Inside Out II
MSRI Publications
Volume **60**, 2012

Enclosure methods for Helmholtz-type equations

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The inverse problem under consideration is to reconstruct the shape information of obstacles or inclusions embedded in the (inhomogeneous) background medium from boundary measurements of propagating waves. This article is a survey of enclosure-type methods implementing exponential complex geometrical optics waves as boundary illumination. The equations for acoustic waves, electromagnetic waves and elastic waves are considered for a medium with impenetrable obstacles and penetrable inclusions (characterized by a jump discontinuity in the parameters). We also outlined some open problems along this direction of research.

1. Introduction

This paper serves as a survey of enclosure-type methods used to determine the obstacles or inclusions embedded in the background medium from the near-field measurements of propagating waves. A type of complex geometric optics waves that exhibits exponential decay with distance from some critical level surfaces (hyperplanes, spheres or other types of level sets of phase functions) are sent to probe the medium. One can easily manipulate the speed of decay such that the waves can only detect the material feature that is close enough to the level surfaces. As a result of sending such waves with level surfaces moving along each direction, one should be able to pick out those that enclose the inclusion.

The problem that Calderón proposed [1980] was whether one can determine the electrical conductivity by making voltage and current measurements at the boundary of the medium. Such electrical methods are also known as electrical impedance tomography (EIT) and have broad applications in medical imaging, geophysics and so on. A breakthrough in solving the problem was due to Sylvester and Uhlmann [1987], who constructed complex geometric optics (CGO) solutions to the conductivity equation and proved the unique determination of C^∞ isotropic conductivity from the boundary measurements in three- and higher-dimensional spaces. The result has been extended to Lipschitz conductivities [Haberman

The first author was supported in part by the National Science Council of Taiwan.

and Tataru 2011] in three dimensions and L^∞ conductivities in two dimensions [Astala and Päivärinta 2006].

The inverse problem in this paper concerns reconstructing an obstacle or a jump-type inclusion (in three dimensions) embedded in a known background medium, which is not included in the previous results when considering electrostatics. Several methods are proposed to solve the problem based on utilizing, generally speaking, two special types of solutions. The Green's type solutions were considered first in [Isakov 1990], and several sampling methods [Cakoni and Colton 2006; Kirsch and Grinberg 2008; Arens 2004; Arens and Lechleiter 2009] and probing methods [Ikehata 1998; Potthast 2001] were developed. On the other hand, with the CGO solutions at disposal, the enclosure method was introduced by Ikehata [1999a; 2000] with the idea as described in the first paragraph. Another method worth mentioning uses the oscillating-decaying type of solutions and was proved valid for elasticity systems [Nakamura et al. 2005]. It is the enclosure type of methods that is of the presenting paper's interest.

Here we aim to discuss the enclosure method for Helmholtz-type equations. For the enclosure method in the static equations, we refer to [Ikehata 1999a; 2000; Ide et al. 2007; Uhlmann and Wang 2008; Takuwa et al. 2008] for the conductivity equation, to [Uhlmann and Wang 2007; Uhlmann et al. 2009] for the isotropic elasticity. The major difference between the static equations and Helmholtz-type equations is the loss of positivity in the latter equations. It turns out we have to analyze the effect of the reflected solution due to the existence of lower order term in Helmholtz-type equations. For the acoustic equation outside of a cavity having a C^2 boundary (representing and impenetrable obstacle), one can overcome the difficulty by the Sobolev embedding theorem, see [Nakamura and Yoshida 2007] (also see [Ikehata 1999b] for a similar idea). Such a result can be generalized for Maxwell's equations to determine impenetrable electromagnetic obstacles [Zhou 2010]. However, in the inclusion case, i.e., penetrable obstacles, the coefficient is merely *piecewise* smooth. The Sobolev embedding theorem does not work because the solution is not smooth enough. To tackle the problem, a Hölder type estimate for the second order elliptic equation with coefficients having jump discontinuity based on the result of Li and Vogelius [2000] was developed by Nagayasu, Uhlmann, and the first author in [Nagayasu et al. 2011]. Later, this result was improved by Sini and Yoshida [2010] using L^p estimate for the second order elliptic equation in divergence form developed by Meyers [1963]. Recently, Kuan [2012] extended Sini and Yoshida's method to the elastic wave equations.

The paper is organized as follows. In Section 2, we discuss the enclosure method for the acoustic and electromagnetic equations with impenetrable obstacles. In Section 3, we to survey results in the inclusion case (penetrable obstacle) for the acoustic and elastic waves. Some open problems are listed in Section 4.

2. Enclosing obstacles using acoustic and electromagnetic waves

In this section, we give more precise descriptions of the enclosure methods to identify impenetrable obstacles of acoustic or electromagnetic equations. In particular, we are interested in the results in [Ikehata 1999a] and [Nakamura and Yoshida 2007] for both convex and nonconvex sound hard obstacles using complex geometrical optics (CGO) solutions for Helmholtz equations and the result in [Zhou 2010] for perfect magnetic conducting (PMC) obstacles using CGO solutions for Maxwell's equations.

2A. ***Nonconvex sound hard obstacles.*** In [Ikehata 1999a] and [Nakamura and Yoshida 2007], the authors consider the inverse scattering problem of identifying a sound hard obstacle $D \subset \mathbb{R}^n$, $n \geq 2$ in a homogeneous medium from the far field pattern. It can be reformulated as an equivalent inverse boundary value problem with near-field measurements described as follows. Given a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary and such that $\overline{D} \subset \Omega$ and $\Omega \setminus \overline{D}$ is connected, the underlying boundary value problem for acoustic wave propagation in the known homogeneous medium in $\Omega \setminus \overline{D}$ with no source is given by

$$\begin{cases} (\Delta + k^2)u = 0 \quad \text{in } \Omega \setminus \overline{D}, \\ u\big|_{\partial\Omega} = f, \\ \partial_\nu u\big|_{\partial D} = 0 \end{cases} \tag{2-1}$$

where $k > 0$ is the wave number and ν denotes the unit outer normal of ∂D. At this point, we assume that ∂D is C^2. Suppose k is not a Dirichlet eigenvalue of Laplacian. Given each prescribed boundary sound pressure $f \in H^{1/2}(\partial\Omega)$, there exists a unique solution $u(x) \in H^1(\Omega \setminus \overline{D})$ to (2-1). The inverse boundary value problem consists of reconstructing the obstacle D from the full boundary data that can be encoded in the Dirichlet-to-Neumann (DN) map on $\partial\Omega$:

$$\begin{aligned} \Lambda_D : H^{1/2}(\partial\Omega) &\to H^{-1/2}(\partial\Omega), \\ f &\mapsto \partial_\nu u\big|_{\partial\Omega}. \end{aligned} \tag{2-2}$$

In particular, the enclosure method utilizes the measurements (DN map) for those f taking the traces of CGO solutions to $(\Delta + k^2)u = 0$ in the background domain Ω

$$u_0 = e^{\tau(\varphi(x)-t)+i\psi(\tau;x))}\,(a(x) + r(x;\tau)), \tag{2-3}$$

where $r(x;\tau)$ and its first derivatives are uniformly bounded in τ. As $\tau \to \infty$, u_0 evolves vertical slope at the level set $\{x \mid \varphi(x) = t\}$ for $t \in \mathbb{R}$. Physically speaking, such evanescent waves couldn't detect the change of the material, namely the presence of D in Ω, happening relatively far from the level set. Hence, there is little gap between the associated energies of domains with and without D. On the other hand, if $\overline{D}$ ever intersects the level set, the energy gap is going to be significant for large τ. This implies that the geometric relation between D and the level set $\{x \mid \varphi(x) = t\}$ can be read from the following indicator function describing the energy gap associated to the input $f = u_0|_{\partial\Omega}$:

$$I(\tau, t) := \int_{\partial\Omega} \big(\Lambda_D - \Lambda_\varnothing\big)(u_0|_{\partial\Omega}), \overline{u_0}|_{\partial\Omega}\, dS, \tag{2-4}$$

where $\Lambda_\varnothing$ represents the DN map associated to the background domain Ω without D, hence $\Lambda_\varnothing(u_0|_{\partial\Omega}) = \partial_\nu u_0|_{\partial\Omega}$. When the linear phase $\varphi(x) = x \cdot \omega$ is used, where $\omega \in \mathbb{S}^{n-1}$, the CGO solution (2-3) is the exponential function

$$u_0(x) = e^{\tau(x\cdot\omega - t) + i\sqrt{\tau^2+k^2}\, x\cdot\omega^\perp}$$

where $\omega^\perp \in \mathbb{S}^{n-1}$ satisfies $\omega \cdot \omega^\perp = 0$. The discussion above is verified in the following result by Ikehata to enclose the convex hull of D by reconstructing the support function

$$h_D(\omega) := \sup_{x\in D} x \cdot \omega.$$

Theorem 2.1 [Ikehata 1999a]. *Assume that the set $\{x \in \mathbb{R}^n \mid x\cdot\omega = h_D(\omega)\} \cap \partial D$ consists of one point and the Gaussian curvature of ∂D is not vanishing at that point. Then the support function $h_D(\omega)$ can be reconstructed by the formula*

$$h_D(\omega) = \inf\big\{t \in \mathbb{R} \mid \lim_{\tau\to\infty} I(\tau, t) = 0\big\}. \tag{2-5}$$

This result shows that a strictly convex obstacle can be identified by an envelope surface of planes. Geometrically, this appears as the planes are enclosing the obstacle from every direction, justifying the name “enclosure method”.

It is natural to expect that the method can be generalized to recover some nonconvex part of the shape of D by using CGO solutions with nonlinear phase. Based on a Carleman estimate approach, such solutions were constructed in [Kenig et al. 2007] (or see [Dos Santos Ferreira et al. 2007]) for the Schrödinger operator (or the conductivity operator) in $\mathbb{R}^3$, with φ being one of a few limiting Carleman weights (LCW)

$$\varphi(x) = \ln|x - x_0|, \quad x_0 \in \mathbb{R}^3 \setminus \overline{\Omega},$$

which bears spherical level sets, and therefore were called complex spherical waves (CSW). Then such CSW were used into the enclosure method in [Ide et al. 2007] to identify nonconvex inclusions in a conductive medium. In $\mathbb{R}^2$, there are more candidates for limiting Carleman weights than in $\mathbb{R}^3$: all harmonic functions with nonvanishing gradient are LCW (see [Dos Santos Ferreira et al. 2009] for more descriptions of LCW). Then the similar reconstruction scheme is available in [Uhlmann and Wang 2008] for more generalized two-dimensional systems by using level curves of harmonic polynomials.

Below we present the result of Nakamura and Yoshida that adopts the CSW described in the following proposition to enclose a nonconvex sound hard obstacle.

Proposition 2.2 [Dos Santos Ferreira et al. 2007]. *Choose $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$ and let $\omega_0 \in \mathbb{S}^{n-1}$ be a vector such that*

$$\{x \in \mathbb{R}^n \mid x - x_0 = m\omega_0,\ m \in \mathbb{R}\} \cap \partial\Omega = \varnothing.$$

Then there exists a solution to the Helmholtz equation in Ω of the form

$$u_0(x; \tau, t, x_0, \omega_0) = e^{\tau(t - \ln|x - x_0|) - i\tau\psi(x)}\big(a(x) + r(x; \tau, t, x_0, \omega_0)\big), \quad \text{(2-6)}$$

where $\tau > 0$ and $t \in \mathbb{R}$ are parameters, $a(x)$ is a smooth function on $\overline{\Omega}$ and $\psi(x)$ is a function defined by

$$\psi(x) := d_{\mathbb{S}^{n-1}}\left(\frac{x - x_0}{|x - x_0|}, \omega_0\right),$$

with the metric function $d_{\mathbb{S}^{n-1}}(\cdot, \cdot)$ on $\mathbb{S}^{n-1}$. Moreover, the remainder function r is in $H^1(\Omega)$ and satisfies

$$\|r\|_{H^1(\Omega)} = O(\tau^{-1}) \quad \text{as } \tau \to \infty.$$

The corresponding support function is given by

$$h_D(x_0) = \inf_{x \in D} \ln|x - x_0|, \quad x_0 \in \mathbb{R}^n \setminus \overline{\Omega},$$

and can be reconstructed based on the following result.

Theorem 2.3 [Nakamura and Yoshida 2007]. *Let $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$. Assume that the set $\{x \in \mathbb{R}^n \mid |x - x_0| = e^{h_D(x_0)}\} \cap \partial D$ consists of finitely many points and the relative curvatures of ∂D at these points are positive. Then there are two characterizations of $h_D(x_0)$:*

$$h_D(x_0) = \sup\{t \in \mathbb{R} \mid \liminf_{\tau \to \infty} |I(\tau, t)| = 0\} \quad \text{(2-7)}$$

and

$$t - h_D(x_0) = \lim_{\tau \to \infty} \frac{\ln|I(\tau, t)|}{2\tau}, \quad \text{(2-8)}$$

where $I(\tau, t)$ is defined by (2-4) *with u_0 given by* (2-6).

Remark. The relative curvature in the theorem refers to the Gaussian curvature after a change of coordinates that stretches the sphere onto flat space. For a more rigorous definition, we refer to [Nakamura and Yoshida 2007].

For completeness, we provide briefly the steps of the proof. The proof of (2-7) involves showing that

$$\lim_{\tau\to\infty} |I(\tau,t)| = 0 \quad \text{when } t < h_D(x_0), \tag{2-9}$$

that is, when the level sphere $S_{t,x_0} := \{x \in \mathbb{R}^n \mid |x-x_0| = e^t\}$ has no intersection with $\overline{D}$, and showing that

$$\liminf_{\tau\to\infty} |I(\tau,t)| > C > 0 \quad \text{when } t \geq h_D(x_0), \tag{2-10}$$

namely, when S_{t,x_0} intersects $\overline{D}$. These two statements can be shown by establishing proper upper and lower bounds of $I(\tau,t)$ from the key equality

$$-I(\tau,t) = \int_{\Omega\setminus\overline{D}} |\nabla w|^2\,dx + \int_D |\nabla u_0|^2\,dx - k^2\int_{\Omega\setminus\overline{D}} |w|^2\,dx - k^2\int_D |u_0|^2\,dx, \tag{2-11}$$

where $w := u - u_0$ is the reflected solution and u is the solution to (2-1) with $f = u_0|_{\partial\Omega}$. Since w is a solution to

$$\begin{cases} (\Delta + k^2)w = 0 & \text{in } \Omega\setminus\overline{D}, \\ w|_{\partial\Omega} = 0, \\ \partial_\nu w|_{\partial D} = -\partial_\nu u_0|_{\partial D}, \end{cases} \tag{2-12}$$

and by (2-11), one has the upper bound

$$|I(\tau,t)| \leq C\|u_0\|^2_{H^1(D)}$$

for some constant $C > 0$ (throughout the article we use the same letter C to denote various constants). As a consequence of plugging in the CGO solution (2-6), the first statement (2-9) is obtained since

$$|I(\tau,t)| \leq C\tau^2 \int_D e^{2\tau(t-\ln|x-x_0|)}\,dx \quad (\tau \gg 1).$$

However, difficulties arise in dealing with the second statement, (2-10). Due to the loss of positivity for the associated bilinear form, two negative terms are present in (2-11), which implies that to find a nonvanishing (as $\tau\to\infty$) lower bound for $I(\tau,t)$ is not as easy as the case of the conductivity equations, in which we have $I(\tau,t) \geq C\int_D |\nabla u_0|^2\,dx$. As a remedy, one needs to show that the two negative terms can be absorbed by the positive terms for τ large. To be more specific, first it is not hard to see

$$I(\tau,t) = e^{2\tau(t-h_D(x_0))} I(\tau, h_D(x_0)). \tag{2-13}$$

This implies that it is sufficient to show (2-10) for $t = h_D(x_0)$, which in turn can be derived from (2-11) and the following two inequalities when $t = h_D(x_0)$:

$$\liminf_{\tau\to\infty} \int_D |\nabla u_0|^2\,dx > C > 0 \tag{2-14}$$

and

$$\frac{k^2\int_{\Omega\setminus\overline{D}} |w|^2\,dx + k^2\int_D |u_0|^2\,dx}{\int_D |\nabla u_0|^2\,dx} < \delta < 1 \quad (\tau \gg 1). \tag{2-15}$$

Equation (2-14) is true since

$$\begin{aligned}\int_D |\nabla u_0|^2\,dx &\ge C\tau^2 \int_D e^{-2\tau(\ln|x-x_0|-h_D(x_0))}\,dx\\ &\ge \begin{cases} O(\tau^{1/2}) & n=2\\ O(1) & n=3\end{cases} \quad (\tau \gg 1),\end{aligned} \tag{2-16}$$

given the geometric assumption of the positive relative curvature of ∂D.

As for (2-15), the difficult part is to show that

$$\liminf_{\tau\to\infty} \frac{k^2\int_{\Omega\setminus\overline{D}} |w|^2\,dx}{\int_D |\nabla u_0|^2\,dx} = 0 \tag{2-17}$$

since the property of CGO solutions gives

$$\frac{k^2\int_D |u_0|^2\,dx}{\int_D |\nabla u_0|^2\,dx} = O(\tau^{-2}) \quad (\tau \gg 1).$$

In both [Ikehata 1999a] and [Nakamura and Yoshida 2007], (2-17) is proved by establishing the following estimate.

Lemma 2.4. *Let $S_{h_D(x_0),x_0} \cap \partial D = \{x_1, \dots, x_N\}$ and define for $\alpha \in (0,1)$*

$$I_{x_j,\alpha} := \int_{\partial D} |\partial_\nu u_0|\; |x - x_j|^\alpha\,dS, \quad j = 1, \dots, N.$$

Then

$$\|w\|^2_{L^2(\Omega\setminus\overline{D})} \le C\Big(\sum_{j=1}^N I^2_{x_j,\alpha} + \|u_0\|^2_{L^2(D)}\Big), \quad \alpha \in (0,1) \tag{2-18}$$

Remark. The proof of Lemma 2.4 is based on H^2-regularity theory and the Sobolev embedding theorem for an auxiliary boundary value problem

$$\begin{cases} (\Delta + k^2)p = \overline{w} & \text{in } \Omega\setminus\overline{D},\\ p|_{\partial\Omega} = 0, \\ \partial_\nu p|_{\partial D} = 0.\end{cases}$$

Such estimates of the reflected solution $\|w\|_{L^2(\Omega\setminus\overline{D})}$ for the impenetrable obstacle case and $\|w\|_{L^2(\Omega)}$ for the penetrable inclusion case, which will be reviewed in the next section, are usually crucial for the justification of the enclosure methods. Several improvements of the result and removal of geometric assumptions are basically due to the development of different estimates, which we will see shortly.

In particular, choosing $\alpha = \frac{1}{2}$ for $n = 3$ and $\alpha = \frac{3}{4}$ when $n = 2$, one can show that

$$I^2_{x_j,\alpha} \le \begin{cases} \sqrt{\varepsilon}\; O(\tau^{1/2}) & \text{if } n = 2, \\ O(\tau^{-1/2}) & \text{if } n = 3, \end{cases}$$

for arbitrary small ε, again by the assumption that the relative curvature is positive. Combined with (2-16), this immediately yields (2-15).

Lastly, the formula (2-8) is directly derived from (2-13) and the fact that

$$|I(\tau, h_D(x_0))| \le C\tau^2, \quad (\tau \gg 1).$$

Remark. The result can be easily extended to the case with inhomogeneous background medium in $\Omega \setminus \overline{D}$, where the CSW in Proposition 2.2 is available.

2B. ***Electromagnetic PMC obstacles.*** This section is devoted to reviewing the enclosure method for Maxwell's equations [Zhou 2010] to identify perfect magnetic conducting (PMC) obstacles. The same reconstruction scheme works for identifying perfect electric conducting (PEC) obstacles and more generalized impenetrable obstacles.

In a bounded domain $\Omega \subset \mathbb{R}^3$ with an obstacle D such that $\overline{D} \subset \Omega$ with ∂D being C^2 and $\Omega \setminus \overline{D}$ connected, the electric-magnetic field (E, H) satisfies the Maxwell equations

$$\begin{cases} \nabla \times E = ik\mu H, \quad \nabla \times H = -ik\varepsilon E, & \text{in } \Omega \setminus \overline{D}, \\ \nu \times E|_{\partial\Omega} = f, & \\ \nu \times H|_{\partial D} = 0 \quad \text{(PMC condition)}, & \end{cases} \tag{2-19}$$

where k is the frequency and $\mu(x)$ and $\varepsilon(x)$ describe the isotropic (inhomogeneous) background electromagnetic medium and satisfy the following assumptions: there are positive constants $\varepsilon_m, \varepsilon_M, \mu_m, \mu_M, \varepsilon_c$ and μ_c such that for all $x \in \Omega$

$$\varepsilon_m \le \varepsilon(x) \le \varepsilon_M, \;\; \mu_m \le \mu(x) \le \varepsilon_M, \;\; \sigma(x) = 0$$

and $\varepsilon - \varepsilon_c, \mu - \mu_c \in C_0^3(\Omega)$. Given that k is not a resonant frequency, we have a well-defined boundary impedance map

$$\begin{aligned}\Lambda_D : \; TH^{1/2}(\partial\Omega) \;&\to\; TH^{1/2}(\partial\Omega),\\ f = \nu \times E|_{\partial\Omega} \;&\mapsto\; \nu \times H|_{\partial\Omega}.\end{aligned}$$

To show that D can be determined by the impedance map Λ_D using the enclosure method, we first notice an analogue of the identity (2-11) for Maxwell's equations:

$$\begin{aligned}i\omega \int_{\partial\Omega} (\nu \times E_0) \cdot \big(\overline{(\Lambda_D - \Lambda_\varnothing)(\nu \times E_0)} \times \nu\big)\, dS \\ = \int_{\Omega\setminus\overline{D}} \mu|\tilde{H}|^2 - \omega^2\varepsilon|\tilde{E}|^2\, dx + \int_D \mu|H_0|^2 - \omega^2\varepsilon|E_0|^2\, dx, \quad \text{(2-20)}\end{aligned}$$

where $(\tilde{E}, \tilde{H}) := (E - E_0, H - H_0)$ denotes the reflected solutions, (E, H) is the solution to (2-19), (E_0, H_0) is the solution to the Maxwell's equations

$$\nabla \times E_0 = ik\mu H_0, \quad \nabla \times H_0 = -ik\varepsilon E_0 \qquad \text{in } \Omega, \qquad \text{(2-21)}$$

and $\nu \times E|_{\partial\Omega} = \nu \times E_0|_{\partial\Omega}$.

One would encounter the same difficulty as that for Helmholtz equations due to the loss of positivity of the system. We recall that this was actually overcome by the property that the CGO solution u_0 shares different asymptotic speed (τ^2 slower) from ∇u_0. More specifically, this is because of the H^1 boundedness of the remainder $r(x;\tau)$ with respect to τ in (2-3). The natural question to ask is then whether this key ingredient: such CGO type of solutions, can be constructed for the background Maxwell's system.

The construction of CGO solutions for the Maxwell's equations has been extensively studied in [Ola et al. 1993; Ola and Somersalo 1996; Colton and Päivärinta 1992]. The work in [Zhou 2010] adopts the construction approach in [Ola and Somersalo 1996] by reducing the Maxwell's equations into a matrix Schrödinger equation. Finally, to guarantee that the CGO solution for the reduced matrix Schrödinger operator derives the CGO solution (E_0, H_0) for the Maxwell's equations and at the same time that the electric field E_0 and H_0 have different asymptotic speeds as $\tau \to \infty$, the incoming constant field corresponding to $a(x)$ in (2-3) has to be chosen very carefully. To summarize, one has

Proposition 2.5. *Let $\omega, \omega^\perp \in \mathbb{S}^2$ with $\omega \cdot \omega^\perp = 0$. Set*

$$\zeta = -i\tau\omega + \sqrt{\tau^2 + k^2}\,\omega^\perp$$

where $k_1 = k(\varepsilon_0\mu_0)^{1/2}$. *Choose* $a \in \mathbb{R}^3$ *such that*

$$a \perp \omega, \quad a \perp \omega^{\perp} \text{ and } b = \frac{1}{\sqrt{2}}\overline{(-i\omega + \omega^{\perp})}.$$

Then, given

$$\theta := \frac{1}{|\zeta|}\big(-(\zeta\cdot a)\zeta - k_1\zeta\times b + k_1^2 a\big), \quad \eta := \frac{1}{|\zeta|}\big(k_1\zeta\times a - (\zeta\cdot b)\zeta + k_1^2 b\big),$$

for $t \in \mathbb{R}$ *and* $\tau > 0$ *large enough, there exists a unique complex geometric optics solution* $(E_0, H_0) \in H^1(\Omega)^3 \times H^1(\Omega)^3$ *of Maxwell's equations* (2-21) *having the form*

$$E_0 = \varepsilon(x)^{-1/2} e^{\tau(x\cdot\omega - t) + i\sqrt{\tau^2+k^2}\,x\cdot\omega^{\perp}}(\eta + R(x)),$$
$$H_0 = \mu(x)^{-1/2} e^{\tau(x\cdot\omega - t) + i\sqrt{\tau^2+k^2}\,x\cdot\omega^{\perp}}(\theta + Q(x)).$$

Moreover, we have

$$\eta = \mathbb{O}(1), \quad \theta = \mathbb{O}(\tau) \text{ for } \tau \gg 1,$$

and $R(x)$ *and* $Q(x)$ *are bounded in* $(L^2(\Omega))^3$ *for* $\tau \gg 1$.

Plugging (E_0, H_0) into the indicator function defined by

$$I(\tau,t) := i\omega \int_{\partial\Omega} (\nu\times E_0)\cdot\big(\overline{(\Lambda_D - \Lambda_\varnothing)(\nu\times E_0)}\times\nu\big)\, dS,$$

a similar argument as for Helmholtz equations follows using identity (2-20) and we have:

Theorem 2.6 [Zhou 2010]. *There is a subset* $\Sigma \subset \mathbb{S}^2$ *of measure zero such that, when* $\omega \in \mathbb{S}^2 \setminus \Sigma$, *the support function*

$$h_D(\omega) := \sup_{x\in D} x\cdot\omega$$

can be recovered by

$$h_D(\omega) = \inf\big\{t \in \mathbb{R} \mid \lim_{\tau\to\infty} I(\tau,t) = 0\big\}.$$

Moreover, if D *is strictly convex, one can reconstruct* D.

On the other hand, the construction of a proper CGO solution with nonlinear weight for the Maxwell's equations has not been successful based on the Carleman estimate argument. An alternative approach to reconstruct nonconvex part of the shape of D would be to introduce some transformation. For example, one can utilize the Kelvin transformation

$$T_{x_0,R} : x \mapsto R^2\frac{x - x_0}{|x - x_0|^2} + x_0 := y,$$

which is the inversion transformation with respect to the sphere $S(x_0, R)$ for $R > 0$ and $x_0 \in \mathbb{R}^3 \setminus \overline{\Omega}$. $T_{x_0,R}$ maps generalized spheres (spheres and planes) into generalized spheres. Geometrically, fixing a reference circle $S(x_0, R)$, enclosing D with spheres passing through x_0 corresponds to enclosing $\hat{D}_{x_0,R} = T_{x_0,R}(D)$ with planes, where the reconstruction scheme in Theorem 2.6 applies. A rigorous proof consists of showing that the Maxwell's equations are invariant under the transformation and computing the impedance map $\hat{I}(\tau, t)$ associated to the image domain. It is worth mentioning the byproduct of this method is the complex spherical wave

$$\hat{E}(y) = \hat{E}_j\, dy^j = \big((DT^{-1}_{x_0,R})^k_j(y) E_k(T^{-1}_{x_0,R}(y))\big)\, dy^j, \quad y = T_{x_0,R}(x),$$

with nonlinear limiting Carleman weight

$$\varphi(x) = \left(R^2 \frac{x - x_0}{|x - x_0|^2} + x_0\right) \cdot \omega, \quad \omega \in \mathbb{S}^2.$$

Therefore, the corresponding support function is given by

$$\hat{h}_D(x_0, R, \omega) = \sup_{x \in D} \left\{ R^2 \left(\frac{x - x_0}{|x - x_0|^2}\right) \cdot \omega + x_0 \cdot \rho \right\}.$$

Theorem 2.7 [Zhou 2010]. *Given $x_0 \in \mathbb{R}^3 \setminus \overline{\Omega}$ and $R > 0$ such that $\overline{\Omega} \subset B(x_0, R)$, there is a zero measure subset Σ of $\mathbb{S}^2$, s.t., when $\omega \in \mathbb{S}^2 \setminus \Sigma$, we have*

$$\hat{h}_D(x_0, R, \omega) = \inf\{t \in \mathbb{R} \mid \lim_{\tau \to \infty} \hat{I}(\tau, t) = 0\}.$$

3. Enclosing inclusions using acoustic and elastic waves

In this section we will consider the enclosure method for the case where the unknown domain is an inclusion by using acoustic and elastic waves. In other words, the obstacle is a penetrable one. In this situation, the reflected solution will satisfy the elliptic equation with discontinuous coefficients. Unlike the case of impenetrable obstacle, the Sobolev embedding theorem is not sufficient to provide us estimates of the reflected solution we need. In the case of acoustic waves, the difficulty was overcome in [Nagayasu et al. 2011] for dimension $n = 2$, using estimates from [Li and Vogelius 2000]. The extension to $n = 3$ was accomplished in [Yoshida 2010]. Sini and Yoshida [2010] then improved the result in [Nagayasu et al. 2011] with the help of Meyers' L^p estimate and the sharp Friedrichs inequality. Kuan [2012] extended Sini and Yoshida's result to elastic waves.

3A. ***Acoustic penetrable obstacle.*** Here we will review the result in [Nagayasu et al. 2011] for $n = 2$. For $n = 3$, one simply replaces CGO solutions in $n = 2$

by complex spherical waves [Yoshida 2010]. We assume $D \Subset \Omega \subset \mathbb{R}^2$. For technical simplicity, we suppose that both D and Ω have C^2 boundaries. Let $\gamma_D \in C^2(\overline{D})$ satisfy $\gamma_D \geq c_\gamma$ for some positive constant c_γ and $\widetilde{\gamma} := 1 + \gamma_D \chi_D$, where χ_D is the characteristic function of D. Let $k > 0$ and consider the steady state acoustic wave equation in Ω with Dirichlet condition

$$\begin{cases} \nabla \cdot (\widetilde{\gamma} \nabla v) + k^2 v = 0 & \text{in } \Omega, \\ v = f & \text{on } \partial\Omega. \end{cases} \tag{3-1}$$

We assume that k^2 is not a Dirichlet eigenvalue of the operator $-\nabla \cdot (\widetilde{\gamma} \nabla \cdot)$. Let $\Lambda_D : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ be the associated Dirichlet-to-Neumann map. As before, our aim is to reconstruct the shape of D by Λ_D. The key in the enclosure method is the CGO solutions. To construct the CGO solutions to the Helmholtz equation for $n = 2$, we begin with the CGO solutions with polynomial phases to the Laplacian operator, then apply the Vekua transform [1967, page 58].

More precisely, let us define $\eta(x) := c_*\big((x_1 - x_{*,1}) + i(x_2 - x_{*,2})\big)^N$ as the phase function, where $c_* \in \mathbb{C}$ satisfies $|c_*| = 1$, N is a positive integer, and $x_* = (x_{*,1}, x_{*,2}) \in \mathbb{R}^2 \setminus \overline{\Omega}$. Without loss of generality we may assume that $x_* = 0$ using an appropriate translation. Denote $\eta_{\mathrm{R}}(x) := \operatorname{Re} \eta(x)$ and note that

$$\eta_{\mathrm{R}}(x) = r^N \cos N(\theta - \theta_*) \text{ for } x = r(\cos\theta, \sin\theta) \in \mathbb{R}^2.$$

It is readily seen that $\eta_{\mathrm{R}}(x) > 0$ for all $x \in \Gamma$, where

$$\Gamma := \left\{ r(\cos\theta, \sin\theta) : |\theta - \theta_*| < \frac{\pi}{2N} \right\},$$

i.e., a cone with opening angle π/N.

Given any $h > 0$, $\check{V}_\tau(x) := \exp(\tau\eta(x))$ is a harmonic function. Following [Vekua 1967], we define a map T_k on any harmonic function $\check{V}(x)$ by

$$\begin{aligned} T_k \check{V}(x) &:= \check{V}(x) - \int_0^1 \check{V}(tx) \frac{\partial}{\partial t} \Big\{ J_0\big(k|x|\sqrt{1-t}\big) \Big\} dt \\ &= \check{V}(x) - k|x| \int_0^1 \check{V}\big((1-s^2)x\big) \, J_1(k|x|s) \, ds, \end{aligned}$$

where J_m is the Bessel function of the first kind of order m. We now set $V_\tau^\sharp(x) := T_k \check{V}_\tau(x)$. Then $V_\tau^\sharp(x)$ satisfies the Helmholtz equation $\Delta V_\tau^\sharp + k^2 V_\tau^\sharp = 0$ in $\mathbb{R}^2$. One can show that $V_\tau^\sharp$ satisfies the following estimate in Γ:

Lemma 3.1 [Nagayasu et al. 2011]. *We have*

$$V_\tau^\sharp(x) = \exp(\tau\eta(x)) \big(1 + R_0(x)\big) \textit{ in } \Gamma, \tag{3-2}$$

where $R_0(x) = R_0(x;\tau)$ *satisfies*

$$|R_0(x)| \le \frac{1}{\tau}\,\frac{k^2|x|^2}{4\eta_{\mathrm{R}}(x)}, \qquad \left|\frac{\partial R_0}{\partial x_j}(x)\right| \le \frac{Nk^2|x|^{N+1}}{4\eta_{\mathrm{R}}(x)} + \frac{1}{\tau}\,\frac{k^2|x_j|}{2\eta_{\mathrm{R}}(x)} \quad \textit{in } \Gamma.$$

Notice that here $V_\tau^\sharp(x)$ is only defined in $\Gamma \cap \Omega$. We now extend it to the whole domain Ω by using an appropriate cut-off. Let $l_s := \{x \in \Gamma : \eta_{\mathrm{R}}(x) = 1/s\}$ for $s > 0$. For $\varepsilon > 0$ small enough and $t^\sharp > 0$ large enough, we choose a function $\phi_t \in C^\infty(\mathbb{R}^2)$ satisfying

$$\phi_t(x) = \begin{cases} 1 & \text{for } x \in \overline{\bigcup_{0<s<t+\varepsilon/2} l_s},\ t \in [0,t^\sharp], \\ 0 & \text{for } x \in \mathbb{R}^2 \setminus \bigcup_{0<s<t+\varepsilon} l_s,\ t \in [0,t^\sharp], \end{cases}$$

and

$$|\partial_x^\alpha \phi_t(x)| \le C_\phi \quad \text{for } |\alpha| \le 2,\ x \in \Omega,\ t \in [0,t^\sharp]$$

for some positive constant C_ϕ depending only on Ω, N, $t^\sharp$ and ε. Next we define the function $V_{t,\tau}$ by

$$V_{t,\tau}(x) := \phi_t(x)\, \exp\left(-\frac{\tau}{t}\right) V_\tau^\sharp(x) \text{ for } x \in \overline{\Omega}.$$

Then we know by Lemma 3.1 that the dominant parts of $V_{t,\tau}$ and its derivatives are as follows:

$$V_{t,\tau}(x) = \begin{cases} 0 \text{ for } x \in \Omega \setminus \bigcup\limits_{0<s<t+\varepsilon} l_s, \\ \exp\Big(\tau\Big(-\dfrac{1}{t} + \eta(x)\Big)\Big)\big(\phi_t(x) + S_0(x)h\big) \\ \qquad\qquad\qquad\qquad \text{for } x \in \Omega \cap \bigcup\limits_{0<s<t+\varepsilon} l_s, \end{cases} \tag{3-3}$$

$$\nabla V_{t,\tau}(x) = \begin{cases} 0 \text{ for } x \in \Omega \setminus \bigcup\limits_{0<s<t+\varepsilon} l_s, \\ \tau \exp\Big(\tau\Big(-\dfrac{1}{t} + \eta(x)\Big)\Big)\big(\phi_t(x)\nabla\eta(x) + \boldsymbol{S}(x)h\big) \\ \qquad\qquad\qquad\qquad \text{for } x \in \Omega \cap \bigcup\limits_{0<s<t+\varepsilon} l_s \end{cases} \tag{3-4}$$

for $t \in (0,t^\sharp]$ and $\tau^{-1} \in (0,1]$, where $S_0(x) = S_0(x;t,\tau)$ and $\boldsymbol{S}(x) = \boldsymbol{S}(x;t,\tau)$ satisfy

$$|S_0(x)|,\ |\boldsymbol{S}(x)| \le C_{\mathrm{V}} \quad \text{for any } x \in \Omega \cap \bigcup_{0<s<t+\varepsilon} l_s,\ t \in (0,t^\sharp],\ \tau^{-1} \in (0,1]$$

with a positive constant C_{V} depending only on Ω, N, $t^\sharp$, ε and k. It should be remarked that the function $V_{t,\tau}$ does not satisfy the Helmholtz equation in Ω. Nonetheless, if we let $v_{0,t,\tau}$ be the solution to the Helmholtz equation in Ω

with boundary value $f_{t,\tau} := V_{t,\tau}|_{\partial\Omega}$, then the error between $V_{t,\tau}$ and $v_{0,t,\tau}$ is exponentially small.

Lemma 3.2. *There exist constants $C_0, C_0' > 0$ and $a > 0$ such that*

$$\|v_{0,t,\tau} - V_{t,\tau}\|_{H^2(\Omega)} \le \tau C_0' e^{-\tau a_t} \le C_0 e^{-\tau a}$$

for any $\tau^{-1} \in (0, 1]$, where the constants C_0 and C_0' depend only on $\Omega, k, N, t^\sharp$ and ε; the constant a depends only on $t^\sharp$ and ε; and we set $a_t := 1/t - 1/(t+\varepsilon/2)$.

This lemma can be proved in the same way as Lemma 4.1 in [Uhlmann and Wang 2008].

Now we consider the energy gap

$$I(\tau, t) = \int_{\partial\Omega} (\Lambda_D - \Lambda_\varnothing) f_{t,\tau} \, \overline{f}_{t,\tau} \, dS.$$

It can be shown that

$$I(\tau, t) \le k^2 \int_\Omega |w_{t,\tau}|^2 \, dx + \int_D \gamma_D \, |\nabla v_{0,t,\tau}|^2 \, dx, \tag{3-5}$$

$$I(\tau, t) \ge \int_D \frac{\gamma_D}{1+\gamma_D} |\nabla v_{0,t,\tau}|^2 \, dx - k^2 \int_\Omega |w_{t,\tau}|^2 \, dx, \tag{3-6}$$

where $v_{0,t,\tau}$ satisfies the Helmholtz equation in Ω with Dirichlet condition $v_{0,t,\tau}|_{\partial\Omega} = f_{t,\tau}$ and $w_{t,\tau} = v_{t,\tau} - v_{0,t,\tau}$ is the reflected solution, i.e.,

$$\begin{cases} \nabla \cdot (\tilde{\gamma} \nabla w_{t,\tau}) + k^2 w_{t,\tau} = -\nabla \cdot ((\tilde{\gamma} - 1) \nabla v_{0,t,\tau}) \text{ in } \Omega, \\ w_{t,\tau} = 0 \text{ on } \partial\Omega \end{cases} \tag{3-7}$$

(see [Nagayasu et al. 2011, Lemma 4.1]). It is easy to see that

$$\int_\Omega |w_{t,\tau}|^2 \, dx \le C \int_D |\nabla v_{0,t,\tau}|^2 \, dx.$$

In other words, in view of (3-5), the upper bound of $I(t, \tau)$ solely depends on $\int_D |\nabla v_{0,t,\tau}|^2 \, dx$.

To estimate the lower bound of $I(\tau, t)$, we proceed as above and introduce

$$I_{x_0,\alpha} := \int_{\partial D} \left|\partial_\nu v_{0,t,\tau}(x)\right| |x - x_0|^\alpha \, dS$$

for any $x_0 \in \Omega$ and $0 < \alpha < 1$. The following estimate is crucial in determining the behavior of $I(\tau, t)$ when the level curve of η_R intersects D.

Lemma 3.3 [Nagayasu et al. 2011, Lemma 3.7]. *For any $x_0 \in \Omega$, $0 < \alpha < 1$ and $2 < q \le 4$, we have*

$$\int_\Omega |w_{t,\tau}|^2 \, dx \le C_{q,\alpha} \big(I_{x_0,\alpha}^2 + I_{x_0,\alpha} \|\nabla v_{0,t,\tau}\|_{L^q(D)} + \|v_{0,t,\tau}\|_{L^2(D)}^2 \big). \tag{3-8}$$

It should be noted that $w_{t,\tau}$ satisfies an elliptic equation with coefficients having jump interfaces. To get the desired estimate (3-8), we make use of the Hölder estimate of Li and Vogelius [2000] for the this type of equations.

The enclosure method is now based on the following theorem regarding the behavior of $I(\tau, t)$.

Theorem 3.4 [Nagayasu et al. 2011, Theorem 4.1]. *Assume $D \cap \Gamma \neq \varnothing$. Suppose that $\{x \in \Gamma : \eta_{\mathrm{R}}(x) = \Theta_D\} \cap \partial D$ consists only of one point x_0 and the relative curvature (see [Nagayasu et al. 2011] for the definition) to $\eta_{\mathrm{R}}(x) = \Theta_D$ of ∂D at x_0 is not zero. Then there exist positive constants C_1, c_1 and τ_1 such that for any $0 < t \leq t^\sharp$ and $\tau \geq \tau_1$ the following holds*:

(I) *if* $1/t > \Theta_D$ *then*

$$|I(\tau, t)| \leq \begin{cases} C_1 \tau^2 \exp\Big(2\tau\Big(-\frac{1}{t} + \frac{1}{t+\varepsilon/2}\Big)\Big) \text{ if } \Theta_D \leq \frac{1}{t+\varepsilon/2}, \\ C_1 \tau^2 \exp\Big(2\tau\Big(-\frac{1}{t} + \Theta_D\Big)\Big) \text{ if } \frac{1}{t+\varepsilon/2} < \Theta_D < \frac{1}{t}. \end{cases}$$

(II) *if* $1/t \leq \Theta_D$ *then*

$$I(\tau, t) \geq c_1 \exp\Big(2\tau\Big(-\frac{1}{t} + \Theta_D\Big)\Big)\tau^{1/2}.$$

The proof of this theorem relies on estimates we obtained above. Moreover, even though we impose some restriction on the curvature of ∂D at x_0, one can show that the curvature assumption is always satisfied as long as N is large enough for C^2 boundary ∂D.

3B. ***An improvement by Sini and Yoshida.*** In the enclosure method discussed above (for impenetrable or penetrable obstacles), two conditions are assumed, that is, the level curve of real part of the phase function in CGO solutions touches ∂D at one point and the nonvanishing of the relative curvature at the touching point. These two assumptions are removed in [Sini and Yoshida 2010]. Roughly speaking, the authors use following estimates for the reflected solution w

$$\|w\|_{L^2(\Omega)} \leq C_p \|v\|_{W^{1,p}(D)} \quad \text{with} \quad p < 2 \tag{3-9}$$

for a penetrable obstacle, and

$$\|w\|_{L^2(\Omega \setminus \bar{D})} \leq C_t \|v\|_{H^{-t+\frac{3}{2}}(D)} \quad \text{with} \quad t < 1 \tag{3-10}$$

for an impenetrable obstacle. Here v satisfies the Helmholtz equation in Ω.

The derivation of (3-9) is based on Meyers' theorem [1963] and the sharp Friedrichs inequality, while, the proof of (3-10) relies on layer potential techniques on Sobolev spaces and integral estimates of the p-powers of Green's function. We refer to [Sini and Yoshida 2010] for details. Here we would like to see how (3-9) and (3-10) lead to the characteristic behaviors of the energy gap in the enclosure method. To illustrate the ideas, we follow Sini and Yoshida in considering only CGO solutions with linear phases, i.e.,

$$v(x;\tau,t)=e^{\tau(x\cdot\omega-t)+i\sqrt{\tau^2+k^2}\,x\cdot\omega^{\perp}}.$$

It is clear that v is a solution of the Helmholtz equation. Denote the energy gap by

$$I(\tau,t)=\int_{\partial\Omega}(\Lambda_D-\Lambda_{\varnothing})v\bar{v}\,dS.$$

The following behavior of I can be obtained.

Theorem 3.5 [Sini and Yoshida 2010, Theorem 2.4]. *Let $D\Subset\Omega$ with Lipschitz boundary ∂D. For both the penetrable and impenetrable cases, we have*:

(i) $\lim_{\tau\to\infty} I(\tau,t)=0$ *if* $t>h_D(\omega)$,

$\liminf_{\tau\to\infty}|I(\tau,h_D(\omega))|=\infty\ (n=2),\quad \liminf_{\tau\to\infty}|I(\tau,h_D(\omega))|>0\ (n=3),$

$\lim_{\tau\to\infty}|I(\tau,t)|=\infty$ *if* $t<h_D(\omega)$.

(ii)
$$h_D(\omega)-t=\lim_{\tau\to\infty}\frac{\ln|I(\tau,t)|}{2\tau}.$$

To prove Theorem 3.5, it is enough to estimate the lower bound of $I(\tau,t)$ at $t=h_D(\omega)$ for $n=3$. Let $y\in\partial D\cap\{x\cdot\omega=h_D(\omega)\}:=K$. Since K is compact, there exist $y_1,\dots,y_N\in K$ such that

$$K\subset D_\delta\quad\text{for }\delta>0\text{ sufficiently small,}$$

where

$$D_\delta=\bigcup_{j=1}^{N}(D\cap B(y_j,\delta)).$$

It is obvious that $\int_{D\setminus D_\delta}|\nabla^m v|^p\,dx$ is exponentially small in τ for $m=0,1$. Therefore, to obtain the behaviors of $\int_D|\nabla^m v|^p\,dx$ in τ, it suffices to study the integrals over D_δ. Using the change of coordinates, it is tedious but not difficult to show that

$$\|v\|^2_{L^2(D)}\ge C\tau^{-2},\qquad \frac{\|\nabla v\|^2_{L^2(D)}}{\|v\|^2_{L^2(D)}}\ge C\tau^2 \tag{3-11}$$

and

$$\frac{\|v\|^2_{L^p(D)}}{\|v\|^2_{L^2(D)}}\le C\tau^{1-2/p},\qquad \frac{\|\nabla v\|^2_{L^p(D)}}{\|v\|^2_{L^2(D)}}\le C\tau^{3-2/p} \tag{3-12}$$

with $\max\{2-\varepsilon, 6/5\} < p \le 2$ (see [Sini and Yoshida 2010, pages 6–9]). Using (3-9) we get from (3-12) that

$$\frac{\|w\|^2_{L^p(D)}}{\|v\|^2_{L^2(D)}} \le C\tau^{3-2/p}. \tag{3-13}$$

Recall that

$$I(\tau,t) \ge \int_D \frac{\gamma_D}{1+\gamma_D}|\nabla v|^2\,dx - k^2\int_\Omega |w|^2\,dx.$$

Thus, combining (3-11) and (3-13) implies that

$$I(\tau, h_D(\omega)) \ge C\tau^2\|v\|^2_{L^2(D)} \ge C' > 0$$

As for an impenetrable obstacle (sound hard), we recall that

$$-I(\tau,t) \ge \int_D |\nabla v|^2 dx - k^2\int_{\Omega\setminus\bar{D}} |w|^2 \tag{3-14}$$

(see for example [Ikehata 1999a, Lemma 4.1]). Let $s=\frac{3}{2}-t$, then $\frac{1}{2} < s \le \frac{3}{2}$ if $0 \le t < 1$. From (3-10) and (3-14), we have that

$$-I(\tau,t) \ge \int_D |\nabla v|^2 dx - C\|v\|^2_{H^s(D)}.$$

Using the interpolation and Young's inequalities, one can choose appropriate parameters such that

$$-I(\tau,t) \ge C\int_D |\nabla v|^2 dx - C'\int_D |v|^2 dx$$

and thus

$$-I(\tau, h_D(\omega)) > 0$$

follows from (3-11).

3C. ***Elastic penetrable obstacles.*** Recently, Kuan [2012] extended the enclosure method to the reconstruction of a penetrable obstacle using elastic waves. Her result is in 2 dimensions, but it can be generalized to 3 dimensions without serious difficulties. Consider the elastic waves in $\Omega \subset \mathbb{R}^2$ with smooth boundary $\partial\Omega$

$$\nabla\cdot(\sigma(u)) + k^2 u = 0 \quad \text{in } \Omega, \tag{3-15}$$

where u is the displacement vector and

$$\sigma(u) = \lambda(\nabla\cdot u)I_2 + 2\mu\epsilon(u)$$

is the stress tensor. Here $\epsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ denotes the infinitesimal strain tensor. Assume that

$$\lambda = \lambda_0 + \lambda_D \chi_D \quad \text{and} \quad \mu = \mu_0 + \mu_D \chi_D,$$

where D is an open subset of Ω with $\bar{D} \subset \Omega$ and λ_D, μ_D belong to $L^\infty(D)$. Assume that

$$\lambda_0 + \mu_0 > 0,\ \mu_0 > 0,\ \lambda + \mu > 0,\ \mu > 0 \quad \text{in } \Omega.$$

We would like to discuss the reconstruct of the shape of D from boundary measurements in the spirit of enclosure method.

Assume that $-k^2$ is not a Dirichlet eigenvalue of the Lamé operator $\nabla \cdot (\sigma(\,\cdot\,))$. Define the Dirichlet-to-Neumann (displacement-to-traction) map

$$\Lambda_D : u|_{\partial\Omega} \to \sigma(u)\nu|_{\partial\Omega}.$$

Let v satisfy the Lamé equation with Lamé coefficients λ_0, μ_0, i.e.,

$$\nabla \cdot (\sigma(v)) + k^2 v = 0 \quad \text{in } \Omega, \tag{3-16}$$

with

$$\sigma(v) = \lambda_0(\nabla \cdot v) I_2 + 2\mu_0 \epsilon(v).$$

Likewise, we assume that $-k^2$ is not a Dirichlet eigenvalue of the free Lamé operator. We then define the corresponding Dirichlet-to-Neumann map

$$\Lambda_\varnothing : v|_{\partial\Omega} \to \sigma(v)\nu|_{\partial\Omega}.$$

Similar as above, in the enclosure method, we need to construct the CGO solutions for the Lamé equation (3-16). For simplicity, we assume that both λ_0 and μ_0 are constants. To construct the CGO solutions in this case, we take advantage of the Helmholtz decomposition and consider two Helmholtz equations

$$\begin{cases} \Delta\varphi + k_1^2 \varphi = 0, \\ \Delta\psi + k_2^2 \psi = 0, \end{cases} \tag{3-17}$$

where

$$k_1 = \left(\frac{k^2}{\lambda_0 + 2\mu_0} \right)^{1/2} \quad \text{and} \quad k_2 = \left(\frac{k^2}{\mu_0} \right)^{1/2}.$$

Then $v = \nabla\varphi + \nabla^\perp \psi$ solves (3-16). Here $\nabla^\perp \psi := (-\partial_2 \psi, \partial_1 \psi)^T$. For (3-17), we can construct the CGO solutions having linear or polynomial phases, which will give us the CGO solutions v for (3-16).

We will not repeat the construction of CGO solutions here. We simply denote $v(\tau, t)$ the CGO solution. Similarly, we define the energy gap

$$I(\tau, t) = \int_{\partial\Omega} (\Lambda_D - \Lambda_\varnothing) f_{\tau,t} \cdot \bar{f}_{\tau,t} \, dS,$$

where $f_{\tau,t} = v(\tau, t)|_{\partial\Omega}$. Let Γ be the domain where the real part of the phase function of v, denoted by $\rho(x)$, is positive. Let

$$h_D = \begin{cases} \sup_{x \in D\cap\Gamma} \rho(x) & \text{if } D \cap \Gamma \neq \varnothing, \\ 0 & \text{if } D \cap \Gamma = \varnothing. \end{cases}$$

Assume appropriate jump conditions on λ_D and μ_D. Then the following behaviors of $I(\tau, t)$ are obtained in [Kuan 2012].

Theorem 3.6. (i) $\lim_{\tau\to\infty} I(\tau, t) = 0$ *if* $t > h_D$.
(ii) *If* $t = h_D$ *and* $\partial D \in C^{0,\alpha}$, $1/3 < \alpha \le 1$, *then* $\liminf_{\tau\to\infty} |I(\tau, h_D(\omega))| = \infty$.
(iii) *If* $t < h_D$ *and* $\partial D \in C^0$, *then* $\lim_{\tau\to\infty} |I(\tau, t)| = \infty$.

The proof of Theorem 3.6 is based on the following inequalities for the energy gap:

$$I(\tau, t) \le \int_D (\lambda_D + \mu_D) |\nabla\cdot v|^2 dx + 2\int_D \mu_D \big|\epsilon(v) - \tfrac{1}{2}(\nabla\cdot v) I_2\big|^2 dx + k^2 \|w\|^2_{L^2(\Omega)},$$

$$I(\tau, t) \ge \int_D \frac{(\lambda_0 + \mu_0)(\lambda_D + \mu_D)}{\lambda + \mu} |\nabla\cdot v|^2 dx + 2\int_D \frac{\mu_0 \mu_D}{\mu} \big|\epsilon(v) - \tfrac{1}{2}(\nabla\cdot v) I_2\big|^2 dx - k^2 \|w\|^2_{L^2(\Omega)},$$

where w is the reflected solution. The rest of the proof is similar to that in [Sini and Yoshida 2010], which relies on the following L^p estimate:

Lemma 3.7 [Kuan 2012, Lemma 4.2]. *There exist constants* $C > 0$ *and* p_0 *in* $[1, 2)$ *such that, for* $p_0 < p \le 2$,

$$\|w\|_{L^2(\Omega)} \le C \|\nabla v\|_{L^p(D)}.$$

Lemma 3.7 can be proved adapting arguments from [Meyers 1963].

4. Open problems

The enclosure method in the electromagnetic waves we discussed in Section 2B is for the case of an impenetrable obstacle. Therefore, it is a legitimate project to study the penetrable case for the electromagnetic waves. However, the tools used in the acoustic waves, i.e., Li–Vogelius type estimates or Meyers type L^p estimates, are not available in the electromagnetic waves. The derivation of these estimates itself is an interesting problem.[1] Another interesting problem is to extend the enclosure method to the plate or shell equations. The distinct feature of these equations is the appearance of the biharmonic operator Δ^2.

Finally, it is desirable to design stable and efficient algorithms for the enclosure method. Attempts of numerical implementations have been made for the conductivity equation [Brühl and Hanke 2000; Ide et al. 2007; Ikehata and Siltanen 2000; Uhlmann and Wang 2008] and for the 2D static elastic equation [Uhlmann et al. 2009]. There are two obvious difficulties. On one hand, the boundary data involves large parameter which gives rise to highly oscillatory functions. On the other hand, a reliable way of numerically determining whether $I(\tau, t)$ decays or blows up as $\tau \to \infty$ is yet to be found.

References

[Arens 2004] T. Arens, "Why linear sampling works", *Inverse Problems* **20**:1 (2004), 163–173. MR 2005b:35036 Zbl 1055.35131

[Arens and Lechleiter 2009] T. Arens and A. Lechleiter, "The linear sampling method revisited", *J. Integral Equations Appl.* **21**:2 (2009), 179–202. MR 2011a:35564 Zbl 1237.65118

[Astala and Päivärinta 2006] K. Astala and L. Päivärinta, "Calderón's inverse conductivity problem in the plane", *Ann. of Math.* (2) **163**:1 (2006), 265–299. MR 2007b:30019 Zbl 1111.35004

[Brühl and Hanke 2000] M. Brühl and M. Hanke, "Numerical implementation of two noniterative methods for locating inclusions by impedance tomography", *Inverse Problems* **16**:4 (2000), 1029–1042. MR 2001d:65080 Zbl 0955.35076

[Cakoni and Colton 2006] F. Cakoni and D. Colton, *Qualitative methods in inverse scattering theory: an introduction*, Springer, Berlin, 2006. MR 2008c:35334 Zbl 1099.78008

[Calderón 1980] A.-P. Calderón, "On an inverse boundary value problem", pp. 65–73 in *Seminar on numerical analysis and its applications to continuum physics* (Rio de Janeiro, 1980), Soc. Brasil. Mat., Rio de Janeiro, 1980. MR 81k:35160

[Colton and Päivärinta 1992] D. Colton and L. Päivärinta, "The uniqueness of a solution to an inverse scattering problem for electromagnetic waves", *Arch. Rational Mech. Anal.* **119**:1 (1992), 59–70. MR 93h:78009 Zbl 0756.35114

[Dos Santos Ferreira et al. 2007] D. Dos Santos Ferreira, C. E. Kenig, J. Sjöstrand, and G. Uhlmann, "Determining a magnetic Schrödinger operator from partial Cauchy data", *Comm. Math. Phys.* **271**:2 (2007), 467–488. MR 2008a:35044 Zbl 1148.35096

[Dos Santos Ferreira et al. 2009] D. Dos Santos Ferreira, C. E. Kenig, M. Salo, and G. Uhlmann, "Limiting Carleman weights and anisotropic inverse problems", *Invent. Math.* **178**:1 (2009), 119–171. MR 2010h:58033 Zbl 1181.35327

[Haberman and Tataru 2011] B. Haberman and D. Tataru, "Uniqueness in Calderón's problem for Lipschitz conductivities", preprint, 2011. arXiv 1108.6068

[1] After the completion of this article, the enclosure type method for the Maxwell equations with a penetrable obstacle was established by Manars Kar and Mourad Sini in the paper "Reconstruction of interfaces using CGO solutions for the Maxwell equations".

[Ide et al. 2007] T. Ide, H. Isozaki, S. Nakata, S. Siltanen, and G. Uhlmann, "Probing for electrical inclusions with complex spherical waves", *Comm. Pure Appl. Math.* **60**:10 (2007), 1415–1442. MR 2008j:35186

[Ikehata 1998] M. Ikehata, "Reconstruction of the shape of the inclusion by boundary measurements", *Comm. Partial Differential Equations* **23**:7-8 (1998), 1459–1474. MR 99f:35222 Zbl 0915.35114

[Ikehata 1999a] M. Ikehata, "How to draw a picture of an unknown inclusion from boundary measurements. Two mathematical inversion algorithms", *J. Inverse Ill-Posed Probl.* **7**:3 (1999), 255–271. MR 2000d:35254 Zbl 0928.35207

[Ikehata 1999b] M. Ikehata, "Reconstruction of obstacle from boundary measurements", *Wave Motion* **30**:3 (1999), 205–223. MR 2000i:35214 Zbl 1067.35506

[Ikehata 2000] M. Ikehata, "Reconstruction of the support function for inclusion from boundary measurements", *J. Inverse Ill-Posed Probl.* **8**:4 (2000), 367–378. MR 2002g:35212

[Ikehata and Siltanen 2000] M. Ikehata and S. Siltanen, "Numerical method for finding the convex hull of an inclusion in conductivity from boundary measurements", *Inverse Problems* **16**:4 (2000), 1043–1052. MR 2001g:65164 Zbl 0956.35133

[Isakov 1990] V. Isakov, "On uniqueness in the inverse transmission scattering problem", *Comm. Partial Differential Equations* **15**:11 (1990), 1565–1587. MR 91i:35203 Zbl 0728.35148

[Kenig et al. 2007] C. E. Kenig, J. Sjöstrand, and G. Uhlmann, "The Calderón problem with partial data", *Ann. of Math.* (2) **165**:2 (2007), 567–591. MR 2008k:35498 Zbl 1127.35079

[Kirsch and Grinberg 2008] A. Kirsch and N. Grinberg, *The factorization method for inverse problems*, Oxford Lecture Series in Mathematics and its Applications **36**, Oxford University Press, 2008. MR 2009k:35322 Zbl 1222.35001

[Kuan 2012] R. Kuan, "Reconstruction of penetrable inclusions in elastic waves by boundary measurements", *J. Differential Equations* **252**:2 (2012), 1494–1520. MR 2012j:35451 Zbl 1243.35174

[Li and Vogelius 2000] Y. Y. Li and M. Vogelius, "Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients", *Arch. Ration. Mech. Anal.* **153**:2 (2000), 91–151. MR 2001m:35083 Zbl 0958.35060

[Meyers 1963] N. G. Meyers, "An L^p e-estimate for the gradient of solutions of second order elliptic divergence equations", *Ann. Scuola Norm. Sup. Pisa* (3) **17** (1963), 189–206. MR 28 #2328

[Nagayasu et al. 2011] S. Nagayasu, G. Uhlmann, and J.-N. Wang, "Reconstruction of penetrable obstacles in acoustic scattering", *SIAM J. Math. Anal.* **43**:1 (2011), 189–211. MR 2012b:35369 Zbl 1234.35315

[Nakamura and Yoshida 2007] G. Nakamura and K. Yoshida, "Identification of a non-convex obstacle for acoustical scattering", *J. Inverse Ill-Posed Probl.* **15**:6 (2007), 611–624. MR 2009j:35387 Zbl 1126.35103

[Nakamura et al. 2005] G. Nakamura, G. Uhlmann, and J.-N. Wang, "Oscillating-decaying solutions, Runge approximation property for the anisotropic elasticity system and their applications to inverse problems", *J. Math. Pures Appl.* (9) **84**:1 (2005), 21–54. MR 2005i:35289 Zbl 1067.35151

[Ola and Somersalo 1996] P. Ola and E. Somersalo, "Electromagnetic inverse problems and generalized Sommerfeld potentials", *SIAM J. Appl. Math.* **56**:4 (1996), 1129–1145. MR 97b:35194 Zbl 0858.35138

[Ola et al. 1993] P. Ola, L. Päivärinta, and E. Somersalo, "An inverse boundary value problem in electrodynamics", *Duke Math. J.* **70**:3 (1993), 617–653. MR 94i:35196 Zbl 0804.35152

[Potthast 2001] R. Potthast, *Point sources and multipoles in inverse scattering theory*, Research Notes in Mathematics **427**, CRC, Boca Raton, FL, 2001. MR 2002j:35313 Zbl 0985.78016

[Sini and Yoshida 2010] M. Sini and K. Yoshida, "On the reconstruction of interfaces using CGO solutions for the acoustic case", preprint, 2010, Available at www.ricam.oeaw.ac.at/people/page/sini/Sini-Yoshida-preprint.pdf.

[Sylvester and Uhlmann 1987] J. Sylvester and G. Uhlmann, "A global uniqueness theorem for an inverse boundary value problem", *Ann. of Math.* (2) **125**:1 (1987), 153–169. MR 88b:35205 Zbl 0625.35078

[Takuwa et al. 2008] H. Takuwa, G. Uhlmann, and J.-N. Wang, "Complex geometrical optics solutions for anisotropic equations and applications", *J. Inverse Ill-Posed Probl.* **16**:8 (2008), 791–804. MR 2010d:35408 Zbl 1152.35519

[Uhlmann and Wang 2007] G. Uhlmann and J.-N. Wang, "Complex spherical waves for the elasticity system and probing of inclusions", *SIAM J. Math. Anal.* **38**:6 (2007), 1967–1980. MR 2008a:35295 Zbl 1131.35088

[Uhlmann and Wang 2008] G. Uhlmann and J.-N. Wang, "Reconstructing discontinuities using complex geometrical optics solutions", *SIAM J. Appl. Math.* **68**:4 (2008), 1026–1044. MR 2009d:35347 Zbl 1146.35097

[Uhlmann et al. 2009] G. Uhlmann, J.-N. Wang, and C.-T. Wu, "Reconstruction of inclusions in an elastic body", *J. Math. Pures Appl.* (9) **91**:6 (2009), 569–582. MR 2010d:35409 Zbl 1173.35123

[Vekua 1967] I. N. Vekua, *New methods for solving elliptic equations*, Applied Mathematics and Mechanics **1**, North-Holland, Amsterdam, 1967. MR 35 #3243 Zbl 0146.34301

[Yoshida 2010] K. Yoshida, "Reconstruction of a penetrable obstacle by complex spherical waves", *J. Math. Anal. Appl.* **369**:2 (2010), 645–657. MR 2011i:65192 Zbl 1196.35229

[Zhou 2010] T. Zhou, "Reconstructing electromagnetic obstacles by the enclosure method", *Inverse Probl. Imaging* **4**:3 (2010), 547–569. MR 2011f:35366 Zbl 1206.35262

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