limit argument it suffices to consider the case when S is noetherian. Furthermore, to verify that a morphism $f : \mathbb{G}_{m,S} \to A$ is constant we may pass to the local rings of S at geometric points, and may therefore assume that S is strictly henselian local. Reducing modulo powers of the maximal ideal, we are then reduced to the case when S is the spectrum of an artinian local ring R with algebraically closed residue field k. Let

$$f : \mathbb{G}_{m,R} \to A$$

be a morphism. Then the reduction of f modulo the maximal ideal of R is a constant morphism by the case of a field. It follows that for each integer n invertible in k the restriction of f to $\mu_{n,R} \subset \mathbb{G}_{m,R}$ is constant, as $\mu_{n,R}$ is étale over R and must have image in the étale group scheme $A[n]$ of n-torsion points of A (and a map of étale schemes over R is determined by its reduction modulo the maximal ideal). It follows that the preimage of the identity $f^{-1}(e) \subset \mathbb{G}_{m,R}$ is a closed subscheme which contains all the subgroup schemes $\mu_{n,R}$ for n invertible in k. From this it follows that $f^{-1}(e) = \mathbb{G}_{m,R}$. □

3.1.4. In particular, in the setting of 3.1.2 any homomorphism $\mathbb{G}_m \to G$ factors through the subtorus $T \subset G$. This implies that the subtorus $T \subset G$ is canonically defined. Indeed if Y denotes the sheaf

$$\underline{\mathrm{Hom}}(\mathbb{G}_m, G),$$

then from above we conclude that Y is a locally constant sheaf of finitely generated abelian groups, and the natural map

$$Y \otimes_{\mathbb{Z}} \mathbb{G}_m \to G, \quad \rho \otimes u \mapsto \rho(u)$$

is a closed immersion with image T.

Note that this implies in particular that if G/k is a smooth group scheme such that the base change $G_{\bar{k}}$ to an algebraic closure is a semiabelian variety, then G is also a semiabelian variety as the subtorus $T_{\bar{k}} \subset G_{\bar{k}}$ descends to G.

3.1.5. For a general base scheme S, we define a *semiabelian scheme over S* to be a smooth commutative group scheme G/S all of whose fibers are semiabelian varieties. Semiabelian schemes arise as degenerations of abelian varieties. The basic theorem in this regard is the following:

Theorem 3.1.6 (Semistable reduction theorem [SGA 1972, IX.3.6]). *Let V be a regular noetherian local ring of dimension* 1, *with field of fractions K, and let A_K be an abelian scheme over K. Then there exists a finite extension K'/K such that the base change $A_{K'}$ of K' extends to a semiabelian scheme G over the integral closure V' of V in K'.*

3.2. ***Fourier expansions and quadratic forms.*** The key to understanding degenerations of abelian varieties and how it relates to moduli, is the connection with quadratic forms. This connection was originally established in the algebraic context by Mumford in [Mumford 1972], and then developed more fully for partial degenerations in [Faltings and Chai 1990]. In this section we explain from the algebraic point of view the basic idea of why quadratic forms are related to degenerations.

3.2.1. First we need some facts about line bundles on tori. Let R be a complete noetherian local ring with maximal ideal $\mathfrak{m} \subset R$ and reside field k. Let G/R be a smooth commutative group scheme such that the reduction G_k is a torus. Assume further that the character group sheaf X of G_k is constant (so G_k is isomorphic to $\mathbb{G}_m^g$ for some g), and write also X for the free abelian group $\Gamma(\mathrm{Spec}(k), X)$. For every integer n, let G_n denote the reduction of G modulo $\mathfrak{m}^{n+1}$, and let T_n denote the torus over $R_n := R/\mathfrak{m}^{n+1}$ defined by the group X. By [SGA 1970, chapitre IX, théorème 3.6] there exists for every $n \geq 0$ a unique isomorphism of group schemes

$$\sigma_n : T_n \to G_n$$

restricting to the identity over k.

Suppose now that L_n is a line bundle on T_n. Then L_n is a trivial line bundle. Indeed since T_0 has trivial Picard group and T_n is affine, there exists a global section $s \in \Gamma(T_n, L_n)$ whose pullback to T_0 is a basis. By Nakayama's lemma this implies that s defines an isomorphism $\mathcal{O}_{T_n} \simeq L_n$.

In particular, the line bundle L_n admits a T_n-linearization. Recall that such a linearization is given by an isomorphism

$$\alpha : m^* L_n \to \mathrm{pr}_1^* L_n$$

over $T_n \times_{\mathrm{Spec}(R_n)} T_n$, where

$$m : T_n \times_{\mathrm{Spec}(R_n)} T_n \to T_n$$

is the group law, and such that over

$$T_n \times_{\mathrm{Spec}(R_n)} T_n \times_{\mathrm{Spec}(R_n)} T_n$$

the diagram

$$\begin{array}{ccc}
(m \times 1)^* m^* L & = & (1 \times m)^* m^* L \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\
(m \times 1)^* \mathrm{pr}_2^* L & & (1 \times m)^* \mathrm{pr}_2^* L \\
\| & & \| \\
\mathrm{pr}_3^* L & & \mathrm{pr}_{23}^* m^* L \\
& \searrow\!\!\!= \quad \swarrow{\scriptstyle \alpha} & \\
& \mathrm{pr}_{23}^* \mathrm{pr}_2^* L &
\end{array}$$

commutes, where we write

$$\mathrm{pr}_{23} : T_n \times_{\mathrm{Spec}(R_n)} T_n \times_{\mathrm{Spec}(R_n)} T_n \to T_n \times_{\mathrm{Spec}(R_n)} T_n$$

for the projection onto the second two components etc.

Since T_n is affine a T_n-linearization can also be described as follows. Let M_n denote $\Gamma(T_n, L_n)$ which is a module over $A_n := \Gamma(T_n, \mathcal{O}_{T_n}) \simeq R_n[X]$ (the group ring on X). Note that since A_n is canonically identified with the group ring on X we have a grading

$$A_n = \bigoplus_{\chi \in X} A_{n,\chi}.$$

Then giving a T_n-linearization on L_n is equivalent to giving a decomposition

$$M = \bigoplus_{\chi \in X} M_\chi$$

of M into submodules indexed by X which is compatible with the X-grading on A_n in the sense that for every $\chi, \eta \in X$ the map

$$A_{n,\chi} \otimes M_\eta \to M$$

has image in $M_{\chi+\eta}$.

Note that if $\chi_0 \in X$ is a fixed element, then we obtain a new T_n-linearization

$$M = \bigoplus_{\chi \in X} (M^{(\chi_0)})_\chi,$$

by setting

$$(M^{(\chi_0)})_\chi := M_{\chi+\chi_0}.$$

We call this new T_n-linearization the χ_0-*translate* of the original one.

Lemma 3.2.2. (i) *Translation by elements of X gives the set of T_n-linearizations on L_n the structure of an X-torsor.*

(ii) *For any T_n-linearization of L_n corresponding to a decomposition $M = \bigoplus_\chi M_\chi$ each of the modules M_χ is a free module over R_n of rank* 1. *Moreover, if $I \subset A_n$ denotes the ideal of the identity section of T_n, then for every $\chi \in X$ the composite map*

$$M_\chi \hookrightarrow M \to M/IM$$

is an isomorphism.

(iii) *Any T_{n-1}-linearization on the reduction L_{n-1} of L_n to T_{n-1} lifts uniquely to a T_n-linearization on L_n.*

Proof. Suppose

$$\alpha, \alpha' : m^* L_n \to \mathrm{pr}_1^* L_n$$

are two T_n-linearizations of L_n.

For any R_n-scheme S let $T_{n,S}$ denote the base change of T_n to S, and let $L_{n,S}$ denote the pullback of L_n to $T_{n,S}$. For any point $u \in T_n(S)$, let

$$t_u : T_{n,S} \to T_{n,S}$$

denote translation by u, and let

$$\alpha_u, \alpha'_u : t_u^* L_{n,S} \to L_{n,S}$$

be the two isomorphisms obtained by pulling back α and α' along the map

$$T_{n,S} = T_n \times_{\mathrm{Spec}(R_n)} S \xrightarrow{\mathrm{id}\times u} T_n \times_{\mathrm{Spec}(R_n)} T_n.$$

The map $\alpha'_u \circ \alpha_u^{-1}$ is then an automorphism of $L_{n,S}$ over $T_{n,S}$, and hence is specified by a global section

$$s_u \in \Gamma(T_{n,S}, \mathbb{O}^*_{T_{n,S}}) = \mathbb{G}_m(S) \times X.$$

By sending $u \in T(S)$ to s_u we therefore obtain a natural transformation of functors

$$s : T_n \to \mathbb{G}_m \times X,$$

or equivalently by Yoneda's lemma a morphism of schemes. Since T_n is connected this map has connected image, and since the identity in T_n goes to the identity in $\mathbb{G}_m \times X$, the map s in fact has image in

$$\mathbb{G}_m \hookrightarrow \mathbb{G}_m \times X, \quad u \mapsto (u, 0).$$

Now the fact that α and α' are compatible with composition implies that the map

$$s : T_n \to \mathbb{G}_m$$

is a homomorphism, whence given by a character $\chi_0 \in X$. From this and the correspondence between T_n-linearizations and gradings on M, we get that α' is obtained from α by translation by χ_0.

This shows that the translation action of X on the set of T_n-linearizations of L_n is transitive. In particular, to verify (ii) it suffices to verify it for a single choice of T_n-linearization, as the validity of (ii) is clearly invariant under translation by elements of X. To verify (ii) it therefore suffices to consider $L_n = \mathbb{O}_{T_n}$ with the standard linearization, where the result is immediate.

Now once we know that each M_χ has rank 1, then it also follows that the action in (i) is simply transitive, as the character χ_0 is determined by the image of M_0.

Finally (iii) follows immediately from (i). □

3.2.3. Consider again the setting of 3.2.1, and let L be a line bundle on G. For every $n \geq 0$ we then get by reduction (and using the isomorphisms σ_n) compatible line bundles L_n on T_n. Fix the following data:

A. A trivialization $t : R \simeq e^*L$, where $e : \mathrm{Spec}(R) \to G$ is the identity section.

B. A T_0-linearization α_0 of L_0.

By 3.2.2 (iii) the T_0-linearization α_0 lifts uniquely to a compatible system of T_n-linearizations $\{\alpha_n\}$. For every $n \geq 0$ and $\chi \in X$, we then get by 3.2.2 (ii) an isomorphism

$$\Gamma(T_n, L_n)_\chi \simeq e^*L_n \simeq R_n$$

where the second isomorphism is given by t. We therefore obtain a compatible system of basis elements $f_{n,\chi} \in \Gamma(T_n, L_n)_\chi$ defining an isomorphism

$$\Gamma(T_n, L_n) \simeq \textstyle\bigoplus_{\chi \in X} R_n \cdot f_{n,\chi}.$$

Passing to the inverse limit we get an isomorphism

$$\varprojlim_n \Gamma(T_n, L_n) \simeq \textstyle\prod'_{\chi \in X} R \cdot f_\chi,$$

where

$$\textstyle\prod'_{\chi \in X} R \cdot f_\chi \subset \prod_{\chi \in X} R \cdot f_\chi$$

denotes the submodule of elements $(g_\chi \cdot f_\chi)_{\chi \in X}$ such that for every $n \geq 0$ almost all $g_\chi \in \mathfrak{m}^{n+1}$.

For any $\mu \in X$, we get by composing the natural map $\Gamma(G, L) \to \varprojlim_n \Gamma(T_n, L_n)$ with the projection

$$\textstyle\prod'_{\chi \in X} R \cdot f_\chi \to R \cdot f_\mu$$

a map

$$\sigma_\mu : \Gamma(G, L) \to R.$$

If $m \in \Gamma(G, L)$ then we write

$$m = \sum_\chi \sigma_\chi(m) \cdot f_\chi$$

for the resulting expression in $\prod'_\chi R \cdot f_\chi$. We call this the *Fourier expansion of* m.

If R is an integral domain with field of fractions K, then we can tensor the maps σ_μ with K to get maps

$$\Gamma(G_K, L_K) \to K,$$

which we again denote by σ_μ. Note that for any $m \in \Gamma(G_K, L_K)$ the elements $\sigma_\mu(m)$ have bounded denominators in the sense that for any $n \geq 0$ we have $\sigma_\mu(m) \in \mathfrak{m}^{n+1}$ for all but finitely many μ.

3.2.4. Suppose

$$t' : R \simeq e^*L$$

is a second choice of trivialization, and α_0' is a second T_0-linearization of L_0. Let

$$\sigma_\mu' : \Gamma(G, L) \to R$$

be the maps obtained using this second choice. Suppose

$$t'(-) = vt(-),$$

for some unit $v \in R^*$ and that α_0' is the χ_0-translate of α_0 for some $\chi_0 \in X$. Then the collections $\{\sigma_\mu\}$ and $\{\sigma_\mu'\}$ are related by the formula

$$\sigma_\mu'(-) = v\sigma_{\mu+\chi_0}(-).$$

3.2.5. Suppose now that our complete noetherian local ring R is also normal, and let K be the field of fractions. Let G/R be a semiabelian scheme whose generic fiber G_K is an abelian variety, and assume as above that the closed fiber G_k is a split torus. As before let X denote the character group of G_k.

Assume given an ample line bundle L_K on G_K, and let

$$\lambda_K : G_K \to G_K^t$$

be the induced polarization, where G_K^t denotes the dual abelian variety of G_K. As explained in [Faltings and Chai 1990, Chapter II, §2], the abelian scheme G_K^t extends uniquely to a semiabelian scheme G^t/R, and the map λ_K extends uniquely to a homomorphism

$$\lambda : G \to G^t.$$

Moreover, the closed fiber G_k^t is also a split torus, say Y is the character group of G_k^t. The map λ defines an inclusion

$$\phi : Y \hookrightarrow X.$$

Since G/R is smooth, the line bundle L_K extends to a line bundle L on G, unique up to isomorphism. Fix a trivialization

$$t : R \simeq e^*L$$

and a T_0-linearization α_0 on L_0, so we get maps

$$\sigma_\mu(-) : \Gamma(G_K, L_K) \to K.$$

Theorem 3.2.6 [Faltings and Chai 1990, Chapter II, 4.1]. *There exist unique functions*

$$a : Y \to K^*, \quad b : Y \times X \to K^*$$

such that the following hold:

(i) *The map b is bilinear.*

(ii) *For any $\mu \in X$ and $y \in Y$ we have*

$$\sigma_{\mu+\phi(y)}(-) = a(y)b(y,\mu)\sigma_\mu(-).$$

(iii) *For any $y, y' \in Y$ we have*

$$b(y,\phi(y')) = b(y',\phi(y)).$$

(iv) *For $y, y' \in Y$ we have*

$$a(y+y') = b(y,\phi(y'))a(y)a(y').$$

(v) *For every nonzero $y \in Y$ we have $b(y,\phi(y)) \in \mathfrak{m}$, and for every $n \geq 0$ we have $a(y) \in \mathfrak{m}^n$ for all but finitely many $y \in Y$.*

Remark 3.2.7. If we choose a different trivialization t' of e^*L and a different T_0-linearization α_0', then we get new functions a' and b', which differ from a and b as follows. By 3.2.4 there exists a unit $v \in R^*$ and an element χ_0 such that

$$\sigma'_\mu(-) = v\sigma_{\mu+\chi_0}(-)$$

for all $\mu \in X$. From this we get that for any $\mu \in X$ and $y \in Y$ we have

$$\sigma'_{\mu+\phi(y)}(-) = a(y)b(y,\mu+\chi_0)\sigma'_\mu(-).$$

Since b is bilinear we have

$$b(y,\mu+\chi_0) = b(y,\mu)b(y,\chi_0).$$

It follows that

$$a'(y) = a(y)b(y,\chi_0), \quad b'(y,x) = b(y,x).$$

3.2.8. In particular, if R is a discrete valuation ring, then we also have a valuation map

$$\nu : K^* \to \mathbb{Z}.$$

Let A (resp. B) denote the composite of a (resp. b) with ν, so we have functions

$$A : Y \to \mathbb{Z}, \quad B : Y \times X \to \mathbb{Z}.$$

If we fix a uniformizer $\pi \in R$ then we also get functions

$$\alpha : Y \to R^*, \quad \beta : Y \times X \to R^*$$

such that

$$a(y) = \alpha(y)\pi^{A(y)}, \quad b(y,x) = \beta(y,x)\pi^{B(y,x)}.$$

Now observe that since G and G^t have the same dimension, the map ϕ induces an isomorphism upon tensoring with $\mathbb{Q}$, so B induces a map

$$B_{\mathbb{Q}} : X_{\mathbb{Q}} \times X_{\mathbb{Q}} \to \mathbb{Q}$$

which is a positive definite quadratic form by 3.2.6(v). Note also that the difference

$$L : Y \to \mathbb{Q}, \quad y \mapsto A(y) - \tfrac{1}{2}B(y, \phi(y))$$

is a linear form on Y, and that B can be recovered from A by the formula

$$B(y, \phi(y')) = A(y + y') - A(y) - A(y').$$

Note that by 3.2.7 the functions B is independent of the choice of (t, α_0), and for different choices of (t, α_0) the corresponding A-functions differ by a linear form.

3.2.9. The situation when G is not totally degenerate (i.e., the closed fiber G_k has an abelian part) is more complicated, and the functions a and b in the above get replaced with data involving the theory of biextensions. We will not go through that here (the interested reader should consult [Faltings and Chai 1990, Chapter II, §5] and [Olsson 2008, proof of 4.7.2]). One important thing to know about this, however, is that even in this case one obtains a positive semidefinite quadratic form

$$B : X_{\mathbb{Q}} \times X_{\mathbb{Q}} \to \mathbb{Q}$$

on the character group X of the maximal torus in G_k. We will use this in what follows.

4. Compactifications

4.1. *Toroidal.* The toroidal compactifications of $\mathscr{A}_g$ defined in [Faltings and Chai 1990] depend on some auxiliary choice of data, which we now explain.

4.1.1. Let X be a free finitely generated abelian group of rank g. For $A = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$, let $B(X_A)$ denote the space of A-valued quadratic forms on X

$$B(X_A) := \mathrm{Hom}(S^2 X, A).$$

For a bilinear form $b \in B(X_{\mathbb{R}})$ the *radical of* b, denoted $\mathrm{rad}(b)$, is defined to be the kernel of the map

$$X_{\mathbb{R}} \to \mathrm{Hom}(X_{\mathbb{R}}, \mathbb{R}), \quad y \mapsto b(y, -).$$

Let $C(X) \subset B(X_{\mathbb{R}})$ denote the subset of positive semidefinite bilinear forms b such that $\mathrm{rad}(b)$ is defined over $\mathbb{Q}$. Then $C(X)$ is a convex cone in the real vector space $B(X_{\mathbb{R}})$, and its interior $C(X)^\circ \subset C(X)$ is the set of positive definite forms b.

Note also that there is an action of $\mathrm{GL}(X)$ on $B(X_{\mathbb{R}})$ induced by the action on X, and $C(X)$ and $C(X)^\circ$ are invariant under this action.

4.1.2. Degenerations of abelian varieties give subsets of $C(X)$ as follows.

Let S be an irreducible normal scheme with generic point η. Let G/S be a semiabelian scheme, and assume that the generic fiber G_η of G is an abelian scheme of dimension g. Suppose further given a principal polarization λ_η on G_η. Then for any complete discrete valuation ring V with algebraically closed residue field and morphism

$$\rho : \mathrm{Spec}(V) \to S \tag{6}$$

sending the generic point of $\mathrm{Spec}(V)$ to η, we can pull back G to get a semiabelian scheme G_ρ/V whose generic fiber is a principally polarized abelian variety. As mentioned in 3.2.9, we therefore get a quadratic form

$$B_\rho \in C(X_{s,\mathbb{Q}}),$$

where X_s denotes the character group of the torus part of the closed fiber G_s of G_ρ. Choosing any surjection $X \to X_s$ we get an element $B'_\rho \in C(X_{\mathbb{Q}})$, well-defined up to the natural $\mathrm{GL}(X)$-action on $C(X_{\mathbb{Q}})$.

4.1.3. An *admissible cone decomposition of* $C(X)$ is a collection $\Sigma = \{\sigma_\alpha\}_{\alpha \in J}$ (where J is some indexing set) as follows:

(1) Each σ_α is a subcone of $C(X)$ of the form

$$\sigma_\alpha = \mathbb{R}_{>0} \cdot v_1 + \cdots + \mathbb{R}_{>0} \cdot v_r$$

for some elements $v_1, \ldots, v_r \in B(X_{\mathbb{Q}})$, and such that σ_α does not contain any line.

(2) $C(X)$ is equal to the disjoint union of the σ_α, and the closure of each σ_α is a disjoint union of σ_β's.

(3) For any $g \in \mathrm{GL}(X)$ and $\alpha \in J$ we have $g(\sigma_\alpha) = \sigma_\beta$ for some $\beta \in J$, and the quotient $J/\mathrm{GL}(X)$ of the set of cones J by the induced action of $\mathrm{GL}(X)$ is finite.

4.1.4. An admissible cone decomposition Σ of $C(X)$ is called *smooth* if for every $\sigma_\alpha \in \Sigma$ we can write

$$\sigma_\alpha = \mathbb{R}_{>0} \cdot v_1 + \cdots + \mathbb{R}_{>0} \cdot v_r$$

where $v_1, \ldots, v_r \in B(X_{\mathbb{Z}})$ can be extended to a $\mathbb{Z}$-basis for $B(X_{\mathbb{Z}})$.

4.1.5. Let Σ be an admissible cone decomposition of $C(X)$ and let B be a regular scheme (the case of interest is when B is the spectrum of a field or $\mathbb{Z}$). A *toroidal compactification of $\mathscr{A}_g$ with respect to Σ over B* is a Deligne–Mumford stack $\mathscr{A}_{g,\Sigma}$ over B together with a dense open immersion $j : \mathscr{A}_{g,B} \hookrightarrow \mathscr{A}_{g,\Sigma}$ over B such that the following hold:

(1) $\mathscr{A}_{g,\Sigma}$ is an irreducible normal algebraic stack, which is smooth over B if Σ is smooth.

(2) The universal abelian scheme $X \to \mathscr{A}_{g,B}$ extends to a semiabelian scheme $X_\Sigma \to \mathscr{A}_{g,\Sigma}$.

(3) Let S be an irreducible normal B-scheme and let G/S be a semiabelian scheme of relative dimension g whose generic fiber G_η is abelian with a principal polarization λ_η. Let $U \subset S$ be a dense open subset such that (G_η, λ_η) defines a morphism

$$f_U : U \to \mathscr{A}_g.$$

Then f_U extends to a (necessarily unique) morphism $f : S \to \mathscr{A}_{g,\Sigma}$ if and only if the following condition holds: For any point $s \in S$ there exists $\alpha \in J$ and a surjection $X \to X_{\bar{s}}$ such that for any morphism (6) sending the closed point of $\mathrm{Spec}(V)$ to s the element $B'_\rho \in C(X_{\mathbb{Q}})$ lies in σ_α.

Remark 4.1.6. The extension X_Σ of X in (2) is unique up to unique isomorphism by [Faltings and Chai 1990, I.2.7].

Remark 4.1.7. Properties (1), (2), and (3) characterize the stack $\mathscr{A}_{g,\Sigma}$ up to unique isomorphism. Indeed suppose we have another irreducible normal algebraic stack $\mathscr{A}'_g$ over B (this stack could be just an Artin stack, and doesn't have to be Deligne–Mumford) together with a dense open immersion $j' : \mathscr{A}_{g,B} \hookrightarrow \mathscr{A}'_g$ and an extension $X' \to \mathscr{A}'_g$ of the universal abelian scheme $X/\mathscr{A}_{g,B}$ to a semiabelian scheme over $\mathscr{A}'_g$. Suppose further that for any smooth morphism $g : W \to \mathscr{A}'_g$ the pullback $X_W \to W$ of X' to W satisfies the condition in (3). We then get a unique extension

$$\tilde{f} : W \to \mathscr{A}_{g,\Sigma}$$

of the map induced by X_W over the preimage of $\mathscr{A}_{g,B}$. Moreover, the two arrows

$$W \times_{\mathscr{A}'_g} W \to \mathscr{A}_{g,\Sigma}$$

obtained by composing the two projections with $\tilde{f}$ are canonically isomorphic by the uniqueness part of (3). In addition, the usual cocycle condition over $W \times_{\mathscr{A}'_g} W \times_{\mathscr{A}'_g} W$ holds again by the uniqueness. The map $\tilde{f}$ therefore descends to a unique morphism

$$f : \mathscr{A}'_g \to \mathscr{A}_{g,\Sigma}$$

compatible with the inclusions of $\mathscr{A}_{g,B}$. In particular, if $\mathscr{A}'_g$ is also a Deligne–Mumford stack satisfying (1), (2), and (3) then we also get a map

$$g : \mathscr{A}_{g,\Sigma} \to \mathscr{A}'_g$$

such that $f \circ g = \mathrm{id}_{\mathscr{A}_{g,\Sigma}}$ and $g \circ f = \mathrm{id}_{\mathscr{A}'_g}$.

One of the main results of [Faltings and Chai 1990] is then the following:

Theorem 4.1.8 [Faltings and Chai 1990, IV.5.7]. *For any smooth admissible cone decomposition Σ of $C(X)$, there exists a toroidal compactification of $\mathscr{A}_g$ with respect to Σ over* Spec($\mathbb{Z}$). *Moreover, for any regular scheme B, the base change $\mathscr{A}_{g,\Sigma,B}$ of $\mathscr{A}_{g,\Sigma}$ to B is a toroidal compactification of $\mathscr{A}_g$ with respect to Σ over B.*

Over $\mathbb{C}$, the smoothness assumption on the cone Σ can be omitted. This follows from Mumford's theory of toroidal embeddings [Ash et al. 1975]. A more accessible discussion in the case of $\mathscr{A}_g$ can be found in [Namikawa 1980].

Theorem 4.1.9. *For any admissible cone decomposition Σ of $C(X)$, there exists a toroidal compactification of $\mathscr{A}_g$ with respect to Σ over* Spec($\mathbb{C}$).

Remark 4.1.10. It seems widely believed that for any admissible cone decomposition Σ of $C(X)$ there exists a toroidal compactification of $\mathscr{A}_g$ with respect to Σ over Spec($\mathbb{Z}$), and it should have the property that for any regular scheme B the base change $\mathscr{A}_{g,\Sigma,B}$ is again a toroidal compactification of $\mathscr{A}_{g,B}$ with respect to Σ over B. However, no proof seems to be available in the literature.

4.2. *Alexeev's compactification $\overline{\mathscr{A}}_g^{Alex}$.*

4.2.1. Alexeev's compactification of $\mathscr{A}_g$ arises from considering $\mathscr{A}_g$ as the moduli stack of quadruples (A, P, L, θ), where A is an abelian variety, P is an A-torsor, L is an ample line bundle on P defining a principal polarization, and θ is a nonzero global section of L (see 2.2.7).

4.2.2. To get a sense for Alexeev's compactification let us consider a 1-parameter degeneration, and explain how the quadratic form obtained in 3.2.8 defines a degeneration of the whole quadruple (A, P, L, θ). So let V be a complete discrete valuation ring, let S denote Spec(V), and let η (resp. s) denote the generic (resp. closed) point of S. Let G/S be a semiabelian scheme with G_η an abelian variety and G_s a split torus. Assume further given a line bundle L on G whose restriction L_η to G_η is ample and defines a principal polarization. Let X denote the character group of G_s and let T denote the torus over V defined by X. Fix a trivialization $t : V \simeq e^*L$ (where $e \in G(V)$ is the identity section) and a T-linearization of L_s (the pullback of L to G_s). Finally let $\theta_\eta \in \Gamma(G_\eta, L_\eta)$ be a global section.

Let P_η denote G_η viewed as a trivial G_η-torsor. We can then construct a degeneration of the quadruple $(G_\eta, P_\eta, L_\eta, \theta_\eta)$ as follows.

4.2.3. Let

$$A : X \to \mathbb{Z}$$

be the quadratic function defined as in 3.2.8 (and using the identification $Y \simeq X$ defined by ϕ, which is an isomorphism since L_η is a principal polarization). Let

$$S := \{(x, A(x))|x \in X\} \subset X_{\mathbb{R}} \oplus \mathbb{R}$$

be the graph of A, and let $S_{\mathbb{R}} \subset X_{\mathbb{R}} \oplus \mathbb{R}$ denote the convex hull of the set S. Then the projection

$$S_{\mathbb{R}} \to X_{\mathbb{R}}$$

is a bijection, and therefore $S_{\mathbb{R}}$ is the graph of a unique function

$$g : X_{\mathbb{R}} \to \mathbb{R}.$$

This function is piece-wise linear in the sense that there exists a unique collection $\Sigma = \{\omega\}$ of polytopes $\omega \subset X_{\mathbb{R}}$ such that the following hold:

(1) For any two elements $\omega, \eta \in \Sigma$ the intersection $\omega \cap \eta$ is also in Σ.

(2) Any face of a polytope $\omega \in \Sigma$ is also in Σ.

(3) $X_{\mathbb{R}} = \cup_{\omega\in\Sigma}\omega$ and for any two distinct elements $\omega, \eta \in \Sigma$ the interiors of ω and η are disjoint.

(4) For any bounded subset $W \subset X_{\mathbb{R}}$ there are only finitely many $\omega \in \Sigma$ with $\omega \cap W \neq \varnothing$.

(5) The top-dimensional polytopes $\omega \in \Sigma$ are precisely the domains of linearity of the function g.

A decomposition Σ of $X_{\mathbb{R}}$ into polytopes which arises from a quadratic function $A : X \to \mathbb{Z}$ by the construction above is called an *integral regular paving of* $X_{\mathbb{R}}$.

Note that the paving Σ is invariant under the action of elements of X acting by translation on $X_{\mathbb{R}}$. Indeed for $x, y \in X$ we have

$$A(x + y) = A(x) + A(y) + B(x, y) \tag{7}$$

so if $t_y : X_{\mathbb{R}} \to X_{\mathbb{R}}$ denotes translation by y, then the composite function

$$X_{\mathbb{R}} \xrightarrow{t_y} X_{\mathbb{R}} \xrightarrow{g} \mathbb{R}$$

is equal to

$$x \mapsto g(x) + B(x, y) + A(y),$$

which differs from g by the linear function $B(-, y) + A(y)$.

Remark 4.2.4. Note that any positive definite quadratic form

$$B : S^2X \to \mathbb{Q}$$

defines an X-invariant paving of $X_{\mathbb{R}}$ by the construction above. If more generally we allow also infinite polytopes in the definition of paving then we can also consider the pavings associated to positive semidefinite quadratic forms.

4.2.5. We use the function g to define a graded V-subalgebra

$$\mathcal{R} \subset K[X \oplus \mathbb{N}].$$

For $\omega \in \Sigma$ let $C_\omega \subset X \oplus \mathbb{N}$ be the integral points of the cone over $\omega \times \{1\} \subset X_{\mathbb{R}} \oplus \mathbb{R}$, so C_ω is the set of elements $(x, d) \in X \oplus \mathbb{N}$ such that the element $(1/d) \cdot x \in X_{\mathbb{Q}}$ lies in ω. Since g is a linear function on ω it extends uniquely to an additive function

$$g_\omega : C_\omega \to \mathbb{Q}, \quad (x, d) \mapsto d \cdot g((1/d) \cdot x).$$

These functions define a function

$$\tilde{g} : X \oplus \mathbb{N}_{>0} \to \mathbb{Q}$$

by sending (x, d) to $g_\omega(x, d)$ for any $\omega \in \Sigma$ such that $(x, d) \in C_\omega$ (note that this is independent of the choice of ω).

Let $C'_\omega \subset C_\omega$ be the submonoid generated by degree 1 elements. Then $C'^{\mathrm{gp}}_\omega \subset C^{\mathrm{gp}}_\omega$ has finite index, say N_ω. Now using property (4) and the translation invariance of the paving, we see that there exists an integer N such that for every $\omega \in \Sigma$ the index of C'^{gp}_ω in C^{gp}_ω divides N. In particular, the function g_ω has image in $(1/N) \cdot \mathbb{Z}$ for all ω.

Also observe that making a base change $V \to V'$ with ramification e in the construction above has the effect of multiplying the function g by e. Therefore, after possibly replacing V by a ramified extension, we may assume that all the g_ω's, and hence also $\tilde{g}$, are integer valued.

Let

$$\mathcal{R} \subset K[X \oplus \mathbb{N}]$$

be the graded V-subalgebra generated by the elements

$$\xi^{(x,d)} := \pi^{\tilde{g}(x,d)} e^{(x,d)},$$

where we write $e^{(x,d)} \in K[X \oplus \mathbb{N}]$ for the element corresponding to $(x, d) \in X \oplus \mathbb{N}$.

Then $\mathcal{R}$ is a graded V-algebra and we can consider the V-scheme

$$\widetilde{P} := \mathrm{Proj}(\mathcal{R}).$$

This scheme comes equipped with a line bundle $L_{\widetilde{P}}$, and we usually consider the pair $(\widetilde{P}, L_{\widetilde{P}})$.

4.2.6. There is a natural action of T on $(\widetilde{P}, L_{\widetilde{P}})$ induced by the X-grading on $\mathcal{R}$.

4.2.7. There is an action of X on $(\widetilde{P}, L_{\widetilde{P}})$ defined as follows. Let

$$\alpha : X \to V^*, \quad \beta : X \times X \to V^*$$

be the maps defined in 3.2.8. Recall that for $x, y \in X$ we have

$$\alpha(x+y) = \beta(x, y)\alpha(x)\alpha(y).$$

The action of $y \in X$ is then given by

$$\xi^{(x,d)} \mapsto \alpha(y)^d \beta(y, x)\xi^{(x+dy,d)}.$$

Note that the actions of T and X on $\widetilde{P}$ commute, but that if $\chi \in T$ (a scheme-valued point) and $y \in X$ then the induced automorphism of $L_{\widetilde{P}}$

$$(T_y \circ S_\chi)^{-1} \circ (S_\chi \circ T_y)$$

is equal to multiplication by $\chi(y)$.

4.2.8. The generic fiber of $\widetilde{P}$ is isomorphic to T_K with the standard action of T_K and trivial action of X.

4.2.9. The closed fiber $\widetilde{P}_0$ of $\widetilde{P}$ has the following description. Note first of all that for any $(x, d), (y, e) \in X \oplus \mathbb{N}_{>0}$ we have

$$\tilde{g}(x+y, d+e) - \tilde{g}(x, d) - \tilde{g}(y, e) < 0$$

unless (x, d) and (y, e) lie in the same C_ω for some $\omega \in \Sigma$. Therefore

$$\xi^{(x,d)} \cdot \xi^{(y,e)} \equiv 0 \pmod{\pi}$$

if (x, d) and (y, e) lie in different cones. We therefore get a map

$$\mathcal{R} \otimes_V k \to k[C_\omega]$$

by sending $\xi^{(d,e)}$ to 0 unless $(d, e) \in C_\omega$ in which case we send $\xi^{(d,e)}$ to the element $e^{(d,e)}$. In this case we get a closed immersion

$$P_\omega := \mathrm{Proj}(k[C_\omega]) \hookrightarrow \widetilde{P}_0,$$

and it follows from the construction that $\widetilde{P}_0$ is equal to the union of the P_ω's glued along the natural inclusions $P_\eta \hookrightarrow P_\omega$, whenever η is a face of ω. Moreover, the T-action on P_ω is given by the natural T-action on each P_ω, and the translation action of $y \in X$ is given by the isomorphisms

$$P_\omega \to P_{\omega+y}$$

given by the natural identification of C_ω and $C_{\omega+y}$ given by the translation invariance of the paving.

Remark 4.2.10. Similarly, for every integer s and $\omega \in \Sigma$, there exists only finitely many cones $\eta \in \Sigma$ such that there exists $(x, d) \in C_\omega$ and $(y, e) \in C_\eta$ with the property

$$\xi^{(x,d)} \cdot \xi^{(\eta,e)} \neq 0 \pmod{\pi^s}.$$

4.2.11. This description of the closed fiber $\widetilde{P}_0$ implies in particular that the action of X on $\widetilde{P}_0$, and hence also the action on $\widetilde{P}_n := \widetilde{P} \otimes (V/\pi^{n+1})$, is properly discontinuous. We can therefore take the quotient

$$P_n := \widetilde{P}_n / X,$$

which is a finite type $V/(\pi^{n+1})$-scheme. The X-action on $L_{\widetilde{P}}$ gives descent data for the line bundles $L_{\widetilde{P}_n} := L_{\widetilde{P}}|_{\widetilde{P}_n}$, so we get a compatible collection of line bundles L_{P_n} on the schemes P_n. One can show that the line bundles L_{P_n} are in fact ample, so by the Grothendieck existence theorem [EGA 1961, chapitre III, corollaire 5.1.8, p. 151] the projective schemes $\{P_n\}$ are induced by a unique projective scheme P/V with a line bundle L_P inducing the L_{P_n}.

4.2.12. Since the action of T on $\widetilde{P}_n$ commutes with the action of X, there is an action of T on each of the P_n which is compatible with the reduction maps. One can show that there is a unique action of G on P inducing these compatible actions of T on the P_n's (recall that there is a canonical identification $G_n \simeq T$). This is one of the most subtle aspects of the construction. A detailed discussion in this special case can be found in [Mumford 1972, §3].

4.2.13. There is a compatible set of global sections $\theta_n \in \Gamma(P_n, L_{P_n})$ defined as follows. First of all note that since the map

$$\pi_n : \widetilde{P}_n \to P_n$$

is an X-torsor, we have a canonical isomorphism

$$\Gamma(P_n, L_{P_n}) \simeq \Gamma(\widetilde{P}_n, L_{\widetilde{P}_n})^X.$$

It therefore suffices to construct an X-invariant section

$$\tilde{\theta}_n \in \Gamma(\widetilde{P}_n, L_{\widetilde{P}_n}).$$

For $x \in X$ let $D(x)_n \subset \widetilde{P}_n$ denote the open subset defined by $\xi^{(x,1)}$, so

$$D(x)_n = \operatorname{Spec}(\mathcal{R}_{n,\xi^{(x,1)}})_0,$$

where $(\mathcal{R}_{n,\xi^{(x,1)}})_0$ denotes the degree 0 elements in $\mathcal{R}_{n,\xi^{(x,1)}}$. Then the $D(x)_n$ cover $\widetilde{P}_n$. Now for every x, all but finitely many $\xi^{(1,y)}$ map to zero in $\mathcal{R}_{n,\xi^{(x,1)}}$

by 4.2.10. Therefore the sum in $R_{n,\xi^{(x,1)}}$

$$\sum_{y \in Y} \alpha(y)\xi^{(y,1)}$$

is finite and defines a section $\tilde{\theta}_n \in \Gamma(D(x), L_{\widetilde{P}_n})$. These sections clearly glue to define the section $\tilde{\theta}_n \in \Gamma(\widetilde{P}_n, L_{\widetilde{P}_n})$. The relation

$$\alpha(x+y) = \alpha(x)\alpha(y)\beta(y,x), \quad x, y \in X$$

and the definition of the X-action on $(\widetilde{P}, L_{\widetilde{P}})$ implies that the section $\tilde{\theta}_n$ is X-invariant and therefore defines the section $\theta_n \in \Gamma(P_n, L_{P_n})$.

Finally since

$$\Gamma(P, L_P) = \varprojlim_n \Gamma(P_n, L_{P_n})$$

the sections $\{\theta_n\}$ are induced by a unique section $\theta \in \Gamma(P, L)$.

4.2.14. Summarizing the preceding discussion, we started with the quadruple $(G_\eta, P_\eta, L_\eta, \theta_\eta)$ over the fraction field K of V, and ended up with a quadruple (G, P, L, θ) as follows:

(1) G is a semiabelian scheme over V;

(2) P is a proper V-scheme with action of G;

(3) L is an ample line bundle on P;

(4) $\theta \in \Gamma(P, L)$ is a global section.

It follows from [Faltings and Chai 1990, Chapter III, 6.4] that the restriction of this quadruple to Spec(K) is canonically isomorphic to the original quadruple $(G_\eta, P_\eta, L_\eta, \theta_\eta)$. The collection (G, P, L, θ) should be viewed as the degeneration of $(G_\eta, P_\eta, L_\eta, \theta_\eta)$.

4.2.15. A careful investigation of this construction, as well as its generalization to the case when G is not totally degenerate, is the starting point for the definition of Alexeev's moduli problem which gives his compactification $\overline{\mathscr{A}}_g^{\text{Alex}}$ of $\mathscr{A}_g$. The end result of this investigation is the following.

4.2.16. Following [Alexeev 2002, 1.1.3.2], define a *stable semiabelic variety* over an algebraically closed field k to be a proper scheme P/k with an action of a semiabelian variety G/k such that the following hold:

(1) The dimension of each irreducible component of P is equal to the dimension of G.

(2) There are only finitely many orbits for the G-action.

(3) The stabilizer group scheme of every point of P is connected, reduced, and lies in the toric part of G.

(4) The scheme P is seminormal (recall that this means that the following property holds: If $f : P' \to P$ is a proper bijective morphism with P' reduced and with the property that for any $p' \in P'$ the map on residue fields $k(f(p')) \to k(k)$ is an isomorphism, then f is an isomorphism).

A *stable semiabelic pair* is a stable semiabelic variety P and a pair (L, θ), where L is an ample line bundle on P and $\theta \in H^0(P, L)$ is a global section whose zero locus does not contain any G-orbits.

Remark 4.2.17. If G is an abelian variety, then condition (3) implies that P is a disjoint union of G-torsors. If, moreover, we have a stable semiabelic pair (G, P, L, θ) with G abelian and $H^0(P, L)$ of dimension 1, then P must be connected so P is a G-torsor.

4.2.18. If S is a general base scheme, we define a *stable semiabelic pair over S* to be a quadruple (G, P, L, θ), where

(1) G/S is a semiabelian scheme.

(2) $f : P \to S$ is a projective flat morphism and G acts on P over S.

(3) L is a relatively ample invertible sheaf on P.

(4) $\theta \in H^0(P, L)$ is a global section.

(5) For every geometric point $\bar{s} \to S$, the geometric fiber $(G_{\bar{s}}, P_{\bar{s}}, L_{\bar{s}}, \theta_{\bar{s}})$ of this data is a stable semiabelic pair over the field $k(\bar{s})$.

Remark 4.2.19. It follows from cohomology and base change and [Alexeev 2002, 5.2.6] that if (G, P, L, θ) is a stable semiabelic pair over a scheme S as above, then f_*L is a locally free sheaf of finite rank on S whose formation commutes with arbitrary base change $S' \to S$. We define the *degree* of L to be the the rank of f_*L (a locally constant function on S).

Definition 4.2.20. Let $\mathscr{A}_g^{\mathrm{Alex}}$ be the stack over the category of schemes, whose fiber over a scheme S is the groupoid of semiabelic pairs (G, P, L, θ) over S with G of dimension g and L of degree 1.

4.2.21. By 2.2.7, there is a morphism of stacks

$$j : \mathscr{A}_g \to \mathscr{A}_g^{\mathrm{Alex}}$$

identifying $\mathscr{A}_g$ with the substack of semiabelic pairs (G, P, L, θ) with G an abelian scheme.

Theorem 4.2.22 [Alexeev 2002, 5.10.1]. *The stack $\mathscr{A}_g^{\mathrm{Alex}}$ is an Artin stack of finite type over $\mathbb{Z}$ with finite diagonal, and the map j is an open immersion.*

Example 4.3. The quadruple (G, P, L, θ) constructed starting in 4.2.2 is a semiabelic pair of degree 1 over $\mathrm{Spec}(V)$ (i.e., a V-point of $\mathscr{A}_g^{\mathrm{Alex}}$). Indeed note that the closed fiber P_0 of P can be described as follows.

Let

$$\widetilde{P}_0 \to P_0$$

be the X-torsor which is the reduction of the scheme $\widetilde{P}$, so as in 4.2.9 the scheme $\widetilde{P}_0$ is equal to a union of the toric varieties P_ω $(\omega \in \Sigma)$. Since $\widetilde{P}_0$ is reduced so is P_0, and the irreducible components of $\widetilde{P}_0$ are the subschemes P_ω with ω top dimensional. From this it follows that each irreducible component of P has dimension equal to the dimension of $G_0 = T$. Also note that the orbits for the T-action on P are in bijection with Σ/X, and hence is finite. To compute the stabilizer group schemes, note that if $\tilde{x} \in \widetilde{P}_0$ is a point in $\widetilde{P}_0$ with image $x \in P_0$, then the stabilizer group scheme of $\tilde{x}$ is equal to the stabilizer group scheme of x. Since each P_ω is a toric variety it follows that the stabilizer of any point of P_0 is a subtorus of T.

That the scheme P_0 is seminormal can be seen as follows. Let $f : Q \to P_0$ be a proper bijective morphism with Q reduced and the property that for any $q \in Q$ the map on residue fields $k(f(q)) \to k(q)$ is an isomorphism, and let $\mathscr{A}$ be the coherent sheaf of $\mathscr{O}_{P_0}$-algebras corresponding to Q. Since P_0 is reduced the map

$$\mathscr{O}_{P_0} \to \mathscr{A}$$

is injective, and we must show that it is also surjective. Let $\widetilde{\mathscr{A}}$ be the pullback of $\mathscr{A}$ to $\widetilde{P}_0$. Then $\widetilde{\mathscr{A}}$ is a coherent sheaf of $\mathscr{O}_{\widetilde{P}_0}$-algebras with an X-action lifting the X-action on $\widetilde{P}_0$. For each $\omega \in \Sigma$, let

$$j_\omega : P_\omega \hookrightarrow \widetilde{P}_0$$

be the inclusion. We construct an X-invariant morphism $s : \widetilde{\mathscr{A}} \to \mathscr{O}_{\widetilde{P}_0}$ such that the composite map

$$\mathscr{O}_{\widetilde{P}_0} \longrightarrow \widetilde{\mathscr{A}} \xrightarrow{\ s\ } \mathscr{O}_{\widetilde{P}_0}$$

is the identity. This will prove the seminormality of P_0, for by the X-invariance the map s descends to a morphism of algebras

$$\bar{s} : \mathscr{A} \to \mathscr{O}_{P_0}.$$

The kernel of this homomorphism is an ideal $\mathscr{I} \subset \mathscr{A}$ which is nilpotent since the map $Q \to P_0$ is bijective. Since Q is assumed reduced this implies that $\mathscr{I}$ is the zero ideal.

To construct the map s, proceed as follows. For each $\omega \in \Sigma$ let

$$i_\omega : P_\omega \hookrightarrow \widetilde{P}_0$$

be the inclusion. Let $\mathcal{S} \subset \Sigma$ be the subset of top-dimensional simplices, and choose an ordering of $\mathcal{S}$. We then have a map

$$\partial : \prod_{\omega \in \mathcal{S}} i_{\omega *}\mathcal{O}_{P_\omega} \to \prod_{\substack{\omega < \omega' \\ \omega, \omega' \in \mathcal{S}}} i_{\omega \cap \omega' *}\mathcal{O}_{P_{\omega \cap \omega'}},$$

defined by sending a local section $(\xi_\omega)_{\omega \in \mathcal{S}}$ to the section of the product whose image in the factor corresponding to $\omega < \omega'$ is

$$\xi_{\omega'}|_{P_{\omega \cap \omega'}} - \xi_\omega|_{P_{\omega \cap \omega'}}.$$

Then a straightforward verification, using the grading on the ring $\mathcal{R}$, shows that the natural map

$$\mathcal{O}_{\widetilde{P}_0} \to \mathrm{Ker}(\partial)$$

is an isomorphism of rings. To construct the map s it therefore suffices to construct compatible maps from $\widetilde{\mathcal{A}}$ to the $i_{\omega *}\mathcal{O}_{P_\omega}$. To construct these maps, note that since P_ω is normal the composite map

$$\mathrm{Spec}(i_\omega^* \widetilde{\mathcal{A}})_{\mathrm{red}} \hookrightarrow \mathrm{Spec}(i_\omega^* \widetilde{\mathcal{A}}) \to P_\omega$$

is an isomorphism, and hence we get maps

$$i_\omega^* \widetilde{\mathcal{A}} \to \mathcal{O}_{P_\omega}$$

which define maps

$$\widetilde{\mathcal{A}} \to i_{\omega *}\mathcal{O}_{P_\omega}$$

which are clearly compatible.

Finally we need to verify that the zero locus of the section $\theta_0 \in \Gamma(P_0, L_0)$ does not contain any T-orbit. For this let L_ω be the pullback of L_0 to P_ω and let $\theta_\omega \in \Gamma(P_\omega, L_\omega)$ be the pullback of θ. Then it suffices to show that the zero locus of θ_ω in P_ω does not contain any T-orbits. For this recall that we have

$$P_\omega = \mathrm{Proj}(k[C_\omega]),$$

and L_ω is equal to $\mathcal{O}_{P_\omega}(1)$. It follows that

$$\Gamma(P_\omega, L_\omega)$$

is isomorphic to the k-vector space with basis $\xi^{(x,1)}$, with $x \in \omega$. In terms of this basis the section θ_ω is by construction given by the sum of the elements $\alpha(x)\xi^{(x,1)}$. From this it follows immediately that the restriction of θ_ω to any T-invariant subset of P_ω is nonzero.

Remark 4.3.1. The stack $\mathcal{A}_g^{\mathrm{Alex}}$ is not irreducible. Explicit examples illustrating this is given in [Alexeev 2001]. In [Olsson 2008] we gave a modular interpretation of the closure of $\mathcal{A}_g$ in $\mathcal{A}_g^{\mathrm{Alex}}$ which we will describe in Section 4.5.

4.4. *Canonical compactification $\mathscr{A}_g \subset \overline{\mathscr{A}}_g$ and the second Voronoi compactification.*

4.4.1. Let $\overline{\mathscr{A}}_g$ denote the normalization of the closure of $\mathscr{A}_g$ in $\mathscr{A}_g^{\mathrm{Alex}}$. We call $\overline{\mathscr{A}}_g$ the *canonical compactification of* $\mathscr{A}_g$ (in Section 4.5 below we discuss a modular interpretation of $\overline{\mathscr{A}}_g$)).

4.4.2. Consider again the lattice X of rank g, and the integral regular paving Σ defined in 4.2.3. View Σ as a category whose objects are the polytopes $\omega \in \Sigma$ and in which the set of morphisms $\omega \to \eta$ is the unital set if $\omega \subset \eta$ and the empty set otherwise. We have a functor

$$P_{\cdot} : \Sigma \to \text{Monoids}$$

sending ω to the monoid C_ω. Taking the associated group we also obtain a functor

$$P_{\cdot}^{\mathrm{gp}} : \Sigma \to \text{Abelian groups}$$

by sending ω to C_ω^{gp}. Consider the inductive limit

$$\varinjlim P_{\cdot}^{\mathrm{gp}}.$$

For every $\omega \in \Sigma$ define

$$\rho_\omega : C_\omega \to \varinjlim P_{\cdot}^{\mathrm{gp}}$$

to be the composite map

$$C_\omega \hookrightarrow C_\omega^{\mathrm{gp}} \to \varinjlim P_{\cdot}^{\mathrm{gp}}.$$

Note that if $\eta \subset \omega$ then the diagram

$$\begin{array}{ccc} C_\eta & \hookrightarrow & C_\omega \\ & \searrow^{\rho_\eta} & \downarrow^{\rho_\omega} \\ & & \varinjlim P_{\cdot}^{\mathrm{gp}} \end{array}$$

commutes. In particular, the $\{\rho_\omega\}$ define a set map

$$\rho : P \to \varinjlim P_{\cdot}^{\mathrm{gp}},$$

where P denotes the integral points of the cone

$$\mathrm{Cone}(1, X_{\mathbb{R}}) \subset \mathbb{R} \oplus X_{\mathbb{R}}.$$

Define

$$\widetilde{H}_\Sigma \subset \varinjlim P_{\cdot}^{\mathrm{gp}}$$

to be the submonoid generated by elements of the form

$$\rho(p)+\rho(q)-\rho(p+q), \quad p, q \in P.$$

4.4.3. There is a natural action of X on $\mathbb{R} \oplus X_{\mathbb{R}}$ given by

$$y * (a, x) := (a, ay + x).$$

Since the paving Σ is X-invariant, this action induces actions of X on $\varinjlim P^{\text{gp}}_{\cdot}$, P, and $\widetilde{H}_\Sigma$.

Let H_Σ denote the quotient (in the category of integral monoids)

$$H_\Sigma := \widetilde{H}_\Sigma / X,$$

and let

$$\pi : \widetilde{H}_\Sigma \to H_\Sigma$$

be the projection. For elements $p, q \in P$ define

$$p * q := \pi(\rho(p)+\rho(q)-\rho(p+q)).$$

By [Olsson 2008, 4.1.6] the monoid H_Σ is finitely generated.

4.4.4. We have a monoid

$$P \rtimes H_\Sigma$$

defined as follows. As a set, $P \rtimes H_\Sigma$ is equal to the product $P \times H_\Sigma$, but the monoid law is given by

$$(p, \alpha) + (q, \beta) := (p+q, \alpha+\beta+p*q).$$

With this definition we get a commutative integral monoid $P \rtimes H_\Sigma$.

There is a natural projection

$$P \rtimes H_\Sigma \to P, \quad (p, \alpha) \mapsto p,$$

and therefore we get a grading on $P \rtimes H_\Sigma$ from the $\mathbb{N}$-grading on P. The scheme

$$\widetilde{\mathcal{P}} := \mathrm{Proj}(\mathbb{Z}[P \rtimes H_\Sigma])$$

over $\mathrm{Spec}(\mathbb{Z}[H_\Sigma])$ generalizes the scheme $\widetilde{P}$ in 4.2.5.

Lemma 4.4.5. *There exists a morphism of monoids*

$$h : H_\Sigma \to \mathbb{N}$$

sending all nonzero elements of H_Σ to strictly positive numbers. In particular, the monoid H_Σ is unit-free.

Proof. Let

$$\tilde{g} : \mathbb{N}_{>0} \oplus X \to \mathbb{Q}$$

be the function defined in 4.2.5. The function $\tilde{g}$ is linear on each C_ω, and therefore induces a function

$$\tilde{h} : \varinjlim P_{\cdot}^{\mathrm{gp}} \to \mathbb{Q}.$$

This function has the property that whenever $p, q \in P$ lies in different cones of Σ then we have

$$\tilde{h}(\rho(p) + \rho(q) - \rho(p+q)) > 0.$$

In particular, we get a morphism of monoids

$$\tilde{h} : \widetilde{H}_\Sigma \to \mathbb{Q}_{\geq 0}$$

sending all nonzero generators, and hence also all nonzero elements, to $\mathbb{Q}_{>0}$. Now observe that if $p = (d, x)$ and $q = (e, y)$ are two elements of P, and if $z \in X$ is an element, then an exercise using (7), which we leave to the reader, shows that

$$\tilde{h}(\rho(d, x+dz) + \rho(e, y+ez) - \rho(d+e, x+y-(d+e)z)) \\ = \tilde{h}(\rho(d, x) + \rho(e, y) - \rho(d+e, x+y)).$$

The map $\tilde{h}$ therefore descends to a homomorphism

$$h : H_\Sigma \to \mathbb{Q}_{\geq 0}.$$

Now since H_Σ is finitely generated, we can by replacing h with Nh for suitable N assume that this has image in $\mathbb{N}$, which gives the desired morphism of monoids. □

4.4.6. In particular, there is a closed immersion

$$\mathrm{Spec}(\mathbb{Z}) \to \mathrm{Spec}(\mathbb{Z}[H_\Sigma]) \tag{8}$$

induced by the map

$$\mathbb{Z}[H_\Sigma] \to \mathbb{Z}$$

sending all nonzero elements of H_Σ to 0. Let $\mathbb{Z}[[H_\Sigma]]$ be the completion of $\mathbb{Z}[[H_\Sigma]]$ with respect to the ideal $J \subset \mathbb{Z}[H_\Sigma]$ defining this closed immersion. Let $\mathscr{V}$ denote the spectrum of $\mathbb{Z}[[H_\Sigma]]$, and for $n \geq 0$ let $\mathscr{V}_n$ denote the closed subscheme of $\mathscr{V}$ defined by J^{n+1}.

As before let T denote the torus associated to X. We define a compatible family of projective schemes with T-action

$$(\mathscr{P}_n, L_{\mathscr{P}_n})$$

over the schemes $\mathscr{V}_n$ as follows. Let $\widetilde{\mathscr{P}}_n$ denote the pullback of $\widetilde{P}$ to $\mathscr{V}_n$, and let $L_{\widetilde{P}_n}$ denote the pullback of $\mathbb{O}_{\widetilde{P}}(1)$. Note that the scheme $\widetilde{\mathscr{P}}_0$ over Spec($\mathbb{Z}$) can be described as in 4.2.9 as the union of the toric varieties $\mathrm{Spec}(\mathbb{Z}[C_\omega])$ for $\omega \in \Sigma$, glued along the natural closed immersions

$$\mathrm{Spec}(\mathbb{Z}[C_\eta]) \hookrightarrow \mathrm{Spec}(\mathbb{Z}[C_\omega])$$

for $\eta \subset \omega$. This implies in particular that the natural X-action on $\widetilde{\mathscr{P}}_n$ is free, and hence we can form the quotient of $(\widetilde{P}_n, L_{\widetilde{\mathscr{P}}_n})$ to get a compatible system of projective schemes $\{(\mathscr{P}_n, L_{\mathscr{P}_n})\}$ over the $\mathscr{V}_n$.

There is a T-action on $\widetilde{\mathscr{P}}$ defined as follows. For this note that the inclusion

$$P \hookrightarrow \mathbb{Z} \oplus X$$

induces an isomorphism $P^{\mathrm{gp}} \simeq \mathbb{Z} \oplus X$, so the projection $P \to \mathbb{N}$ defines a morphism of monoids

$$P \rtimes H_\Sigma \to \mathbb{Z} \oplus X.$$

This defines an action of $\mathbb{G}_m \times T$ on the affine scheme

$$\mathrm{Spec}(\mathbb{Z}[P \rtimes H_\Sigma]).$$

Since

$$\widetilde{\mathscr{P}} = (\mathrm{Spec}(\mathbb{Z}[P \rtimes H_\Sigma]) - \{\text{zero section}\})/\mathbb{G}_m$$

we therefore get an action of T on $\widetilde{\mathscr{P}}$. By construction this action commutes with the X-action, and hence we get also compatible actions of T on the $\mathscr{P}_n$.

Each of the line bundles $L_{\mathscr{P}_n}$ is ample on $\mathscr{P}_n$, so by the Grothendieck existence theorem the compatible system $\{(\mathscr{P}_n, L_{\mathscr{P}_n})\}$ is induced by a unique projective scheme $\mathscr{P}/\mathscr{V}$ with ample line bundle $L_{\mathscr{P}}$.

If $f : \mathscr{P} \to \mathscr{V}$ is the structure morphism, then $f_* L_{\mathscr{P}}$ is a locally free sheaf of rank 1 on $\mathscr{V}$ whose formation commutes with arbitrary base change (this follows from cohomology and base change and [Alexeev and Nakamura 1999, 4.4]). If we choose a nonzero global section $\theta \in f_* L_{\mathscr{P}}$, we then get a compatible family of objects

$$(T_{\mathscr{V}_n}, \mathscr{P}_n, L_{\mathscr{P}_n}, \theta_n) \in \mathscr{A}_g^{\mathrm{Alex}}(\mathscr{V}_n),$$

which induce a morphism

$$\mathrm{Spec}(\mathscr{V}) \to \mathscr{A}_g^{\mathrm{Alex}}. \tag{9}$$

We conclude that there exists a semiabelian scheme $G/\mathscr{V}$ with abelian generic fiber and closed fiber T which acts on $\mathscr{P}$ such that

$$(G, \mathscr{P}, L_{\mathscr{P}}, \theta)$$

defines a point of $\mathscr{A}_g^{\mathrm{Alex}}(\mathscr{V})$.

Remark 4.4.7. The discussion above is a bit circular, and it would be better to construct G using the theory of degenerations discussed in [Faltings and Chai 1990, Chapters II and III]. In fact, this theory enters into the construction of $\mathscr{A}_g^{\text{Alex}}$.

4.4.8. Let H_Σ^{sat} denote the saturation of the monoid H_Σ, and let $\mathscr{V}^{\text{sat}}$ denote the fiber product

$$\mathscr{V}^{\text{sat}} := \mathscr{V} \times_{\text{Spec}(\mathbb{Z}[H_\Sigma])} \text{Spec}(\mathbb{Z}[H_\Sigma^{\text{sat}}]).$$

Note that the map

$$\mathbb{Z}[H_\Sigma] \to \mathbb{Z}[H_\Sigma^{\text{sat}}]$$

is finite so the coordinate ring of the affine scheme $\mathscr{V}^{\text{sat}}$ is J-adically complete.

Let $\overline{\mathscr{A}}_g$ denote the normalization of the scheme-theoretic closure of $\mathscr{A}_g$ in $\mathscr{A}_g^{\text{Alex}}$ (below we shall give a modular interpretation of this stack). Then the map (9) induces a map

$$\mathscr{V}^{\text{sat}} \to \overline{\mathscr{A}}_g, \tag{10}$$

since $\mathscr{V}^{\text{sat}}$ is normal and the restriction of $(G, \mathscr{P}, L_{\mathscr{P}}, \theta)$ to the generic fiber of $\mathscr{V}^{\text{sat}}$ defines a point of $\mathscr{A}_g$.

This map (10) is étale (a more general result is given in [Olsson 2008, 4.5.20]).

4.4.9. The relationship between H_Σ and quadratic forms is the following. Consider the exact sequence

$$0 \to \widetilde{H}_\Sigma^{\text{gp}} \to (P \rtimes \widetilde{H}_\Sigma)^{\text{gp}} \to P^{\text{gp}} \to 0. \tag{11}$$

Now by the universal property of the group associated to a monoid, we have

$$H_\Sigma^{\text{gp}} = (\widetilde{H}_\Sigma/X)^{\text{gp}} = (\widetilde{H}_\Sigma^{\text{gp}})/X.$$

In particular, the long exact sequence of group homology arising from (11) defines a morphism

$$H_1(X, P^{\text{gp}}) \to H_0(X, \widetilde{H}_\Sigma^{\text{gp}}) = H_\Sigma^{\text{gp}}. \tag{12}$$

Now we have a short exact sequence of groups with X-action

$$0 \to X \to P^{\text{gp}} \to \mathbb{Z} \to 0,$$

where the inclusion $X \hookrightarrow P^{\text{gp}}$ is the identification of X with the degree 0 elements of P^{gp}, and the X-action on X and $\mathbb{Z}$ is trivial. We therefore obtain a map

$$H_1(X, \mathbb{Z}) \otimes X \to H_1(X, P^{\text{gp}}),$$

and hence by composing with (12) a map

$$H_1(X, \mathbb{Z}) \otimes X \to H_\Sigma^{\text{gp}}.$$

Now as explained in [Olsson 2008, 5.8.4] there is a natural identification of $H_1(X, \mathbb{Z})$ with X, and hence we get a map

$$X \otimes X \to H_\Sigma^{\mathrm{gp}}.$$

As explained in [Olsson 2008, 5.8.8] this map is equal to the map sending $x \otimes y \in X \otimes X$ to

$$(1, x+y) * (1, 0) - (1, x) * (1, y).$$

In particular, the map is symmetric and therefore defines a map

$$\tau : S^2 X \to H_\Sigma^{\mathrm{gp}}.$$

By [Olsson 2008, 5.8.15] this map induces an isomorphism after tensoring with $\mathbb{Q}$.

4.4.10. In particular we get an inclusion

$$\mathrm{Hom}(H_\Sigma, \mathbb{Q}_{\geq 0}) \hookrightarrow \mathrm{Hom}(S^2 X, \mathbb{Q})$$

of the rational dual of H_Σ into the space of quadratic forms on X. By [Olsson 2008, 5.8.16] this identifies the cone $\mathrm{Hom}(H_\Sigma, \mathbb{Q}_{\geq 0})$ with the cone

$$U(\Sigma) \subset \mathrm{Hom}(S^2 X, \mathbb{Q})$$

of positive semidefinite quadratic forms whose associated paving is coarser than the paving Σ.

4.4.11. As we now discuss, this description of H_Σ leads naturally to the *second Voronoi decomposition* of the space of quadratic forms. As explained in [Namikawa 1976, 2.3] there exists a unique admissible cone decomposition Σ^{Vor} of $C(X)$ (notation as in 4.1.1), called the *second Voronoi decomposition*, such that two quadratic forms $B, B' \in C(X)$ lie in the same $\sigma \in \Sigma^{\mathrm{Vor}}$ if and only if the pavings of $X_\mathbb{R}$ defined by B and B' as in 4.2.3 are equal. This paving is known to be smooth if $g \leq 4$, but for $g > 4$ is not smooth (see [Alexeev and Nakamura 1999, 1.14]). Let

$$\mathscr{A}_g^{\mathrm{Vor}}$$

denote the corresponding toroidal compactification over $\mathbb{C}$.

4.4.12. If V is a complete discrete valuation ring and

$$\rho : \mathrm{Spec}(V) \to \mathscr{V}$$

is a morphism sending the closed point of $\mathrm{Spec}(V)$ to a point in $\mathscr{V}_0$ and the generic point to the open subset of $\mathscr{V}$ over which G is an abelian scheme, then the pullback of G to V defines by the discussion in 3.2.8 a quadratic form

$$B_\rho : S^2X \to \mathbb{Q}.$$

It follows the construction that this quadratic form is equal to the composite map

$$S^2X \xrightarrow{\tau} H_\Sigma^{\mathrm{gp}} \xrightarrow{\rho^*} K^* \xrightarrow{\mathrm{val}} \mathbb{Z}.$$

In particular, it follows from 4.1.7 that the inclusion

$$\mathscr{A}_{g,\mathbb{C}} \hookrightarrow \mathscr{A}_g^{\mathrm{Vor}}$$

extends to some neighborhood of the image of $\mathscr{V}_{\mathbb{C}}^{\mathrm{sat}}$ in $\overline{\mathscr{A}}_{g,\mathbb{C}}$.

A similar description of the versal deformation space of partial degenerations (as discussed in [Olsson 2008, §4.5]), and again using 4.1.7, shows that in fact the inclusion $\mathscr{A}_{g,\mathbb{C}} \subset \mathscr{A}_g^{\mathrm{Vor}}$ extends to a morphism of stacks

$$\pi : \overline{\mathscr{A}}_{g,\mathbb{C}} \to \mathscr{A}_g^{\mathrm{Vor}}. \tag{13}$$

4.4.13. The local description of the map π is the following.

Let $\mathscr{V}_{\mathbb{C}}$ denote the spectrum of the completion of $\mathbb{C}[H_\Sigma]$ with respect to the morphism to $\mathbb{C}$ defined by (8), and let $\mathscr{V}_{\mathbb{C}}^{\mathrm{sat}}$ denote the base change

$$\mathscr{V}_{\mathbb{C}} \times_{\mathrm{Spec}(\mathbb{C}[H_\Sigma])} \mathrm{Spec}(\mathbb{C}[H_\Sigma^{\mathrm{sat}}]).$$

Consider the composite map

$$\mathscr{V}_{\mathbb{C}}^{\mathrm{sat}} \longrightarrow \overline{\mathscr{A}}_{g,\mathbb{C}} \longrightarrow \mathscr{A}_g^{\mathrm{Vor}}.$$

Let $Q \subset S^2X$ be the cone of elements $q \in S^2X$ such that for every $B \in U(\Sigma)$ we have

$$B(q) \geq 0.$$

Note that by 4.4.10 we have a natural inclusion

$$Q \hookrightarrow H_\Sigma^{\mathrm{sat}}.$$

Let $\mathscr{W}$ denote the spectrum of the completion of $\mathbb{C}[Q]$ with respect to the kernel of the composite map

$$\mathbb{C}[Q] \to \mathbb{C}[H_\Sigma^{\mathrm{sat}}] \to \mathbb{C}[H_\Sigma^{\mathrm{sat}}]/J^{\mathrm{sat}},$$

where $J^{\mathrm{sat}} \subset \mathbb{C}[H_\Sigma^{\mathrm{sat}}]$ is the ideal induced by $J \subset \mathbb{C}[H_\Sigma]$. The inclusion $Q \hookrightarrow H_\Sigma^{\mathrm{sat}}$ induces a map

$$\lambda : \mathscr{V}_{\mathbb{C}}^{\mathrm{sat}} \to \mathscr{W}.$$

4.4.14. By construction of the toroidal compactification $\mathscr{A}_g^{\mathrm{Vor}}$ we then have a formally étale map

$$\mathscr{W} \to \mathscr{A}_g^{\mathrm{Vor}},$$

and it follows from the construction of the toroidal compactification (see [Faltings and Chai 1990, Chapter IV, §3]) that the resulting diagram

$$\begin{array}{ccc} \mathscr{V}^{\mathrm{sat}}_{\mathbb{C}} & \xrightarrow{\lambda} & \mathscr{W} \\ \downarrow & & \downarrow \\ \overline{\mathscr{A}}_{g,\mathbb{C}} & \longrightarrow & \mathscr{A}_g^{\mathrm{Vor}} \end{array}$$

commutes.

4.4.15. This implies in particular that in a neighborhood of any totally degenerate point of $\overline{\mathscr{A}}_{g,\mathbb{C}}$ the map (13) is étale locally quasifinite, whence quasifinite. A suitable generalization of the preceding discussion to the partially degenerate case, gives that in fact that map (13) is a quasifinite morphism. This together with the fact that $\mathscr{A}_g^{\mathrm{Vor}}$ is normal implies that the map (13) identifies $\mathscr{A}_g^{\mathrm{Vor}}$ with the relative coarse moduli space of the morphism (13), in the sense of [Abramovich et al. 2011, §3].

This implies in particular that the map (13) induces an isomorphism on coarse moduli spaces.

The map (13) is not, however, in general an isomorphism. This can be seen from the fact that the map $Q \hookrightarrow H_\Sigma^{\mathrm{sat}}$ is not in general an isomorphism. The stack $\overline{\mathscr{A}}_g$ has some additional "stacky structure" at the boundary.

4.4.16. Granting that one has also a toroidal compactification of $\mathscr{A}_g$ over $\mathbb{Z}$ with respect to the second Voronoi decomposition over $\mathbb{Z}$ (this is known if $g \leq 4$), the preceding discussion applies verbatim over $\mathbb{Z}$ as well. Here one can see the difference between $\overline{\mathscr{A}}_g$ and $\mathscr{A}_g^{\mathrm{Vor}}$ even more clearly, for while $\mathscr{A}_g^{\mathrm{Vor}}$ is a Deligne–Mumford stack, the stack $\overline{\mathscr{A}}_g$ is only an Artin stack with finite diagonal, as the stabilizer group schemes in positive characteristic may have a diagonalizable local component.

4.5. *Modular interpretation of* $\overline{\mathscr{A}}_g$.

4.5.1. The key to giving $\overline{\mathscr{A}}_g$ a modular interpretation is to systematically use the toric nature of the construction in 4.4.6 using logarithmic geometry. We will assume in this section that the reader is familiar with the basic language of logarithmic geometry (the basic reference is [Kato 1989]).

4.5.2. Consider again the family

$$\tilde{f} : \widetilde{\mathscr{P}} \to \mathscr{V}$$

constructed in 4.4.6. The natural map

$$H_\Sigma \to \mathbb{O}_{\mathcal{V}}$$

defines a fine log structure $M_{\mathcal{V}}$ on M. Moreover, there is a fine log structure $M_{\widetilde{\mathcal{P}}}$ on $\widetilde{\mathcal{P}}$ and a morphism

$$f^b : f^* M_{\mathcal{V}} \to M_{\widetilde{\mathcal{P}}}$$

such that the induced morphism of fine log schemes

$$(f, f^b) : (\widetilde{\mathcal{P}}, M_{\widetilde{\mathcal{P}}}) \to (\mathcal{V}, M_{\mathcal{V}})$$

is log smooth. Moreover, the T-action on $\widetilde{\mathcal{P}}$ extends naturally to a T-action on the log scheme $(\widetilde{\mathcal{P}}, M_{\widetilde{\mathcal{P}}})$ over $(\mathcal{V}, M_{\mathcal{V}})$.

This log structure $M_{\widetilde{\mathcal{P}}}$ can be constructed as follows. The scheme $\widetilde{\mathcal{P}}$ is equal to the quotient of

$$\mathrm{Spec}(\mathbb{Z}[P \rtimes H_\Sigma]) - \{\text{zero section}\}$$

by the action of $\mathbb{G}_m$ defined by the $\mathbb{N}$-grading on $P \rtimes H_\Sigma$. The action of $\mathbb{G}_m$ extends naturally to an action on the log scheme

$$(\mathrm{Spec}(\mathbb{Z}[P \rtimes H_\Sigma]), \text{log structure associated to } P \rtimes H_\Sigma \to \mathbb{Z}[P \rtimes H_\Sigma])$$

over the log scheme

$$(\mathrm{Spec}(\mathbb{Z}[H_\Sigma]), \text{log structure associated to } H_\Sigma \to \mathbb{Z}[H_\Sigma]).$$

Passing to the quotient by this $\mathbb{G}_m$-action and base changing to $\mathcal{V}$, we therefore get the map

$$(f, f^b) : (\widetilde{\mathcal{P}}, M_{\widetilde{\mathcal{P}}}) \to (\mathcal{V}, M_{\mathcal{V}}).$$

Note that the X-action on $\widetilde{\mathcal{P}}$ extends naturally to an action of X on the log scheme $(\widetilde{\mathcal{P}}, M_{\widetilde{\mathcal{P}}})$ over $(\mathcal{V}, M_{\mathcal{V}})$. In particular, base changing to $\mathcal{V}_n$ and passing to the quotient by the X-action we get the log structure $M_{\mathcal{P}_n}$ on $\mathcal{P}_n$ and a morphism of log schemes

$$(\mathcal{P}_n, M_{\mathcal{P}_n}) \to (\mathcal{V}_n, M_{\mathcal{V}_n}). \tag{14}$$

4.5.3. Let H_Σ^{sat} be the saturation of H_Σ, and let $\mathcal{V}^{\text{sat}}$ be as in 4.4.8. Define $M_{\mathcal{V}^{\text{sat}}}$ to be the log structure on $\mathcal{V}^{\text{sat}}$ defined by the natural map

$$H_\Sigma^{\text{sat}} \to \mathbb{O}_{\mathcal{V}^{\text{sat}}}$$

so we have a morphism of log schemes

$$(\mathcal{V}^{\text{sat}}, M_{\mathcal{V}^{\text{sat}}}) \to (\mathcal{V}, M_{\mathcal{V}}).$$

If $\mathcal{V}_n^{\mathrm{sat}}$ denotes $\mathcal{V}^{\mathrm{sat}} \times_{\mathcal{V}} \mathcal{V}_n$, then we get by base change a compatible collection of morphisms

$$(\mathcal{P}_n^{\mathrm{sat}}, M_{\mathcal{P}_n^{\mathrm{sat}}}) \to (\mathcal{V}_n^{\mathrm{sat}}, M_{\mathcal{V}_n^{\mathrm{sat}}})$$

from the collection (14).

4.5.4. If k is a field, define a *totally degenerate standard family* over k to be a collection of data

$$\big(M_k, T, f : (P, M_P) \to (\mathrm{Spec}(k), M_k), L_P\big)$$

as follows:

(1) M_k is a fine saturated log structure on $\mathrm{Spec}(k)$;

(2) T is a torus over k of dimension g;

(3) $f : (P, M_P) \to (\mathrm{Spec}(k), M_k)$ is a log smooth morphism with P/k proper, together with a T-action on (P, M_P) over $(\mathrm{Spec}(k), M_k)$.

(4) L_P is an ample line bundle on P such that $H^0(P, L_P)$ has dimension 1.

(5) The data is isomorphic to the collection obtained from the closed fiber of the family constructed in 4.5.3.

More generally, as explained in [Olsson 2008, §4.1] given a semiabelian scheme G/k with toric part X, a paving of X corresponding to a quadratic form etc., there is a generalization of the preceding construction which gives a fine saturated log structure M_k on $\mathrm{Spec}(k)$ and a log smooth morphism

$$f : (P, M_P) \to (\mathrm{Spec}(k), M_k),$$

where P/k is proper, and G acts on (P, M_P) over $(\mathrm{Spec}(k), M_k)$. Moreover, the construction gives a line bundle L_P on P which is ample and such that $H^0(P, L_P)$ has dimension 1. We define a *standard family* over k to be a collection of data

$$(M_k, G, f : (P, M_P) \to (\mathrm{Spec}(k), M_k), L_P)$$

obtained in this way (so the G-action on (P, M_P) is part of the data of a standard family).

For an arbitrary scheme S define $\overline{\mathcal{T}}_g(S)$ as the groupoid of collections of data

$$\big(M_S, G, f : (P, M_P) \to (S, M_S), L_P\big) \tag{15}$$

as follows:

(1) M_S is a fine saturated log structure on S.

(2) G/S is a semiabelian scheme of dimension g.

(3) $f : (P, M_P) \to (S, M_S)$ is a log smooth morphism with P/S proper.

(4) L_P is a relatively ample invertible sheaf on P.

(5) For every geometric point $\bar{s} \to S$, the collection of data over $\bar{s}$

$$(M_{\bar{s}}, G_{\bar{s}}, f_{\bar{s}} : (P_{\bar{s}}, M_{P_{\bar{s}}}) \to (\bar{s}, M_{\bar{s}}), L_{P_{\bar{s}}})$$

obtained by pullback, is a standard family in the preceding sense.

By definition a morphism

$$\big(M_S, G, f : (P, M_P) \to (S, M_S), L_P\big) \to \big(M'_S, G', f' : (P', M_{P'}) \to (S, M'_S), L_{P'}\big)$$

between two objects of $\overline{\mathcal{T}}_g$ consists of the following data:

(1) An isomorphism $\sigma : M'_S \to M_S$ of log structures on S.

(2) An isomorphism of fine log schemes

$$\tilde{\sigma} : (P, M_P) \to (P', M_{P'})$$

such that the square

$$\begin{array}{ccc} (P, M_P) & \xrightarrow{\tilde{\sigma}} & (P', M_{P'}) \\ \downarrow f & & \downarrow f' \\ (S, M_S) & \xrightarrow{(\mathrm{id}, \sigma)} & (S, M'_S) \end{array}$$

commutes.

(3) An isomorphism $\tau : G \to G'$ of semiabelian group schemes over S such that the diagram

$$\begin{array}{ccc} G \times_S (P, M_P) & \xrightarrow{\text{action}} & (P, M_P) \\ \downarrow \tau \times \tilde{\sigma} & & \downarrow \tilde{\sigma} \\ G' \times_S (P', M_{P'}) & \xrightarrow{\text{action}} & (P', M_{P'}) \end{array}$$

commutes.

(4) $\lambda : \tilde{\sigma}^* L_{P'} \to L_P$ is an isomorphism of line bundles on P.

In particular, for any object (15) of $\overline{\mathcal{T}}_g(S)$ and element $u \in \mathbb{G}_m(S)$ we get an automorphism of (15) by taking $\sigma = \mathrm{id}$, $\tilde{\sigma} = \mathrm{id}$, $\tau = \mathrm{id}$, and λ equal to multiplication by u.

With the natural notion of pullback we then get a stack $\overline{\mathcal{T}}_g$ over the category of schemes, together with an inclusion

$$\mathbb{G}_m \hookrightarrow \mathcal{I}_{\overline{\mathcal{T}}_g}$$

of $\mathbb{G}_m$ into the inertia stack of $\overline{\mathcal{T}}_g$.

Theorem 4.5.5 [Olsson 2008, 4.6.2]. *The stack $\overline{\mathcal{T}}_g$ is algebraic and there is a natural map $\overline{\mathcal{T}}_g \to \overline{\mathcal{A}}_g$ identifying $\overline{\mathcal{A}}_g$ with the rigidification of $\overline{\mathcal{T}}_g$ with respect to the subgroup $\mathbb{G}_m$ of the inertia stack.*

4.5.6. In fact the map $\overline{\mathcal{T}}_g \to \overline{\mathcal{A}}_g$ has a section. Consider the stack $\overline{\mathcal{A}}'_g$ whose fiber over a scheme S is the groupoid of data

$$(M_S, G, f : (P, M_P) \to (S, M_S), L_P, \theta),$$

where

$$(M_S, G, f : (P, M_P) \to (S, M_S), L_P) \in \overline{\mathcal{T}}_g(S)$$

is an object and $\theta \in f_*L_P$ is a section which is nonzero in every fiber. So $\overline{\mathcal{A}}'_g$ is the total space of the $\mathbb{G}_m$-torsor over $\overline{\mathcal{T}}_g$ corresponding to the line bundle defined by the sheaves f_*L_P (which are locally free of rank 1 and whose formation commutes with arbitrary base change). Then it follows, by an argument similar to the one proving 2.3.5, that the composite map

$$\overline{\mathcal{A}}'_g \to \overline{\mathcal{T}}_g \to \overline{\mathcal{A}}_g$$

is an isomorphism. So $\overline{\mathcal{A}}_g$ can be viewed as the stack whose fiber over a scheme S is the groupoid of collections of data

$$(M_S, G, f : (P, M_P) \to (S, M_S), L_P, \theta)$$

as above. In particular, from the log structures M_S in this collection, we get a natural log structure $M_{\overline{\mathcal{A}}_g}$ on $\overline{\mathcal{A}}_g$, whose open locus of triviality is the stack $\mathcal{A}_g$.

5. Higher degree polarizations

5.0.7. One advantage of the approach to $\overline{\mathcal{A}}_g$ using $\overline{\mathcal{T}}_g$ and rigidification is that it generalizes well to higher degree polarizations and moduli spaces for abelian varieties with level structure.

Fix an integer $d \geq 1$, and let $\mathcal{A}_{g,d}$ be the stack of abelian schemes of dimension g with polarization of degree d. Let $\mathcal{T}_{g,d}$ be the stack defined in 2.3.9, so that $\mathcal{A}_{g,d}$ is the rigidification of $\mathcal{A}_{g,d}$ with respect to the universal theta group $\mathcal{G}$ over $\mathcal{T}_{g,d}$. To compactify $\mathcal{A}_{g,d}$, we first construct a dense open immersion $\mathcal{T}_{g,d} \hookrightarrow \overline{\mathcal{T}}_{g,d}$ and an extension of the universal theta group over $\mathcal{T}_{g,d}$ to a subgroup $\overline{\mathcal{G}} \subset \mathcal{I}_{\overline{\mathcal{T}}_{g,d}}$, and then $\overline{\mathcal{A}}_{g,d}$ will be obtained as the rigidification of $\overline{\mathcal{T}}_{g,d}$ with respect to $\overline{\mathcal{G}}$. Though the stack $\overline{\mathcal{T}}_{g,d}$ is not separated, it should be viewed as a compactification of $\mathcal{T}_{g,d}$ as it gives a proper stack $\overline{\mathcal{A}}_{g,d}$ after rigidifying.

5.1. ***Standard families.***

5.1.1. To get a sense for the boundary points of $\overline{\mathcal{T}}_{g,d}$, let us again consider the case of maximal degeneration. Let V be a complete discrete valuation ring, $S = \mathrm{Spec}(V)$, and let G/V be a semiabelian scheme over V whose generic fiber G_η is an abelian variety and whose closed fiber is a split torus T. Let X denote the character group of T. Assume further given a polarization

$$\lambda : G_\eta \to G_\eta^t$$

of degree d. In this case we again get by 3.2.8 a quadratic form on $X_{\mathbb{Q}} = Y_{\mathbb{Q}}$, where Y is as in 3.2.5. Note that in this case we only get a quadratic function

$$A : X \to \mathbb{Q},$$

but after making a suitable base change of V we may assume that this function actually takes values in $\mathbb{Z}$. We then get a paving Σ of $X_{\mathbb{R}}$ by considering the convex hull of the set of points

$$\{(x, A(x)) | x \in X\} \subset X_{\mathbb{R}} \oplus \mathbb{R}.$$

Just as before we get a paving Σ of $X_{\mathbb{R}}$ and we can consider the scheme

$$\widetilde{P} \to \mathrm{Spec}(V)$$

defined in the same way as in 4.2.5. The main difference is that now we get an action of Y on $\widetilde{P}$ as opposed to an action of X. Taking the quotient of the reductions of $\widetilde{Y}$ by this Y-action, and algebraizing as before we end up with a projective V-scheme P/V with G-action and an ample line bundle L_P on P, such that the generic fiber P_η is a torsor under G_η, and the map

$$G_\eta \to G_\eta^t$$

defined by the line bundle L_P is equal to λ.

5.1.2. The construction of the logarithmic structures in 4.5.2 also generalizes to the case of higher degree polarization by the same construction. From the construction in 4.5.2 we therefore obtain candidates for the boundary points of $\overline{\mathcal{T}}_{g,d}$ over an algebraically closed field k as collections of data

$$(M_k, G, f : (P, M_P) \to (\mathrm{Spec}(k), M_k), L_P),$$

where

(1) M_k is a fine saturated log structure on $\mathrm{Spec}(k)$.

(2) f is a log smooth morphism of fine saturated log schemes such that P/k is proper.

(3) G is a semiabelian variety over k which acts on (P, M_P) over $(\mathrm{Spec}(k), M_k)$.

(4) L_P is an ample line bundle on P.

(5) This data is required to be isomorphic to the data arising from a paving Σ of $X_{\mathbb{R}}$ coming from a quadratic form as above.

We call such a collection of data over k a *standard family*. More generally, there is a notion of standard family in the case when G is not totally degenerate (see [Olsson 2008, §5.2] for the precise definition).

Over a general base scheme S we define $\overline{\mathcal{T}}_{g,d}(S)$ to be the groupoid of collections of data

$$(M_S, G, f : (P, M_P) \to (S, M_S), L_P),$$

where

(1) M_S is a fine saturated log structure on S.

(2) f is a log smooth morphism whose underlying morphism $P \to S$ is proper.

(3) G/S is a semiabelian scheme which acts on (P, M_P) over (S, M_S).

(4) L_P is a relatively ample invertible sheaf on P.

(5) For every geometric point $\bar{s} \to S$ the pullback

$$(M_{\bar{s}}, G_{\bar{s}}, f_{\bar{s}} : (P_{\bar{s}}, M_{P_{\bar{s}}}) \to (\bar{s}, M_{\bar{s}}), L_{P_{\bar{s}}})$$

is a standard family over $\bar{s}$.

With the natural notion of pullback we get a stack $\overline{\mathcal{T}}_{g,d}$ over S.

Theorem 5.1.3 [Olsson 2008, 5.10.3]. *The stack $\overline{\mathcal{T}}_{g,d}$ is an algebraic stack of finite type. If $M_{\overline{\mathcal{T}}_{g,d}}$ denotes the natural log structure on $\overline{\mathcal{T}}_{g,d}$, then the restriction of $(\overline{\mathcal{T}}_{g,d}, M_{\overline{\mathcal{T}}_{g,d}})$ to $\mathbb{Z}[1/d]$ is log smooth.*

5.2. *The theta group.*

5.2.1. The stack $\overline{\mathcal{T}}_{g,d}$ is not separated, but it does have an extension of the theta group. Namely, for any objects

$$\mathcal{S} = (M_S, G, f : (P, M_P) \to (S, M_S), L_P) \in \overline{\mathcal{T}}_{g,d}(S)$$

over some scheme S, define

$$\mathcal{G}_{\mathcal{S}} : (S\text{-schemes})^{\mathrm{op}} \to (\mathrm{Groups})$$

to be the functor which to any S'/S associates the group of pairs

$$(\rho, \iota),$$

where

$$\rho : (P_{S'}, M_{P_{S'}}) \to (P_{S'}, M_{P_{S'}})$$

is an automorphism of log schemes over $(S', M_{S'})$ (where $M_{S'}$ is the pullback of M_S to $M_{S'}$), and

$$\iota : \rho^* L_{P_{S'}} \to L_{P_{S'}}$$

is an isomorphism of line bundles. We call $\mathcal{G}_{\mathcal{S}}$ the *theta group of* $\mathcal{S}$.

Note that there is a natural inclusion

$$i : \mathbb{G}_m \hookrightarrow \mathcal{G}_S$$

sending a unit u to the automorphism with $\rho = \mathrm{id}$ and ι multiplication by u.

Theorem 5.2.2 [Olsson 2008, 5.4.2]. *The functor $\mathcal{G}_{\mathcal{S}}$ is representable by a flat group scheme over S, which we again denote by $\mathcal{G}_{\mathcal{S}}$. The quotient of $\mathcal{G}_{\mathcal{S}}$ by $\mathbb{G}_m$ is a finite flat commutative group scheme $H_{\mathcal{S}}$ of rank d^2.*

5.2.3. So we have a central extension

$$1 \to \mathbb{G}_m \to \mathcal{G}_S \to H_{\mathcal{S}} \to 1,$$

with $H_{\mathcal{S}}$ commutative. We can then define a skew symmetric pairing

$$e : H_{\mathcal{S}} \times H_{\mathcal{S}} \to \mathbb{G}_m$$

by setting

$$e(x, y) := \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1} \in \mathbb{G}_m,$$

where $\tilde{x}, \tilde{\in}\mathcal{G}_{\mathcal{S}}$ are local lifts of x and y respectively. We call this pairing on $H_{\mathcal{S}}$ the *Weil pairing*. It is shown in [Olsson 2008, 5.4.2] that this pairing is nondegenerate.

5.3. *The stack* $\overline{\mathcal{A}}_{g,d}$.

5.3.1. The theta groups of objects of $\overline{\mathcal{T}}_{g,d}$ define a flat subgroup scheme

$$\mathcal{G} \hookrightarrow \mathcal{I}_{\overline{\mathcal{T}}_{g,d}}$$

of the inertia stack of $\overline{\mathcal{T}}_{g,d}$, and we define

$$\overline{\mathcal{A}}_{g,d}$$

to be the rigidification of $\overline{\mathcal{T}}_{g,d}$ with respect to $\mathcal{G}$.

Theorem 5.3.2 [Olsson 2008, §5.11]. (i) *The stack $\overline{\mathcal{A}}_{g,d}$ is a proper algebraic stack over $\mathbb{Z}$.*

(ii) *The log structure $M_{\overline{\mathcal{T}}_{g,d}}$ on $\overline{\mathcal{T}}_{g,d}$ descends uniquely to a log structure $M_{\overline{\mathcal{A}}_{g,d}}$ on $\overline{\mathcal{A}}_{g,d}$. The restriction of $(\overline{\mathcal{A}}_{g,d}, M_{\overline{\mathcal{A}}_{g,d}})$ to $\mathbb{Z}[1/d]$ is log smooth.*

(iii) *The natural inclusion* $\mathscr{A}_{g,d} \hookrightarrow \overline{\mathscr{A}}_{g,d}$ *is a dense open immersion and identifies* $\mathscr{A}_{g,d}$ *with the open substack of* $\overline{\mathscr{A}}_{g,d}$ *where* $M_{\overline{\mathscr{A}}_{g,d}}$ *is trivial.*

(iv) *The finite flat group scheme* $\mathscr{H} := \mathscr{G}/\mathbb{G}_m$ *with its Weil pairing* e *descends to a finite flat group scheme with perfect pairing (still denoted* $(\mathscr{H}, e)$*) on* $\overline{\mathscr{A}}_{g,d}$*. The restriction of* $\mathscr{H}$ *to* $\mathscr{A}_{g,d}$ *is the kernel of the universal polarization*

$$\lambda : X \to X^t$$

on the universal abelian scheme $X/\mathscr{A}_{g,d}$*.*

5.4. ***Moduli spaces for abelian varieties with level structure.*** Theorem 5.3.2 enables one to give compactifications for moduli spaces of abelian varieties with level structure. We illustrate this with an example.

5.4.1. Let $g \geq 1$ be an integer, let p be a prime, and let $\mathscr{A}_g(p)$ denote the stack over $\mathbb{Z}[1/p]$ which to any scheme S associates the groupoid of pairs

$$(A, \lambda, x : S \to A),$$

where (A, λ) is a principally polarized abelian variety of dimension g, and $x \in A(S)$ is a point of exact order p.

Note that if

$$X[p] \to \mathscr{A}_g$$

denotes the p-torsion subgroup of the universal principally polarized abelian scheme over $\mathscr{A}_g$, then $\mathscr{A}_g(p)$ is equal to the restriction to $\mathbb{Z}[1/p]$ of the complement of the zero section of $X[p]$ (which is finite over $\mathscr{A}_g[1/p]$ since the restriction of $X[p]$ to $\mathscr{A}_g[1/p]$ is finite étale). So we can view the problem of compactifying $\mathscr{A}_g(p)$ as a problem of compactifying the universal p-torsion subgroup scheme over $\mathscr{A}_g$.

5.4.2. For this note first that if (A, λ) is a principally polarized abelian scheme over a scheme S, then the p-torsion subgroup $A[p]$ is the kernel of

$$p\lambda : A \to A^t.$$

Let

$$j : \mathscr{A}_g[1/p] \to \mathscr{A}_{g,p^g}[1/p]$$

be the map sending (A, λ) to $(A, p\lambda)$. By 2.3.12 this map is an open and closed immersion, and if

$$\eta : X \to X^t$$

denotes the universal polarization over $\mathscr{A}_{g,p^g}[1/p]$ then the universal p-torsion subgroup over $\mathscr{A}_g[1/p]$ is the restriction of the finite étale group scheme

$$\mathrm{Ker}(\eta) \to \mathscr{A}_{g,p^g}[1/p].$$

5.4.3. Let

$$\mathcal{H} \to \overline{\mathcal{A}}_{g,p^g}$$

be the finite flat group scheme discussed in 5.3.2(iii). The rank of $\mathcal{H}$ is p^{2g}, so its restriction $\mathcal{H}[1/p]$ to $\overline{\mathcal{A}}_{g,p^g}[1/p]$ is a finite étale group scheme of rank p^{2g}, whose restriction to $\mathcal{A}_{g,p^g}[1/p]$ is $\mathrm{Ker}(\eta)$. We then get a compactification of $\mathcal{A}_g(p)$ by taking the closure of $\mathcal{A}_g(p)$ in the complement of the identity section in $\mathcal{H}[1/p]$. Since $\mathcal{H}[1/p]$ is finite étale over $\overline{\mathcal{A}}_{g,p^g}$ the resulting space $\overline{\mathcal{A}_g(p)}$ is finite étale over $\overline{\mathcal{A}}_{g,p^g}[1/p]$, and in particular is proper over $\mathbb{Z}[1/p]$ with toric singularities.

References

[Abramovich et al. 2003] D. Abramovich, A. Corti, and A. Vistoli, "Twisted bundles and admissible covers", *Comm. Algebra* **31**:8 (2003), 3547–3618.

[Abramovich et al. 2011] D. Abramovich, M. Olsson, and A. Vistoli, "Twisted stable maps to tame Artin stacks", *J. Algebraic Geom.* **20**:3 (2011), 399–477.

[Alexeev 2001] V. Alexeev, "On extra components in the functorial compactification of A_g", pp. 1–9 in *Moduli of abelian varieties* (Texel, 1999), Progr. Math. **195**, Birkhäuser, Basel, 2001.

[Alexeev 2002] V. Alexeev, "Complete moduli in the presence of semiabelian group action", *Ann. of Math.* (2) **155**:3 (2002), 611–708.

[Alexeev and Nakamura 1999] V. Alexeev and I. Nakamura, "On Mumford's construction of degenerating abelian varieties", *Tohoku Math. J.* (2) **51**:3 (1999), 399–420.

[Artin 1969] M. Artin, "Algebraic approximation of structures over complete local rings", *Inst. Hautes Études Sci. Publ. Math.* 36 (1969), 23–58.

[Ash et al. 1975] A. Ash, D. Mumford, M. Rapoport, and Y. Tai, *Smooth compactification of locally symmetric varieties*, Lie Groups: History, Frontiers and Applications **4**, Math. Sci. Press, Brookline, MA, 1975.

[Deligne and Mumford 1969] P. Deligne and D. Mumford, "The irreducibility of the space of curves of given genus", *Inst. Hautes Études Sci. Publ. Math.* 36 (1969), 75–109.

[EGA 1961] A. Grothendieck, "Éléments de géométrie algébrique, III: étude cohomologique des faisceaux cohérents (première partie)", *Inst. Hautes Études Sci. Publ. Math.* **11** (1961), 5–167. Zbl 0153.22301

[Faltings and Chai 1990] G. Faltings and C.-L. Chai, *Degeneration of abelian varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **22**, Springer, Berlin, 1990.

[Kajiwara et al. 2008a] T. Kajiwara, K. Kato, and C. Nakayama, "Logarithmic abelian varieties, I: complex analytic theory", *J. Math. Sci. Univ. Tokyo* **15**:1 (2008), 69–193.

[Kajiwara et al. 2008b] T. Kajiwara, K. Kato, and C. Nakayama, "Logarithmic abelian varieties", *Nagoya Math. J.* **189** (2008), 63–138.

[Kato 1989] K. Kato, "Logarithmic structures of Fontaine–Illusie", pp. 191–224 in *Algebraic analysis, geometry, and number theory* (Baltimore, 1988), Johns Hopkins Univ. Press, Baltimore, MD, 1989.

[Keel and Mori 1997] S. Keel and S. Mori, "Quotients by groupoids", *Ann. of Math.* (2) **145**:1 (1997), 193–213.

[Laumon and Moret-Bailly 2000] G. Laumon and L. Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **39**, Springer, Berlin, 2000.

[Mumford 1965] D. Mumford, *Geometric invariant theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, N.F. **34**, Springer, Berlin, 1965.

[Mumford 1966] D. Mumford, "On the equations defining abelian varieties, I", *Invent. Math.* **1** (1966), 287–354.

[Mumford 1967] D. Mumford, "On the equations defining abelian varieties, II and III", *Invent. Math.* **3** (1967), 71–135 and 215–244.

[Mumford 1970] D. Mumford, *Abelian varieties*, Tata Studies in Mathematics **5**, Tata Institute of Fundamental Research, Bombay, 1970.

[Mumford 1972] D. Mumford, "An analytic construction of degenerating abelian varieties over complete rings", *Compositio Math.* **24** (1972), 239–272.

[Namikawa 1976] Y. Namikawa, "A new compactification of the Siegel space and degeneration of Abelian varieties, I", *Math. Ann.* **221**:2 (1976), 97–141.

[Namikawa 1980] Y. Namikawa, *Toroidal compactification of Siegel spaces*, Lecture Notes in Mathematics **812**, Springer, Berlin, 1980.

[Olsson 2008] M. C. Olsson, *Compactifying moduli spaces for abelian varieties*, Lecture Notes in Mathematics **1958**, Springer, Berlin, 2008.

[Oort 1971] F. Oort, "Finite group schemes, local moduli for abelian varieties, and lifting problems", *Compositio Math.* **23** (1971), 265–296.

[SGA 1970] M. Demazure and A. Grothendieck (editors), *Schémas en groupes, II: Groupes de type multiplicatif, et structure des schémas en groupes généraux* (Séminaire de Géométrie Algébrique du Bois Marie 1962/64 = SGA 3 II), Lecture Notes in Math. **152**, Springer, Berlin, 1970.

[SGA 1972] A. Grothendieck (editor), *Groupes de monodromie en géométrie algébrique, I* (Séminaire de Géométrie Algébrique du Bois Marie 1967/69 = SGA 7 I), Lecture Notes in Mathematics **288**, Springer, Berlin, 1972.

molsson@math.berkeley.edu *Department of Mathematics, University of California, Berkeley, CA 94720, United States*