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Basic results on irregular varieties via Fourier–Mukai methods

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Recently Fourier–Mukai methods have proved to be a valuable tool in the study of the geometry of irregular varieties. The purpose of this paper is to illustrate these ideas by revisiting some basic results. In particular, we show a simpler proof of the Chen–Hacon birational characterization of abelian varieties. We also provide a treatment, along the same lines, of previous work of Ein and Lazarsfeld. We complete the exposition by revisiting further results on theta divisors. Two preliminary sections of background material are included.

In recent years the systematic use of the classical Fourier–Mukai transform between dual abelian varieties, and of related integral transforms, has proved to be a valuable tool for investigating the geometry of irregular varieties. An especially interesting point is the interplay between vanishing notions naturally arising in the Fourier–Mukai context, as weak index theorems, and the generic vanishing theorems of Green and Lazarsfeld. This naturally leads to the notion of *generic vanishing sheaves* (GV-sheaves for short). The purpose of this paper is to exemplify these ideas by revisiting some basic results.

To be precise, we focus on the theorem of Chen and Hacon [2001a] characterizing (birationally) abelian varieties by means of the conditions $q(X) = \dim X$ and $h^0(K_X) = h^0(2K_X) = 1$; this is stated as Lemma 4.2 below. We show that the Fourier–Mukai/Generic Vanishing package, in combination with Kollár's theorems on higher direct images of canonical bundles, produces a surprisingly quick and transparent proof of this result. Along the way, we provide a unified Fourier–Mukai treatment of most of the results of [Ein and Lazarsfeld 1997], where both the original and the present proof of the Chen–Hacon theorem find their roots.[1] We complete the exposition with a refinement of Hacon's cohomological characterization of desingularizations of theta divisors, as it fits well in the same framework.

Although many of the results treated here have led to further developments (see, for example, [Chen and Hacon 2002; Hacon and Pardini 2002; Jiang 2011; Debarre and Hacon 2007]), we have not attempted to recover the latter with

[1] However, our treatment of the results of Ein and Lazarsfeld differs from the original arguments only in some aspects.

the present approach. However, we hope that the point of view illustrated in this paper will be useful in the further study of irregular varieties with low invariants. In particular, the main lemma used to prove the Chen–Hacon theorem (see Lemma 4.2 and Scholium 4.3) is new, as far as I know, and can be useful in other situations. Moreover the present version of Hacon's characterization of desingularized theta divisors (Theorem 5.1) improves slightly the ones appearing in the literature.

The paper is organized as follows: there are two preliminary sections where the background material is recalled and informally discussed at some length. The first one is about the Fourier–Mukai transform, related integral transforms, and GV-sheaves. The second one is on generic vanishing theorems, including Hacon's generic vanishing theorem for higher direct images of canonical sheaves, which is already one of the most relevant applications of the Fourier–Mukai methods in the present context. The last three sections are devoted respectively to the work of Ein and Lazarsfeld, to the Chen–Hacon characterization of abelian varieties, and to Hacon's characterization of desingularizations of theta divisors.

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1. Fourier–Mukai transform, cohomological support loci, and GV-sheaves

Unless otherwise stated, in this paper we will deal with smooth complex projective varieties (but, as it will be pointed out in the sequel, some results work more generally for complex Kähler manifolds and some others for smooth projective varieties on any algebraically closed field). By *sheaf* we will mean always coherent sheaf.

Given a smooth complex projective variety X, its *irregularity* is

$$q(X) := h^0(\Omega^1_X) = h^1(\mathcal{O}_X) = \tfrac{1}{2}b_1(X),$$

and X is said to be *irregular* if $q(X) > 0$. Its *Albanese variety*

$$\operatorname{Alb} X := H^0(\Omega^1_X)^*/H_1(X, \mathbb{Z})$$

is a $q(X)$-dimensional complex torus which, since X is assumed to be projective, is an abelian variety. The *Albanese morphism* $\operatorname{alb} : X \to \operatorname{Alb} X$ is defined by making sense of $x \to (\omega \mapsto \int_{x_0}^{x} \omega)$, where x_0 is a fixed point of X. Note that alb is defined up to translation in $\operatorname{Alb} X$. The Albanese morphism is a universal morphism of X to abelian varieties (or, more generally, complex tori). The *Albanese dimension* of X is the dimension of the image of its Albanese map. It is easily seen that the Albanese dimension of X is positive as soon as X is irregular (we refer to [Ueno 1975, §9] for a thorough treatment of the Albanese morphism). X is said *of maximal Albanese dimension* if $\dim \mathrm{alb}(X) = \dim X$.

The dual abelian variety of the Albanese variety is

$$\mathrm{Pic}^0 X = H^1(\mathbb{O}_X)/H^1(X, \mathbb{Z}).$$

The exponential sequence shows that $\mathrm{Pic}^0 X$ parametrizes those line bundles on X whose first Chern class vanishes [Griffiths and Harris 1978, p. 313]. Hence $\mathrm{Pic}^0 X$ is a (smooth and compact) space of deformations of the structure sheaf of X. So all sheaves $\mathcal{F}$ on X have a common family of deformations: $\{\mathcal{F}\otimes\alpha\}_{\alpha\in\mathrm{Pic}^0 X}$. Since Riemann it has been natural to consider, rather than the cohomology $H^*(X, \mathcal{F})$ of $\mathcal{F}$, the full family $\{H^*(X, \mathcal{F}\otimes\alpha)\}_{\alpha\in\mathrm{Pic}^0 X}$. For example, a good part of the geometry of curves is captured by the Brill–Noether varieties $W^r_d(C) = \{\alpha \in \mathrm{Pic}^0 C \mid h^0(L\otimes\alpha) \geq r+1\}$, where L is a line bundle on C of degree d [Arbarello et al. 1985]. In fact, it is often convenient to do a related thing. Since $\mathrm{Pic}^0 X$ is a fine moduli space; i.e., $X \times \mathrm{Pic}^0 X$ carries a universal line bundle P, the Poincaré line bundle, one can consider the *integral transform*

$$\mathbf{R}q_*(p^*(\,\cdot\,)\otimes P) : \mathbf{D}(X) \to \mathbf{D}(\mathrm{Pic}^0 X),$$

where p and q are respectively the projections on X and $\mathrm{Pic}^0 X$. Given a sheaf $\mathcal{F}$, the cohomology sheaves of $\mathbf{R}q_*(p^*(\mathcal{F})\otimes P)$ are isomorphic to $R^i q_*(p^*(\mathcal{F})\otimes P)$. They are naturally related to the family of cohomology groups $H^i(X, \mathcal{F}\otimes\alpha)_{\alpha\in\mathrm{Pic}^0 X}$ via base change (see 1.3 below).

The pullback map $\mathrm{alb}^* : \mathrm{Pic}^0(\mathrm{Alb}\, X) \to \mathrm{Pic}^0 X$ is an isomorphism [Griffiths and Harris 1978, p. 332], and, via this identification, the Poincaré line bundle on $X \times \mathrm{Pic}^0 X$ is the pullback of the Poincaré line bundle on $\mathrm{Alb}\, X \times \mathrm{Pic}^0(\mathrm{Alb}\, X)$. Therefore the above transform should be thought as a tool for studying the part of the geometry of X coming from its Albanese morphism.

Integral transform associated to the Poincaré line bundle, cohomological support loci, GV_{-k}-sheaves. In practice it is convenient to consider an integral transform as above for an arbitrary morphism from X to an abelian variety.

Definition 1.1 (Integral transforms associated to Poincaré line bundles). Let X be a projective variety of dimension d, equipped with a morphism to a q-dimensional abelian variety

$$a : X \to A.$$

Let $\mathcal{P}$ (script) be a Poincaré line bundle on $A \times \mathrm{Pic}^0 A$. We will denote

$$P_a = (a \times \mathrm{id}_{\mathrm{Pic}^0 A})^*(\mathcal{P})$$

and p, q the two projections of $X \times \mathrm{Pic}^0 A$. Given a sheaf $\mathcal{F}$ on X, we define

$$\Phi_{P_a}(\mathcal{F}) = q_*(p^*(\mathcal{F}) \otimes P_a).$$

We consider the derived functor of the functor Φ_{P_a}:

$$\mathbf{R}\Phi_{P_a} : \mathbf{D}(X) \to \mathbf{D}(\mathrm{Pic}^0 A). \tag{1}$$

Sometimes we will have to consider the analogous derived functor $\mathbf{R}\Phi_{P_a^{-1}}$: $\mathbf{D}(X) \to \mathbf{D}(\mathrm{Pic}^0 A)$ as well. Since $\mathcal{P}^{-1} \cong (1_A \times (-1)_{\mathrm{Pic}^0 A})^* \mathcal{P}$, there is not much difference between the two:

$$\mathbf{R}\Phi_{P_a^{-1}} = (-1_{\mathrm{Pic}^0 A})^* \mathbf{R}\Phi_{P_a} \tag{2}$$

Finally, when the map a is the Albanese map of X, denoted by $\mathrm{alb} : X \to \mathrm{Alb}\, X$, the map alb^* identifies $\mathrm{Pic}^0(\mathrm{Alb} X)$ with $\mathrm{Pic}^0 X$ and the line bundle P_{alb} is identified with the Poincaré line bundle on $X \times \mathrm{Pic}^0 X$. We will simply set $P_{\mathrm{alb}} := P$. When X is an abelian variety then its Albanese morphism is (up to translation) the identity. In this case the transform $\mathbf{R}\Phi_{\mathcal{P}}$ is called the *Fourier–Mukai transform* (see below).

In the sequel, we will adopt the following notation: given a line bundle $\alpha \in \mathrm{Pic}^0 A$, we will denote $[\alpha]$ the point of $\mathrm{Pic}^0 A$ parametrizing α (via the Poincaré line bundle $\mathcal{P}$). In other words $\alpha = \mathcal{P}_{|A \times [\alpha]}$.

Definition 1.2 (Cohomological support loci). Given a coherent sheaf $\mathcal{F}$ on X, its *i-th cohomological support locus with respect to a* is

$$V^i_a(X, \mathcal{F}) = \{[\alpha] \in \mathrm{Pic}^0 A \mid h^i(X, \mathcal{F} \otimes a^*\alpha) > 0\}$$

In the special case when a is the Albanese map of X, we omit the reference to the map, writing

$$V^i(X, \mathcal{F}) = \{[\alpha] \in \mathrm{Pic}^0 X \mid h^i(X, \mathcal{F} \otimes \alpha) > 0\}.$$

As is customary for cohomology groups, when possible we will omit the variety X in the notation, writing simply $V^i_a(\mathcal{F})$ or $V^i(\mathcal{F})$ instead of $V^i_a(X, \mathcal{F})$ and $V^i(X, \mathcal{F})$.

Finally, we will adopt the notation

$$R\Delta(\mathcal{F}) = \mathbf{R}\mathcal{H}om(\mathcal{F}, \omega_X).$$

1.3 (Hyper)cohomology and base change. Given a sheaf, or more generally, a complex of sheaves $\mathcal{G}$ on X, the sheaf $R^i\Phi_{P_a}(\mathcal{G})$ is said to *have the base change property at a given point* $[\alpha]$ *of* $\mathrm{Pic}^0 X$ if the natural map $R^i\Phi_{P_a}(\mathcal{G}) \otimes \mathbb{C}([\alpha]) \to H^i(X, \mathcal{G} \otimes a^*\alpha)$ is an isomorphism, where $\mathbb{C}([\alpha])$ denotes the residue field at the point $[\alpha] \in \mathrm{Pic}^0 X$. We will frequently use the following well-known base-change result (applied to our setting): given a sheaf (or, more generally, a bounded complex) $\mathscr{G}$ on X, *if* $h^{i+1}(X, \mathscr{G} \otimes a^*\alpha)$ *is constant in a neighborhood of* $[\alpha]$,[2] *then both* $R^{i+1}\Phi_{P_a}(\mathscr{G})$ *and* $R^i\Phi_{P_a}(\mathscr{G})$ *have the base-change property in a neighborhood of* $[\alpha]$. When $\mathscr{G}$ is a sheaf this is well known; see [Mumford 1970, Corollary 2, p. 52], for instance. For the general case of complexes see [EGA 1963, §7.7, pp. 65–72]. It follows that, if $\mathscr{F}$ is a *sheaf*, then $R^i\Phi_{P_a}(\mathscr{F})$ and $R^i\Phi_{P_a}(R\Delta\mathscr{F})$ vanish for $i > \dim X$, and both $R^d\Phi_{P_a}(\mathscr{F})$ and $R^d\Phi_{P_a}(R\Delta\mathscr{F})$ have the base change property at all $[\alpha] \in \mathrm{Pic}^0 A$.

The following basic result, whose proof will be outlined in the next page, compares two types of vanishing notions. The first one is *generic vanishing of cohomology groups* i.e., roughly speaking, that certain cohomological support loci $V^i(\mathscr{F})$ are *proper* closed subsets of $\mathrm{Pic}^0 A$. The second one is the *vanishing of cohomology sheaves* of the transform of the derived dual of $\mathscr{F}$:

Theorem 1.4 [Pareschi and Popa 2011a, Theorem A; Pareschi and Popa 2009, Theorem 2.2]. *For* $\mathscr{F}$ *a sheaf on* X *and* k *a nonnegative integer*, *equivalence holds between*[3]

(*a*) $$\mathrm{codim}_{\mathrm{Pic}^0 A}\, V^i_a(\mathscr{F}) \geq i - k \quad \textit{for all } i \geq 0$$

and

(*b*) $$R^i\Phi_{P_a}(R\Delta\mathscr{F}) = 0 \quad \textit{for all } i \notin [d-k, d].$$

Definition 1.5 (GV_{-k}-sheaves). When one of the two equivalent conditions of Theorem 1.4 holds, the sheaf $\mathscr{F}$ is said to be a *GV_{-k}-sheaf with respect to the morphism a*. When possible, we will omit the reference to the morphism a.

GV-sheaves. We focus on the special case $k = 0$ in Theorem 1.4. For sake of brevity, a GV_0-sheaf will be simply referred to as a GV-sheaf (with respect to the morphism a). Note that in this case it follows from condition (a) of Theorem 1.4 that, for *generic* $\alpha \in \mathrm{Pic}^0 A$, the cohomology groups $H^i(\mathscr{F} \otimes a^*\alpha)$ vanish for all $i > 0$. The second equivalent condition of Theorem 1.4 says that, for a GV-sheaf $\mathscr{F}$, the full transform $\mathbf{R}\Phi_{P_a}(R\Delta(\mathscr{F}))$ is a sheaf concentrated in degree $d = \dim X$:

$$\mathbf{R}\Phi_{P_a}(R\Delta\mathscr{F}) = R^d\Phi_{P_a}(R\Delta\mathscr{F})[-d]$$

(in the terminology of Fourier–Mukai theory, "$R\Delta\mathscr{F}$ satisfies the weak index theorem with index d"). In this situation one usually writes

$$R^d\Phi_{P_a}(R\Delta\mathscr{F}) = \widehat{R\Delta\mathscr{F}}.$$

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[2]By semicontinuity, this holds if $h^{i+1}(X, \mathscr{G} \otimes a^*\alpha) = 0$.

[3]If $V^1_a(\mathscr{F})$ is empty we declare that its codimension is ∞.

The following proposition provides two basic properties of the sheaf $\widehat{R\Delta\mathcal{F}}$.

Proposition 1.6. *Let $\mathcal{F}$ be a GV-sheaf on X, with respect to a.*

(a) *The rank of $\widehat{R\Delta\mathcal{F}}$ equals $\chi(\mathcal{F})$.*

(b) $\mathcal{E}xt^i_{\mathcal{O}_{\mathrm{Pic}^0 A}}(\widehat{R\Delta\mathcal{F}}, \mathcal{O}_{\mathrm{Pic}^0 A}) \cong (-1_{\mathrm{Pic}^0 A})^* R^i\Phi_{P_a}(\mathcal{F})$.

Proof. (a) By Serre duality and base change, the rank of $\widehat{R\Delta\mathcal{F}}$ at a general point is the generic value of $h^0(\mathcal{F}\otimes a^*\alpha)$, which coincides with $\chi(\mathcal{F}\otimes a^*\alpha)$ (the higher cohomology vanishes for generic $\alpha\in\mathrm{Pic}^0 X$). Then (a) follows from the deformation invariance of the Euler characteristic.
(b) In the context of Definition 1.1, Grothendieck duality says that

$$\mathbf{R}\mathcal{H}om(R\Phi_{P_a}(\mathcal{F}), \mathcal{O}_{\mathrm{Pic}^0 A}) \cong \mathbf{R}\Phi_{P_a^{-1}}(R\Delta\mathcal{F})[d]; \tag{3}$$

see [Pareschi and Popa 2011a, Lemma 2.2]. Therefore Theorem 1.4(b), combined with (2), yields

$$\mathbf{R}\mathcal{H}om(R\Phi_{P_a}(\mathcal{F}), \mathcal{O}_{\mathrm{Pic}^0 A}) \cong (-1_{\mathrm{Pic}^0 X})^* \widehat{R\Delta\mathcal{F}}$$

Since $\mathbf{R}\mathcal{H}om(\,\cdot\,, \mathcal{O}_{\mathrm{Pic}^0 X})$ is an involution on $\mathbf{D}(\mathrm{Pic}^0 X)$, we have also

$$\mathbf{R}\Phi_{P_a}(\mathcal{F}) \cong (-1_{\mathrm{Pic}^0 X})^* \mathbf{R}\mathcal{H}om(\widehat{R\Delta\mathcal{F}}, \mathcal{O}_{\mathrm{Pic}^0 X}) \tag{4}$$

which proves (b). □

Outline of proof of Theorem 1.4. The implication (b) ⇒ (a) of Theorem 1.4 in the case $k=0$ is proved as follows. Recall that, by the Auslander–Buchsbaum–Serre formula, if $\mathcal{G}$ is a sheaf on $\mathrm{Pic}^0 A$, then the support of $\mathcal{E}xt^i(\mathcal{G}, \mathcal{O}_{\mathrm{Pic}^0 A})$ has codimension $\geq i$ in $\mathrm{Pic}^0 A$ (see [Okonek et al. 1980, Lemma II.1.1.2], for instance). Applying this to the sheaf $\widehat{R\Delta\mathcal{F}}$, from Proposition 1.6(b) (which is a consequence of hypothesis (b) of Theorem 1.4) we get that

$$\mathrm{codim}_{\mathrm{Pic}^0 A}\, \mathrm{Supp}\, R^i\Phi_{P_a}(\mathcal{F}) \geq i \quad \text{for all } i\geq 0. \tag{5}$$

To show that (5) is equivalent to Theorem 1.4(a), we argue by descending induction on i. For $i=d$ this is immediate since $R^d\Phi_{P_a}(\mathcal{F})$ has the base-change property. Now suppose that $\mathrm{codim}\, V_a^{\bar{i}}(\mathcal{F}) < \bar{i}$ for a given $\bar{i}<d$, and let $[\alpha]$ be a general point of a component of $V_a^{\bar{i}}(\mathcal{F})$ achieving the dimension. Because of (5) it must be that $R^{\bar{i}}\Phi_{P_a}(\mathcal{F})$ does not have the base-change property in the neighborhood of $[\alpha]$. Hence, by 1.3, such component has to be contained in $V_a^{\bar{i}+1}(\mathcal{F})$. Therefore $\mathrm{codim}\, V_a^{\bar{i}+1}(\mathcal{F}) < \bar{i}$, violating the inductive hypothesis. Hence $\mathrm{codim}\, V_a^i(\mathcal{F})\geq i$ for all $i\geq 0$. This proves the implication (b) ⇒ (a) for $k=0$. For arbitrary k one uses the same argument, replacing Proposition 1.6(b) with the spectral sequence arising from (4).

The implication (a) $\Rightarrow$ (b) can be proved, with more effort, using the same ingredients (Grothendieck duality, Auslander–Buchsbaum–Serre formula and base change; see [Pareschi and Popa 2009, Theorem 2.2]). $\square$

A peculiar property of GV-sheaves is the following:

Lemma 1.7 [Hacon 2004, Corollary 3.2]. *Let $\mathcal{F}$ be a GV-sheaf on X, with respect to a. Then*

$$V_a^d(\mathcal{F}) \subseteq \cdots \subseteq V_a^1(\mathcal{F}) \subseteq V_a^0(\mathcal{F}).$$

Proof. Let $i > 0$ and assume that

$$[\alpha] \in V_a^i(\mathcal{F}) = -V_a^{d-i}(R\Delta(\mathcal{F}))$$

(the equality follows from Serre duality). Since $R^{d-i}\Phi_{P_a}(R\Delta(\mathcal{F})) = 0$, it follows by base change that $[\alpha] \in -V_a^{d-i+1}(R\Delta\mathcal{F}) = V_a^{i-1}(\mathcal{F})$. $\square$

The usefulness of the concept of GV-sheaf stems from the fact that some features of the cohomology groups $H^i(\mathcal{F} \otimes a^*\alpha)$ and of the cohomological support loci $V_a^i(\mathcal{F})$ can be detected by local and sheaf-theoretic properties of the transform $\widehat{R\Delta\mathcal{F}}$. The following simple example will be repeatedly used in Sections 3 and 4.

Lemma 1.8 [Pareschi and Popa 2011a, Proposition 3.15][4]. *Let $\mathcal{F}$ be a GV-sheaf on X with respect to a, let W be an irreducible component of $V_a^0(\mathcal{F})$, and let $k = \operatorname{codim}_{\operatorname{Pic}^0 A} W$. Then W is also a component of $V_a^k(\mathcal{F})$. Therefore* $\dim X \geq k$. *In particular, if $[\alpha]$ is an isolated point of $V_a^0(\mathcal{F})$ then $[\alpha]$ is also an isolated point of $V_a^q(\mathcal{F})$* (here $q = \dim A$). *Therefore* $\dim X \geq \dim A$.

Proof. Since $\widehat{R\Delta\mathcal{F}}$ has the base-change property, it is supported at $V_a^d(R\Delta\mathcal{F}) = -V_a^0(\mathcal{F})$. Hence $-W$ is a component of the support of $\widehat{R\Delta\mathcal{F}}$. Let $[\alpha]$ be a general point of W. Since

$$R^i\Phi_{P_a}(\mathcal{F}) = (-1_{\operatorname{Pic}^0 A})^* \mathcal{E}xt^i(\widehat{R\Delta\mathcal{F}}, \mathcal{O}_{\operatorname{Pic}^0 A}),$$

from well-known properties of $\mathcal{E}xt$'s it follows that, in a suitable neighborhood in $\operatorname{Pic}^0 A$ of $[\alpha]$, $R^i\Phi_{P_a}(\mathcal{F})$ vanishes for $i < k$ and is supported at W for $i = k$. Therefore, by base change (see 1.3), W is contained in $V_a^k(\mathcal{F})$ (and in fact it is a component since, again by Theorem 1.4, $\operatorname{codim} V_a^k(\mathcal{F}) \geq k$). $\square$

From the previous lemma it follows that, if $\mathcal{F}$ is a GV-sheaf, then either $V_a^0(\mathcal{F}) = \operatorname{Pic}^0 A$ or there is a positive i such that $\operatorname{codim} V_a^i(\mathcal{F}) = i$, i.e., such that equality is achieved in condition (a) of Theorem 1.4. This can be rephrased as follows:

[4] In that reference, this result appears with an unnecessary hypothesis.

Corollary 1.9. (a) *If $\mathcal{F}$ is a nonzero GV-sheaf, there exists $i \geq 0$ such that* $\operatorname{codim} V^i_a(\mathcal{F}) = i$.

(b) *Let $\mathcal{F}$ be any sheaf on X. Then either $V^i_a(\mathcal{F}) = \varnothing$ for all $i \geq 0$ or there is an $i \geq 0$ such that* $\operatorname{codim} V^i_a(\mathcal{F}) \leq i$.

Proof. (a) follows immediately from the previous lemma.

(b) If not all of the cohomological support loci of $\mathcal{F}$ are empty then either $\mathcal{F}$ is a GV-sheaf, in which case part (a) applies, or there is an i such that $\operatorname{codim} V^i_a(\mathcal{F}) < i$. □

Lemma 1.8 is a particular instance of a wider and more precise picture. In fact the condition that the cohomological support loci $V^0_a(\mathcal{F})$ is a *proper* subvariety of $\operatorname{Pic}^0 A$ is equivalent, by Serre duality and base change, to the fact that the generic rank of $\widehat{R\Delta\mathcal{F}}$ is zero; i.e., $\widehat{R\Delta\mathcal{F}}$ is a torsion sheaf on $\operatorname{Pic}^0 A$. Under this condition Lemma 1.8 says that there is an $i > 0$ achieving the bound of Theorem 1.4(a), i.e., such that $\operatorname{codim} V^i(\mathcal{F}) = i$. There is the following converse (which will be in use in the proof of Hacon's characterization of theta divisors in Section 5). In what follows we will say that a sheaf *has torsion* if it is not torsion-free.

Theorem 1.10 [Pareschi and Popa 2009, Corollary 3.2; 2011b, Proposition 2.8]. *Let $\mathcal{F}$ be a GV-sheaf on X. The following are equivalent*:

(a) *There is an $i > 0$ such that* $\operatorname{codim} V^i_a(\mathcal{F}) = i$.

(b) *$\widehat{R\Delta\mathcal{F}}$ has torsion.*

Proof. We start by recalling a general commutative algebra result. Let $\mathcal{G}$ be a sheaf on a smooth variety Y. Then $\mathcal{G}$ is torsion-free if and only if

$$\operatorname{codim}_Y \operatorname{Supp}(\mathcal{E}xt^i_Y(\mathcal{G}, \mathcal{O}_Y)) > i \quad \text{for all } i > 0 \tag{6}$$

(see, for example, [Pareschi and Popa 2009, Proposition 6.4] or [Pareschi and Popa 2011b, Lemma 2.9]). We apply this to the sheaf $\widehat{R\Delta\mathcal{F}}$ on the smooth variety $\operatorname{Pic}^0 X$. From (6) and Proposition 1.6(b) it follows that: $\widehat{R\Delta\mathcal{F}}$ has torsion if and only if there exists an $i > 0$ such that

$$\operatorname{codim}_{\operatorname{Pic}^0(X)} \operatorname{Supp}(R^i\Phi_{P_a}\mathcal{F}) = i. \tag{7}$$

By base change (see §1.3), condition (7) implies that there exists a $i > 0$ such that $\operatorname{codim} V^i_a(\mathcal{F}) \leq i$. Hence, since $\mathcal{F}$ is GV, $\operatorname{codim} V^i_a(\mathcal{F}) = i$. Conversely, the same argument as in the indication of proof of Theorem 1.4 proves that if (a) holds, then in any case there is a $j \geq i$ such that $\operatorname{codim} \operatorname{Supp}(R^j\Phi_{P_a}\mathcal{F}) = j$. Therefore, by (6), $\widehat{R\Delta\mathcal{F}}$ has torsion. □

Mukai's equivalence of derived categories of abelian varieties. Nonvanishing. Assume that X coincides with the abelian variety A (and the map a is the identity). In this special case, according to Notation 1.1, the Poincaré line bundle on $A \times \operatorname{Pic}^0 A$ is denoted by $\mathcal{P}$. A well known theorem of Mukai asserts that $\mathbf{R}\Phi_{\mathcal{P}}$ is an equivalence of categories. More precisely, set $q = \dim A$ and denote the "opposite" functor $\mathbf{R}p_*(q^*(\,\cdot\,) \otimes \mathcal{P})$ by

$$\mathbf{R}\Psi_{\mathcal{P}} : \mathbf{D}(\operatorname{Pic}^0 A) \to \mathbf{D}(A).$$

Theorem 1.11 [Mukai 1981, Theorem 2.2]. *Let A be an abelian variety (over any algebraically closed field k). Then*

$$\mathbf{R}\Psi_{\mathcal{P}} \circ \mathbf{R}\Phi_{\mathcal{P}} = (-1_A)^*[-q], \quad \mathbf{R}\Phi_{\mathcal{P}} \circ \mathbf{R}\Psi_{\mathcal{P}} = (-1_{\operatorname{Pic}^0 A})^*[-q].$$

Mukai's theorem can be used to provide nonvanishing criteria for spaces of global sections. Here are some immediate ones:

Lemma 1.12 (nonvanishing). *Let $\mathcal{F}$ be a nonzero sheaf on an abelian variety X.*

(a) *If $\mathcal{F}$ is a GV-sheaf then $V^0(\mathcal{F})$ is nonempty.*

(b) *If* $\operatorname{codim} V^i(\mathcal{F}) > i$ *for all* $i > 0$, *then* $V^0(\mathcal{F}) = \operatorname{Pic}^0 X$.

Proof. (a) By base change, $\widehat{R\Delta\mathcal{F}} = R^d\Phi_{\mathcal{P}}(R\Delta\mathcal{F})$ is supported at $-V^0(\mathcal{F})$. Therefore if $V^0(\mathcal{F}) = \varnothing$ then $R^d\Phi_{\mathcal{P}}(R\Delta\mathcal{F})$ is zero, i.e., by Theorem 1.4, $\mathbf{R}\Phi_{\mathcal{P}}(R\Delta\mathcal{F})$ is zero. Then, by Mukai's theorem, $R\Delta\mathcal{F}$ is zero. Therefore $\mathcal{F}$ itself is zero, since $R\Delta$ is an involution on the derived category.

(b) If $V^0(\mathcal{F})$ is a *proper* subvariety of $\operatorname{Pic}^0 X$ then, by Lemma 1.8, there is at least one $i > 0$ such that $\operatorname{codim} V^i(\mathcal{F}) = i$. □

In the context of irregular varieties, Mukai's theorem is frequently used via the following proposition, whose proof is an exercise.

Proposition 1.13. *In the notation of Definition 1.1,*

$$\mathbf{R}\Phi_{P_a} \cong \mathbf{R}\Phi_{\mathcal{P}} \circ \mathbf{R}a_*.$$

Going back to Mukai's Theorem 1.11, the key point of its proof is the verification of the statement for the one-dimensional skyscraper sheaf at the identity point, namely that

$$\mathbf{R}\Phi_{\mathcal{P}}(\mathbf{R}\Psi_{\mathcal{P}}(k(\hat{0}))) = k(\hat{0})[-q].$$

Since $\mathbf{R}\Psi_{\mathcal{P}}(k(\hat{0})) = \mathcal{O}_A$, this amounts to proving that $\mathbf{R}\Phi_{\mathcal{P}}(\mathcal{O}_A) = k(\hat{0})[-q]$, or equivalently

$$R^i\Phi_{\mathcal{P}}(\mathcal{O}_A) = 0 \quad \text{for } i < q \qquad \text{and} \qquad R^q\Phi_{\mathcal{P}}(\mathcal{O}_A) = k(\hat{0}). \tag{8}$$

Since $V^i(\mathcal{O}_A) = \{\hat{0}\}$ for all i such that $0 \le i \le q$, the first part follows from easily from Theorem 1.4. Concerning the second part of (8), it does not follow from base change, and has to be proved with a different argument. Over the complex numbers this can be done easily using the explicit description of the Poincaré line bundle on an abelian variety (see [Kempf 1991, Theorem 3.15] or [Birkenhake and Lange 2004, Corollary 14.1.6]). In arbitrary characteristic it is proved in [Mumford 1970, p. 128]. Another proof can be found in [Huybrechts 2006, p. 202].

Now let X be an irregular variety of dimension d. The next proposition is a generalization of the second part of (8) to any smooth variety and is proved via an argument similar to Mumford's.

Proposition 1.14 [Barja et al. 2012, Proposition 6.1]. *Let X be a smooth projective variety* (*over any algebraically closed field k*), *equipped with a morphism to an abelian variety $a : X \to A$ such that the pullback map $a^* : \mathrm{Pic}^0 A \to \mathrm{Pic}^0 X$ is an embedding. Then*

$$R^d \Phi_{P_a}(\omega_X) \cong k(\hat{0}).$$

Notes 1.15. (1) All results in this section work in any characteristic. They work for compact Kähler manifolds as well.

(2) The implication (b) $\Rightarrow$ (a) of Theorem 1.4 and also Lemma 1.7 were already observed by Hacon [Hacon 2004, Theorem 1.2 and Corollary 3.2]. While the converse implication of Theorem 1.4 makes the picture conceptually more clear — and is also useful in various applications as Proposition 2.4 below — the careful reader will note that in the proof of the Chen–Hacon theorem we are only using the implication (b) $\Rightarrow$ (a).

(3) Theorem 1.10 is a particular case of a much more general statement: *the sheaf $\widehat{R\Delta\mathcal{F}}$ is not a k-syzygy sheaf if and only if there is a $i > 0$ such that* $\mathrm{codim}\, V^i(\mathcal{F}) = i + k$. An example of an application of these ideas is the higher dimensional Castelnuovo–de Franchis inequality [Pareschi and Popa 2009].

(4) The result about k-the syzygy sheaves mentioned in the previous note, together with Theorems 1.4 and 1.10, hold in a much more general setting. In the first place they work not only for sheaves, but also for objects in the bounded derived category of X. Secondly, they are not specific to the transforms (1) but they work for all integral transforms $\Phi_P : \mathbf{D}(X) \to \mathbf{D}(Y)$ whose kernel P is a perfect object of $\mathbf{D}(X \times Y)$, where X is a Cohen–Macaulay equidimensional perfect scheme and Y is a locally noetherian scheme, both defined over a field k. A thorough analysis of the implications at the derived category level of these results is carried out in [Popa 2009].

(5) The hypothesis of Lemma 1.12(b), also called *M-regularity*, has many applications to global generation properties. In fact more than the conclusion of Lemma 1.12(b) holds: besides $V_0(\mathcal{F})$ being the whole $\mathrm{Pic}^0 X$, $\mathcal{F}$ is also

continuously globally generated. A survey on M-regularity and its applications is [Pareschi and Popa 2008]. A more recent development where the concept of M-regularity is relevant is the result of [Barja et al. 2012] on the bicanonical map of irregular varieties.

2. Generic vanishing theorems for the canonical sheaf and its higher direct images

Kollár's theorems on higher direct images of canonical sheaves. The following theorems will be of ubiquitous use in what follows.

Theorem 2.1 [Kollár 1986a, Theorem 2.1; 1986b, Theorem 3.1]. *Let X and Y be complex projective varieties of dimension d and $d-k$, with X smooth, and let $f : X \to Y$ a surjective map. Then*:

(a) *$R^i f_*\omega_X$ is torsion-free for all $i \geq 0$.*

(b) *$R^i f_*\omega_X = 0$ if $i > k$.*

(c) *Let L be an ample line bundle on Y. Then*

$$H^j(L \otimes R^i f_*\omega_X) = 0 \quad \textit{for all } i \geq 0 \textit{ and } j > 0;$$

(d) *in the derived category of Y*,

$$\mathbf{R}f_*\omega_X \cong \bigoplus_{i=0}^{k} R^i f_*\omega_X[-i].$$

In the next section, Theorem 2.1 will be used in the following variant, which is also a very particular case of more general formulations in [Kollár 1986b, §3].

Variant 2.2. *In the hypothesis and notation of Theorem 2.1, ω_X can be replaced by $\omega_X \otimes \beta$, where $[\beta]$ is a torsion point of $\operatorname{Pic}^0 X$.*

Generic vanishing theorems: Green–Lazarsfeld and Hacon. According to the previous terminology, a *generic vanishing theorem* is the statement that a certain sheaf is a GV_{-k}-sheaf. Within such terminology, the Green–Lazarsfeld generic vanishing theorem [Green and Lazarsfeld 1987], arisen independently of the theory of Fourier–Mukai transforms, can be stated as follows

Theorem 2.3 [Green and Lazarsfeld 1987, Theorem 1; Ein and Lazarsfeld 1997, Remark 1.6]. *Let $a : X \to A$ be a morphism from X to an abelian variety A, and let $k = \dim X - \dim a(X)$. Then ω_X is a GV_{-k}-sheaf (with respect to a). In particular, if a is generically finite onto its image then ω_X is a GV-sheaf.*

In fact Theorem 2.3 is sharp, as shown by the next proposition.

Proposition 2.4 [Barja et al. 2012, Proposition 2.7] (a similar result appears in [Lazarsfeld and Popa 2010, Proposition 1.2 and Remark 1.4]). *With the hypothesis and notation of Theorem 2.3, ω_X is a GV_{-k}-sheaf and it is not a $GV_{-(k-1)}$-sheaf.*

Proof. By the Green–Lazarsfeld generic vanishing theorem, ω_X is a GV_{-k}-sheaf. The fact that ω_X is not a $\mathrm{GV}_{-(k-1)}$-sheaf means that there is a $j \geq 0$ (in fact a $j \geq k$) such that

$$\operatorname{codim} V^j(\omega_X) = j - k. \tag{9}$$

By Theorem 2.1(b,d), $H^d(\omega_X) = \oplus_{i=0}^k H^{d-i}(R^i a_* \omega_X)$. Since $H^d(\omega_X) \neq 0$, it follows that $H^{d-k}(R^k a_* \omega_X) \neq 0$. Hence, by Corollary 1.9, there is a $\bar{i} \geq 0$ such that

$$\operatorname{codim} V^{\bar{i}}(R^k a_* \omega_X) \leq \bar{i}. \tag{10}$$

Again by Theorem 2.1(b),(d), and projection formula

$$H^i(X, \omega_X \otimes a^* \alpha) = \bigoplus_{h=0}^{min\{i,k\}} H^{i-h}(A, R^h a_* \omega_X \otimes \alpha) \tag{11}$$

Therefore

$$V_a^{\bar{i}+k}(X, \omega_X) \supseteq V^{\bar{i}}(A, R^k a_* \omega_X).$$

Hence (10) yields $\operatorname{codim} V_a^{\bar{i}+k}(\omega_X) \leq \bar{i} = (\bar{i} + k) - k$. In fact equality holds, since ω_X is a GV_{-k}-sheaf. Therefore (9) is proved. □

In the argument of [Green and Lazarsfeld 1987] the GV_{-k} condition is verified by proving condition (a) of Theorem 1.4, i.e., the bound on the codimension of the cohomological support loci $V_a^i(\omega_X)$. This is achieved via an infinitesimal argument, based on Hodge theory. In fact, the Green–Lazarsfeld theorem holds, more generally, in the realm of compact Kähler varieties. Using the theory of Fourier–Mukai transforms, Hacon extended Theorem 2.3 to higher direct images of dualizing sheaves (in the case of smooth projective varieties). Hacon's result can be stated in several slightly different variants. A simple one, which is enough for the application of the present paper, is this:

Theorem 2.5 [Hacon 2004, Corollary 4.2]. *Let X be a smooth projective variety and let $a : X \to A$ be a morphism to an abelian variety. Then $R^i a_* \omega_X$ is a GV-sheaf on A for all $i \geq 0$.*

In fact, the Green–Lazarsfeld theorem follows from Hacon's via Kollár's theorems:

Proof that Theorem 2.5 implies Theorem 2.3. It follows from (11) that

$$V^i_a(X, \omega_X) = \bigcup_{h=0}^{\min\{i,k\}} V^{i-h}(A, R^h a_* \omega_X). \tag{12}$$

By Theorem 2.5, codim $V^{i-h}(R^h a_* \omega_X) \geq i - h \geq i - k$. □

Hacon's generic vanishing theorem will also be used in a variant form:

Variant 2.6 [Hacon and Pardini 2005, Theorem 2.2]. *Theorem 2.5* (*hence also Theorem 2.3*) *still holds if* ω_X *is replaced by* $\omega_X \otimes \beta$, *where* $[\beta]$ *is a torsion point of* $\operatorname{Pic}^0 X$.

Let us describe briefly the proof of Theorem 2.5, which is completely different from the arguments of Green and Lazarsfeld. Hacon's approach consists in reducing a generic vanishing theorem to a vanishing theorem of Kodaira–Nakano type. Interestingly enough this is done by verifying directly condition (b) of Theorem 1.4, rather than condition (a), the dimensional bound for the cohomological support loci. The argument is as follows. Let L be an ample line bundle on $\operatorname{Pic}^0 A$ and consider, for a positive n, the locally free sheaf on A obtained as the "converse" Fourier–Mukai transform of L^n:

$$\mathbf{R}\Psi_{\mathcal{P}} L^n = R^0 \Psi_{\mathcal{P}} L^n$$

(see the notation above concerning the Fourier–Mukai transform), where the above equality follows from Kodaira vanishing. With an argument similar to the proof of Grauert–Riemenschneider vanishing, Hacon shows that, given a sheaf $\mathcal{F}$ on A, the condition (b) of Theorem 1.4 (namely $R^i \Phi_{\mathcal{P}}(R\Delta\mathcal{F}) = 0$ for $0 \neq \dim X$) is equivalent to the fact that there exist an n_0 such that, for all $n \geq n_0$,

$$H^i(A, (R\Delta\mathcal{F}) \otimes R^0\Psi_{\mathcal{P}} L^n) = 0 \quad \text{for all } i < q, \qquad \text{where} q = \dim A.$$

By Serre duality, this is equivalent to

$$H^i(\mathcal{F} \otimes (R^0\Psi_{\mathcal{P}} L^n)^*) = 0 \quad \text{for all } i > 0. \tag{13}$$

On the other hand, it is well known that, up to an isogeny, $R^0\Psi_{\mathcal{P}} L^n$ is the direct sum of copies of negative line bundles. More precisely, let $\phi_{L^n} : \operatorname{Pic}^0 A \to A$ be the *polarization* associated to L^n. Then (see [Mukai 1981, Proposition 3.11(1)], for example)

$$\phi^*_{L^n}(R^0\Psi_{\mathcal{P}} L^n) \cong H^0(\operatorname{Pic}^0 A, L^n) \otimes L^{-n} \tag{14}$$

Therefore, putting together (13) and (14) it turns out that, to prove that $\mathcal{F}$ is a GV-sheaf, it is enough to prove that, for n big enough,

$$H^i(\operatorname{Pic}^0 A, \phi^*_{L^n}(\mathcal{F}) \otimes L^n) = 0 \quad \text{for all } i > 0. \tag{15}$$

Condition (15) is certainly satisfied by the sheaves $\mathscr{F}$ enjoying the following property: *for each isogeny $\pi : B \to A$, and for each ample line bundle N on B, $H^i(B, \pi^*\mathscr{F} \otimes N) = 0$ for all $i > 0$.* Such property is satisfied by a higher direct image of a canonical sheaf since, via étale base extension, its pullback via an étale cover is still a higher direct image of a canonical sheaf, so that Kollár's Theorem 2.1(c) applies.

Notes 2.7. In [Pareschi and Popa 2011a] it is shown that Hacon's approach works in greater generality, and does not need the ambient variety to be an abelian variety. This yields other "generic vanishing theorems". For example, Green–Lazarsfeld's result (Theorem 2.3) works also for line bundles of the form $\omega_X \otimes L$, with L nef (see *loc. cit.* Corollary 5.2 and Theorem 5.8 for a better statement). In *loc. cit.* (Theorem A) is also shown that a part of Hacon's approach works in the setting of arbitrary integral transforms.

The subtorus theorems of Green, Lazarsfeld and Simpson. The geometry of the loci $V^i_a(\omega_X)$ is described by the Green–Lazarsfeld subtorus theorem, with an important addition due to Simpson.

Theorem 2.8 [Green and Lazarsfeld 1991, Theorem 0.1; Ein and Lazarsfeld 1997, Proof of Theorem 3, p. 249; Simpson 1993]. *Let X be a compact Kähler manifold, and W a component of $V^i(\omega_X)$ for some i. Then*
(*a*) *There exists a torsion point $[\beta]$ and a subtorus B of $\operatorname{Pic}^0 X$ such that $W = [\beta] + B$.*
(*b*) *Let $g := \pi \circ \operatorname{alb} : X \to \operatorname{Pic}^0 B$, where $\pi : \operatorname{Alb} X \to \operatorname{Pic}^0 B$ is the dual map of the embedding $B \hookrightarrow \operatorname{Pic}^0 X$. Then $\dim g(X) \le \dim X - i$.*

Simpson's result (conjectured by Beauville and Catanese) is that $[\beta]$ is a torsion point. In [Simpson 1993] there are also, among other things, other different proofs of part (a) of the theorem. It is worth mentioning that, admitting part (a), the dimensional bound (b) is a direct consequence of the generic vanishing theorem:
Proof of (b). By Theorem 2.8(a) it follows that $V^i_g(\omega_X \otimes \beta) = \operatorname{Pic}^0(\operatorname{Pic}^0 B) = B$. Therefore, by definition, $\omega_X \otimes \beta$ is a GV_{-h}-sheaf with $h \ge i$, with respect to g. Hence, by Variant 2.6, $\dim X - \dim g(X) \ge i$. □

Notes 2.9. (a) One defines the loci $V^i_m(\mathscr{F}) = \{[\alpha] \in \operatorname{Pic}^0 A \mid h^i(X, \mathscr{F} \otimes \alpha) \ge m\}$ and the Green–Lazarsfeld–Simpson result (Theorem 2.8) holds more generally for these loci as well. As noted in [Hacon and Pardini 2005, Theorem 2.2(b)], this implies that Theorem 2.8(a) holds replacing ω_X with higher direct images of $R^i f_*\omega_X$, where f is a morphism $f : X \to Y$, where Y is a smooth irregular variety (for example, an abelian variety).

(b) An explicit description and classification of all possible (positive-dimensional) components of the loci $V^i(\omega_X)$ is known only for $i = \dim X - 1$, by [Beauville 1992, Corollary 2.3]. Note that in this case, by Theorem 2.3(b), the image $g(X)$ is a curve.

3. Some results of Ein and Lazarsfeld

The content of this section is composed of four basic results of L. Ein and R. Lazarsfeld. We provide their proofs with the tools described here, both for sake of self-containedness, and also because they are good examples of application of the general principles of the previous section. The first two of them will be basic steps in the proof of the Chen–Hacon theorem appearing in the next section (they appear also in the original argument), while the last two will be used in the characterization of desingularizations of theta divisors (Section 5).

A theorem of Kawamata [1981], conjectured in [Ueno 1975], asserts that the Albanese map of a complex projective variety X of Kodaira dimension is zero is surjective and has connected fibers. As a consequence, one has Kawamata's characterization of abelian varieties as varieties of Kodaira dimension zero such that $q(X) = \dim X$. Subsequently Kollár [1986a; 1993; 1995] addressed the problem of giving effective versions of such results, replacing the hypothesis on the Kodaira dimension with the knowledge of finitely many plurigenera. In fact, he proved that the surjectivity conclusion of Kawamata's theorem and the characterization of abelian varieties held under the weaker assumption that $p_m(X) := h^0(\omega_X^m) = 1$ for some $m \geq 3$, and conjectured that $m = 2$ would suffice. The surjectivity part of Kawamata's theorem was settled by Ein and Lazarsfeld in the result quoted below as Theorem 3.1(b), while the characterization of abelian varieties is the content of the Chen–Hacon theorem (see the next section).[5]

Theorem 3.1 [Ein and Lazarsfeld 1997, Theorem 4]. *Let X be a smooth projective variety such that $p_1(X) = p_2(X) = 1$.*

(a) (Kollár) *There is no positive-dimensional subvariety Z of* $\operatorname{Pic}^0 X$ *such that both Z and $-Z$ are contained in $V^0(\omega_X)$.*

(b) *The Albanese map of X is surjective.*

Proof. (a) (This is as in [Kollár 1993, Theorem 17.10]; the proof is included here for self-containedness). Assume that there is a positive-dimensional subvariety Z of $\operatorname{Pic}^0 X$ as in the statement. The images of the multiplication maps of global sections

$$H^0(\omega_X \otimes \gamma) \otimes H^0(\omega_X \otimes \gamma^{-1}) \to H^0(\omega_X^2)$$

[5]Concerning an effective version of the connectedness part of Kawamata's theorem, recently Zhi Jiang has proved that if $p_1(X) = p_2(X) = 1$ then the Albanese morphism has connected fibers [Jiang 2011, Theorem 1.3]. The proof uses also the theorem of Chen and Hacon.

are nonzero for all $\gamma \in T$. Therefore $h^0(\omega_X^2) > 1$, since otherwise the only effective divisor in $|\omega_X^2|$ would have infinitely many components.

(b) By hypothesis, the identity point $\hat{0}$ of $\mathrm{Pic}^0 X$ belongs to $V^0(\omega_X)$. It must be an isolated point, since otherwise, by Theorem 2.8, a positive-dimensional component Z of $V^0(\omega_X)$ containing $\hat{0}$ would be a subtorus, thus contradicting (a). Therefore $\hat{0}$ is also an isolated point of $V^0(\mathrm{Alb}\, X, \mathrm{alb}_* \omega_X)$. Since $\mathrm{alb}_* \omega_X$ is a GV-sheaf on $\mathrm{Alb}\, X$ (Theorem 2.5), $\hat{0}$ is also a isolated point of $V^{q(X)}(\mathrm{Alb}\, X, \mathrm{alb}_* \omega_X)$ (Lemma 1.8). In particular $H^{q(X)}(\mathrm{Alb}\, X, \mathrm{alb}_* \omega_X)$ is nonzero. Therefore alb, the Albanese map of X, is surjective. □

Next, we provide a different proof of the following characterization of abelian varieties. The same type of argument will be applied in the characterization of theta divisors of Section 5.

Theorem 3.2 (Ein and Lazarsfeld; see [Chen and Hacon 2001a, Theorem 1.8]). *Let X be a smooth projective variety of maximal Albanese dimension such that* $\dim V^0(\omega_X) = 0$. *Then X is birational to an abelian variety.*

Proof. By Theorem 2.3, ω_X is a GV-sheaf (with respect to the Albanese morphism). By Lemma 1.8, the hypothesis yields that $\dim X = q(X)$ and $V^0(\omega_X) = \{\hat{0}\}$. Using Proposition 1.14, $\mathbb{C}(\hat{0}) = R^q\Phi_P(\omega_X)$. Therefore, by Proposition 1.6, $\mathcal{E}xt^q(\widehat{\mathcal{O}_X}, \mathcal{O}_{\mathrm{Pic}^0 X}) = \mathbb{C}(\hat{0})$. Moreover, since $\widehat{\mathcal{O}_X}$ is supported at a finite set, $\mathcal{E}xt^i(\widehat{\mathcal{O}_X}, \mathcal{O}_{\mathrm{Pic}^0 X}) = 0$ for $i < q$. Summarizing: $R\Delta(\widehat{\mathcal{O}_X}) = \mathbb{C}(\hat{0})[-q]$. Since the functor $R\Delta$ is an involution, $\widehat{\mathcal{O}_X} = \mathbb{C}(\hat{0})$. In conclusion

$$\mathbf{R}\Phi_P(\mathcal{O}_X) = \mathbb{C}(\hat{0})[-q].$$

By Proposition 1.13 this means that $\mathbf{R}\Phi_{\mathcal{P}}(\mathbf{R}\,\mathrm{alb}_* \mathcal{O}_X) = \mathbb{C}(\hat{0})[-q]$. Then, by Mukai's inversion theorem (Theorem 1.11), $\mathbf{R}\,\mathrm{alb}_* \mathcal{O}_X = \mathcal{O}_{\mathrm{Alb}\, X}$. In particular $\mathrm{alb}_* \mathcal{O}_X = \mathcal{O}_{\mathrm{Alb}\, X}$. Since alb is assumed to be generically finite, this means that it is birational onto $\mathrm{Alb}\, X$. □

The next result concerns varieties for which the Albanese dimension is maximal and $\chi(\omega_X) = 0$. Similarly to Theorem 3.1, here the proof is only partially different from the original argument of [Ein and Lazarsfeld 1997].

Theorem 3.3 [Ein and Lazarsfeld 1997, Theorem 3]. *Let X be a smooth projective variety of maximal Albanese dimension such that $\chi(\omega_X) = 0$. Then the image of the Albanese map of X is fibered by translates of abelian subvarieties of* $\mathrm{Alb}\, X$.

Proof. Since ω_X is GV, the condition $\chi(\omega_X) = 0$ is equivalent to the fact that $V^0(\omega_X)$ is a proper subvariety of $\mathrm{Pic}^0 X$ (Proposition 1.6(a)). Hence, by Lemma 1.8, there is a positive k such that $V^k(\omega_X)$ has a component W of codimension k. At this point the proof is that of Ein and Lazarsfeld: the subtorus theorem (Theorem 1.10) says that $W = [\beta] + T$, where T is a subtorus of $\mathrm{Pic}^0 X$, and provides the diagram

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{alb}} & \mathrm{Alb}\, X \\ & \searrow^{g} & \downarrow{\scriptstyle \pi} \\ & & B = \mathrm{Pic}^0 T \end{array} \quad ,$$

where $\dim g(X) \le \dim X - k$, π is surjective, and the fibers of π are translates of k-dimensional subtori of $\mathrm{Alb}\, X$. Since alb is generically finite, it follows that $\dim g(X) = \dim X - k$ and that a generic fiber of g surjects onto a generic fiber of π. □

Notes 3.4. (a) We recall that Theorem 3.3 settled a conjecture of Kollár, asserting that a variety X of general type and maximal Albanese dimension should have $\chi(\omega_X) > 0$. Ein and Lazarsfeld in [Ein and Lazarsfeld 1997] disproved the conjecture, producing a threefold X of general type, maximal Albanese dimension and $\chi(\omega_X) = 0$. But, at the same time, with Theorem 3.3, they showed that if $\chi(\omega_X) = 0$ then (a desingularization of) the Albanese image of X can't be of general type. However, the structure of varieties of general type and maximal Albanese dimension X with $\chi(\omega_X) = 0$ still remain mysterious. Results in this direction are due to Chen and Hacon [2001b; 2004].

(b) Corollary 5.1 of [Pareschi and Popa 2009] extends Theorem 3.3 to varieties with low $\chi(\omega_X)$ as follows: let X be a variety of maximal Albanese dimension. Then the image of the Albanese map of X is fibered by h-codimensional subvarieties of subtori of $AlbX$, with $h \le \chi(\omega_X)$ (see *loc. cit.* for a more precise statement). The proof uses k-syzygy sheaves and the Evans–Griffith syzygy theorem.

(c) Theorems 3.2 and 3.3, as well as the extension mentioned in (b) above, work also in the compact Kähler setting. Moreover, the present proof of Theorem 3.2 is algebraic, so that the results holds over any algebraically closed field.

We conclude with Ein–Lazarsfeld's result on the singularities of theta divisors. Here the difference with the original argument is that adjoint ideals are not invoked.

Theorem 3.5 [Ein and Lazarsfeld 1997, Theorem 1]. *Let Θ be an irreducible theta divisor of a principally polarized abelian variety A. Then Θ is normal with rational singularities.*

Proof. Let $a : X \to \Theta$ be a resolution of singularities of X. It is well known that, under our hypotheses, the fact that Θ is normal with rational singularities is equivalent to the fact that the trace map $t : a_*\omega_X \to \omega_\Theta$ is an isomorphism. We have that $a_*\omega_X$ is a GV-sheaf on A (this follows by Hacon's generic vanishing or, more simply, by the Green–Lazarsfeld generic vanishing and the fact that, by Grauert–Riemenschneider vanishing, $V^i(X,\omega_X)=V^i(A,a_*\omega_X)$). Moreover, an immediate calculation with the adjunction formula shows that $V^i(\omega_\Theta)=\{\hat{0}\}$ for all $i>0$. We consider the exact sequence (the trace map t is injective)

$$0\to a_*\omega_X\xrightarrow{t}\omega_\Theta\to\operatorname{coker}t\to 0. \tag{16}$$

Tensoring with $\alpha\in\operatorname{Pic}^0A$ and taking cohomology it follows that

$$\operatorname{codim}V^i(A,\operatorname{coker}t)>i\quad\text{for all }\ i>0.$$

By Lemma1.12(b), it follows that if $\operatorname{coker}t\neq 0$ then $V^0(A,\operatorname{coker}t)=\operatorname{Pic}^0A$, or again, by Proposition 1.6(a), that $\chi(\operatorname{coker}t)>0$. Then, since $\chi(\omega_\Theta)=1$, from (16) it follows that $\chi(a_*\omega_X)=0$, that is, $\chi(\omega_X)=0$. At this point one concludes as in [Ein and Lazarsfeld 1997]. In fact, by Theorem 3.3, Θ would be fibered by subtori of A, which is not the case since Θ is ample and irreducible. □

In [Ein and Lazarsfeld 1997] there are also results on the singularities of pluri-theta divisors, extending previous seminal results of Kollár in [Kollár 1993]. These, together with Theorem 3.5, have been extended to other polarizations of low degree, especially in the case of simple abelian varieties, by Debarre and Hacon [2007].

4. The Chen–Hacon birational characterization of abelian varieties

The goal of this section is to supply a new proof of the Chen–Hacon characterization of abelian varieties. We refer to the previous section for a short history and motivation.

Theorem 4.1 [Chen and Hacon 2001a]. *Let X be smooth complex projective variety. Then X is birational to an abelian variety if and only if $q(X)=\dim X$, and $h^0(\omega_X)=h^0(\omega_X^2)=1$.*

Via the approach of Kollár and Ein–Lazarsfeld, the Chen–Hacon theorem will be a consequence of the following:

Lemma 4.2. *Let X be a projective variety of maximal Albanese dimension. If $\dim V^0(\omega_X)>0$ then there exists a positive-dimensional subvariety Z of Pic^0X such that both Z and $-Z$ are contained in $V^0(\omega_X)$.*

Proof of Theorem 4.1. Let X be a smooth projective variety such that $p_1(X)=p_2(X)=1$ and $q(X)=\dim X$. By Theorem 3.1(b) the Albanese map of X is surjective, hence generically finite. By Lemma 4.2, combined with Theorem 3.1(a), it follows that $\dim V^0(\omega_X)=0$. Therefore, thanks to the characterization provided by Theorem 3.2, X must be birational to an abelian variety. □

Proof of Lemma 4.2. Let W be a positive-dimensional component of $V^0(\omega_X)$. If W contains the identity point then it is a subtorus by Theorem 2.8(a), and the conclusion of the Lemma is obviously satisfied. If W does not contain the identity point then, again by Theorem 2.8(a), $W = [\beta] + T$ where $[\beta]$ is a torsion point of $\mathrm{Pic}^0 X$ and T is a subtorus of $\mathrm{Pic}^0 X$ not containing $[\beta]$. To prove Lemma 4.2 it is enough to show then that there is a positive-dimensional subvariety Z of $[\beta^{-1}] + B$ which is contained in $V^0(\omega_X)$.

Let $d = \dim X$, $q = q(X)$, and $k = \mathrm{codim}_{\mathrm{Pic}^0 X} W$. We have the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ \mathrm{alb}\ } & \mathrm{Alb}\, X \\ & \searrow^{g} & \downarrow \pi \\ & & B = \mathrm{Pic}^0 T \end{array}$$

and, as in the proof of Theorem 3.3,

$$\dim g(X) = d - k. \tag{17}$$

Next, we claim that

$$R^k g_*(\omega_X \otimes \beta) \neq 0. \tag{18}$$

Indeed, by Kollár splitting (Variant 2.2(d)) and the projection formula, we have (replacing $g(X)$ with B) that for all $\alpha \in T = \mathrm{Pic}^0 B$,

$$H^k(X, \omega_X \otimes \beta \otimes g^*\alpha) = \bigoplus_{i=0}^{k} H^{k-i}(B, R^i g_*(\omega_X \otimes \beta) \otimes \alpha) \tag{19}$$

We know that $H^k(X, \omega_X \otimes \beta \otimes g^*\alpha) > 0$ for all $\alpha \in \mathrm{Pic}^0 B$ (in other words: $[\beta] + g^* \mathrm{Pic}^0 B$ is contained in $V^k(X, \omega_X)$). By (19), this means that

$$\mathrm{Pic}^0 B = \bigcup_{i=0}^{k} V^{k-i}(B, R^i g_*(\omega_X \otimes \beta)).$$

But, by Hacon's generic vanishing theorem (as in Variant 2.6), all sheaves $R^i g_*(\omega_X \otimes \beta)$ are GV-sheaves (on B). In particular their $V^{k-i}(\,)$ are *proper* subvarieties of $\mathrm{Pic}^0 B$ for $k - i > 0$. Therefore $T = \mathrm{Pic}^0 B$ must be equal to $V^0(B, R^k g_*(\omega_X \otimes \beta))$. This implies (18). By Kollár's torsion-freeness result (Variant 2.2(a)), $R^k g_*(\omega_X \otimes \beta)$ is torsion-free on $g(X)$.

Let $X \xrightarrow{f} Y \xrightarrow{a} g(X)$ be the Stein factorization of the morphism g. It follows from (18) that $R^k f_*(\omega_X \otimes \beta) \neq 0$. Therefore, denoting F a general fiber of f, $H^k(\omega_F \otimes \beta) > 0$. Since, by (17), the dimension of a general fiber F of f is k, $[\beta]$ must belong to the kernel of the restriction map $\mathrm{Pic}^0 X \to \mathrm{Pic}^0 F$. Hence so does $[\beta^{-1}]$. Therefore $R^k f_*(\omega_X \otimes \beta^{-1})$ is nonzero (in fact, again by Variant 2.2, it is torsion-free on Y). Hence

$$R^k g_*(\omega_X \otimes \beta^{-1}) \neq 0.$$

Finally, we claim that

$$\dim V^0(B, R^k g_*(\omega_X \otimes \beta^{-1})) > 0 \tag{20}$$

(in fact it turns out that $V^0(B, R^k g_*(\omega_X \otimes \beta^{-1}))$ has no isolated point). Granting (20) for the time being, we conclude the proof. By (19) with β^{-1} instead of β, the positive-dimensional subvariety $V^0(B, R^k g_*(\omega_X \otimes \beta^{-1}))$ induces a positive-dimensional subvariety, say Z, of $[\beta^{-1}] + B$, which is contained in $V^k(X, \omega_X)$[6]. By base change (Lemma 1.7), Z is contained in $V^0(X, \omega_X)$. This proves Lemma 4.2.

It remains to prove (20). Again by Hacon's generic vanishing (Variant 2.6), $R^k g_*(\omega_X \otimes \beta^{-1})$ is a GV-sheaf on B. Therefore, by the nonvanishing result of Lemma 1.12, the variety $V^0(B, R^k g_*(\omega_X \otimes \beta^{-1}))$ is nonempty. If $V^0(B, R^k g_*(\omega_X \otimes \beta^{-1}))$ had an isolated point, say $[\bar{\alpha}]$, then $[\bar{\alpha}]$ would belong also to $V^{q-k}(A, R^k g_*(\omega_X \otimes \beta^{-1}))$ (Lemma 1.8, recalling that $\dim B = q - k$). It would follow that $d - k = q - k$, and so $d = q$. Hence

$$H^{d-k}(A, R^k g_*(\omega_X \otimes \beta^{-1}) \otimes \bar{\alpha}) \neq 0.$$

Once again, by Kollár splitting as in (19), it would follow that

$$H^d(X, \omega_X \otimes \beta^{-1} \otimes g^*\bar{\alpha}) > 0,$$

implying that the line bundle $\beta^{-1} \otimes g^*\bar{\alpha}$ is trivial. But this is impossible since β^{-1} does not belong to $B = g^* \operatorname{Pic}^0 A$. □

It is perhaps worth mentioning that slightly more has been proved:

Scholium 4.3. *Let X be a variety of maximal Albanese dimension such that* $\dim V^0(\omega_X) > 0$. *Given a positive-dimensional component $[\beta] + B$ of $V^0(\omega_X)$, where $[\beta]$ is of order $n > 1$ and B is a subtorus of* $\operatorname{Pic}^0 X$, *then, for all $k = 1, \dots, n-1$ coprime with n, there is a positive-dimensional subtorus C_k of B such that $[\beta^k] + C_k$ is contained in $V^0(\omega_X)$.*

5. On Hacon's characterization of theta divisors

Let Θ be an irreducible theta divisor in a principally polarized abelian variety A, and let $X \to \Theta$ be a desingularization. Thanks to the fact that Θ has rational singularities (Theorem 3.5), $V^i(X, \omega_X) = V^i(\Theta, \omega_\Theta)$. Hence the following conditions hold:

[6] In fact Z is a translate of a subtorus; see Note 2.9(a).

(a) $V^i(X, \omega_X) = \{\hat{0}\}$ for all $i > 0$.
(b) $h^0(\omega_X \otimes \alpha) = 1$ for all $[\alpha] \in \operatorname{Pic}^0 X$ such that $[\alpha] \neq \hat{0}$.

In particular, it follows that $\chi(\omega_X) = 1$ and $\operatorname{codim} V^i(\omega_X) > i + 1$ for all i such that $0 < i < \dim X$.

The following refinement of a theorem of Hacon's shows that desingularizations of theta divisors can be characterized — among varieties such that $\dim X < q(X)$ — by conditions (a) and (b). The proof illustrates in a simple case a principle — already mentioned before Lemma 1.8 — often appearing in arguments based on generic vanishing theorems and Fourier–Mukai transform: the interplay between the size of the cohomological support loci $V^i(\mathcal{F})$, where $\mathcal{F}$ is a GV-sheaf, and the sheaf-theoretic properties of transform $\widehat{R\Delta\mathcal{F}}$. The statement and the argument provided here are modeled on Proposition 3.1 of [Barja et al. 2012].

Theorem 5.1. *Let X be a smooth projective variety such that*:
(*a*) $\chi(\omega_X) = 1$;
(*b*) $\operatorname{codim} V^i(\omega_X) > i + 1$ *for all i such that* $0 < i < \dim X$;
(*c*) $\dim X < q(X)$.
Then X is birational to a theta divisor.

Proof. Let us denote, as usual, $d = \dim X$ and $q = q(X)$. Conditions (b) and (c) imply that ω_X is a GV-sheaf. Therefore the Albanese map of X is generically finite (Proposition 2.4). Not only: (b) and (c) imply that $\operatorname{codim} V^i(\omega_X) > i$ for all $i > 0$. Therefore, by Theorem 1.10, the sheaf $\widehat{\mathcal{O}_X}$ is torsion-free. Since its generic rank is $\chi(\omega_X) = 1$, it has to be an ideal sheaf twisted by a line bundle on $\operatorname{Pic}^0 X$:

$$\widehat{\mathcal{O}_X} = \mathcal{I}_Z \otimes L.$$

Next, we claim that, for each (non-embedded) component W of Z

$$\operatorname{codim}_{\operatorname{Pic}^0 X} W = d + 1 \tag{21}$$

To prove this we note that, for $i > 1$,

$$\mathcal{E}xt^i(\widehat{\mathcal{O}_X}, \mathcal{O}_{\operatorname{Pic}^0 X}) = \mathcal{E}xt^i(\mathcal{I}_Z \otimes L, \mathcal{O}_{\operatorname{Pic}^0 X}) \cong \mathcal{E}xt^{i+1}(L \otimes \mathcal{O}_Z, \mathcal{O}_{\operatorname{Pic}^0 X}) \tag{22}$$

Let W be one such component of Z, and let $j+1$ be its codimension. We have that the support of $\mathcal{E}xt^{j+1}(L \otimes \mathcal{O}_Z, \mathcal{O}_{\operatorname{Pic}^0 X})$ contains W. Hence, combining (22), Grothendieck duality (Proposition 1.6(b)), and the Auslander–Buchsbaum–Serre formula, it follows that $R^j\Phi_P(\omega_X)$ is supported in codimension $j+1$. This implies, by base-change, that $\operatorname{codim} V^j(\omega_X) \leq j+1$. Because of hypothesis (b), it must be that $j = \dim X$. This proves (21). Next, we claim that

$$R\mathcal{H}om(\widehat{\mathcal{O}_X}, \mathcal{O}_{\operatorname{Pic}^0 X}) = R\mathcal{H}om(\mathcal{I}_Z \otimes L, \mathcal{O}_{\operatorname{Pic}^0 X}) = \mathbb{C}(\hat{0})[-d] \quad \text{and} \quad d = q - 1 \tag{23}$$

Indeed, arguing as in the proof of (21), from (21) it follows that $\mathcal{E}xt^i(\widehat{\mathcal{O}_X}, \mathcal{O}_{\mathrm{Pic}^0 X})$ is zero for $i < \mathrm{codim}_{\mathrm{Pic}^0 X} Z = d+1$. By Proposition 1.6(b) again, this means that $R^i\Phi_P(\omega_X)$ is zero for $i < d+1 = \mathrm{codim}_{\mathrm{Pic}^0 X} Z$. Since we know from Proposition 1.14 that $R^d\Phi_P(\omega_X) \cong \mathbb{C}(\hat{0})$, we get the first part of (23). Since the dualization functor is an involution, it follows that Z itself is the reduced point $\hat{0}$ and that $d = q-1$, completing the proof of (23). At this point the proof is exactly as in [Barja et al. 2012, Proposition 3.1]. We report it here for the reader's convenience. By Proposition 1.13,

$$R\Phi_P(\mathcal{O}_X) = R\Phi_{\mathcal{P}}(R\ \mathrm{alb}_* \mathcal{O}_X) = \mathcal{I}_{\hat{0}} \otimes L[-q+1].$$

Therefore, by Mukai's inversion theorem (Theorem 1.11),

$$R\Psi_{\mathcal{P}}(\mathcal{I}_{\hat{0}} \otimes L) = (-1)^*_{\mathrm{Pic}^0 X} R\ \mathrm{alb}_* \mathcal{O}_X[-1]. \tag{24}$$

In particular,

$$R^0\Psi_{\mathcal{P}}(\mathcal{I}_{\hat{0}} \otimes L) = 0 \quad \text{and} \quad R^1\Phi_{\mathcal{P}}(\mathcal{I}_{\hat{0}} \otimes L) \cong \mathrm{alb}_* \mathcal{O}_X. \tag{25}$$

Applying $\Psi_{\mathcal{P}}$ to the standard exact sequence

$$0 \to \mathcal{I}_{\hat{0}} \otimes L \to L \to \mathcal{O}_{\hat{0}} \otimes L \to 0, \tag{26}$$

and using (25) we get

$$0 \to R^0\Psi_{\mathcal{P}}(L) \to \mathcal{O}_{\mathrm{Alb}\, X} \to \mathrm{alb}_* \mathcal{O}_X, \tag{27}$$

whence $R^0\Psi_{\mathcal{P}}(L)$ is supported everywhere (since $\mathrm{alb}_* \mathcal{O}_X$ is supported on a divisor). It is well known that this implies that L is *ample*. Therefore $R^i\Psi_{\mathcal{P}}(L) = 0$ for $i > 0$. Hence, by sequence (26), $R^i\Psi_{\mathcal{P}}(\mathcal{I}_{\hat{0}} \otimes L) = 0$ for $i > 1$. By (24) and (25), this implies that $R^i\ \mathrm{alb}_*(\mathcal{O}_X) = 0$ for $i > 0$. Furthermore, (27) implies easily that $h^0(\mathrm{Pic}^0 X, L) = 1$; that is, L is a *principal* polarization on $\mathrm{Pic}^0 X$. Therefore, via the identification $\mathrm{Alb}(X) \cong \mathrm{Pic}^0(X)$ provided by L, we have $R^0\Psi_{\mathcal{P}}(L) \cong L^{-1}$ (see [Mukai 1981, Proposition 3.11(1)]). Since the arrow on the right in (27) is onto, it follows that $\mathrm{alb}_* \mathcal{O}_X = \mathcal{O}_\Theta$, where Θ is the only effective divisor in the linear series $|L|$. As we already know that alb is generically finite, this implies that alb is a birational morphism onto Θ. □

Notes 5.2. (1) The cohomological characterization of theta divisors is due to Hacon [2000], who proved it under some extra hypotheses, subsequently refined in [Hacon and Pardini 2002]. A further refinement was proved in [Barja et al. 2012, Proposition 3.1] and [Lazarsfeld and Popa 2010, Proposition 3.8(ii)]. The present approach is the one in [Barja et al. 2012].

(2) Concerning the significance of the hypothesis of the above theorem, note that, removing the hypothesis $\dim X < q$ there are varieties nonbirational to theta

divisors satisfying (a) and (b) (e.g., sticking to varieties of maximal Albanese dimension, the double cover of a p.p.a.v. ramified on a smooth divisor $D \in |2\Theta|$). Moreover products of (desingularized) theta divisors are examples of varieties satisfying conditions (a) and (c), but not (b).

(3) Theorem 5.1 and its proof hold assuming, more generally, that X is compact Kähler. The argument works also for projective varieties over any algebraically closed field, except for the fact that Proposition 2.4 is used to ensure the maximal Albanese dimension. Therefore, up to adding the hypothesis that X is of maximal Albanese dimension and replacing condition (b) with the condition $\dim X < \dim \operatorname{Alb} X$, Theorem 5.1 holds in any characteristic.

(4) With the same argument one can prove the following characterization of abelian varieties, valid in any characteristic: *Assume that X is a smooth projective variety of maximal Albanese dimension such that*

(a) $\chi(\omega_X) = 0$ *and*

(b) $\operatorname{codim} V^i(\omega_X) > i$ *for all i such that* $0 < i < \dim X$.

Then X is birational to an abelian variety.

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