Inside Out II
MSRI Publications
Volume **60**, 2012

Inverse problems for connections

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We discuss various recent results related to the inverse problem of determining a unitary connection from its parallel transport along geodesics.

1. Introduction

Let (M, g) be a compact oriented Riemannian manifold with smooth boundary, and let $SM = \{(x, v) \in TM\ ;\ |v| = 1\}$ be the unit tangent bundle with canonical projection $\pi : SM \to M$. The geodesics going from ∂M into M can be parametrized by the set $\partial_+(SM) = \{(x, v) \in SM\ ;\ x \in \partial M, \langle v, \nu\rangle \leq 0\}$, where ν is the outer unit normal vector to ∂M. For any $(x, v) \in SM$ we let $t \mapsto \gamma(t, x, v)$ be the geodesic starting from x in direction v. We assume that (M, g) is nontrapping, which means that the time $\tau(x, v)$ when the geodesic $\gamma(t, x, v)$ exits M is finite for each $(x, v) \in SM$. The scattering relation

$$\alpha = \alpha_g : \partial_+(SM) \to \partial_-(SM)$$

maps a starting point and direction of a geodesic to the end point and direction, where $\partial_-(SM) = \{(x, v) \in SM\ ;\ x \in \partial M, \langle v, \nu\rangle \geq 0\}$.

Suppose now that $E \to M$ is a Hermitian vector bundle of rank n over M and ∇ is a unitary connection on E. Associated with ∇ there is the following additional piece of scattering data: given $(x, v) \in \partial_+(SM)$, let $P(x, v) = P_\nabla(x, v) : E(x) \to E(\pi \circ \alpha(x, v))$ denote the parallel transport along the geodesic $\gamma(t, x, v)$. This map is a linear isometry and the main inverse problem we wish to discuss here is the following:

Question. Does P determine ∇?

The first observation is that the problem has a natural gauge equivalence. Let ψ be a gauge transformation, that is, a smooth section of the bundle of automorphisms Aut$E$. The set of all these sections naturally forms a group (known as the gauge group) which acts on the space of unitary connections by the rule

$$(\psi^*\nabla)s := \psi\nabla(\psi^{-1}s)$$

where s is any smooth section of E. If in addition $\psi|_{\partial M} = \mathrm{Id}$, then it is a simple exercise to check that

$$P_\nabla = P_{\psi^*\nabla}.$$

Thus we can rephrase the question above more precisely as follows:

Question I (manifolds with boundary). Let ∇_1 and ∇_2 be two unitary connections with $P_{\nabla_1} = P_{\nabla_2}$. Does there exist a gauge transformation ψ with $\psi|_{\partial M} = \mathrm{Id}$ and $\psi^*\nabla_1 = \nabla_2$?

There is a version of this question which makes sense also for closed manifolds, that is, $\partial M = \varnothing$. Let $\gamma : [0, T] \to M$ be a closed geodesic and let $P_\nabla(\gamma) : E(\gamma(0)) \to E(\gamma(0))$ be the parallel transport along γ.

Question II (closed manifolds). Let ∇_1 and ∇_2 be two unitary connections and suppose there is a connection ∇ gauge equivalent to ∇_1 such that $P_\nabla(\gamma) = P_{\nabla_2}(\gamma)$ for every closed geodesic γ. Are ∇_1 and ∇_2 gauge equivalent?

A connection ∇ is said to be *transparent* if $P_\nabla(\gamma) = \mathrm{Id}$ for all closed geodesics γ. Understanding the set of transparent connections modulo gauge is an important special case of Question II.

To make further progress on Questions I and II we need to impose some conditions on the manifold (M, g).

In the case of manifolds with boundary a typical hypothesis is that of *simplicity*. A compact Riemannian manifold with boundary is said to be *simple* if for any point $x \in M$ the exponential map $\exp_x$ is a diffeomorphism onto M, and if the boundary is strictly convex. The notion of simplicity arises naturally in the context of the boundary rigidity problem [Michel 1981]. For the case of closed manifolds there are two reasonable disjoint options. One is to assume that (M, g) is a Zoll manifold, that is, a Riemannian manifold all of whose geodesics are closed, but we shall not really discuss this case in any detail here. The other is to assume that the geodesic flow is *Anosov*. Recall that the geodesic flow ϕ_t is Anosov if there is a continuous splitting $TSM = E^0 \oplus E^u \oplus E^s$, where E^0 is the flow direction, and there are constants $C > 0$ and $0 < \rho < 1 < \eta$ such that for all $t > 0$ we have

$$\|d\phi_{-t}|_{E^u}\| \le C\,\eta^{-t} \quad \text{and} \quad \|d\phi_t|_{E^s}\| \le C\,\rho^t.$$

It is very well known that the geodesic flow of a closed negatively curved Riemannian manifold is a contact Anosov flow [Katok and Hasselblatt 1995]. The Anosov property automatically implies that the manifold is free of conjugate points [Klingenberg 1974; Anosov 1985; Mañé 1987]. Simple manifolds are also free of conjugate points (this follows directly from the definition) and both conditions (simplicity and Anosov) are open conditions on the metric. It is remarkable that similar results exist in both situations.

It is easy to see from the definition that a simple manifold must be diffeomorphic to a ball in $\mathbb{R}^n$. Therefore any bundle over such M is necessarily trivial. For most of this paper we shall consider Questions I and II only for the case of trivial bundles; this will make the presentation clearer without removing substantial content.

Question I arises naturally when considering the hyperbolic Dirichlet-to-Neumann map associated to the Schrödinger equation with a connection. It was shown in [Finch and Uhlmann 2001] that when the metric is Euclidean, the scattering data for a connection can be determined from the hyperbolic Dirichlet-to-Neumann map. A similar result holds true on simple Riemannian manifolds: a combination of the methods in [Finch and Uhlmann 2001; Uhlmann 2004] shows that the hyperbolic Dirichlet-to-Neumann map for a connection determines the scattering data P_∇.

2. Elementary background on connections

Consider the trivial bundle $M \times \mathbb{C}^n$. For us a connection A will be a complex $n \times n$ matrix whose entries are smooth 1-forms on M. Another way to think of A is to regard it as a smooth map $A : TM \to \mathbb{C}^{n\times n}$ which is linear in $v \in T_x M$ for each $x \in M$.

Very often in physics and geometry one considers *unitary* or *Hermitian* connections. This means that the range of A is restricted to skew-Hermitian matrices. In other words, if we denote by $\mathfrak{u}(n)$ the Lie algebra of the unitary group $U(n)$, we have a smooth map $A : TM \to \mathfrak{u}(n)$ which is linear in the velocities. There is yet another equivalent way to phrase this. The connection A induces a covariant derivative d_A on sections $s \in C^\infty(M, \mathbb{C}^n)$ by setting $d_A s = ds + As$. Then A being Hermitian or unitary is equivalent to requiring compatibility with the standard Hermitian inner product of $\mathbb{C}^n$ in the sense that

$$d\langle s_1, s_2\rangle = \langle d_A s_1, s_2\rangle + \langle s_1, d_A s_2\rangle$$

for any pair of functions s_1, s_2.

Given two unitary connections A and B we shall say that A and B are gauge equivalent if there exists a smooth map $u : M \to U(n)$ such that

$$B = u^{-1} du + u^{-1} A u. \tag{1}$$

It is an easy exercise to check that this definition coincides with the one given in the previous section if we set $\psi = u^{-1}$.

The *curvature* of the connection is the 2-form F_A with values in $\mathfrak{u}(n)$ given by

$$F_A := dA + A \wedge A.$$

If A and B are related by (1) then:

$$F_B = u^{-1}\, F_A\, u.$$

Given a smooth curve $\gamma : [a,b] \to M$, the *parallel transport* along γ is obtained by solving the linear differential equation in $\mathbb{C}^n$:

$$\begin{cases} \dot{s} + A(\gamma(t), \dot{\gamma}(t))s = 0, \\ s(a) = w \in \mathbb{C}^n. \end{cases} \tag{2}$$

The isometry $P_A(\gamma) : \mathbb{C}^n \to \mathbb{C}^n$ is defined as $P_A(\gamma)(w) := s(b)$. We may also consider the fundamental unitary matrix solution $U : [a,b] \to U(n)$ of (2). It solves

$$\begin{cases} \dot{U} + A(\gamma(t), \dot{\gamma}(t))U = 0, \\ U(a) = \mathrm{Id}. \end{cases} \tag{3}$$

Clearly $P_A(\gamma)(w) = U(b)w$.

3. The transport equation and the attenuated ray transform

Consider now the case of a compact simple Riemannian manifold. We would like to pack the information provided by (3) along every geodesic into one PDE in SM. For this we consider the vector field X associated with the geodesic flow ϕ_t and we look at the unique solution $U_A : SM \to U(n)$ of

$$\begin{cases} X(U_A) + A(x,v)U_A = 0, \ (x,v) \in SM \\ U_A|_{\partial_+(SM)} = \mathrm{Id}. \end{cases} \tag{4}$$

The scattering data of the connection A is now the map $C_A : \partial_-(SM) \to U(n)$ defined as $C_A := U_A|_{\partial_-(SM)}$.

We can now rephrase Question I as follows:

Question I (manifolds with boundary). Let A and B be two unitary connections with $C_A = C_B$. Does there exist a smooth map $U : M \to U(n)$ with $U|_{\partial M} = \mathrm{Id}$ and $B = U^{-1}dU + U^{-1}AU$?

Suppose $C_A = C_B$ and define $U := U_A(U_B)^{-1} : SM \to U(n)$. One easily checks that U satisfies:

$$\begin{cases} XU + AU - UB = 0, \\ U|_{\partial(SM)} = \mathrm{Id}. \end{cases}$$

If we show that U is in fact smooth *and* it only depends on the base point $x \in M$ we would have an answer to Question I, since the equation above reduces to $dU + AU - UB = 0$ and $U|_{\partial M} = \mathrm{Id}$ which is exactly gauge equivalence. Showing that U only depends of x is not an easy task and it often is the crux of the matter in these type of problems. To tackle this issue we will rephrase the problem in terms of an *attenuated ray transform*.

Consider $W := U - \mathrm{Id} : SM \to \mathbb{C}^{n\times n}$, where as before $\mathbb{C}^{n\times n}$ stands for the set of all $n \times n$ complex matrices. Clearly W satisfies

$$XW + AW - WB = B - A, \tag{5}$$

$$W|_{\partial(SM)} = 0. \tag{6}$$

We introduce a new connection $\hat{A}$ on the trivial bundle $M \times \mathbb{C}^{n\times n}$ as follows: given a matrix $R \in \mathbb{C}^{n\times n}$ we define $\hat{A}(R) := AR - RB$. One easily checks that $\hat{A}$ is Hermitian if A and B are. Then equations (5) and (6) are of the form:

$$\begin{cases} Xu + Au = -f, \\ u|_{\partial(SM)} = 0. \end{cases}$$

where A is a unitary connection, $f : SM \to \mathbb{C}^N$ is a smooth function linear in the velocities, $u : SM \to \mathbb{C}^N$ is a function that we would like to prove smooth and only dependent on $x \in M$ and $N = n \times n$. As we will see shortly this amounts to understanding which functions f linear in the velocities are in the kernel of the attenuated ray transform of the connection A.

First recall that in the scalar case, the attenuated ray transform $I_a f$ of a function $f \in C^\infty(SM, \mathbb{C})$ with attenuation coefficient $a \in C^\infty(SM, \mathbb{C})$ can be defined as the integral

$$I_a f(x, v) := \int_0^{\tau(x,v)} f(\varphi_t(x, v)) \exp\left[\int_0^t a(\varphi_s(x, v))\, ds\right] dt, \qquad (x, v) \in \partial_+(SM).$$

Alternatively, we may set $I_a f := u|_{\partial_+(SM)}$ where u is the unique solution of the *transport equation*

$$Xu + au = -f \ \text{ in } SM, \quad u|_{\partial_-(SM)} = 0.$$

The last definition generalizes without difficulty to the case of connections. Assume that A is a unitary connection and let $f \in C^\infty(SM, \mathbb{C}^n)$ be a vector valued function. Consider the following transport equation for $u : SM \to \mathbb{C}^n$,

$$Xu + Au = -f \ \text{ in } SM, \quad u|_{\partial_-(SM)} = 0.$$

On a fixed geodesic the transport equation becomes a linear ODE with zero initial condition, and therefore this equation has a unique solution $u = u^f$.

Definition 3.1. The attenuated ray transform of $f \in C^\infty(SM, \mathbb{C}^n)$ is given by

$$I_A f := u^f|_{\partial_+(SM)}.$$

We note that I_A acting on sums of 0-forms and 1-forms always has a nontrivial kernel, since

$$I_A(dp + Ap) = 0 \text{ for any } p \in C^\infty(M, \mathbb{C}^n) \text{ with } p|_{\partial M} = 0.$$

Thus from the ray transform $I_A f$ one only expects to recover f up to an element having this form.

The transform I_A also has an integral representation. Consider the unique matrix solution $U_A : SM \to U(n)$ from above. Then it is easy to check that

$$I_A f(x, v) = \int_0^{\tau(x,v)} U_A^{-1}(\phi_t(x, v)) f(\phi_t(x, v))\, dt.$$

We are now in a position to state the next main question:

Question III (kernel of I_A). Let (M, g) be a compact simple Riemannian manifold and let A be a unitary connection. Assume that $f : SM \to \mathbb{C}^n$ is a smooth function of the form $F(x) + \alpha_j(x)v^j$, where $F : M \to \mathbb{C}^n$ is a smooth function and α is a $\mathbb{C}^n$-valued 1-form. If $I_A(f) = 0$, is it true that $F = 0$ and $\alpha = d_A p = dp + Ap$, where $p : M \to \mathbb{C}^n$ is a smooth function with $p|_{\partial M} = 0$?

As explained above a positive answer to Question III gives a positive answer to Question I. The next recent result provides a full answer to Question III in the two-dimensional case:

Theorem 3.2 [Paternain et al. 2011a]. *Let M be a compact simple surface. Assume that $f : SM \to \mathbb{C}^n$ is a smooth function of the form $F(x) + \alpha_j(x)v^j$, where $F : M \to \mathbb{C}^n$ is a smooth function and α is a $\mathbb{C}^n$-valued 1-form. Let also $A : TM \to \mathfrak{u}(n)$ be a unitary connection. If $I_A(f) = 0$, then $F = 0$ and $\alpha = d_A p$, where $p : M \to \mathbb{C}^n$ is a smooth function with $p|_{\partial M} = 0$.*

Let us explicitly state the positive answer to Question I in the case of surfaces:

Theorem 3.3 [Paternain et al. 2011a]. *Assume M is a compact simple surface and let A and B be two unitary connections. Then $C_A = C_B$ implies that there exists a smooth $U : M \to U(n)$ such that*

$$U|_{\partial M} = \mathrm{Id} \quad \textit{and} \quad B = U^{-1}dU + U^{-1}AU.$$

We will provide a sketch of the proof of Theorem 3.2 in the next section, but first we survey some prior results on this topic.

In the case of Euclidean space with the Euclidean metric the attenuated ray transform is the basis of the medical imaging technology of SPECT and has been extensively studied; see [Finch 2003] for a review. We remark that in connection with injectivity results for ray transforms, there is great interest in reconstruction procedures and inversion formulas. For the attenuated ray transform in $\mathbb{R}^2$ with Euclidean metric and scalar attenuation function, an explicit inversion formula was proved by R. Novikov [2002a]. A related formula also including 1-form attenuations appears in [Boman and Strömberg 2004], inversion formulas for matrix attenuations in Euclidean space are given in [Eskin 2004; Novikov 2002b], and the case of hyperbolic space $\mathbb{H}^2$ is considered in [Bal 2005].

In our general geometric setting an essential contribution is made in the paper [Salo and Uhlmann 2011] in which it is was shown that the attenuated ray transform is injective in the scalar case with $a \in C^\infty(M, \mathbb{C})$ for simple two dimensional manifolds. This paper also contains the proof of existence of holomorphic integrating factors of a for arbitrary simple surfaces; a result that extends to the case when a is a 1-form and that will be crucial in the proof of Theorem 3.2.

Various versions of Theorem 3.3 have been proved in the literature. Sharafutdinov [2000] proves the theorem assuming that the connections are C^1 close to another connection with small curvature (but in any dimension). In the case of domains in the Euclidean plane the theorem was proved by Finch and Uhlmann [2001] assuming that the connections have small curvature and by G. Eskin [2004] in general. R. Novikov [2002b] considers the case of connections which are not compactly supported (but with suitable decay conditions at infinity) and establishes local uniqueness of the trivial connection and gives examples in which global uniqueness fails (existence of "ghosts"). His examples are based on a remarkable connection between the Bogomolny equation in Minkowski $(2+1)$-space and the scattering data associated with the transport equation considered above. As explained in [Ward 1988] (see also [Dunajski 2010, Section 8.2.1]), certain soliton solutions A have the property that when restricted to space-like planes the scattering data is trivial. In this way one obtains connections in $\mathbb{R}^2$ with the property of having trivial scattering data but which are not gauge equivalent to the trivial connection. Of course these pairs are not compactly supported in $\mathbb{R}^2$ but they have a suitable decay at infinity. Motivated by this L. Mason obtained a full classification of $U(n)$ transparent connections for the round metric on S^2 (unpublished) using methods from twistor theory as in [Mason 2006].

4. Sketch of proof of Theorem 3.2

Let (M, g) be a compact oriented two dimensional Riemannian manifold with smooth boundary ∂M. As before SM will denote the unit circle bundle which is a compact 3-manifold with boundary given by $\partial(SM) = \{(x, v) \in SM : x \in \partial M\}$. Since M is assumed oriented there is a circle action on the fibers of SM with infinitesimal generator V called the *vertical vector field*. It is possible to complete the pair X, V to a global frame of $T(SM)$ by considering the vector field $X_\perp := [X, V]$. There are two additional structure equations given by $X = [V, X_\perp]$ and $[X, X_\perp] = -KV$ where K is the Gaussian curvature of the surface. Using this frame we can define a Riemannian metric on SM by declaring $\{X, X_\perp, V\}$ to be an orthonormal basis and the volume form of this metric will be denoted by $d\Sigma^3$. The fact that $\{X, X_\perp, V\}$ are orthonormal together with the commutator formulas implies that the Lie derivative of $d\Sigma^3$ along the three vector fields vanishes.

Given functions $u, v : SM \to \mathbb{C}^n$ we consider the inner product

$$(u, v) = \int_{SM} \langle u, v\rangle_{\mathbb{C}^n} \, d\Sigma^3.$$

Since $X, X_\perp, V$ are volume preserving we have $(Vu, v) = -(u, Vv)$ for $u, v \in C^\infty(SM, \mathbb{C}^n)$, and if additionally $u|_{\partial(SM)} = 0$ or $v|_{\partial(SM)} = 0$ then also $(Xu, v) = -(u, Xv)$ and $(X_\perp u, v) = -(u, X_\perp v)$.

The space $L^2(SM, \mathbb{C}^n)$ decomposes orthogonally as a direct sum

$$L^2(SM, \mathbb{C}^n) = \bigoplus_{k\in\mathbb{Z}} H_k$$

where H_k is the eigenspace of $-iV$ corresponding to the eigenvalue k. A function $u \in L^2(SM, \mathbb{C}^n)$ has a Fourier series expansion

$$u = \sum_{k=-\infty}^{\infty} u_k,$$

where $u_k \in H_k$. Let $\Omega_k = C^\infty(SM, \mathbb{C}^n) \cap H_k$.

An important ingredient is the fiberwise *Hilbert transform* $\mathcal{H}$. This can be introduced in various ways (see [Pestov and Uhlmann 2005; Salo and Uhlmann 2011]), but perhaps the most informative approach is to indicate that it acts fiberwise and for $u_k \in \Omega_k$,

$$\mathcal{H}(u_k) = -\operatorname{sgn}(k)\, i u_k$$

where we use the convention $\operatorname{sgn}(0)=0$. Moreover, $\mathcal{H}(u)=\sum_k \mathcal{H}(u_k)$. Observe that

$$(\mathrm{Id}+i\mathcal{H})u = u_0 + 2\sum_{k=1}^{\infty} u_k \quad \text{and} \quad (\mathrm{Id}-i\mathcal{H})u = u_0 + 2\sum_{k=-\infty}^{-1} u_k.$$

Definition 4.1. A function $u : SM \to \mathbb{C}^n$ is said to be holomorphic if

$$(\mathrm{Id}-i\mathcal{H})u = u_0.$$

Equivalently, u is holomorphic if $u_k = 0$ for all $k < 0$. Similarly, u is said to be antiholomorphic if $(\mathrm{Id}+i\mathcal{H})u = u_0$ which is equivalent to saying that $u_k = 0$ for all $k > 0$.

As in previous works the following commutator formula from [Pestov and Uhlmann 2005] will come into play:

$$[\mathcal{H}, X]u = X_\perp u_0 + (X_\perp u)_0, \quad u \in C^\infty(SM, \mathbb{C}^n). \tag{7}$$

We will give a proof of this formula in Lemma 6.7.

It is easy to extend this bracket relation so that it includes a connection A. We often think of A as a function restricted to SM. We also think of A as acting on smooth functions $u \in C^\infty(SM, \mathbb{C}^n)$ by multiplication. Note that $V(A)$ is a new function on SM which can be identified with the restriction of $-\star A$ to SM, so we will simply write $V(A) = -\star A$. Here $\star$ denotes the Hodge star operator of the metric g.

Lemma 4.2. *For any smooth function u we have*

$$[\mathcal{H}, X + A]u = (X_\perp + \star A)(u_0) + \{(X_\perp + \star A)(u)\}_0.$$

The proof makes use of this regularity result:

Proposition 4.3 [Paternain et al. 2011a, Proposition 5.2]. *Let $f : SM \to \mathbb{C}^n$ be smooth with $I_A(f) = 0$. Then $u^f : SM \to \mathbb{C}^n$ is smooth.*

The next proposition will provide the holomorphic integrating factors in the scalar case.

Proposition 4.4 [Paternain et al. 2011a, Theorem 4.1]. *Let (M, g) be a simple two-dimensional manifold and $f \in C^\infty(SM, \mathbb{C})$. The following conditions are equivalent.*

(a) *There exist a holomorphic $w \in C^\infty(SM, \mathbb{C})$ and an antiholomorphic $\tilde{w} \in C^\infty(SM, \mathbb{C})$ such that $Xw = X\tilde{w} = -f$.*

(b) *$f(x, v) = F(x) + \alpha_j(x)v^j$ where F is a smooth function on M and α is a 1-form.*

The existence of holomorphic and antiholomorphic solutions for the case $\alpha = 0$ was first proved in [Salo and Uhlmann 2011], but here we will need the case in which $F = 0$, but α is nonzero.

Another key ingredient is an energy identity or a "Pestov type identity", which generalizes the standard Pestov identity [Sharafutdinov 1994] to the case where a connection is present. There are several predecessors for this formula [Vertgeĭm 1991; Sharafutdinov 2000] and its use for simple surfaces is in the spirit of [Sharafutdinov and Uhlmann 2000; Dairbekov and Paternain 2007]. Recall that the curvature F_A of the connection A is defined as $F_A = dA + A \wedge A$ and $\star F_A$ is a function $\star F_A : M \to \mathfrak{u}(n)$.

Lemma 4.5 (energy identity). *If $u : SM \to \mathbb{C}^n$ is a smooth function such that $u|_{\partial(SM)} = 0$, then*

$$\|(X+A)Vu\|^2 - (K\,Vu, Vu) - (\star F_A u, Vu) = \|V(X+A)(u)\|^2 - \|(X+A)u\|^2.$$

Remark 4.6. The same energy identity holds true for *closed* surfaces.

To use the energy identity we need to control the signs of various terms. The first easy observation is the following:

Lemma 4.7. *Assume $(X + A)u = F(x) + \alpha_j(x)v^j$, where $F : M \to \mathbb{C}^n$ is a smooth function and α is a $\mathbb{C}^n$-valued 1-form. Then*

$$\|V(X + A)u\|^2 - \|(X + A)u\|^2 = -\|F\|^2 \le 0.$$

Proof. It suffices to note the identities

$$\|V(X + A)u\|^2 = \|V\alpha\|^2 = \|\alpha\|^2 \quad \text{and} \quad \|F + \alpha\|^2 = \|\alpha\|^2 + \|F\|^2. \quad \square$$

Next we have the following lemma due to the absence of conjugate points on simple surfaces (compare with [Dairbekov and Paternain 2007, Theorem 4.4]):

Lemma 4.8. *Let M be a compact simple surface. If $u : SM \to \mathbb{C}^n$ is a smooth function such that $u|_{\partial(SM)} = 0$, then*

$$\|(X + A)Vu\|^2 - (K\,Vu, Vu) \ge 0.$$

Proof. Consider a smooth function $a : SM \to \mathbb{R}$ which solves the Riccati equation $X(a) + a^2 + K = 0$. These exist by the absence of conjugate points (see for example [Sharafutdinov 1999, Theorem 6.2.1] or the proof of Lemma 4.1 in [Sharafutdinov and Uhlmann 2000]). Set for simplicity $\psi = V(u)$. Clearly $\psi|_{\partial(SM)} = 0$.

Using that A is skew-Hermitian, we compute

$$\begin{aligned}|(X + A)(\psi) - a\psi|^2_{\mathbb{C}^n} &= |(X + A)(\psi)|^2_{\mathbb{C}^n} - 2\Re\langle (X + A)(\psi), a\psi\rangle_{\mathbb{C}^n} + a^2|\psi|^2_{\mathbb{C}^n}\\ &= |(X + A)(\psi)|^2_{\mathbb{C}^n} - 2a\Re\langle X(\psi), \psi\rangle_{\mathbb{C}^n} + a^2|\psi|^2_{\mathbb{C}^n}.\end{aligned}$$

Using the Riccati equation we have

$$X(a|\psi|^2) = (-a^2 - K)|\psi|^2 + 2a\Re\langle X(\psi), \psi\rangle_{\mathbb{C}^n};$$

thus

$$|(X+A)(\psi) - a\psi|^2_{\mathbb{C}^n} = |(X+A)(\psi)|^2_{\mathbb{C}^n} - K|\psi|^2_{\mathbb{C}^n} - X(a|\psi|^2_{\mathbb{C}^n}).$$

Integrating this equality with respect to $d\Sigma^3$ and using that ψ vanishes on $\partial(SM)$ we obtain

$$\|(X+A)(\psi)\|^2 - (K\psi, \psi) = \|(X+A)(\psi) - a\psi\|^2 \geq 0. \qquad \square$$

Theorem 4.9. *Let $f : SM \to \mathbb{C}^n$ be a smooth function. Suppose $u : SM \to \mathbb{C}^n$ satisfies*

$$\begin{cases} Xu + Au = -f, \\ u|_{\partial(SM)} = 0. \end{cases}$$

Then if $f_k = 0$ for all $k \leq -2$ and $i\star F_A(x)$ is a negative definite Hermitian matrix for all $x \in M$, the function u must be holomorphic. Moreover, if $f_k = 0$ for all $k \geq 2$ and $i\star F_A(x)$ is a positive definite Hermitian matrix for all $x \in M$, the function u must be antiholomorphic.

Proof. Let us assume that $f_k = 0$ for $k \leq -2$ and $i\star F_A$ is a negative definite Hermitian matrix; the proof of the other claim is similar.

We need to show that $(\mathrm{Id} - i\mathcal{H})u$ only depends on x. We apply $X + A$ to it and use Lemma 4.2 together with $(\mathrm{Id} - i\mathcal{H})f = f_0 + 2f_{-1}$ to derive

$$\begin{aligned}
(X+A)&[(\mathrm{Id} - i\mathcal{H})u] \\
&= -f - i(X+A)(\mathcal{H}u) \\
&= -f - i(\mathcal{H}((X+A)(u)) - (X_\perp + \star A)(u_0) - \{(X_\perp + \star A)(u)\}_0) \\
&= -(\mathrm{Id} - i\mathcal{H})(f) + i(X_\perp + \star A)(u_0) + i\{(X_\perp + \star A)(u)\}_0 \\
&= -f_0 - 2f_{-1} + i(X_\perp + \star A)(u_0) + i\{(X_\perp + \star A)(u)\}_0 \\
&= F(x) + \alpha_x(v),
\end{aligned}$$

where $F : M \to \mathbb{C}^n$ and α is a $\mathbb{C}^n$-valued 1-form. Now we are in good shape to use the energy identity from Lemma 4.5. We will apply it to

$$v = (\mathrm{Id} - i\mathcal{H})u = u_0 + 2\sum_{k=-\infty}^{-1} u_k.$$

We know from Lemma 4.7 that its right-hand side is ≤ 0 and using Lemma 4.8 we deduce

$$(\star F_A v, Vv) \geq 0.$$

On the other hand,

$$(\star F_A v, Vv) = -4 \sum_{k=-\infty}^{-1} k(i \star F_A u_k, u_k)$$

and since $i\star F_A$ is negative definite this forces $u_k = 0$ for all $k < 0$. □

Outline of proof of Theorem 3.2. Consider the area form ω_g of the metric g. Since M is a disk there exists a smooth 1-form φ such that $\omega_g = d\varphi$. Given $s \in \mathbb{R}$, consider the Hermitian connection

$$A_s := A - is\varphi \,\mathrm{Id}.$$

Clearly its curvature is given by

$$F_{A_s} = F_A - is\omega_g \mathrm{Id};$$

therefore

$$i\star F_{A_s} = i\star F_A + s\mathrm{Id},$$

from which we see that there exists $s_0 > 0$ such that for $s > s_0$, $i\star F_{A_s}$ is positive definite and for $s < -s_0$, $i\star F_{A_s}$ is negative definite.

Since $I_A(f) = 0$, Proposition 4.3 implies that there is a *smooth* $u : SM \to \mathbb{C}^n$ such that $(X + A)(u) = -f$ and $u|_{\partial(SM)} = 0$ (to abbreviate the notation we write u instead of u^f).

Let e^{sw} be an integrating factor of $-is\varphi$. In other words $w : SM \to \mathbb{C}$ satisfies $X(w) = i\varphi$. By Proposition 4.4 we know we can choose w to be holomorphic or antiholomorphic. Observe now that $u_s := e^{sw}u$ satisfies $u_s|_{\partial(SM)} = 0$ and solves

$$(X + A_s)(u_s) = -e^{sw} f.$$

Choose w to be holomorphic. Since $f = F(x) + \alpha_j(x)v^j$, the function $e^{sw}f$ has the property that its Fourier coefficients $(e^{sw}f)_k$ vanish for $k \le -2$. Choose s such that $s < -s_0$ so that $i\star F_{A_s}$ is negative definite. Then Theorem 4.9 implies that u_s is holomorphic and thus $u = e^{-sw}u_s$ is also holomorphic.

Choosing w antiholomorphic and $s > s_0$ we show similarly that u is antiholomorphic. This implies that $u = u_0$ which together with $(X + A)u = -f$, gives $d_A u_0 = -f$. If we set $p = -u_0$ we see right away that $F \equiv 0$ and $\alpha = d_A p$ as desired. □

5. Applications to tensor tomography

In this section we explain how the ideas of the previous section can be used to tackle a well-known inverse problem which is a priori unrelated with unitary connections.

We consider the geodesic ray transform acting on symmetric m-tensor fields on M. When the metric is Euclidean and $m = 0$ this transform reduces to the usual X-ray transform obtained by integrating functions along straight lines. More generally, given a symmetric (covariant) m-tensor field

$$f = f_{i_1 \cdots i_m}\, dx^{i_1} \otimes \cdots \otimes dx^{i_m}$$

on M, we define the corresponding function on SM by

$$f(x, v) = f_{i_1 \cdots i_m} v^{i_1} \cdots v^{i_m}.$$

The ray transform of f is defined by

$$If(x, v) = \int_0^{\tau(x,v)} f(\phi_t(x, v))\, dt, \quad (x, v) \in \partial_+(SM),$$

where ϕ_t denotes the geodesic flow of the Riemannian metric g. If h is a symmetric $(m-1)$-tensor field, its inner derivative dh is a symmetric m-tensor field defined by $dh = \sigma \nabla h$, where σ denotes symmetrization and ∇ is the Levi-Civita connection. A direct calculation in local coordinates shows that

$$dh(x, v) = Xh(x, v),$$

where X as before is the geodesic vector field associated with ϕ_t. If additionally $h|_{\partial M} = 0$, then one clearly has $I(dh) = 0$. The ray transform on symmetric m-tensors is said to be s-injective if these are the only elements in the kernel. The terminology arises from the fact that any tensor field f may be written uniquely as $f = f^s + dh$, where f^s is a symmetric m-tensor with zero divergence and h is an $(m-1)$-tensor with $h|_{\partial M} = 0$ (see [Sharafutdinov 1994]). The tensor fields f^s and dh are called respectively the *solenoidal* and *potential* parts of the tensor f. Saying that I is s-injective is saying precisely that I is injective on the set of solenoidal tensors.

The next result shows that the ray transform on simple surfaces is s-injective for tensors of any rank. This settles a long standing question in the two-dimensional case [Pestov and Sharafutdinov 1988; Sharafutdinov 1994, Problem 1.1.2].

Theorem 5.1 [Paternain et al. 2011b]. *Let (M, g) be a simple surface and let $m \geq 0$. If f is a smooth symmetric m-tensor field on M which satisfies $If = 0$, then $f = dh$ for some smooth symmetric $(m-1)$-tensor field h on M with $h|_{\partial M} = 0$. (If $m = 0$, then $f = 0$.)*

It is not the objective of this article to discuss the vast literature on the tensor tomography problem for simple manifolds. Instead we refer the reader to [Sharafutdinov 1994] and to the references in [Paternain et al. 2011b] and we limit ourselves to supplying a proof of Theorem 5.1 based on the ideas of the previous section. The proof reduces to proving the next result. We say that $f \in C^\infty(SM, \mathbb{C})$ has degree m if $f_k = 0$ for $|k| \geq m+1$ and $m \geq 0$ is the smallest nonnegative integer with that property.

Proposition 5.2. *Let (M, g) be a simple surface. Assume that $u \in C^\infty(SM, \mathbb{C})$ satisfies $Xu = -f$ in SM with $u|_{\partial(SM)} = 0$. If $f \in C^\infty(SM, \mathbb{C})$ has degree $m \geq 1$, then u has degree $m-1$. If f has degree 0, then $u = 0$.*

Proof of Theorem 5.1. Let f be a symmetric m-tensor field on SM and suppose that $If = 0$. We write

$$u(x, v) := \int_0^{\tau(x,v)} f(\phi_t(x, v)), \quad (x, v) \in SM.$$

Then $u|_{\partial(SM)} = 0$, and also $u \in C^\infty(SM)$ by Proposition 4.3.

Now f has degree m, and u satisfies $Xu = -f$ in SM with $u|_{\partial(SM)} = 0$. Proposition 5.2 implies that u has degree $m-1$ (and $u = 0$ if $m = 0$). We let $h := -u$. It is not hard to see that h gives rise to a symmetric $(m-1)$-tensor still denoted by h. Since $X(h) = f$, this implies that dh and f agree when restricted to SM and thus $dh = f$. This proves the theorem. □

Proposition 5.2 is in turn an immediate consequence of the next two results.

Proposition 5.3. *Let (M, g) be a simple surface. Assume that $u \in C^\infty(SM, \mathbb{C})$ satisfies $Xu = -f$ in SM with $u|_{\partial(SM)} = 0$. If $m \geq 0$ and if $f \in C^\infty(SM, \mathbb{C})$ is such that $f_k = 0$ for $k \leq -m-1$, then $u_k = 0$ for $k \leq -m$.*

Proposition 5.4. *Let (M, g) be a simple surface. Assume that $u \in C^\infty(SM, \mathbb{C})$ satisfies $Xu = -f$ in SM with $u|_{\partial(SM)} = 0$. If $m \geq 0$ and if $f \in C^\infty(SM, \mathbb{C})$ is such that $f_k = 0$ for $k \geq m+1$, then $u_k = 0$ for $k \geq m$.*

We will only prove Proposition 5.3, the proof of the other result being completely analogous. We shall need the following result from [Salo and Uhlmann 2011, Proposition 5.1]:

Proposition 5.5. *Let (M, g) be a simple surface and let f be a smooth holomorphic (antiholomorphic) function on SM. Suppose $u \in C^\infty(SM, \mathbb{C})$ satisfies*

$$Xu = -f \text{ in } SM, \quad u|_{\partial(SM)} = 0.$$

Then u is holomorphic (antiholomorphic) and $u_0 = 0$.

Proof of Proposition 5.3. Suppose that u is a smooth solution of $Xu = -f$ in SM where $f_k = 0$ for $k \leq -m-1$ and $u|_{\partial(SM)} = 0$. We choose a nonvanishing function $r \in \Omega_m$ and define the 1-form

$$A := -r^{-1} Xr.$$

Then ru solves the problem

$$(X+A)(ru) = -rf \text{ in } SM, \quad ru|_{\partial(SM)} = 0.$$

Note that rf is a holomorphic function. Next we employ a holomorphic integrating factor: by Proposition 4.4 there exists a holomorphic $w \in C^\infty(SM, \mathbb{C})$ with $Xw = A$. The function $e^w ru$ then satisfies

$$X(e^w ru) = -e^w rf \text{ in } SM, \quad e^w ru|_{\partial(SM)} = 0.$$

The right-hand side, $e^w rf$, is holomorphic. Now Proposition 5.5 implies that the solution $e^w ru$ is also holomorphic and $(e^w ru)_0 = 0$. Looking at Fourier coefficients shows that $(ru)_k = 0$ for $k \leq 0$, and therefore $u_k = 0$ for $k \leq -m$ as required. □

Finally, let us explain the choice of r and A in the proof in more detail. Since M is a disk we can consider global isothermal coordinates (x, y) on M such that the metric can be written as $ds^2 = e^{2\lambda}(dx^2 + dy^2)$ where λ is a smooth real-valued function of (x, y). This gives coordinates (x, y, θ) on SM where θ is the angle between a unit vector v and $\partial/\partial x$. Then Ω_m consists of all functions $a(x, y)e^{im\theta}$ where $a \in C^\infty(M, \mathbb{C})$. We choose the specific nonvanishing function

$$r(x, y, \theta) := e^{im\theta}.$$

In the (x, y, θ) coordinates the geodesic vector field X is given by:

$$X = e^{-\lambda}\left(\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} + \left(-\frac{\partial\lambda}{\partial x}\sin\theta + \frac{\partial\lambda}{\partial y}\cos\theta\right)\frac{\partial}{\partial\theta}\right). \tag{8}$$

The connection $A = -Xr/r$ has the form

$$A = ime^{-\lambda}\left(-\frac{\partial\lambda}{\partial y}\cos\theta + \frac{\partial\lambda}{\partial x}\sin\theta\right) = im\left(-\frac{\partial\lambda}{\partial y}dx + \frac{\partial\lambda}{\partial x}dy\right).$$

Here as usual we identify A with $A(x, v)$ where $(x, v) \in SM$. This shows that the connection A is essentially the Levi-Civita connection of the metric g on the tensor power bundle $TM^{\otimes m}$, and since $(X+A)r = 0$ we have that r corresponds to a section of the pull-back bundle $\pi^*(TM^{\otimes m})$ whose covariant derivative along the geodesic vector field vanishes (here $\pi : SM \to M$ is the standard projection).

A second proof of Proposition 5.4 in the same spirit may be found in [Paternain et al. 2011b].

6. Closed manifolds

In this section we will discuss Question II, but before embarking into that we need some preliminary discussion on cocycles with values in a Lie group over a flow ϕ_t.

Let N be a closed manifold and $\phi_t : N \to N$ a smooth flow with infinitesimal generator X. Let G be a compact Lie group; for our purposes it is enough to think of G as a compact matrix group like $U(n)$.

Definition 6.1. A G-valued cocycle over the flow ϕ_t is a map $C : N \times \mathbb{R} \to G$ that satisfies

$$C(x, t+s) = C(\phi_t x, s)\, C(x, t)$$

for all $x \in N$ and $s, t \in \mathbb{R}$.

In this paper the cocycles will always be smooth. In this case C is determined by its infinitesimal generator $B : N \to \mathfrak{g}$ given by

$$B(x) := -\left.\frac{d}{dt}\right|_{t=0} C(x, t).$$

The cocycle can be recovered from B as the unique solution to

$$\frac{d}{dt} C(x,t) = -dR_{C(x,t)}(B(\phi_t x)), \quad C(x, 0) = \mathrm{Id},$$

where R_g is right translation by $g \in G$. We will indistinctly use the word "cocycle" for C or its infinitesimal generator B.

Definition 6.2. The cocycle C is said to be *cohomologically trivial* if there exists a smooth function $u : N \to G$ such that

$$C(x,t) = u(\phi_t x) u(x)^{-1}$$

for all $x \in N$ and $t \in \mathbb{R}$.

Observe that the condition of being cohomologically trivial can be equivalently expressed in terms of the infinitesimal generator B of the cocycle by saying that there exists a smooth function $u : N \to G$ that satisfies the equation

$$d_x u(X(x)) + d_{Id} R_{u(x)}(B(x)) = 0$$

for all $x \in N$. If G is a matrix group we can write this more succinctly as

$$Xu + Bu = 0$$

where it is understood that differentiation and multiplication is in the set of matrices.

Definition 6.3. A cocycle C is said to satisfy the periodic orbit obstruction condition if $C(x,T) = \mathrm{Id}$ whenever $\phi_T x = x$.

Obviously a cohomologically trivial cocycle satisfies the periodic orbit obstruction condition. The converse turns out to be true for transitive Anosov flows: this is one of the celebrated Livšic theorems [Livšic 1971; 1972; Nițică and Török 1998].

Theorem 6.4 (smooth Livšic periodic data theorem). *Suppose ϕ_t is a smooth transitive Anosov flow. Let C be a smooth cocycle such that $C(x,T) = \mathrm{Id}$ whenever $\phi_T x = x$. Then C is cohomologically trivial.*

Given two G-valued cocycles C_1 and C_2 we shall say that they are *cohomologous* (or X-cohomologous) if there is a smooth function $u : N \to G$ such that

$$C_1(x,t) = u(\phi_t x) C_2(x,t) u(x)^{-1}$$

for all $x \in N$ and $t \in \mathbb{R}$. Clearly if C_1 and C_2 are cohomologous, $C_1(x,T) = u(x)C_2(x,T)u(x)^{-1}$, whenever $\phi_T x = x$. An extension of the Livšic theorem due to W. Parry [1999] together with the regularity result from [Nițică and Török 1998] gives the following extension of Theorem 6.4:

Theorem 6.5 (smooth Livšic periodic data theorem for two cocycles). *Suppose ϕ_t is a smooth transitive Anosov flow. Let C_1 and C_2 be two smooth cocycles such that there is a Hölder continuous function $u : N \to G$ for which $C_1(x,T) = u(x)C_2(x,T)u(x)^{-1}$ whenever $\phi_T x = x$. Then C and D are cohomologous.*

Observe that if G is a matrix group then two cocycles C_1 and C_2 are cohomologous if and only if their infinitesimal generators B_1 and B_2 are related by a smooth function $u : N \to G$ such that

$$Xu + B_1 u - u B_2 = 0$$

or equivalently

$$B_2 = u^{-1} X u + u^{-1} B_1 u.$$

Note the formal similarity of this equation with the one that defines gauge equivalent connections. One could take the viewpoint that the main question raised in this paper is to decide when it is possible to go from cohomology defined by the operator X to cohomology defined by d in the geometric situation when X is the geodesic vector field. Let us be a bit more precise about this.

Let (M,g) be a closed Riemannian manifold with unit tangent bundle SM and projection $\pi : SM \to M$. The geodesic flow ϕ_t acts on SM with infinitesimal generator X.

Consider the trivial bundle $M \times \mathbb{C}^n$ and let $\mathcal{A}$ stand for the set of all unitary connections. Given $A \in \mathcal{A}$, we have a pull-back connection $\pi^* A$ on the bundle $SM \times \mathbb{C}^n$ and we denote by $\pi^*\mathcal{A}$ the set of all such connections.

Each connection A gives rise to a cocycle over the geodesic flow whose generator is $\pi^*A(X) : SM \to \mathfrak{u}(n)$. Note that $\pi^*A(X)(x,v) = A(x,v)$, in words, $\pi^*A(X)$ is the restriction of $A : TM \to \mathfrak{u}(n)$ to SM.

The cocycle C associated with this generator is nothing but parallel transport along geodesics, so that $C : SM \times \mathbb{R} \to U(n)$ solves

$$\frac{d}{dt}C(x,v,t) + A(\phi_t(x,v))C(x,v,t) = 0, \quad C(x,v,0) = \mathrm{Id}\,.$$

On the set $\pi^*\mathcal{A}$ we impose the equivalence relation $\sim X$ of being X-cohomologous and on $\mathcal{A}$ we have the equivalence relation $\sim$ given by gauge equivalence. There is a natural map induced by π:

$$\mathcal{A}/\!\sim\ \mapsto \pi^*\mathcal{A}/\!\sim X. \tag{9}$$

Suppose now we have two connections A_1 and A_2 as in Question II and the geodesic flow is Anosov. Then Theorem 6.5 implies that $\pi^*A_1(X)$ and $\pi^*A_2(X)$ are cohomologous cocycles, that is, there is a smooth map $u : SM \to U(n)$ such that on SM we have the *cohomological equation*

$$A_2 = u^{-1}Xu + u^{-1}A_1 u. \tag{10}$$

This is the main dynamical input, that allows the passage from closed geodesics to X-cohomology. What is left is the geometric problem of deciding if the map in (9) is injective. Suppose for a moment that for some reason we can show that $u(x,v)$ only depends on x. Then (10) means exactly that A_1 and A_2 are gauge equivalent since $Xu = du$. Thus understanding the dependence of u in the velocities is crucial and this often can be achieved using Pestov type identities and/or Fourier analysis as in the sketch of proof of Theorem 3.2 before. Let us see a good example of this in the simplest possible case in which $n = 1$. Since $U(1) = S^1$ is abelian we can reduce (10) to the cohomologically trivial case

$$Xu + Au = 0$$

where $A = A_1 - A_2$ and $u : SM \to S^1$. Write $A = i\theta$, where θ is an ordinary real-valued 1-form. Then

$$du(X) + i\theta u = 0. \tag{11}$$

The function u gives rise to a real-valued closed 1-form in SM given by $\varphi := \frac{du}{iu}$. Since $\pi^* : H^1(M,\mathbb{R}) \to H^1(SM,\mathbb{R})$ is an isomorphism when M is different from the 2-torus, there exists a closed 1-form ω in M and a smooth function $f : SM \to \mathbb{R}$ such that

$$\varphi = \pi^*\omega + df.$$

(It is easy to see that if ϕ_t is Anosov, then M cannot be a 2-torus since for example, $\pi_1(M)$ must grow exponentially.) When this equality is applied to X and combined with (11) one obtains

$$-\theta_x(v) - \omega_x(v) = df(X(x,v)) = X(f)(x,v)$$

for all $(x,v) \in SM$. It is known that this implies that $\theta + \omega$ is exact and that f only depends on x. This was proved by V. Guillemin and D. Kazhdan [1980a] for surfaces of negative curvature, by C. Croke and Sharafutdinov [1998] for arbitrary manifolds of negative curvature and by N.S. Dairbekov and Sharafutdinov [2003] for manifolds whose geodesic flow is Anosov. It follows easily now that u only depends on x and hence A_1 and A_2 must be gauge equivalent and thus for $n = 1$ we have a full answer. Before going further let us explain why if we have a smooth solution u to the cohomological equation

$$Xu = \theta$$

where θ is a 1-form, then θ is exact and u only depends on x. We can see this for $\dim M = 2$ using the energy identity from Lemma 4.5 for the case $A = 0$. Since θ is a 1-form, the right-hand side is zero as in Lemma 4.7; thus

$$\|XVu\|^2 - (KVu, Vu) = 0. \tag{12}$$

If the flow is Anosov there are two solutions $r^{s,u}$ of the Riccati equation $X(r) + r^2 + K = 0$. These solutions are related to the stable and unstable bundles as follows: $-X_\perp + r^{s,u}V \in E^{s,u}$ and $r^s - r^u$ never vanishes; for an account of these results we refer to [Paternain 1999]. Hence using the proof of Lemma 4.8 and (12) we deduce:

$$XVu - r^{s,u}Vu = 0$$

from which it follows that $(r^s - r^u)Vu = 0$ and thus $Vu = 0$. This shows that u only depends on x and therefore θ is exact.

Now that we have a better understanding of the abelian case $n = 1$, let us go back to the general equation (10). As in Section 3 we can introduce a new unitary connection $\hat{A}$ on the trivial bundle $M \times \mathbb{C}^{n\times n}$ as follows: given a matrix $R \in \mathbb{C}^{n\times n}$ we define $\hat{A}(R) := AR - RB$. Then (10) is the form

$$Xu + Au = 0$$

at the price of course, of increasing the rank of our trivial vector bundle. Note that $F_{\hat{A}}(R) = F_A R - RF_B$.

This suggests that in general we should study the following problem on closed manifolds. Given a unitary connection A on $M \times \mathbb{C}^n$ and $f : SM \to \mathbb{C}^n$ a smooth function of the form $F(x) + \alpha_j(x)v^j$, where $F : M \to \mathbb{C}^n$ is a smooth function and α is a $\mathbb{C}^n$-valued 1-form, describe the set of smooth solutions $u : SM \to \mathbb{C}^n$ to the equation

$$Xu + Au = -f. \tag{13}$$

Unfortunately we know very little about (13) in the general Anosov case. However for closed surfaces of negative curvature we have the following fundamental result which should be regarded as an extension of [Guillemin and Kazhdan 1980a, Theorem 3.6].

The Fourier analysis that we set up in Section 4 works equally well in the case of closed oriented surfaces. Given $u \in C^\infty(SM, \mathbb{C}^n)$, we write $u = \sum_{m \in \mathbb{Z}} u_m$, where $u_m \in \Omega_m$. We will say that u has degree N, if N is the smallest nonnegative integer such that $u_m = 0$ for all m with $|m| \geq N + 1$.

Theorem 6.6 [Paternain 2009, Theorem 5.1]. *If M is a closed surface of negative curvature and $f : SM \to \mathbb{C}^n$ has finite degree, then any smooth solution u of $Xu + Au = -f$ has finite degree.*

Below we shall sketch the proof of this theorem, but first we need some preliminaries. As in [Guillemin and Kazhdan 1980a] we introduce the first-order elliptic operators

$$\eta_+, \eta_- : C^\infty(SM, \mathbb{C}^n) \to C^\infty(SM, \mathbb{C}^n)$$

given by

$$\eta_+ := (X + iX_\perp)/2, \quad \eta_- := (X - iX_\perp)/2.$$

Clearly $X = \eta_+ + \eta_-$. We have

$$\eta_+ : \Omega_m \to \Omega_{m+1}, \quad \eta_- : \Omega_m \to \Omega_{m-1}, \quad (\eta_+)^* = -\eta_-.$$

Before going further, let us use these operators to give a short proof of the bracket relation (7):

Lemma 6.7. *The following formula holds*:

$$[\mathcal{H}, X]u = X_\perp u_0 + (X_\perp u)_0, \quad u \in C^\infty(SM, \mathbb{C}^n).$$

Proof. It suffices to show that

$$[\mathrm{Id} + i\mathcal{H}, X]u = iX_\perp u_0 + i(X_\perp u)_0.$$

Since $X=\eta_+ + \eta_-$ we need to compute $[\mathrm{Id}+i\mathcal{H}, \eta_\pm]$, so let us find $[\mathrm{Id}+i\mathcal{H}, \eta_+]u$, where $u=\sum_k u_k$. Recall that $(\mathrm{Id}+i\mathcal{H})u = u_0 + 2\sum_{k\geq 1} u_k$. We find:

$$(\mathrm{Id}+i\mathcal{H})\eta_+ u = \eta_+ u_{-1} + 2\sum_{k\geq 0} \eta_+ u_k,$$
$$\eta_+(\mathrm{Id}+i\mathcal{H})u = \eta_+ u_0 + 2\sum_{k\geq 1} \eta_+ u_k.$$

Thus

$$[\mathrm{Id}+i\mathcal{H}, \eta_+]u = \eta_+ u_{-1} + \eta_+ u_0.$$

Similarly we find

$$[\mathrm{Id}+i\mathcal{H}, \eta_-]u = -\eta_- u_0 - \eta_- u_1.$$

Therefore using that $iX_\perp = \eta_+ - \eta_-$ we obtain

$$[\mathrm{Id}+i\mathcal{H}, X]u = iX_\perp u_0 + i(X_\perp u)_0$$

as desired. □

To deal with the equation $Xu+Au=-f$, we introduce the "twisted" operators

$$\mu_+ := \eta_+ + A_1, \quad \mu_- := \eta_- + A_{-1},$$

where $A = A_{-1} + A_1$ and

$$A_1 := \frac{A - iV(A)}{2} \in \Omega_1, \quad A_{-1} := \frac{A + iV(A)}{2} \in \Omega_{-1}.$$

This decomposition corresponds precisely with the usual decomposition of $\mathfrak{u}(n)$-valued 1-forms on a surface:

$$\Omega^1(M, \mathfrak{u}(n)) \otimes \mathbb{C} = \Omega^{1,0}(M, \mathfrak{u}(n)) \oplus \Omega^{0,1}(M, \mathfrak{u}(n)),$$

where $\star = -i$ on $\Omega^{1,0}$ and $\star = i$ on $\Omega^{0,1}$ (here $\star$ is the Hodge star operator of the metric).

We also have

$$\mu_+ : \Omega_m \to \Omega_{m+1}, \quad \mu_- : \Omega_m \to \Omega_{m-1}, \quad (\mu_+)^* = -\mu_-.$$

The equation $Xu + Au = -f$ is now $\mu_+(u) + \mu_-(u) = -f$.

Sketch of proof of Theorem 6.6. We shall use the following equality proved in [Paternain 2009, Corollary 4.4]. Given $u \in C^\infty(SM, \mathbb{C}^n)$ we have

$$\|\mu_+ u\|^2 = \|\mu_- u\|^2 + \frac{i}{2}\big((K\,Vu, u) + (\star F_A u, u)\big),$$

where K is the Gaussian curvature of the metric and F_A is the curvature of A. This L^2 identity is a close relative of the identity in Lemma 4.5. For $u_m \in \Omega_m$ we have

$$\|\mu_+ u_m\|^2 = \|\mu_- u_m\|^2 + \frac{1}{2}\big((i\star F_A - mK\ \mathrm{Id})u_m, u_m\big).$$

Hence if $K < 0$, there exist a constant $c > 0$ and a positive integer ℓ such that

$$\|\mu_+ u_m\|^2 \geq \|\mu_- u_m\|^2 + c\|u_m\|^2 \tag{14}$$

for all $m \geq \ell$. Projecting the equation $Xu + Au = -f$ onto Ω_m-components we obtain

$$\mu_+(u_{m-1}) + \mu_-(u_{m+1}) = -f_m \tag{15}$$

for all $m \in \mathbb{Z}$. Since f has finite degree, combining (15) and (14) we obtain

$$\|\mu_+(u_{m+1})\| \geq \|\mu_+(u_{m-1})\| \tag{16}$$

for all m sufficiently large. Since the function u is smooth, $\mu_+(u_m)$ must tend to zero in the L^2-topology as $m \to \infty$. It follows from (16) that $\mu_+(u_m) = 0$ for all m sufficiently large. However, (14) implies that μ_+ is injective for m large enough and thus $u_m = 0$ for all m large enough.

A similar argument shows that $u_m = 0$ for all m sufficiently large and negative thus concluding that u has finite degree as desired. □

A glance at the proof shows that we can obtain the same finiteness result under the following weaker hypothesis: $K \leq 0$ and the support of $\star F_A$ is contained in the region where $K < 0$. A more careful inspection shows the following:

Corollary 6.8. *Suppose that the Hermitian matrix $\pm i\star F_A(x) - K(x)\ \mathrm{Id}$ is positive definite for all $x \in M$ and that f has degree N. Then, any solution u of $Xu + Au = -f$ must have degree $N - 1$. If $N = 0$, then $f = 0$ and $u = u(x)$ with $d_A u = 0$.*

Let us apply these ideas to show that for closed negatively curved surfaces, the map (9) is locally injective at flat connections. If A and B are two connections with sufficiently small curvatures, then $F_{\hat{A}}$ will be small enough so that the hypothesis of Corollary 6.8 is satisfied. Hence the map u solving (10) depends only on $x \in M$ and A and B must be gauge equivalent. Putting everything together we have shown:

Theorem 6.9. *Let M be a closed negatively curved surface. There is $\varepsilon > 0$ such that if A and B are two connections as in Question II with $\|F_A\|_{C^0}, \|F_B\|_{C^0} < \varepsilon$, then A and B are gauge equivalent.*

When A is the trivial connection, this is essentially [Paternain 2009, Theorem A].

Let us see an easy example which shows that the result in the theorem fails for $n = 2$ without assumptions on the smallness of the curvature. The tangent bundle of an orientable Riemannian surface M is naturally a Hermitian line bundle. It is certainly not trivial in general, but it carries the Levi-Civita connection which is easily seen to be transparent. Indeed, the parallel transport along a closed geodesic γ must fix $\dot{\gamma}(0)$ and consequently any vector orthogonal to it since the parallel transport is an isometry and the surface is orientable. The Levi-Civita connection on T^*M is also transparent and thus we obtain a transparent unitary connection on $TM \oplus T^*M$. But $TM \oplus T^*M$ has zero first Chern class and thus it is unitarily equivalent to the trivial bundle $M \times \mathbb{C}^2$. In this way we obtain a transparent connection on $M \times \mathbb{C}^2$ which in general is not equivalent to the trivial connection (it is nonflat if the Gaussian curvature is not identically zero). Taking higher tensor powers of TM and T^*M and adding them we obtain more examples of transparent connections all arising from the Levi-Civita connection. It turns out that these are not the only examples, but the failure of the uniqueness can be fully understood at least in some important cases. This will be the content of the next section, but before that, we would like to take another look at Theorem 6.6 in the abelian case $n = 1$.

When $n = 1$, $A = i\theta$ and the set Ω_m can be identified with the set of smooth sections of $\kappa^{\otimes m}$, where κ is the canonical line bundle of M. In this case, well known results on the theory of Riemann surfaces imply that μ_- is surjective for $m \geq 2$ (see for example [Duistermaat 1972]) since μ_- is essentially a $\bar{\partial}_A$-operator (we are assuming here that M has genus ≥ 2); see (25) below. It follows that μ_+ is injective for $m \geq 1$. Hence if f has degree N and u has finite degree and solves $Xu + i\theta u = -f$, then u must have degree $N - 1$. Thus in the abelian case we have:

Theorem 6.10. *Suppose that M is a closed surface of negative curvature and $n = 1$. If f has degree N, then any solution u of $Xu + i\theta u = -f$ must have degree $N - 1$. If $N = 0$, then $f = 0$ and $u = u(x)$ with $du + i\theta u = 0$.*

When $n \geq 2$, the operators μ_+ could have nontrivial kernels for $m \geq 1$, and this precisely gives room for the existence of transparent connections.

7. Transparent connections

We start with some motivation for the constructions in this section. How can we construct a cohomologically trivial connection on $M \times \mathbb{C}^2$? Let us suppose that we start with the simplest possible nontrivial u. This would be a smooth map $u : SM \to \mathrm{SU}(2)$ such that $u = u_{-1} + u_1$. We would need $A = -X(u)u^{-1}$ to be a connection, thus its Fourier expansion should have only terms of degree ± 1.

Writing $X = \eta_+ + \eta_-$ we discover that A is a connection if and only if

$$\eta_+(u_1)u_{-1}^* = \eta_-(u_{-1})u_1^* = 0.$$

In fact $\eta_-(u_{-1})u_1^* = 0$ implies $\eta_+(u_1)u_{-1}^* = 0$ and vice versa. This can be seen simply by conjugating each relation. So we need to ensure that:

$$\eta_-(u_{-1})u_1^* = 0. \tag{17}$$

What does this mean? Since u is unitary we have $u_{-1}u_1^* = 0$ from which we see that $L := \mathrm{Ker}(u_{-1}) = \mathrm{Im}(u_1^*)$ is a line subbundle of $\mathbb{C}^2$. Now (17) can be rewritten as

$$u_{-1}\eta_-(u_1^*) = 0$$

so if we pick $0 \neq \xi \in \mathbb{C}^2$, then $s := u_1^*\xi \in L$ and the equation above says $\eta_-(s) \in L$. In (22) below we will write an equation for η_- in local coordinates which shows that it is essentially a $\bar{\partial}$-operator and hence (17) is saying that L must be a *holomorphic* line bundle. But there is an ample supply of these: it is equivalent to providing a meromorphic function on M. Now we can ask, given a holomorphic line bundle L can we find a function $u : SM \to \mathrm{SU}(2)$ such that $u = u_{-1} + u_1$ and $\mathrm{Ker}(u_{-1}) = L$? We will see below that this is indeed the case, but here is one way to think about it. Given the line bundle L consider the unique map $f : M \to \mathfrak{su}(2)$ with $\det f = 1$ (so that it hits the unit sphere in $\mathfrak{su}(2)$) such that L is the eigenspace of f corresponding to the eigenvalue i. The map u in local coordinates is now

$$u(x,\theta) = \cos\theta\ \mathrm{Id} + \sin\theta\ f(x).$$

Thus for every meromorphic function we obtain a cohomologically trivial connection. Are these all? Not quite, there are many more in which u has higher-order dependence on velocities as we will see below.

We now provide details and we begin by giving a general classification result for cohomologically trivial connections on any surface.

As before let M be an oriented surface with a Riemannian metric and let SM be its unit tangent bundle. Let

$$\mathcal{A} := \{A : SM \to \mathfrak{u}(n) : \ V^2(A) = -A\}.$$

The set $\mathcal{A}$ is identified with the set of all unitary connections on the trivial bundle $M \times \mathbb{C}^n$. Indeed, a function A satisfying $V^2(A) + A = 0$ extends to a function on TM depending linearly on the velocities.

Recall from the previous section that A is said to be cohomologically trivial if there exists a smooth $u : SM \to U(n)$ such that $C(x,v,t) = u(\phi_t(x,v))u(x,v)^{-1}$.

Differentiating with respect to t and setting $t = 0$ this is equivalent to

$$Xu + Au = 0. \tag{18}$$

Let $\mathcal{A}_0$ be the set of all cohomologically trivial connections, that is, the set of all $A \in \mathcal{A}$ such that there exists $u : SM \to U(n)$ for which (18) holds.

Given a vector field W in SM, let G_W be the set of all $u : SM \to U(n)$ such that $W(u) = 0$, that is, first integrals of W. Note that G_V is nothing but the group of gauge transformations of the trivial bundle $M \times \mathbb{C}^n$.

We wish to understand $\mathcal{A}_0/G_V$. Now let $\mathcal{D}$ be the set of all $f : SM \to \mathfrak{u}(n)$ such that

$$-X_\perp(f) + VX(f) = [X(f), f]$$

and there is $u : SM \to U(n)$ such that $f = u^{-1}V(u)$. It is easy to check that G_X acts on $\mathcal{D}$ by $f \mapsto a^{-1} f\, a + a^{-1}V(a)$ where $a \in G_X$.

Theorem 7.1. *There is a one-to-one correspondence between $\mathcal{A}_0/G_V$ and $\mathcal{D}/G_X$.*

Proof. Forward direction: a cohomologically trivial connection A comes with a u such that $Xu + Au = 0$. If we set $f := u^{-1}V(u)$, then $f \in \mathcal{D}$, that is, f satisfies the PDE $-X_\perp(f) + VX(f) = [X(f), f]$. This a calculation (see [Paternain 2009, Theorem B] for details), but for the reader's convenience we explain the geometric origin of this equation. Using u we may define a connection on SM gauge equivalent to π^*A by setting $B := u^{-1}du + u^{-1}\pi^*Au$, where $\pi : SM \to M$ is the foot-point projection. Since π^*A is the pull-back of a connection on M, the curvature F_B of B must vanish when one of the entries is the vertical vector field V. The PDE $-X_\perp(f) + VX(f) = [X(f), f]$ arises by combining the two equations $F_B(X, V) = F_B(X_\perp, V) = 0$ with $B(X) = 0$.

Backward direction: Given f with $uf = V(u)$, set $A := -X(u)u^{-1}$. Then $A \in \mathcal{A}_0$, that is, $V^2(A) = -A$; again this is a calculation done fully in Theorem B in [Paternain 2009].

Now there are two ambiguities here. Going forward, we may change u as long as we solve $Xu + Au = 0$. This changes f by the action of G_X. Going backwards we may change u as long as $uf = V(u)$, this changes A by a gauge transformation, that is, an element in G_V. □

Note that if the geodesic flow is transitive (i.e., there is a dense orbit) the only first integrals are the constants and thus $G_X = U(n)$ acts simply by conjugation. If M is closed and of negative curvature, the geodesic flow is Anosov and therefore transitive.

The fact that the PDE describing cohomologically trivial connections arises from zero curvature conditions is an indication of the "integrable" nature of the problem at hand. The existence of a Bäcklund transformation that we will introduce shortly is another typical feature of integrable systems. Note that the space $\mathfrak{D}/G_X$ is in some sense simpler and larger when the underlying geodesic flow is more complicated, that is, when it is transitive G_X reduces to $U(n)$.

The Bäcklund transformation. For the remainder of this section we restrict to the case in which the structure group is SU(2). This is the simplest nontrivial case.

Suppose there is a smooth map $b : SM \to \mathrm{SU}(2)$ such that $f := b^{-1}V(b)$ solves the PDE:

$$-X_\perp(f) + VX(f) = [X(f), f]. \tag{19}$$

Then, by Theorem 7.1, $A := -X(b)b^{-1}$ defines a cohomologically trivial connection on M and $-\star A = V(A) = -bX(f)b^{-1} + X_\perp(b)b^{-1}$.

Lemma 7.2. *Let $g : M \to \mathfrak{su}(2)$ be a smooth map with* $\det g = 1$ (*i.e.*, $g^2 = -\mathrm{Id}$). *Then there exists $a : SM \to \mathrm{SU}(2)$ such that $g = a^{-1}V(a)$.*

Proof. Let $L(x)$ and $U(x)$ be the eigenspaces corresponding respectively to the eigenvalues i and $-i$ of $g(x)$. We have an orthogonal decomposition $\mathbb{C}^2 = L(x) \oplus U(x)$ for every $x \in M$. Consider sections

$$\alpha \in \Omega^{1,0}(M, \mathbb{C}) \quad \text{and} \quad \beta \in \Omega^{1,0}(M, \mathrm{Hom}(L, U)) = \Omega^{1,0}(M, L^*U)$$

such that $|\alpha|^2 + |\beta|^2 = 1$. Such a pair of sections always exists; for example, we can choose a section $\tilde\beta$ with a finite number of isolated zeros and then choose a $\tilde\alpha$ that does not vanish on the zeros of $\tilde\beta$. Then we set $\alpha := \tilde\alpha/(|\tilde\alpha|^2 + |\tilde\beta|^2)^{1/2}$ and $\beta := \tilde\beta/(|\tilde\alpha|^2 + |\tilde\beta|^2)^{1/2}$. Note that

$$\bar\alpha \in \Omega^{0,1}(M, \mathbb{C}) \quad \text{and} \quad \beta^* \in \Omega^{0,1}(M, \mathrm{Hom}(U, L)) = \Omega^{0,1}(M, U^*L).$$

Using the orthogonal decomposition we define $a : SM \to \mathrm{SU}(2)$ by

$$a(x, v) = \begin{pmatrix} \alpha(x, v) & \beta^*(x, v) \\ -\beta(x, v) & \bar\alpha(x, v) \end{pmatrix}.$$

Clearly $a = a_{-1} + a_1$, where

$$a_1 = \begin{pmatrix} \alpha & 0 \\ -\beta & 0 \end{pmatrix} \quad \text{and} \quad a_{-1} = \begin{pmatrix} 0 & \beta^* \\ 0 & \bar\alpha \end{pmatrix}.$$

It is straightforward to check that $ag = V(a)$. □

Remark 7.3. There is an alternative proof of this lemma along the following lines. Consider an open set U in M over which the circle fibration $\pi : SM \to M$ trivializes as $U \times S^1$, where $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. In this trivialization $V = \partial/\partial\theta$ and any solution to $ag = V(a)$ has the form $a_U := r_U(x)(\cos\theta\,\mathrm{Id} + \sin\theta\, g(x))$, where $r_U : U \to \mathrm{SU}(2)$ is smooth. Consider another set U' which trivializes $\pi : SM \to M$ and which intersects U. We obtain a transition function $\psi_{UU'} : U \cap U' \to S^1$. The functions a_U can be glued to define a global function $a : SM \to \mathrm{SU}(2)$ as long as

$$r_U(x)(\cos\theta\,\mathrm{Id} + \sin\theta\, g(x)) = r_{U'}(x)(\cos\theta'\,\mathrm{Id} + \sin\theta'\, g(x))$$

where $\theta = \theta' + \psi_{UU'}(x)$ and $x \in U \cap U'$. Hence to have a globally defined a we need to show the existence of smooth functions $r_U : U \to \mathrm{SU}(2)$ such that

$$\varphi_{UU'}(x) := \cos(\psi_{UU'}(x))\mathrm{Id} + \cos(\psi_{UU'}(x))g(x) = (r_U(x))^{-1} r_{U'}(x).$$

The key observation is that $\varphi_{UU'}$ defines an SU(2)-cocycle in the sense of principal bundles. Indeed, the cocycle property $\varphi_{UU''}(x) = \varphi_{UU'}(x)\,\varphi_{U'U''}(x)$ follows right away from the fact that $\psi_{UU'}$ is an S^1-cocycle. But an SU(2)-bundle over a surface is trivial. The existence of the functions $r_U : U \to \mathrm{SU}(2)$ follows.

Note that by construction, $\mathrm{Ker}\, a_{\pm 1}$ coincides with the $\mp i$ eigenspace of g.

Now let $u := ab : SM \to \mathrm{SU}(2)$ and let $F := (ab)^{-1}V(ab) = b^{-1} g\, b + f$.

Question. When does F satisfy (19)?

If it does, then it defines (via Theorem 7.1) a new cohomologically trivial connection given by

$$A_F = -X(ab)(ab)^{-1} = -X(a)a^{-1} + aAa^{-1},$$

where A is the cohomologically trivial connection associated to f.

Recall that the connection A defines a covariant derivative $d_A g = dg + [A, g]$.

Lemma 7.4. *F satisfies* (19) *if and only if*

$$-\star d_A g = (d_A g)\, g. \tag{20}$$

Proof. Starting with $F = b^{-1} g\, b + f$ and using that $A = -X(b)b^{-1} = bX(b^{-1})$ we compute

$$X(F) = b^{-1}\left([A, g] + X(g)\right) b + X(f).$$

Similarly, using $X_\perp(b) = -(\star A)b + bX(f)$ we find

$$X_\perp(F) = b^{-1}\left([\star A, g] + X_\perp(g)\right) b - [X(f), b^{-1} g\, b] + X_\perp(f).$$

Now we compute $VX(F)$; here we use that $V(g) = 0$. We obtain

$$VX(F) = [b^{-1}([A, g] + X(g))b, f] + b^{-1}\left([-\star A, g] + VX(g)\right) b + VX(f).$$

The last term we need for (19) is

$$[X(F), F] = b^{-1}\big[[A,g] + X(g), g\big]b + \big[b^{-1}([A,g] + X(g))b, f\big] + [X(f), b^{-1} g\, b] + [X(f), f].$$

Since f satisfies (19) we see that F satisfies (19) if and only if

$$-X_{\perp}(g) + VX(g) - 2[\star A, g] = \big[[A,g] + X(g), g\big].$$

Since g depends only on the base point and $g^2 = -\mathrm{Id}$ we can rewrite this as

$$-2\star(dg + [A,g]) = [dg + [A,g], g] = 2(dg + [A,g])\, g.$$

Thus F satisfies (19) if and only if

$$-\star d_A g = (d_A g)\, g,$$

as claimed. □

We will now rephrase (20) in terms of holomorphic line bundles. Recall that the connection A induces a holomorphic structure on the trivial bundle $M \times \mathbb{C}^2$ and on the endomorphism bundle $M \times \mathbb{C}^{2\times 2}$. We have an operator $\bar\partial_A = (d_A - i\star d_A)/2 = \bar\partial + [A_{-1}, \cdot]$ acting on sections $f : M \to \mathbb{C}^{2\times 2}$.

Set $\pi := (\mathrm{Id} - ig)/2$ and $\pi^{\perp} = (\mathrm{Id} + ig)/2$ so that $\pi + \pi^{\perp} = \mathrm{Id}$. Let $L(x)$ be as above the eigenspace corresponding to the eigenvalue i of $g(x)$. Note that π is the Hermitian orthogonal projection over $L(x) = \mathrm{Image}(\pi(x))$.

Lemma 7.5. *Let* $g : M \to \mathfrak{su}(2)$ *be a smooth map with* $\det g = 1$. *The following conditions are equivalent.*

(1) $-\star d_A g = (d_A g) g$.

(2) *L is a $\bar\partial_A$-holomorphic line bundle.*

(3) $\pi^{\perp}\bar\partial_A \pi = 0$.

Proof. Suppose that (1) holds. Apply $\star$ to obtain $d_A g = (\star d_A g)\, g$. Thus

$$d_A g - i\star d_A g = i(d_A g - i\star d_A g) g.$$

In other words $\bar\partial_A g = i(\bar\partial_A g)\, g = -ig(\bar\partial_A g)$ (recall that $g^2 = -\mathrm{Id}$). Since $\pi = (\mathrm{Id} - ig)/2$, then $\bar\partial_A g = -ig(\bar\partial_A g)$ is equivalent to $\pi^{\perp}\bar\partial_A\pi = 0$ which is (3).

Using the condition $\pi^2 = \pi$, we see that $\pi^{\perp}\bar\partial_A\pi = 0$ is equivalent to $(\bar\partial_A\pi)\pi = 0$. The line bundle L is holomorphic if and only if given a local section ξ of L, then $\bar\partial_A\xi \in L$. Using that $\pi\xi = \xi$ we see that $\bar\partial_A\xi \in L$ if and only if $(\bar\partial_A\pi)\xi = 0$. Clearly, this happens if and only if $(\bar\partial_A\pi)\pi = 0$ and thus (2) holds if and only if (3) holds. □

The next theorem summarizes the Bäcklund transformation that we just introduced and it follows directly from Lemmas 7.4 and 7.5 and Theorem 7.1.

Theorem 7.6. *Let A be a cohomologically trivial connection and let L be a holomorphic line subbundle of the trivial bundle $M \times \mathbb{C}^2$ with respect to the complex structure induced by A. Define a map $g : M \to \mathfrak{su}(2)$ with* $\det g = 1$ *by declaring L to be its eigenspace with eigenvalue i. Consider $a : SM \to \mathrm{SU}(2)$ with $g = a^{-1}V(a)$ as given by Lemma 7.2. Then*

$$A_F := -X(a)a^{-1} + aAa^{-1}$$

defines a cohomologically trivial connection.

Definition 7.7. Let A be a cohomologically trivial connection. Given a map $g : M \to \mathfrak{su}(2)$ with $\det g = 1$ and $-\star d_A g = (d_A g)g$, let $a : SM \to \mathrm{SU}(2)$ be any smooth map with $ag = V(a)$. Then the *Bäcklund transformation* of the connection A with respect to the pair (g, a) is:

$$\mathcal{B}_{g,a}(A) := -X(a)a^{-1} + aAa^{-1}.$$

By Theorem 7.6, $\mathcal{B}_{g,a}(A)$ is a new cohomologically trivial connection.

Remark 7.8. Note that if the geodesic flow is transitive, two solutions u, w of $Xu + Au = 0$ are related by $u = wg$ where g is a constant unitary matrix, because $X(w^{-1}u) = 0$. Thus the degrees of u and w are the same. We can then talk about the "degree" of a cohomologically trivial connection as the degree of any solution of $Xu + Au = 0$.

Remark 7.9. If we let $q := aga^{-1}$, then a simple calculation shows that $V(q) = 0$ and $d_{A_F} q = a(d_A g)a^{-1}$. Moreover, $\star d_{A_F} q = (d_{A_F} q)q$ which means that $-q$ satisfies (20) with respect to A_F. Hence if we run the Bäcklund transformation on A_F with $g' := -q$ and $a' := a^{-1}$ we recover A (note that $a'g' = V(a')$). In other words $\mathcal{B}_{-q,a^{-1}}(\mathcal{B}_{g,a}(A)) = A$. Thus the Bäcklund transformation described in Theorem 7.6 has a natural "inverse".

If we start, for example, with the trivial connection $A = 0$ (which is obviously cohomologically trivial), then a map $g : M \to \mathfrak{su}(2)$ with $\det g = 1$ and $-\star dg = (dg)g$ can be identified with a meromorphic function. The connections of degree one $A_F = -X(a)a^{-1}$ given by Theorem 7.6 were first found in [Paternain 2009] and coincide with the ones described at the beginning of the section. In the next subsection we will show that any cohomologically trivial connection such that the associated u has a finite Fourier series can be built up by successive applications of the transformation described in Theorem 7.6, provided that the geodesic flow is transitive. This will provide a full classification of transparent $\mathrm{SU}(2)$-connections over negatively curved surfaces.

The classification result. Let A be a transparent connection with $A=-X(b)b^{-1}$ and $f=b^{-1}V(b)$, where $b:SM\to \mathrm{SU}(2)$.

We first make some remarks concerning the SU(2)-structure. Let $j:\mathbb{C}^2\to\mathbb{C}^2$ be the antilinear map given by

$$j(z_1,z_2)=(-\bar{z}_2,\bar{z}_1).$$

If we think of a matrix $a\in \mathrm{SU}(2)$ as a linear map $a:\mathbb{C}^2\to\mathbb{C}^2$, then $ja=aj$. This implies that given $b:SM\to\mathrm{SU}(2)$ with $b=\sum_{k\in\mathbb{Z}}b_k$, then $jb_k=b_{-k}j$ for all $k\in\mathbb{Z}$.

Assumption. Suppose b has a finite Fourier expansion, that is, $b=\sum_{k=-N}^{k=N}b_k$, where $N\geq 1$. By Theorem 6.6 we know that this holds if M has negative curvature.

Let us assume also that N is the degree of b, so both b_N and $b_{-N}=-jb_Nj$ are nonzero.

The unitary condition $bb^*=b^*b=\mathrm{Id}$ implies that $b_Nb_{-N}^*=b_{-N}^*b_N=0$. These relations imply that the rank of b_{-N} and b_N is at most one and equals one on an open set, which, as we will see shortly, must be all of M except for perhaps a finite number of points. But first we need some preliminaries.

Consider isothermal coordinates (x,y) on M such that the metric can be written as $ds^2=e^{2\lambda}(dx^2+dy^2)$, where λ is a smooth real-valued function of (x,y). This gives coordinates (x,y,θ) on SM, where θ is the angle between a unit vector v and $\partial/\partial x$. In these coordinates X is given by (8) and $X_\perp$ by:

$$X_\perp=-e^{-\lambda}\left(-\sin\theta\frac{\partial}{\partial x}+\cos\theta\frac{\partial}{\partial y}-\left(\frac{\partial\lambda}{\partial x}\cos\theta+\frac{\partial\lambda}{\partial y}\sin\theta\right)\frac{\partial}{\partial\theta}\right).\tag{21}$$

Consider $u\in\Omega_m$ and write it locally as $u(x,y,\theta)=h(x,y)e^{im\theta}$. Using (8) and (21) a straightforward calculation shows that

$$\eta_-(u)=e^{-(1+m)\lambda}\bar{\partial}(he^{m\lambda})e^{i(m-1)\theta},\tag{22}$$

where $\bar{\partial}=\frac{1}{2}\left(\partial/\partial x+i\,\partial/\partial y\right)$. In order to write μ_- suppose that $A(x,y,\theta)=a(x,y)\cos\theta+b(x,y)\sin\theta$. If we also write $A=A_xdx+A_ydy$, then $A_x=ae^{\lambda}$ and $A_y=be^{\lambda}$. Let $A_{\bar{z}}:=\frac{1}{2}(A_x+iA_y)$. Using the definition of A_{-1} we derive

$$A_{-1}=\tfrac{1}{2}(a+ib)e^{-i\theta}=A_{\bar{z}}d\bar{z}.\tag{23}$$

Putting this together with (22) we obtain

$$\mu_-(u)=e^{-(1+m)\lambda}\big(\bar{\partial}(he^{m\lambda})+A_{\bar{z}}he^{m\lambda}\big)e^{i(m-1)\theta}.\tag{24}$$

Note that Ω_m can be identified with the set of smooth sections of the bundle $(M\times\mathbb{M}_2(\mathbb{C}))\otimes\kappa^{\otimes m}$ where κ is the canonical line bundle. The identification takes

$u = he^{im\theta}$ into $he^{m\lambda}(dz)^m$ $(m \geq 0)$ and $u = he^{-im\theta} \in \Omega_{-m}$ into $he^{m\lambda}(d\bar{z})^m$. The second equality in (23) should be understood using this identification.

Consider now a fixed vector $\xi \in \mathbb{C}^2$ such that $s(x, v) := b_{-N}(x, v)\xi \in \mathbb{C}^2$ is not zero identically. Clearly s can be seen as a section of $(M \times \mathbb{C}^2) \otimes \kappa^{\otimes -N}$. We may write b_{-N} and s in local isothermal coordinates as $b_{-N} = he^{-iN\theta}$ and $s = e^{N\lambda}h\xi(d\bar{z})^N$.

Lemma 7.10. *The local section $e^{-2N\lambda}s$ is $\bar{\partial}_A$-holomorphic.*

Proof. Using the operators $\mu_\pm$ we can write $X(b) + Ab = 0$ as

$$\mu_+(b_{k-1}) + \mu_-(b_{k+1}) = 0$$

for all k. This gives $\mu_+(b_N) = \mu_-(b_{-N}) = 0$. But $\mu_-(b_{-N}) = 0$ is saying that $e^{-2N\lambda}s$ is $\bar{\partial}_A$-holomorphic. Indeed, using (24), we see that $\mu_-(b_{-N}) = 0$ implies

$$\bar{\partial}(he^{-N\lambda}) + A_{\bar{z}}he^{-N\lambda} = 0$$

which in turn implies

$$\bar{\partial}(e^{-N\lambda}h\xi) + A_{\bar{z}}e^{-N\lambda}h\xi = 0.$$

This equation says that $e^{-2N\lambda}s = e^{-N\lambda}h\xi(d\bar{z})^N$ is $\bar{\partial}_A$-holomorphic. □

The section s spans a line bundle L over M which by the previous lemma is $\bar{\partial}_A$-holomorphic. The section s may have zeros, but at a zero z_0, the line bundle extends holomorphically. Indeed, in a neighborhood of z_0 we may write $e^{-2N\lambda(z)}s(z) = (z - z_0)^k w(z)$, where w is a local holomorphic section with $w(z_0) \neq 0$. The section w spans a holomorphic line subbundle which coincides with the one spanned by s off z_0. Therefore L is a $\bar{\partial}_A$-holomorphic line bundle that contains the image of b_{-N} (and $U = jL$ is an antiholomorphic line bundle that contains the image of b_N). We summarize this in a lemma:

Lemma 7.11. *The line bundle L determined by the image of b_{-N} is $\bar{\partial}_A$-holomorphic.*

We will now use the line bundle L to construct an appropriate $g : M \to \mathfrak{su}(2)$ such that when we run the Bäcklund transformation from the previous subsection we obtain a cohomologically trivial connection of degree $\leq N - 1$. But first we need the following lemma. Recall that a matrix-valued function f is said to be *odd* if $f(x, v) = -f(x, -v)$ and *even* if $f(x, v) = f(x, -v)$.

Lemma 7.12. *Assume that the geodesic flow is transitive and let $b : SM \to \mathrm{SU}(2)$ solve $X(b) + Ab = 0$. Then b is either even or odd.*

Proof. Write $b = b_o + b_e$ where b_o is odd and b_e is even. Since the operator $(X + A)$ maps even to odd and odd to even, the equation $X(b) + Ab = 0$ decouples as

$$X(b_o) + Ab_o = 0,$$
$$X(b_e) + Ab_e = 0.$$

A calculation using these equations shows that $X(b_o^* b_o)$, $X(b_e^* b_e)$, $X(b_o^* b_e)$ all vanish. Since the geodesic flow is transitive, these matrices are all constant. Moreover, since $b_o^* b_e$ is odd it must be zero. On the other hand $jb = bj$ implies that $jb_o = b_o j$ and $jb_e = b_e j$, which in turn implies that both b_o and b_e cannot have rank 1. Putting all this together, we see that either b_o or b_e must vanish identically. □

Suppose the geodesic flow is transitive. By Lemma 7.12, $b = b_{-N} + d + b_N$, where d has degree $\leq N - 2$. We now seek $a : SM \to \mathrm{SU}(2)$ of degree one such that $u := ab$ has degree $\leq N - 1$. For this we need $a_1 b_N = a_{-1} b_{-N} = 0$. We take a map $g : M \to \mathfrak{su}(2)$ with $\det g = 1$ such that its i eigenspace is L and its $-i$ eigenspace is U. By Lemmas 7.5 and 7.11, $-\star d_A g = (d_A g)\, g$. The construction of a with $ag = V(a)$ from Lemma 7.2 is precisely such that the kernel of a_{-1} is L and the kernel of a_1 is U, so the needed relations to lower the degree hold.

Finally by Theorem 7.6, u gives rise to a cohomologically trivial connection $-X(u)u^{-1}$. Combining this with Theorem 6.6 we have arrived at the main result of this section:

Theorem 7.13. *Let M be a closed orientable surface of negative curvature. Then any transparent* SU(2)*-connection can be obtained by successive applications of Bäcklund transformations as described in Theorem 7.6.*

We finish this section with some remarks on the operators $\mu_\pm$. Let $\Gamma(M, \kappa^{\otimes m})$ denote the space of smooth sections of the m-th tensor power of the canonical line bundle κ. Locally its elements have the form $w(z)dz^m$ for $m \geq 0$ and $w(z)d\bar{z}^{-m}$ for $m \leq 0$. Given a metric g on M, there is map

$$\varphi_g : \Gamma(M, \kappa^{\otimes m}) \to \Omega_m$$

given by restriction to SM. This map is a complex linear isomorphism. Let us check what this map looks like in isothermal coordinates. An element of $\Gamma(M, \kappa^{\otimes m})$ is locally of the form $w(z)dz^m$. Consider a tangent vector $\dot{z} = \dot{x}_1 + i\dot{x}_2$. It has norm 1 in the metric g if and only if $e^{i\theta} = e^{\lambda}\dot{z}$. Hence the restriction of $w(z)dz^m$ to SM is

$$w(z)e^{-m\lambda}e^{im\theta}$$

as indicated above. Moreover there is also a restriction map

$$\psi_g : \Gamma(M, \kappa^{\otimes m} \otimes \bar{\kappa}) \to \Omega_{m-1}$$

which is an isomorphism. The restriction of $w(z)dz^m \otimes d\bar{z}$ to SM is

$$w(z)e^{-(m+1)\lambda}e^{i(m-1)\theta},$$

because $e^{-i\theta} = e^{\lambda}\bar{\dot{z}}$.

Given any holomorphic bundle ξ over M, there is a $\overline{\partial}$-operator defined on:

$$\overline{\partial} : \Gamma(M, \xi) \to \Gamma(M, \xi \otimes \bar{\kappa}).$$

In particular we can take $\xi = \kappa^{\otimes m}$ (or $\kappa^{\otimes m} \otimes \mathbb{C}^n$). Combining this with (22) we get the following commutative diagram:

$$\begin{array}{ccc} \Gamma(M, \kappa^{\otimes m}) & \xrightarrow{\varphi_g} & \Omega_m \\ \Big\downarrow \overline{\partial} & & \Big\downarrow \eta_- \\ \Gamma(M, \kappa^{\otimes m} \otimes \bar{\kappa}) & \xrightarrow{\psi_g} & \Omega_{m-1} \end{array}$$

In other words:

$$\eta_- = \psi_g \, \overline{\partial} \, \varphi_g^{-1}.$$

This equation exhibits explicitly the relation of η_- with the metric. More generally, if we let $\overline{\partial}_A := \overline{\partial} + A_{\bar{z}}$, then (24) shows that

$$\mu_- = \psi_g \, \overline{\partial}_A \, \varphi_g^{-1}. \tag{25}$$

In particular we see from (25) that the injectivity and surjectivity properties of μ_- only depend on the conformal class of the metric and are the same as those of $\overline{\partial}_A$. Also, the index of μ_- may be computed using Riemann–Roch; see [McDuff and Salamon 2004, Appendix C]. If $\xi = \kappa^{\otimes m} \otimes \mathbb{C}^n$ and g denotes the genus of M, then

$$\operatorname{index}(\mu_-) = n(1 - \mathrm{g}) + c_1(\xi) = (\mathrm{g} - 1)n(2m - 1).$$

For the abelian case $n = 1$, it is a classical result that $\overline{\partial}_A$ is surjective if $\mathrm{g} \geq 2$ and $m \geq 2$.

8. Higgs fields

Virtually everything that we have said above extends when a Higgs field is present. For us, a Higgs field is a smooth matrix-valued function $\Phi : M \to \mathbb{C}^{n\times n}$. Often in gauge theories, the structure group is $U(n)$ and the field Φ is required to take values in $\mathfrak{u}(n)$. We call a Higgs field $\Phi : M \to \mathfrak{u}(n)$ a skew-Hermitian Higgs field. The pairs (A, Φ) often appear in the so-called Yang–Mills–Higgs theories. A good example of this is the Bogomolny equation in Minkowski $(2+1)$-space given by $d_A\Phi = \star F_A$. Here d_A stands for the covariant derivative induced on endomorphism $d_A\Phi = d\Phi + [A, \Phi]$, $F_A = dA + A \wedge A$ is the curvature of A and $\star$ is the Hodge star operator of Minkowski space. The Bogomolny equation appears as a reduction of the self-dual Yang–Mills equation in $(2+2)$-space and has been object of intense study in the literature of solitons and integrable systems; see for instance [Dunajski 2010; Manton and Sutcliffe 2004, Chapter 8; Hitchin et al. 1999, Chapter 4; Mason and Woodhouse 1996].

To include the Higgs field in the discussions above, we consider the following transport equation for $u : SM \to \mathbb{C}^n$,

$$Xu + Au + \Phi u = -f \quad \text{in } SM, \qquad u|_{\partial_-(SM)} = 0.$$

As before, on a fixed geodesic the transport equation becomes a linear system of ODEs with zero initial condition, and therefore this equation has a unique solution $u = u^f$.

Definition 8.1. The geodesic ray transform of $f \in C^\infty(SM, \mathbb{C}^n)$ with attenuation determined by the pair (A, Φ) is given by

$$I_{A,\Phi} f := u^f|_{\partial_+(SM)}.$$

Obviously $I_A = I_{A,0}$ when $\Phi = 0$. The following extension of Theorem 3.2 holds:

Theorem 8.2 [Paternain et al. 2011a]. *Let M be a compact simple surface. Assume that $f : SM \to \mathbb{C}^n$ is a smooth function of the form $F(x) + \alpha_j(x)v^j$, where $F : M \to \mathbb{C}^n$ is a smooth function and α is a $\mathbb{C}^n$-valued 1-form. Let also $A : SM \to \mathfrak{u}(n)$ be a unitary connection and $\Phi : M \to \mathfrak{u}(n)$ a skew-Hermitian matrix function. If $I_{A,\Phi}(f) = 0$, then $F = \Phi p$ and $\alpha = d_A p$, where $p : M \to \mathbb{C}^n$ is a smooth function with $p|_{\partial M} = 0$.*

The introduction of the Higgs field complicates matters from a technical point of view: more terms appear in the Pestov identity, and these need to be carefully controlled, we refer the reader to [Paternain et al. 2011a] for details.

Given a pair (A, Φ) one can also associate to it scattering data. We look at the unique solution $U_{A,\Phi} : SM \to U(n)$ of

$$\begin{cases} X(U_{A,\Phi}) + (A(x,v) + \Phi(x))U_{A,\Phi} = 0, & (x,v) \in SM, \\ U_{A,\Phi}|_{\partial_+(SM)} = \mathrm{Id}. \end{cases}$$

The scattering data of the pair (A, Φ) is now the map $C_{A,\Phi} : \partial_-(SM) \to U(n)$ defined as $C_{A,\Phi} := U_{A,\Phi}|_{\partial_-(SM)}$.

Using Theorem 8.2 we can derive the following result just as we have done for the proof of Theorem 3.3.

Theorem 8.3 [Paternain et al. 2011a]. *Assume M is a compact simple surface, let A and B be two Hermitian connections, and let Φ and Ψ be two skew-Hermitian Higgs fields. Then $C_{A,\Phi} = C_{B,\Psi}$ implies that there exists a smooth $U : M \to U(n)$ such that $U|_{\partial M} = \mathrm{Id}$ and $B = U^{-1}dU + U^{-1}AU$, $\Psi = U^{-1}\Phi U$.*

A Higgs field can also be included for the case of closed manifolds. A classification of SO(3)-transparent pairs (A, Φ) for surfaces of negative curvature may be found in [Paternain 2012].

9. Arbitrary bundles

In this section we briefly discuss the case of closed surfaces and arbitrary (not necessarily trivial) bundles. We begin with some generalities.

Suppose E is a rank n Hermitian vector bundle over a closed manifold N and $\phi_t : N \to N$ is a smooth transitive Anosov flow.

Definition 9.1. A cocycle over ϕ_t is an action of $\mathbb{R}$ by bundle automorphisms which covers ϕ_t. In other words, for each $(x,t) \in N \times \mathbb{R}$, we have a unitary map $C(x,t) : E_x \to E_{\phi_t x}$ such that $C(x, t+s) = C(\phi_t x, s)\, C(x,t)$.

If E admits a unitary trivialization $f : E \to N \times \mathbb{C}^n$, then

$$f\, C(x,t)\, f^{-1}(x,a) = (\phi_t x, D(x,t)a),$$

where $D : N \times \mathbb{R} \to U(n)$ is a cocycle as in Definition 6.1.

Let E^* denote the dual vector bundle to E. If E carries a Hermitian metric h, we have a conjugate isomorphism $\ell_h : E \to E^*$, which induces a Hermitian metric h^* on E^*. Given a cocycle C on E, $C^* := \ell_h\, C\, \ell_h^{-1}$ is a cocycle on (E^*, h^*).

Proposition 9.2. *Let E be a Hermitian vector bundle over N such that $E \oplus E^*$ is a trivial vector bundle. Let C be a smooth cocycle on E such that $C(x,T) = \mathrm{Id}$ whenever $\phi_T x = x$. Then E is a trivial vector bundle.*

Proof. As explained above, the cocycle C on E induces a cocycle C^* on E^*. On the trivial vector bundle $E \oplus E^*$ we consider the cocycle $C \oplus C^*$. Clearly $C \oplus C^*(x,T) = \mathrm{Id}$ every time that $\phi_T x = x$. Choose a unitary trivialization $f : E \oplus E^* \to N \times \mathbb{C}^{2n}$ and write

$$f\, C \oplus C^*(x,t)\, f^{-1}(x,a) = (\phi_t x, D(x,t)a).$$

By Theorem 6.4, there exists a smooth function $u : N \to U(2n)$ such that $D(x,t) = u(\phi_t x)u^{-1}(x)$. Since ϕ_t is a transitive flow, we may choose $x_0 \in N$

with a dense orbit and without loss of generality we may suppose that $u(x_0) = \mathrm{Id}$. Let

$$\{e_1(x_0), \dots, e_n(x_0)\}$$

be a unitary frame at E_{x_0}. Write $f(x_0, e_i(x_0)) = (x_0, a_i)$, where $a_i \in \mathbb{C}^{2n}$. Let

$$e_i(x) := f^{-1}(x, u(x)a_i).$$

Clearly at every $x \in N$, $\{e_1(x), \dots, e_n(x)\}$ is a smooth unitary n-frame of $E_x \oplus E_x^*$. We claim that in fact $e_i(x) \in E_x$ for all $x \in N$. This, of course, implies the triviality of E. Note that

$$\begin{aligned} e_i(\phi_t x_0) &= f^{-1}(\phi_t x_0, u(\phi_t x_0)a_i) \\ &= f^{-1}(\phi_t x_0, D(x_0,t)a_i) = C \oplus C^*(x_0,t)e_i(x_0). \end{aligned}$$

But $e_i(x_0) \in E_{x_0}$, thus $e_i(\phi_t x_0) \in E_{\phi_t x_0}$. It follows that $e_i(x) \in E_x$ for a dense set of points in N. By continuity of e_i, $e_i(x) \in E_x$ for all $x \in N$. □

Remark 9.3. The hypothesis of $E \oplus E^*$ being trivial is not needed in Proposition 9.2. Ralf Spatzier has informed me that it is possible to adapt the proof of the usual Livšic periodic data theorem to show directly that E is trivial. However, this weaker version is all that we will need below.

Let M be a closed orientable surface. In this case, complex vector bundles E over M are classified topologically by the first Chern class $c_1(E) \in H^2(M, \mathbb{Z}) = \mathbb{Z}$. Since $c_1(E^*) = -c_1(E)$ and c_1 is additive with respect to direct sums, we see that $E \oplus E^*$ is the trivial bundle and therefore we will be able to apply Proposition 9.2. In fact we will show:

Theorem 9.4 [Paternain 2009]. *Let M be a closed orientable Riemannian surface of genus* g *whose geodesic flow is Anosov. A complex vector bundle E over M admits a transparent connection if and only if* $2-2\mathrm{g}$ *divides* $c_1(E)$.

Proof. Suppose E admits a transparent connection. As explained above we may apply Proposition 9.2 to deduce that π^*E is a trivial bundle and since $c_1(\pi^*E) = \pi^*c_1(E)$ we conclude that $\pi^*c_1(E) = 0$. Consider now the Gysin sequence of the unit circle bundle $\pi : SM \to M$:

$$\begin{aligned} 0 \to H^1(M,\mathbb{Z}) \xrightarrow{\pi^*} H^1(SM,\mathbb{Z}) \\ \xrightarrow{0} H^0(M,\mathbb{Z}) \xrightarrow{\times(2-2\mathrm{g})} H^2(M,\mathbb{Z}) \xrightarrow{\pi^*} H^2(SM,\mathbb{Z}) \to \cdots. \end{aligned}$$

We see that $\pi^*c_1(E) = 0$ if and only if $c_1(E)$ is in the image of the map $H^0(M,\mathbb{Z}) \to H^2(M,\mathbb{Z})$ given by cup product with the Euler class of the unit circle bundle. Equivalently, $2-2\mathrm{g}$ must divide $c_1(E)$.

Let κ be the canonical line bundle of M. We can think of κ as the cotangent bundle to M; it has $c_1(\kappa) = 2g - 2$. The tensor powers κ^s of κ (positive and negative) generate all possible line bundles with first Chern class divisible by $2 - 2g$ and they all carry the unitary connection induced by the Levi-Civita connection of the Riemannian metric on M. All these connections are clearly transparent. Topologically, all complex vector bundles over M whose first Chern class is divisible by $2 - 2g$ are of the form $\kappa^s \oplus \varepsilon$, where ε is the trivial vector bundle. Since the trivial connection on the trivial bundle is obviously transparent, it follows that every complex vector bundle whose first Chern class is divisible by $2 - 2g$ admits a transparent connection. □

A similar argument shows that if E is a Hermitian *line* bundle with a transparent connection and $\dim M \geq 3$, then E must be trivial and the connection is gauge equivalent to the trivial connection.

10. Open problems

To organize the discussion we will divide the set of open questions into the two cases: compact simple M and closed manifolds with Anosov geodesic flow.

Compact simple manifolds with boundary.

(1) The most important problem here is to decide if Theorem 8.2 (or Theorem 3.2) holds when $\dim M \geq 3$. This will automatically extend Theorem 8.3 to any dimension.

(2) Of equal importance is the tensor tomography problem in dimension ≥ 3. In other words, does Theorem 5.1 extend to any dimension? This problem is explicitly stated in [Sharafutdinov 1994, Problem 1.1.2] and it has been solved by Pestov and Sharafutdinov [1988] for negatively curved manifolds and then by Sharafutdinov [1994] under a weaker curvature condition. It is also known that if "ghosts" exist, they must be regular: on a simple Riemannian manifold, every L^2 solenoidal tensor field belonging to the kernel of the ray transform is C^∞ smooth [Sharafutdinov et al. 2005].

(3) We have only considered unitary connections and skew-Hermitian Higgs fields, mostly because these are the most relevant in physics, but the problems addressed here make sense for any structure group. In particular, does Theorem 8.2 extend to the case of $GL(n, \mathbb{C})$?

(4) The proof of Theorem 3.2 uses in an essential way the existence of holomorphic integrating factors from Proposition 4.4 for scalar 1-forms and

carefully avoids the question of existence of holomorphic integrating factors for matrix valued 1-forms. In other words, suppose A is a $GL(n,\mathbb{C})$-connection with $n \geq 2$. Does there exist a smooth fiberwise holomorphic map $R : SM \to GL(n,\mathbb{C})$ such that $XR + AR = 0$ on SM?

(5) Are there versions of Theorems 8.2 and Theorem 8.3 when the set of geodesics of a simple surface is replaced by another set of distinguished curves? I would expect a positive answer for magnetic geodesics in view of the work in [Dairbekov et al. 2007].

Closed manifolds with Anosov geodesic flow. Here, the lack of answers is more pronounced, even for surfaces, but this is reasonable as one expects this setting to be harder. As we have seen, the appearance of ghosts (nontrivial transparent connections) has to do with the different holomorphic structures that one can have on a complex vector bundle over the surface. For simple surfaces this does not appear because there is essentially only one $\bar{\partial}_A$ operator on a disk.

(1) Perhaps one of the most important questions for surfaces is whether in Theorem 6.6 one can replace "negative curvature" by "Anosov geodesic flow". This question is of great interest even when $A = 0$.

(2) Does Theorem 6.9 extend to higher dimensions? I would expect a positive answer based on the Fourier analysis displayed in [Guillemin and Kazhdan 1980b]. There is virtually nothing known on transparent connections in $\dim M \geq 3$ as the next question shows.

(3) Are there nontrivial transparent connections on $M \times \mathbb{C}^2$, where M is a closed hyperbolic 3-manifold?

(4) Classify transparent $U(n)$-connections (and pairs) over a negatively curved surface using the ideas displayed in Section 7 for SU(2).

(5) Let M be a surface with an Anosov geodesic flow and suppose there is a smooth $u : SM \to \mathbb{R}$ such that $Xu = f$, where f arises from a symmetric m-tensor. Must f be potential? (The tensor tomography problem for an Anosov surface). The proof given in this paper for simple surfaces does not extend since we do not have the analogue of holomorphic integrating factors from Proposition 4.4. The best result available for 2-tensors appears in [Sharafutdinov and Uhlmann 2000] where a positive answer is given assuming in addition that the surface is free of focal points. A solution of this problem for the case of symmetric 2-tensors would give right away infinitesimal spectral rigidity for Anosov surfaces [Guillemin and Kazhdan 1980a].

Acknowledgements

I am very grateful to Gareth Ainsworth, Will Merry, Mikko Salo and the referee for several comments and corrections on an earlier draft.

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