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Elastic-wave inverse scattering based on reverse time migration with active and passive source reflection data

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We develop a comprehensive theory and microlocal analysis of reverse-time imaging — also referred to as reverse-time migration or RTM — for the anisotropic elastic wave equation based on the single scattering approximation. We consider a configuration representative of the seismic inverse scattering problem. In this configuration, we have an interior (point) body-force source that generates elastic waves, which scatter off discontinuities in the properties of earth's materials (anisotropic stiffness, density), and are observed at receivers on the earth's surface. The receivers detect all the components of displacement. We introduce (i) an anisotropic elastic-wave RTM inverse scattering transform, and for the case of mode conversions (ii) a microlocally equivalent formulation avoiding knowledge of the source via the introduction of so-called array receiver functions. These allow a seamless integration of passive source and active source approaches to inverse scattering.

1. Introduction

We develop a program and analysis for elastic wave-equation inverse scattering, based on the single scattering approximation, from two interrelated points of view, known in the seismic imaging literature as "receiver functions" (passive source) and "reverse-time migration" (active source).

We consider an interior (point) body-force source that generates elastic waves, which scatter off discontinuities in the properties of earth's materials (anisotropic stiffness, density), and which are observed at receivers on the earth's surface. The receivers detect all the components of displacement. We decompose the medium into a smooth background model and a singular contrast and assume the single scattering or Born approximation. The inverse scattering problem concerns the reconstruction of the contrast given a background model.

Keywords: elastic wave equation, inverse scattering, receiver function, microlocal analysis.

In this paper, we extend the original reverse-time imaging or migration (RTM) procedure for scalar waves [Whitmore 1983; McMechan 1983; Baysal et al. 1983] to elastic waves. We generalize the analysis developed in [Op 't Root et al. 2012] for inverse scattering based on RTM for scalar waves; part of this analysis contains elements of the original integral formulation of [Schneider 1978] and the inverse scattering integral equation of [Bojarski 1982]. Elastic-wave RTM has recently become a subject of considerable interest. The current developments have been mostly limited to approaches based on certain polarized qP-wave approximations [Sun and McMechan 2001; Zhang et al. 2007; Jones et al. 2007; Lu et al. 2009; Fletcher et al. 2009a; 2009b; Fowler et al. 2010]. In our framework, the RTM imaging condition is connected to a decomposition into polarizations (for an implementation of such a decomposition in quasihomogeneous media, see [Yan and Sava 2007; 2008]).

We develop a comprehensive theory and microlocal analysis of reverse-time imaging for the anisotropic elastic wave equation. We construct a transform that yields inverse scattering up to the contrast-source radiation patterns and which naturally removes the "smooth artifacts" discussed in [Yoon et al. 2004; Mulder and Plessix 2004; Fletcher et al. 2005; Xie and Wu 2006; Guitton et al. 2007]. Our work is based on results presented in [de Hoop and de Hoop 2000; Stolk and de Hoop 2002] while assuming a common-source data acquisition. The main results are: (i) the introduction of an (anisotropic) elastic-wave RTM inverse scattering transform, and (ii) the reformulation of (i) using mode-converted wave constituents removing the knowledge of the source while introducing the notion of array receiver functions, which generalize the notion of receiver functions in planarly layered media. Under the assumption of absence of source caustics (the generation of caustics between the source and scattering points), the RTM inverse scattering transform defines a Fourier integral operator the propagation of singularities of which is described by a canonical graph. The array receiver functions provide a seamless integration of passive source and active source approaches to inverse scattering.

A key application concerns the reconstruction of discontinuities in Earth's upper mantle, such as the Moho (the crust-mantle interface) and the 660 discontinuity (the discontinuity at an approximate depth of 660 km marking the lower boundary of the upper mantle transition zone). In Figure 1 we illustrate the propagation of singularities associated with certain body-wave reflections off and mode conversion at a conormal singularity (a piece of smooth interface) in the transition zone.

Over the past decades, converted seismic waves have been extensively used in global seismology to identify discontinuities in earth's crust, lithosphere-asthenosphere boundary, and mantle transition zone. The method commonly

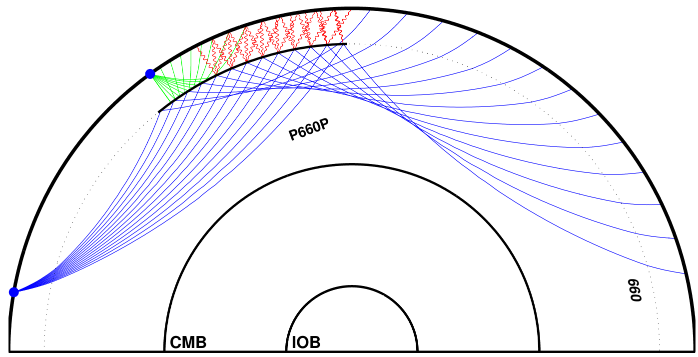

Figure 1. Propagation of singularities (body-wave phases) in Earth's mantle (for illustration purposes we use here a spherically symmetric isotropic model). The dots indicate the locations of seismic events. (We note the absence of source-wave caustics; for underside reflections a caustic is generated between scattering points and receiver networks.) A singular coefficient perturbation is indicated by a curved line segment (representing the 660 discontinuity). To image and characterize this perturbation, we use topside (P-wave) reflections (green ray segments), underside (P- or S-wave) reflections (blue ray segments) and (P-to-S) mode conversions (red squiggly lines). We note the possibility, with limited regions data acquisition, to globally illuminate singularities in Earth's transition zone. (CMB: core-mantle boundary; ICB: inner-core boundary.)

used has been the one of receiver functions, which were introduced and developed in [Vinnik 1977] and [Langston 1979]. In this method, essentially, the converted (scattered) S-wave observation is deconvolved (in time) with the corresponding incident P-wave observation at each available receiver, and assumes a planarly layered earth model. Various refinements have been developed for arrays of receivers. We mention binning according to common-conversion points [Dueker and Sheehan 1997] and diffraction stacking [Revenaugh 1995]. An analysis of (imaging with) receiver functions starting from plane-wave single scattering has been given in [Rydberg and Weber 2000]. (Plane-wave) Kirchhoff migration for mode-converted waves was considered in [Bostock 1999] and [Poppeliers and Pavlis 2003], while its extension to wave-form inversion was developed in [Frederiksen and Revenaugh 2004]. Receiver functions, however, being bilinear in the data (through cross correlation in time), do not fit a description directly in terms of Kirchhoff migration, being linear in the data. We resolve this issue by making precise under which limiting assumptions receiver function imaging is equivalent with (Kirchhoff-style) RTM via the synthesis of source plane waves.

The outline of the paper is as follows. In the next section, we summarize our inverse scattering procedures both for a known source and an unknown source. In Section 3 we discuss various aspects of the parametrix construction for the elastic wave equation, as well as the WKBJ approximation. In Section 4, we introduce the single scattering approximation and the notion of continued scattered field. In Section 5, we discuss and analyze reverse-time continuation from the boundary. In Section 6 we present the inverse scattering analysis, and in Section 7 we construct array receiver functions. We also discuss how receiver functions that are commonly used in global seismology can be recovered from array receiver functions in flat, planarly layered earth models using the WKBJ approximation. In Section 8 we discuss applications in global seismology and conclude with some final remarks.

2. Reverse-time migration based inverse scattering

2A. ***Elastic waves.*** The propagation and scattering of seismic waves is governed by the elastic wave equation, which is written in the form

$$P_{il} u_l = 0, \tag{2-1}$$

$$u_l|_{t=0} = 0, \quad \partial_t u_l|_{t=0} = h_l, \tag{2-2}$$

where

$$u_l = \sqrt{\rho}\,(\text{displacement})_l, \tag{2-3}$$

and

$$P_{il} = \delta_{il}\frac{\partial^2}{\partial t^2} + A_{il}, \quad A_{il} = -\frac{\partial}{\partial x_j}\frac{c_{ijkl}(x)}{\rho(x)}\frac{\partial}{\partial x_k} + \text{l.o.t.}, \tag{2-4}$$

where l.o.t. stands for lower-order terms. Here, $x \in \mathbb{R}^n$ and the subscripts $i, j, k, l \in \{1, \dots, n\}$; $c_{ijkl} = c_{ijkl}(x)$ denotes the stiffness tensor and $\rho = \rho(x)$ the density of mass. The system of partial differential equations is assumed to be of principal type. It supports different wave types (modes). System (2-1) is real, time reversal invariant, and its solutions satisfy reciprocity.

We decompose the medium into a smooth background model and a singular contrast, and assume that the contrast is supported in a bounded subset X of $\mathbb{R}^n$.

Polarizations. We consider here propagation in the background model which has smoothly varying coefficients. Decoupling of the modes is then accomplished by diagonalizing the system. We describe how the system (2-1) can be decoupled by transforming it with appropriate matrix-valued pseudodifferential operators, $Q(x, D_x)_{iM}$, $D_x = -\mathrm{i}\partial/\partial x$; see [Taylor 1975; Ivrii 1979; Dencker 1982]. Since the time derivative in P_{il} is already in diagonal form, it remains only to diagonalize its spatial part, $A_{il}(x, D_x)$. The goal becomes finding Q_{iM} and A_M such that

$$Q(x, D_x)^{-1}_{Mi}\, A_{il}(x, D_x)\, Q(x, D_x)_{lN} = \operatorname{diag}(A_M(x, D_x)\,;\, M = 1, \ldots, n)_{MN}. \tag{2-5}$$

The indices M, N denote the mode of propagation. Then

$$u_M = Q(x, D_x)^{-1}_{Mi} u_i, \quad h_M = Q(x, D_x)^{-1}_{Mi} h_i \tag{2-6}$$

satisfy the uncoupled equations

$$P_M(x, D_x, D_t) u_M = 0, \tag{2-7}$$

$$u_M|_{t=0} = 0, \quad \partial_t u_M|_{t=0} = h_M, \tag{2-8}$$

where $D_t = -\mathrm{i}\,\partial/\partial t$, and in which

$$P_M(x, D_x, D_t) = \partial_t^2 + A_M(x, D_x).$$

Because of the properties of stiffness related to (i) the conservation of angular momentum, (ii) the properties of the strain-energy function, and (iii) the positivity of strain energy, subject to the adiabatic and isothermal conditions, the principal symbol $A^{\text{prin}}_{il}(x, \xi)$ of $A_{il}(x, D_x)$ is a positive, symmetric matrix. Hence, it can be diagonalized by an orthogonal matrix. On the level of principal symbols, composition of pseudodifferential operators reduces to multiplication. Therefore, we let $Q^{\text{prin}}_{iM}(x, \xi)$ be this orthogonal matrix, and we let $A^{\text{prin}}_{M}(x, \xi)$ be the eigenvalues of $A^{\text{prin}}_{il}(x, \xi)$, so that

$$Q^{\text{prin}}_{Mi}(x, \xi)^{-1} A^{\text{prin}}_{il}(x, \xi) Q^{\text{prin}}_{lN}(x, \xi) = \operatorname{diag}(A^{\text{prin}}_{M}(x, \xi))_{MN}. \tag{2-9}$$

The principal symbol $Q^{\text{prin}}_{iM}(x, \xi)$ is the matrix that has as its columns the orthonormalized polarization vectors associated with the modes of propagation. If the $A^{\text{prin}}_{M}(x, \xi)$ are all different, A_{il} can be diagonalized with a unitary operator, that is, $Q(x, D_x)^{-1} = Q(x, D_x)^*$; see Appendix.

We introduce $B_M(x, D_x) = \sqrt{A_M(x, D_x)}$. Furthermore, we introduce boundary normal coordinates, $x = (x', x_n)$; that is, $x' = (x_1, \ldots, x_{n-1})$ and $x_n = 0$ defines the boundary. We also write $z = x_n$ and $\zeta = \xi_n$. We let Σ denote a bounded open subset of the boundary where the receivers are placed. In Section 3C, we also introduce operators $C_\mu(x', z, D_{x'}, D_t)$, the principal parts of the symbols, $C_\mu(x', z, \xi', \tau)$, of which are the solutions for ζ of

$$A^{\text{prin}}_{M}(x', z, \xi', \zeta) = \tau^2. \tag{2-10}$$

2B. ***A known source.*** We introduce the polarized Green's function, G_N, to be the causal solution of the equation

$$P_N(x, D_x, D_t) G_N(x, \tilde{x}, t) = \delta_{\tilde{x}}(x)\delta_0(t); \tag{2-11}$$

G_N is identified as the source or incident field.

We let d_{MN} denote the N-to-M converted data, which, for a given source at $\tilde{x}$, are observed on $\Sigma \times (0, T)$. We introduce the reverse-time continued field, v_r, as the anticausal solution of the equation

$$\begin{aligned}[\partial_t^2 + A_M(x, D_x)]\, v_r(x,t)& \\ = \delta(x_n)\, \mathcal{N}_M(x', D_{x'}, D_t)\, &\Psi_{\mu,\Sigma}(x', t, D_{x'}, D_t) d_{MN}(x', t; \tilde{x}).\end{aligned} \tag{2-12}$$

Here,

$$\mathcal{N}_M(x', D_{x'}, D_t) = -2\mathrm{i} D_t \frac{\partial B_M^{\text{prin}}}{\partial \xi_n}\big(x', 0, D_t^{-1} D_{x'}, C_\mu(0, x', D_t^{-1} D_{x'}, 1)\big), \tag{2-13}$$

and $\Psi_{\mu,\Sigma}$ is a pseudodifferential cutoff, which removes grazing rays. We define first-order partial differential and pseudodifferential operators $\Xi(x, D_x, D_t)$ and $\Theta(x, D_x, D_t)$ with (principal) symbols

$$\Xi_0(x, \xi, \tau) = \tau, \quad \Xi_j(x, \xi, \tau) = \xi_j$$

and

$$\Theta_0(x, \xi, \tau) = \tau, \quad \Theta_j(x, \xi, \tau) = \tau \frac{\partial B_M^{\text{prin}}}{\partial \xi_j}(x, \xi).$$

We then define the operator, H_{MN}, as

$$\begin{aligned}(H_{MN} d_{MN})_{ijkl}(x) = &\\ -\frac{1}{2\pi} \int \frac{2\Omega(\tau)}{\mathrm{i}\tau\, |\widehat{G}_N(x, \tilde{x}, \tau)|^2} &\sum_{p=0}^{n} \left(\frac{\partial}{\partial x_k}\, Q(x, D_x)_{lN}\, \Xi_p(x, D_x, \tau) \overline{\widehat{G}_N(x, \tilde{x}, \tau)} \right) \\ &\times \left(\frac{\partial}{\partial x_j}\, Q(x, D_x)_{iM}\, \Theta_p(x, D_x, \tau) \widehat{v}_r(x, \tau) \right) \mathrm{d}\tau\end{aligned} \tag{2-14}$$

for imaging the contrast in stiffness tensor in X, and similarly for the density contrast (to be indexed by a subscript $_0$) upon replacing $\partial/\partial x_j$ and $\partial/\partial x_k$ by $\mathrm{i}\tau$ and the subscript l by i. In this expression, $\widehat{\ }$ denotes the Fourier transform in time. The Fourier multiplier $\Omega(\tau)$ is a smooth function which is zero in a neighborhood of $\tau = 0$. In Theorem 6.2 we present the inverse scattering properties of operator H_{MN}.

2C. ***An unknown source and array receiver functions.*** In the case of mode conversions ($M \neq N$) one may observe separately the source field; the incident field data are represented by d_N. We then introduce the reverse-time continued field, $w_{\tilde{x};r}$, as the anticausal solution of the equation

$$[\partial_t^2 + A_N(x, D_x)]\, w_{\tilde{x};r}(x,t) \\ = \delta(x_n)\, \mathcal{N}_N(x', D_{x'}, D_t)\, \Psi_{\nu,\Sigma}(x', t, D_{x'}, D_t) d_N(x', t; \tilde{x}). \quad (2\text{-}15)$$

We replace $\widehat{G}_N(x, \tilde{x}, \tau)$ by $\widehat{w}_{\tilde{x};r}(x, \tau)$ in (2-14). In Lemma 7.1, we obtain an operator which is bilinear in the data in as much as it acts on array receiver functions, which we define as

Definition 2.1. For $M \neq N$, the array receiver function (ARF), R_{MN}, is defined as the bilinear map, $d \to \mathsf{R}_{MN}$, with

$$(\mathsf{R}_{MN}(d(\,\cdot\,, \cdot\,; \tilde{x})))(r, t, r') \\ = \int (\mathcal{N}_M \Psi_{\mu,\Sigma} d_{MN})(r, t' + t; \tilde{x})\, (\mathcal{N}_N \Psi_{\nu,\Sigma} d_N)(r', t'; \tilde{x})\, \mathrm{d}t' \quad (2\text{-}16)$$

(no sums over N, M).

3. Parametrix construction

Having assumed that P_{il} is of principal type, the multiplicities of the eigenvalues $A_M^{\text{prin}}(x, \xi)$ are constant, whence the principal symbol $Q_{iM}^{\text{prin}}(x, \xi)$ depends smoothly on (x, ξ) and microlocally Equation (2-9) carries over to an operator equation. Taylor [1975] has shown that if this condition is satisfied, then decoupling can be accomplished to all orders.

The second-order equations (2-7) inherit the symmetries of the original system, such as time-reversal invariance and reciprocity. Time-reversal invariance follows because the operators $Q_{iM}(x, D_x)$, $A_M(x, D_x)$ can be chosen in such a way that $Q_{iM}(x, \xi) = -\overline{Q_{iM}(x, -\xi)}$, $A_M(x, \xi) = \overline{A_M(x, \xi)}$. Then Q_{iM}, A_M are real-valued. Reciprocity for the causal Green's function $G_{ij}(x, x_0, t)$ means that $G_{ij}(x, x_0, t) = G_{ji}(x_0, x, t)$. Such a relationship also holds (modulo smoothing operators) for the Green's function $G_M(x, x_0, t)$ associated with (2-7).

Remark 3.1. In the isotropic case, for $n = 3$, the symbol matrix $A_{il}^{\text{prin}}(x, \xi)$ attains the form

$$\rho A_{il}^{\text{prin}}(x, \xi) = \begin{pmatrix} (\lambda+\mu)\xi_1^2 + \mu|\xi|^2 & (\lambda+\mu)\xi_1\xi_2 & (\lambda+\mu)\xi_1\xi_3 \\ (\lambda+\mu)\xi_1\xi_2 & (\lambda+\mu)\xi_2^2 + \mu|\xi|^2 & (\lambda+\mu)\xi_2\xi_3 \\ (\lambda+\mu)\xi_1\xi_3 & (\lambda+\mu)\xi_2\xi_3 & (\lambda+\mu)\xi_3^2 + \mu|\xi|^2 \end{pmatrix},$$

where $\lambda = \lambda(x)$ and $\mu = \mu(x)$ denote the Lamé parameters. We find that

$$\widetilde{Q}^{\text{prin}} = \widetilde{Q}^{\text{prin}}(\xi) = \begin{pmatrix} | & | & | \\ \widetilde{Q}_{\text{P}} & \widetilde{Q}_{\text{SV}} & \widetilde{Q}_{\text{SH}} \\ | & | & | \end{pmatrix},$$

which is independent of x and where

$$\widetilde{Q}_{\text{P}} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}, \quad \widetilde{Q}_{\text{SH}} = n \times \widetilde{Q}_{\text{P}} = \begin{pmatrix} -\xi_2 \\ \xi_1 \\ 0 \end{pmatrix}, \quad \widetilde{Q}_{\text{SV}} = \widetilde{Q}_{\text{P}} \times \widetilde{Q}_{\text{SH}} = \begin{pmatrix} -\xi_1\xi_3 \\ -\xi_2\xi_3 \\ \xi_1^2 + \xi_2^2 \end{pmatrix},$$

with $n = (0,0,1)^t$, diagonalizes $A_{il}^{\text{prin}}(x,\xi)$:

$$\operatorname{diag}(\rho A_M^{\text{prin}}(x,\xi)\,;\, M = 1,\dots,n) = \begin{pmatrix} (\lambda + 2\mu)\,|\xi|^2 & 0 & 0 \\ 0 & \mu\,|\xi|^2 & 0 \\ 0 & 0 & \mu\,|\xi|^2 \end{pmatrix}.$$

The polarizations are identified as P, SV and SH. Upon normalizing the columns of $\widetilde{Q}^{\text{prin}}$, we obtain the unitary symbol matrix, Q^{prin}, with

$$(Q^{\text{prin}})^{-1} = (Q^{\text{prin}})^* = \begin{pmatrix} \dfrac{\xi_1}{|\xi|} & \dfrac{\xi_2}{|\xi|} & \dfrac{\xi_3}{|\xi|} \\ \dfrac{-\xi_1\xi_3}{(\xi_1^2+\xi_2^2)^{1/2}|\xi|} & \dfrac{-\xi_2\xi_3}{(\xi_1^2+\xi_2^2)^{1/2}|\xi|} & \dfrac{(\xi_1^2+\xi_2^2)^{1/2}}{|\xi|} \\ \dfrac{-\xi_2}{(\xi_1^2+\xi_2^2)^{1/2}} & \dfrac{\xi_1}{(\xi_1^2+\xi_2^2)^{1/2}} & 0 \end{pmatrix}.$$

We note that $\widetilde{Q}_{\text{SV}}$ and $\widetilde{Q}_{\text{SH}}$ are zero if $\xi \parallel n$. This reflects the fact that it is not possible to construct a nonvanishing continuous tangent vector field on S^2 (the Euler characteristic of S^2 is nonvanishing).

With the projections onto P and S, it follows that

$$Q_{i1}^{\text{prin}}(Q^{\text{prin}})^*_{1j}u_j = \Big(-\nabla\big(-\Delta^{-1}\big(\nabla\cdot\big(\, u_1 \;\; u_2 \;\; u_3\,\big)^T\big)\big)\Big)_i$$

and

$$[Q_{i2}^{\text{prin}}(Q^{\text{prin}})^*_{2j} + Q_{i3}^{\text{prin}}(Q^{\text{prin}})^*_{3j}]\,u_j = \Big(\nabla\times\big(-\Delta^{-1}\big(\nabla\times\big(\, u_1 \;\; u_2 \;\; u_3\,\big)^T\big)\big)\Big)_i$$

in accordance with the Helmholtz decomposition of u. Here superscript ^{T} denotes transposition.

3A. ***A particular oscillatory integral representation.*** To evaluate the parametrix, we use the first-order system for u_M that is equivalent to (2-7),

$$\frac{\partial}{\partial t}\begin{pmatrix} u_M \\ \frac{\partial u_M}{\partial t}\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -A_M(x, D_x) & 0 \end{pmatrix}\begin{pmatrix} u_M \\ \partial u_M/\partial t\end{pmatrix}. \tag{3-1}$$

This system can be decoupled also, namely, by the matrix-valued pseudodifferential operators

$$V_M(x, D_x) = \begin{pmatrix} 1 & 1 \\ -\mathrm{i}B_M(x, D_x) & \mathrm{i}B_M(x, D_x) \end{pmatrix},$$
$$\Lambda_M(x, D_x) = \tfrac{1}{2}\begin{pmatrix} 1 & \mathrm{i}B_M(x, D_x)^{-1} \\ 1 & -\mathrm{i}B_M(x, D_x)^{-1} \end{pmatrix},$$

where $B_M(x, D_x) = \sqrt{A_M(x, D_x)}$ is a pseudodifferential operator of order 1 that exists because $A_M(x, D_x)$ is positive definite. The principal symbol of $B_M(x, D_x)$ is given by $B_M^{\mathrm{prin}}(x, \xi) = \sqrt{A_M^{\mathrm{prin}}(x, \xi)}$. (In the isotropic case — see Remark 3.1 — we have $B_{\mathrm{P}}^{\mathrm{prin}}(x, \xi) = \rho^{-1}(\lambda + 2\mu)(x)\,|\xi|$ and $B_{\mathrm{SV}}^{\mathrm{prin}}(x, \xi) = B_{\mathrm{SH}}^{\mathrm{prin}}(x, \xi) = \rho^{-1}\mu(x)\,|\xi|$.) Then

$$u_{M,\pm} = \tfrac{1}{2}u_M \pm \tfrac{1}{2}\mathrm{i}B_M(x, D_x)^{-1}\frac{\partial u_M}{\partial t} \tag{3-2}$$

satisfy the two first-order ("half wave") equations

$$P_{M,\pm}(x, D_x, D_t)u_{M,\pm} = 0, \tag{3-3}$$

where

$$P_{M,\pm}(x, D_x, D_t) = \partial_t \pm \mathrm{i}B_M(x, D_x), \quad P_{M,+}P_{M,-} = P_M, \tag{3-4}$$

supplemented with the initial conditions

$$u_{M,\pm}|_{t=0} = h_{M,\pm}, \quad h_{M,\pm} = \pm\tfrac{1}{2}\mathrm{i}B_M(x, D_x)^{-1}h_M. \tag{3-5}$$

We construct operators $S_{M,\pm}(t)$ that solve the initial value problem (3-3), (3-5): $u_{M,\pm}(y, t) = (S_{M,\pm}(t)h_{M,\pm})(y)$; then

$$u_M(y, t) = ([S_{M,+}(t) - S_{M,-}(t)]\tfrac{1}{2}\mathrm{i}B_M^{-1}h_M)(y).$$

The operators $S_{M,\pm}(t)$ are Fourier integral operators. Their construction is well known; see for example [Duistermaat 1996, Chapter 5]. Microlocally, the solution operator associated with (3-1) can be written in the form

$$S_M(t) = V_M\begin{pmatrix} S_{M,+}(t) & 0 \\ 0 & S_{M,-}(t)\end{pmatrix}\Lambda_M;$$

in this notation, $S_{M,12}(t) = ([S_{M,+}(t) - S_{M,-}(t)]\tfrac{1}{2}\mathrm{i}B_M^{-1}$.

For the later analysis, we introduce the operators $S_M(t,s)$ and $S_{M,\pm}(t,s)$: $S_M(t,s)$ solves the problem

$$\begin{aligned} &P_M(x, D_x, D_t) S_M(\,\cdot\,, s) = 0, \\ &S_M(\,\cdot\,, s)|_{t=s} = 0, \quad \partial_t S_M(\,\cdot\,, s)|_{t=s} = \mathrm{Id}, \end{aligned}$$

so that the solution of

$$P_M(x, D_x, D_t) u_M = f_M, \quad u_M(t<0) = 0,$$

is given by

$$\begin{aligned} u_M(y,t) &= \int_0^t \mathsf{P}_1 S_M(t,s) \begin{pmatrix} 0 \\ f_M(\,\cdot\,, s) \end{pmatrix} (y)\, \mathrm{d}s \\ &= \iint G_M(y, x, t-s) f_M(x,s)\, \mathrm{d}x\, \mathrm{d}s, \end{aligned}$$

where we identified the causal Green's function $G_M(y,x,t-s)$. Here, P_1 is the projection onto the first component. Likewise, $S_{M,+}(t,s)$ solves (for $t \in \mathbb{R}$) the problem

$$\begin{aligned} &P_{M,+}(x, D_x, D_t) S_{M,+}(\,\cdot\,, s) = 0, \\ &S_{M,+}(\,\cdot\,, s)|_{t=s} = \mathrm{Id}, \end{aligned}$$

so that the causal solution of

$$P_{M,+}(x, D_x, D_t) u_{M,+} = f_{M,+}$$

is given by

$$\begin{aligned} u_{M,+}(y,t) &= \int_{-\infty}^t (S_{M,+}(t,s) f_{M,+}(\,\cdot\,, s))(y)\, \mathrm{d}s \\ &= \iint G_{M,+}(y, x, t-s) f_{M,+}(x,s)\, \mathrm{d}x\, \mathrm{d}s, \end{aligned}$$

while the anticausal solution is given by

$$\begin{aligned} u_{M,+}(y,t) &= -\int_t^\infty (S_{M,+}(t,s) f_{M,+}(\,\cdot\,, s))(y)\, \mathrm{d}s \\ &= \iint G_{M,+}(y, x, s-t) f_{M,+}(x,s)\, \mathrm{d}x\, \mathrm{d}s. \end{aligned}$$

A similar construction holds with $+$ replaced by $-$.

For sufficiently small t (in the absence of conjugate points), we obtain the oscillatory integral representation

$$(S_{M,\pm}(t)h_{M,\pm})(y) = (2\pi)^{-n} \iint a_{M,\pm}(y,t,\xi) \exp\bigl(\mathrm{i}\phi_{M,\pm}(y,t,x,\xi)\bigr) h_{M,\pm}(x)\, \mathrm{d}x\, \mathrm{d}\xi, \quad \text{(3-6)}$$

where

$$\phi_{M,\pm}(y,t,x,\xi) = \alpha_{M,\pm}(y,t,\xi) - \langle \xi, x\rangle. \quad \text{(3-7)}$$

We note that $\alpha_{M,-}(y,t,\xi) = -\alpha_{M,+}(y,t,-\xi)$. Singularities are propagated along the bicharacteristics, that are determined by Hamilton's equations generated by the principal symbol $\pm B_M^{\text{prin}}(x,\xi)$

$$\frac{\mathrm{d}y^t}{\mathrm{d}t} = \pm \frac{\partial B_M^{\text{prin}}(y^t,\eta^t)}{\partial \eta}, \qquad \frac{\mathrm{d}\eta^t}{\mathrm{d}t} = \mp \frac{\partial B_M^{\text{prin}}(y^t,\eta^t)}{\partial y}. \quad \text{(3-8)}$$

(In the seismological literature, one refers to "ray tracing".) We denote the solution of (3-8) with the $+$ sign and initial values (x,ξ) at $t=0$ by

$$(y_M^t(x,\xi), \eta_M^t(x,\xi)) = \Phi_M^t(x,\xi).$$

The solution with the $-$ sign is found upon reversing the time direction and is given by $(y_M^{-t}(x,\xi), \eta_M^{-t}(x,\xi))$. Away from conjugate points, y_M^t and ξ determine η_M^t and x; we write $x = x_M^t(y,\xi)$. Then

$$\alpha_{M,+}(y,t,\xi) = \langle \xi, x_M^t(y,\xi)\rangle.$$

To highest order,

$$a_{M,+}(y,t,\xi) = \left| \left.\frac{\partial(y_M^t)}{\partial(x)}\right|_{\xi, x=x_M^t(y,\xi)} \right|^{-1/2}. \quad \text{(3-9)}$$

We consider the perturbations of (y_M^t, η_M^t) with respect to the initial conditions (x,ξ),

$$W_M^t(x,\xi) = \begin{pmatrix} W_{M,1}^t(x,\xi) & W_{M,2}^t(x,\xi) \\ W_{M,3}^t(x,\xi) & W_{M,4}^t(x,\xi) \end{pmatrix} = \begin{pmatrix} \partial_x y_M^t(x,\xi) & \partial_\xi y_M^t(x,\xi) \\ \partial_x \eta_M^t(x,\xi) & \partial_\xi \eta_M^t(x,\xi) \end{pmatrix}. \quad \text{(3-10)}$$

This matrix solves the (linearized) Hamilton–Jacobi equations,

$$\frac{\mathrm{d}W^t}{\mathrm{d}t}(x,\xi) = \begin{pmatrix} \partial_{\eta y} B_M^{\text{prin}}(y^t,\eta^t) & \partial_{\eta\eta} B_M^{\text{prin}}(y^t,\eta^t) \\ -\partial_{yy} B_M^{\text{prin}}(y^t,\eta^t) & -\partial_{y\eta} B_M^{\text{prin}}(y^t,\eta^t) \end{pmatrix} W^t(x,\xi), \quad \text{(3-11)}$$

subject to initial conditions $W^{t=0} = I$. We note that away from conjugate points, the submatrix $W^t_{M,1}$ is invertible. Because

$$x^t_M = \frac{\partial \alpha_{M,+}}{\partial \xi}, \quad \eta^t_M = \frac{\partial \alpha_{M,+}}{\partial y},$$

integration of (3-11) along (y^t, η^t) yields

$$\frac{\partial^2 \alpha_{M,+}}{\partial y \partial \xi}(y^t_M(x,\xi), t, \xi) = (W^t_{M,1}(x,\xi))^{-1}, \tag{3-12}$$

$$\frac{\partial^2 \alpha_{M,+}}{\partial \xi^2}(y^t_M(x,\xi), t, \xi) = (W^t_{M,1}(x,\xi))^{-1} W^t_{M,2}(x,\xi), \tag{3-13}$$

$$\frac{\partial^2 \alpha_{M,+}}{\partial y^2}(y^t_M(x,\xi), t, \xi) = W^t_{M,3}(x,\xi)(W^t_{M,1}(x,\xi))^{-1}, \tag{3-14}$$

which we evaluate at $x = x^t_M(y,\xi)$. It follows that

$$a_{M,+}(y,t,\xi) = \left| \det W^t_{M,1}|_{x=x^t_M(y,\xi),\xi} \right|^{-1/2}.$$

The amplitude of $S_{M,+}(t)\,\frac{1}{2}\mathrm{i}B_M^{-1}$, then becomes

$$\tilde{a}_{M,+}(y,t,\xi) = a_{M,+}(y,t,\xi)\,\tfrac{1}{2}\mathrm{i}B_M^{\text{prin}}(x^t_M(y,\xi),\xi)^{-1} \tag{3-15}$$

to leading order. The amplitude $a_{M,-}$ follows from time reversal:

$$a_{M,-}(y,t,\xi) = \overline{a_{M,+}(y,t,-\xi)}.$$

3B. ***Absence of caustics: The source field.*** In the absence of caustics, we can change phase variables in the oscillatory integral representation of G_N according to

$$\begin{aligned} G_{N,+}(y,x,t) &= (2\pi)^{-1} \int (2\pi)^{-n} \int a_{N,+}(y,t',\xi) \exp\big(\mathrm{i}\phi_{N,+}(y,t',x,\xi)\big)\,\mathrm{d}\xi \\ &\qquad\qquad \times \exp\big(\mathrm{i}\tau(t-t')\big)\,\mathrm{d}t'\,\mathrm{d}\tau \\ &= (2\pi)^{-1} \int a'_{N,+}(y,x,\tau) \exp\big(\mathrm{i}\tau(t - T_N(y,x))\big)\,\mathrm{d}\tau. \end{aligned}$$

We find the leading-order contribution to $a'_{N,+} = \mathscr{A}_{N,+}$ by applying the method of stationary phase in the variables (ξ, t'):

$$\frac{\partial \alpha_{N,+}}{\partial \xi}(y,t',\xi) = x, \tag{3-16}$$

$$\frac{\partial \alpha_{N,+}}{\partial t'}(y,t',\xi) = \tau, \tag{3-17}$$

at $\xi = \xi(y, x, \tau)$, $t' = t'(y, x, \tau) = T_N(y, x)$; $\xi(y, x, \tau)$ is homogeneous of degree 1 in τ, whence $\partial\xi/\partial\tau = \tau^{-1}\xi$. With the matrix product

$$\begin{pmatrix} W^t_{N,1} & 0 \\ 0 & 1 \end{pmatrix}\Bigg|_{t=T_N(y,x),\xi=\xi(y,x,\tau)} \overbrace{\begin{pmatrix} \frac{\partial^2\alpha_{N,+}}{\partial\xi^2} & \frac{\partial^2\alpha_{N,+}}{\partial\xi\partial t} \\ \frac{\partial^2\alpha_{N,+}}{\partial t\partial\xi} & \frac{\partial^2\alpha_{N,+}}{\partial t^2} \end{pmatrix}}^{\Delta(y,t,\xi)}\Bigg|_{t=T_N(y,x),\xi=\xi(y,x,\tau)} = \begin{pmatrix} W^t_{M,2} & \frac{\partial y^t_N}{\partial t} \\ \frac{\partial\tau}{\partial\xi} & \frac{\partial\tau}{\partial t} \end{pmatrix}\Bigg|_{t=T_N(y,x),\xi=\xi(y,x,\tau)},$$

we find that

$$\begin{aligned} |\mathcal{A}_{N,+}(y, x, \tau)| &= (2\pi)^{-n} a_{N,+}(y, T_N(y, x), \xi(y, x, \tau)) \\ &\qquad \times (2\pi)^{(n+1)/2}\big|\det\Delta(y, T_N(y, x), \xi(y, x, \tau))\big|^{-1/2} \\ &= (2\pi)^{-(n-1)/2}\left|\det\frac{\partial(x, \xi, t)}{\partial(y, x, \tau)}\right|^{1/2}. \end{aligned} \tag{3-18}$$

Furthermore, $\phi_{N,+}(y, T_N(y, x), x, \xi(y, x, \tau)) = 0$. Thus the source field can be written in the form

$$G_N(x, \tilde{x}, t) = (2\pi)^{-1}\int a'_N(x, \tilde{x}, \tau)\exp\big(\mathrm{i}\tau(t - T_N(x, \tilde{x}))\big)\,\mathrm{d}\tau. \tag{3-19}$$

Here, $\tilde{x}$ is the source location and T_N is the travel time satisfying the eikonal equation

$$B_N(x, -\partial_x T_N(x, \tilde{x})) = -1; \tag{3-20}$$

to highest order, $a'_N = \mathcal{A}_N$ with

$$|\mathcal{A}_N(x, \tilde{x}, \tau)| = |\mathcal{A}_{N,\pm}(x, \tilde{x}, \tau)|\,\frac{1}{2|\tau|}. \tag{3-21}$$

We introduce

$$n_{\tilde{x}}(x) = \frac{\partial_x T_N(x, \tilde{x})}{|\partial_x T_N(x, \tilde{x})|},$$

and, using (3-20) and the homogeneity of B_N, we can write

$$\partial_x T_N(x, \tilde{x}) = \frac{1}{B_N(x, n_{\tilde{x}}(x))}\,n_{\tilde{x}}(x). \tag{3-22}$$

With a point-body force, $f_k = e_k\delta_{\tilde{x}}\delta_0$, the polarized source field is modeled by

$$\int G_N(x, \tilde{x}', t)\,\mathcal{Q}^{-1}_{Nk'}(\tilde{x}', \tilde{x})e_{k'}\,\mathrm{d}\tilde{x}' \quad \text{(no sum over } N\text{)}.$$

To simplify the analysis, we will consider a polarized source, $f_k = \mathcal{Q}_{kN}(\,\cdot\,, \tilde{x})\delta_0$, where $\mathcal{Q}$ denotes the kernel of Q. Then the source field reduces to $G_N(x, \tilde{x}, t)$.

We also denote the source field as $w_{\tilde{x}}(x,t)$, and use the time-decomposed wavefields

$$\begin{pmatrix} w_{\tilde{x};+} \\ w_{\tilde{x};-} \end{pmatrix} = \Lambda_N \begin{pmatrix} w_{\tilde{x}} \\ \partial_t w_{\tilde{x}} \end{pmatrix}.$$

We suppress the subscript N in $w_{\tilde{x}}$.

3C. *Flat, smoothly layered media.* Here, we make use of results in [Woodhouse 1974; Garmany 1983; 1988; Fryer and Frazer 1984; 1987; Singh and Chapman 1988]. We introduce coordinates $x = (x', z)$ if $x_n = z$ is the (depth) coordinate normal to the surface, and write $c_{jk;il} = (c_{jk})_{il} = c_{ijkl}$. We consider the displacement, $\rho^{-1/2} u_i$, and the traction, $\sum_{k,l=1}^n c_{nk;il} \partial(\rho^{-1/2} u_l)/\partial x_k$, and form

$$W = \begin{pmatrix} \rho^{-1/2} u_i \\ \sum\limits_{k,l=1}^{n} c_{nk;il} \dfrac{\partial(\rho^{-1/2} u_l)}{\partial x_k} \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ f_i \end{pmatrix}, \; i = 1, \dots, n. \tag{3-23}$$

The elastic wave equation — see (2-1)–(2-4) — can then be rewritten as the system of equations,

$$\frac{\partial W_a}{\partial z} = \mathrm{i} \sum_{b=1}^{2n} C_{ab}(x', z, D_{x'}, D_t) W_b + F_a, \tag{3-24}$$

with

$$C_{ab}(x', z, D_{x'}, D_t) \\ = -\mathrm{i} \begin{pmatrix} -\sum\limits_{q=1}^{n-1} \sum\limits_{j=1}^{n} (c_{nn})^{-1}_{ij} c_{nq;jl} \dfrac{\partial}{\partial x_q} & (c_{nn})^{-1}_{il} \\ -\sum\limits_{p,q=1}^{n-1} \dfrac{\partial}{\partial x_p} b_{pq;il} \dfrac{\partial}{\partial x_q} + \rho \delta_{il} \dfrac{\partial^2}{\partial t^2} & -\sum\limits_{p=1}^{n-1} \dfrac{\partial}{\partial x_p} c_{pn;ij} (c_{nn})^{-1}_{jl} \end{pmatrix}_{ab}, \\ i, l = 1, \dots, n, \tag{3-25}$$

where $b_{pq;il} = c_{pq;il} - \sum\limits_{j,k=1}^{n} c_{pn;ij} (c_{nn})^{-1}_{jk} c_{nq;kl}$. Diagonalizing the system, microlocally, involves

$$C_{ab}(x', z, D_{x'}, D_t) = \sum_{\mu,\nu=1}^{2n} L(x', z, D_{x'}, D_t)_{a\mu} \\ \times \operatorname{diag}(C_\mu(x', z, D_{x'}, D_t); \mu = 1, \dots, 2n)_{\mu\nu} L(x', z, D_{x'}, D_t)^{-1}_{\nu b}; \tag{3-26}$$

the principal parts of the symbols $C_\mu(x', z, \xi', \tau)$ are the solutions for ζ of (2-10).

In smoothly layered media one can Fourier transform (3-24) with respect to x' and t and obtain a system of ordinary differential equations for

$$\widetilde{W}(z)=\widetilde{W}(\xi',z,\tau)=\int W(x',z,t)\,\exp\biggl(-\mathrm{i}\biggl(\sum_{j=1}^{n-1}\xi_j x_j+\tau t\biggr)\biggr)\,\mathrm{d}x'\,\mathrm{d}t,$$

namely

$$\frac{\partial \widetilde{W}_a}{\partial z}=\mathrm{i}\sum_{b=1}^{2n}C_{ab}(z,\xi',\tau)\widetilde{W}_b+\widetilde{F}_a. \tag{3-27}$$

We choose the C_μ such that the homogeneity property

$$C_\mu(z,\xi',\tau)=\tau C_\mu(z,\tau^{-1}\xi',1)$$

extends to $\tau<0$. We have

$$L_{a\mu}(z,\xi',\tau) = \begin{pmatrix} Q_{iM(\mu)}(z,(\xi',C_\mu(z,\xi',\tau))) \\ \sum_{k,l=1}^{n} c_{nk;il}(-\mathrm{i})(\xi',C_\mu(z,\xi',\tau))_k\,Q_{lM(\mu)}(z,(\xi',C_\mu(z,\xi',\tau))) \end{pmatrix}_{a\mu}, \tag{3-28}$$

with inverse

$$L^{-1}(z,\xi',\tau)=N(z,\xi',\tau)L^t(z,\xi',\tau)\,J,\quad \text{where } J=\begin{pmatrix} 0 & I_n \\ I_n & 0\end{pmatrix}. \tag{3-29}$$

Here, $N(z,\xi',\tau)$ is a diagonal normalization matrix, $\operatorname{diag}(N_\mu(z,\xi',\tau))_{\mu\nu}$. It follows that

$$\begin{aligned} N_\mu(z,\xi',\tau)^{-1} &= \sum_{i=1}^{n} Q_{iM(\mu)}(z,(\xi',C_\mu(z,\xi',\tau))) \\ &\quad\times\sum_{k,l=1}^{n}(c_{nk;il}+c_{nk;li})\,(-\mathrm{i})(\xi',C_\mu(z,\xi',\tau))_k\,Q_{lM(\mu)}(z,(\xi',C_\mu(z,\xi',\tau))). \end{aligned} \tag{3-30}$$

The index mapping $\mu\to M(\mu)$ assigns the appropriate mode to the depth component of the wave vector.

We cast (3-27) into an equivalent initial value problem. Let $\widetilde{W}_{ab}(z,z_0)$ be the solution to

$$\frac{\partial \widetilde{W}_a}{\partial z}=\mathrm{i}\sum_{b=1}^{2n}C_{ab}(z,\xi',\tau)\widetilde{W}_b,\quad \widetilde{W}(z_0)=I_{2n}.$$

Then $\widetilde{W}_a(z) = \int_{z_0}^{z} \sum_{b=1}^{2n} \widetilde{W}_{ab}(z, z_0)\widetilde{F}_b(z_0)\,\mathrm{d}z_0$ solves (3-27). We introduce

$$\dot{W} = \begin{pmatrix} \rho^{-1/2}u_i \\ \sum_{k,l=1}^{n} c_{nk;il} D_t^{-1} \dfrac{\partial(\rho^{-1/2}u_l)}{\partial x_k} \end{pmatrix}, \quad \dot{F} = \begin{pmatrix} 0 \\ D_t^{-1} f_i \end{pmatrix}; \tag{3-31}$$

with $\xi' = \tau p'$, we make the identification $\tilde{\dot{W}}(p', z, \tau) = \widetilde{W}(\tau p', z, \tau)$, whereas

$$\frac{\partial \tilde{\dot{W}}_a}{\partial z} = \mathrm{i}\tau \sum_{b=1}^{2n} C_{ab}(z, p', 1)\tilde{\dot{W}}_b + \tilde{\dot{F}}_b.$$

In the WKBJ approximation, in the absence of turning rays (the characteristics are nowhere horizontal), we have

$$\begin{aligned} &\tilde{\dot{W}}_{ab}(z, z_0) \\ &\quad\approx \sum_{\mu=1}^{2n} L_{a\mu}(z, p', 1)Y_\mu(z, p', 1) \exp\left(\mathrm{i}\tau \int_{z_0}^{z} C_\mu(\bar{z}, p', 1)\,\mathrm{d}\bar{z}\right) \\ &\qquad\qquad\qquad\qquad \times Y_\mu(z_0, p', 1)^{-1} L_{\mu b}^{-1}(z_0, p', 1) \\ &\quad= \sum_{\mu=1}^{2n} L_{a\mu}(z, p', 1)Y_\mu(z, p', 1) \exp\left(\mathrm{i}\tau \int_{z_0}^{z} C_\mu(\bar{z}, p', 1)\,\mathrm{d}\bar{z}\right) \\ &\qquad\qquad\qquad\qquad \times Y_\mu(z_0, p', 1)(L^t(z_0, p', 1)J)_{\mu b}. \end{aligned}$$

Here, $Y_\mu(z, p', 1) = [N_\mu(z, p', 1)]^{1/2}$. We identify the "vertical" travel time

$$\tau_\mu(z, z_0, p') = -\int_{z_0}^{z} C_\mu(\bar{z}, p', 1)\,\mathrm{d}\bar{z}. \tag{3-32}$$

To obtain the tensor G_{ij}, we substitute a δ source for f_i, yielding $J\widetilde{F} = \begin{pmatrix} I_n \\ 0 \end{pmatrix} \delta(\cdot - z_0)$:

$$\begin{aligned} &G_{ij}(x', z, x_0', z_0, t - t_0) \\ &\approx {\sum_{\mu=1}^{2n}}' \frac{1}{(2\pi)^n} \iint Q_{iM(\mu)}\big(z, (p', C_\mu(z, p', 1))\big) Y_\mu(z, p', 1) \\ &\qquad \times \exp\left(\mathrm{i}\tau\left(-\tau_\mu(z, z_0, p') + \sum_{l=1}^{n-1} p_l'(x' - x_0')_l + t - t_0\right)\right) \\ &\qquad \times Y_\mu(z_0, p', 1)\, Q_{M(\mu)j}^t\big(z_0, (p', C_\mu(z, p', 1))\big)\,\mathrm{d}p'|\tau|^{n-1}\,\mathrm{d}\tau; \end{aligned} \tag{3-33}$$

which values of μ contribute depends on whether $z > z_0$ ("downgoing") or $z < z_0$ ("upgoing"). The (negative) values of the components of p' associated with the ray connecting (z_0, x_0') with (z, x') is the solution of the equation

$$\partial_{p'} \tau_\mu(z, z_0, p') = x' - x_0'.$$

4. Continued scattered field

Here, we introduce and analyze the scattered field. To this end, we consider the contrast formulation, in which the total value of the medium parameters ρ, c_{ijkl} is written as the sum of a smooth background component $\rho(x), c_{ijkl}(x)$ and a singular perturbation $\delta\rho(x), \delta c_{ijkl}(x)$, namely $\rho+\delta\rho$, $c_{ijkl}+\delta c_{ijkl}$; we assume that $\delta\rho, \delta c_{ijkl} \in \mathscr{E}'(X)$ with X a compact subset of $\mathbb{R}^n$. This decomposition induces a perturbation of P_{il} (cf. (2-4)),

$$\delta P_{il} = \delta_{il} \frac{\delta\rho(x)}{\rho(x)} \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial x_j} \frac{\delta c_{ijkl}(x)}{\rho(x)} \frac{\partial}{\partial x_k}.$$

The first-order perturbation, δG_{il}, of the (causal) kernel G_{il} of the solution operator admits the representation

$$\delta G_{jk}(y, \tilde{x}, t) = -\int_0^t \int_X G_{ji}(y, x, t-t')\, \delta P_{il}(x, D_x, D_{t'})\, G_{lk}(x, \tilde{x}, t')\, \mathrm{d}x\, \mathrm{d}t', \quad (4\text{-}1)$$

which is the Born approximation. Here, $\tilde{x}$ denotes a source location as before, and x a scattering point location. We restrict our time window (of observation) to $(0, T)$ for some $0 < T < \infty$.

We introduce the MN contribution, δG_{MN}, to δG_{jk} as follows:

$$\int \delta G_{jk}(y, \tilde{x}, t-\tilde{t})\, f_k(\tilde{x}, \tilde{t})\, \mathrm{d}\tilde{x}\, \mathrm{d}\tilde{t} = Q(y, D_y)_{jM} \int \delta G_{MN}(y, \tilde{x}, t-\tilde{t})\, (Q(\tilde{x}, D_{\tilde{x}})^{-1}_{Nk} f_k)(\tilde{x}, \tilde{t})\, \mathrm{d}\tilde{x}\, \mathrm{d}\tilde{t}. \quad (4\text{-}2)$$

We apply reciprocity in (y, x) to the integrand of the right-hand side and obtain

$$\begin{aligned}
\delta G_{MN}(y, \tilde{x}, t) &= -\int_0^t \int_X (Q(x, D_x)^{-1})^*_{iM} G_M(x, y, t-t') \\
&\qquad \times \delta P_{il}(x, D_x, D_{t'})\, Q(x, D_x)_{lN} G_N(x, \tilde{x}, t')\, \mathrm{d}x\, \mathrm{d}t' \\
&= -\int_0^t \int_X (Q(x, D_x)^{-1})^*_{iM} G_M(x, y, t-t') \frac{\partial}{\partial(t', x_j)} \left(\delta_{il} \frac{\delta\rho(x)}{\rho(x)}, -\frac{\delta c_{ijkl}(x)}{\rho(x)} \right) \\
&\qquad \times \frac{\partial}{\partial(t', x_k)}\, Q(x, D_x)_{lN} G_N(x, \tilde{x}, t')\, \mathrm{d}x\, \mathrm{d}t'
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \int_X \frac{\partial}{\partial(t, x_j)} \,(Q(x, D_x)^{-1})^*_{iM} G_M(x, y, t-t') \left(\delta_{il} \frac{\delta\rho(x)}{\rho(x)}, -\frac{\delta c_{ijkl}(x)}{\rho(x)}\right) \\
&\qquad\qquad\qquad \times \frac{\partial}{\partial(t', x_k)} \, Q(x, D_x)_{lN} G_N(x, \tilde{x}, t') \,\mathrm{d}x \,\mathrm{d}t' \\
&= \int_X \biggl(\int_0^t \frac{\partial}{\partial(t', x_j)} \, Q(x, D_x)_{iM} G_M(x, y, t-t') \\
&\qquad\qquad \times \frac{\partial}{\partial(t', x_k)} \, Q(x, D_x)_{lN} G_N(x, \tilde{x}, t') \,\mathrm{d}t' \biggr) \\
&\qquad\qquad\qquad \times \left(\delta_{il} \frac{\delta\rho(x)}{\rho(x)}, -\frac{\delta c_{ijkl}(x)}{\rho(x)}\right) \mathrm{d}x \qquad (4\text{-}3)
\end{aligned}
$$

upon integration by parts. Reciprocity implies that

$$
G_M(x, y, t-t') = G_M(y, x, t-t') \quad \text{and} \quad G_N(x, \tilde{x}, t') = G_N(\tilde{x}, x, t').
$$

Also, $\delta G_{MN}(x, \tilde{x}, t)$ is the solution to the initial value problem

$$
P_M(x, D_x, D_t) v = Q(x, D_x)^{-1}_{Mi} \delta P_{il}(x, D_x, D_t) \, Q(x, D_x)_{lN} G_N(x, \tilde{x}, t), \quad (4\text{-}4)
$$

$$
v|_{t=0} = 0, \quad \partial_t v|_{t=0} = 0. \qquad (4\text{-}5)
$$

The continued scattered field, v_h, is defined as the solution to a final value problem such that the Cauchy data at $t = T_1$ coincide with the Cauchy data of the scattered field:

$$
P_M(x, D_x, D_t) v_h = 0 \qquad (4\text{-}6)
$$

$$
v_h|_{t=T_1} = v|_{t=T_1}, \quad \partial_t v_h|_{t=T_1} = \partial_t v|_{t=T_1}. \qquad (4\text{-}7)
$$

We assume that the contributions from the scattered field entirely come to pass within the time interval $[T_0, T_1]$; $T_1 < T$. Then, for $t \geq T_1$, $v_h = v$, but these fields differ from one another for $t < T_1$. The corresponding the time-decomposed wavefields are given by

$$
\begin{pmatrix} v_{h,+} \\ v_{h,-} \end{pmatrix} = \Lambda_M \begin{pmatrix} v_h \\ \partial_t v_h \end{pmatrix}.
$$

We suppress the subscripts M, N in v and v_h.

The single scattering operator, $F(t)$, is defined by the map

$$
\left(\frac{\delta\rho}{\rho}, -\frac{\delta c_{ijkl}}{\rho} \right) \mapsto \begin{pmatrix} v_h \\ \partial_t v_h \end{pmatrix}.
$$

We decompose $F(t)$ into operators $F_\pm(t)$ mapping the pair on the left to $v_{h,\pm}(\,\cdot\,, t)$. We carry out the analysis for a small time interval in the neighborhood of a point in the scattering region, X. Let $\{\chi_\iota\}_{\iota \in \mathcal{J}}$ be a finite partition of unity.

The time interval $[t_{0\imath}, t_{1\imath}]$ satisfies $T_N(\operatorname{supp}(\chi_\imath), \tilde{x}) \subset [t_{0\imath}, t_{1\imath}]$. Then

$$F_+(t) = \sum_{\imath \in \mathscr{J}} S_{M,+}(t - t_{1\imath}) F_+(t_{1\imath}) \chi_\imath,$$

and similarly for $F_-(t)$. We construct an oscillatory integral representation for the kernel of $F_+(t_{1\imath})\chi_\imath$ using the representations developed in Section 3, which is enabled by the partition of unity. We omit the subscript $\imath$ below.

From the source field we get an amplitude contribution

$$\mathscr{A}_N(x, \tilde{x}, \tau) Q^{\text{prin}}_{lN}(x, -\tau \partial_x T_N(x, \tilde{x}))\, \mathrm{i}\tau\, (1, -\partial_{x_k} T_N(x, \tilde{x}))$$

to highest order, and from the solution operator we get an amplitude contribution

$$\widetilde{a}_{M,+}(y, t_1 - T_N(x, \tilde{x}), \xi)\, Q^{\text{prin}}_{iM}(x, -\xi)\, \mathrm{i}(\tau, -\xi_j)$$

to highest order; here

$$\tau = \partial_t \alpha_{M,+}(y, t_1 - T_N(x, \tilde{x}), \xi). \tag{4-8}$$

We introduce the radiation patterns $(w_{MN;0}, w_{MN;ijkl})$ as

$$w_{MN;0}(y,t_1,x,\xi) = -Q^{\text{prin}}_{iM}(x,-\xi)\, Q^{\text{prin}}_{iN}(x,-\tau\partial_x T_N(x,\tilde{x}))\, \tau^2, \tag{4-9}$$

$$w_{MN;ijkl}(y,t_1,x,\xi) = \quad Q^{\text{prin}}_{iM}(x,-\xi)\, Q^{\text{prin}}_{lN}(x,-\tau\partial_x T_N(x,\tilde{x}))\, \xi_j \tau \partial_{x_k} T_N(x,\tilde{x}), \tag{4-10}$$

again subject to the substitution (4-8).

Then

$$\begin{aligned} v_{h,+}(y, t_1) = {} & \\ & (2\pi)^{-n} \iint_X A_{F,MN}(y, t_1, x, \xi) \\ & \quad \times \left(w_{MN;0}(y, t_1, x, \xi) \frac{\delta\rho(x)}{\rho(x)} + w_{MN;ijkl}(y, t_1, x, \xi) \frac{\delta c_{ijkl}(x)}{\rho(x)} \right) \chi(x) \\ & \qquad \times \exp\bigl(\mathrm{i}\varphi_{MN}(y, t_1, x, \xi)\bigr)\, \mathrm{d}x\, \mathrm{d}\xi \end{aligned} \tag{4-11}$$

modulo lower-order terms in amplitude, where

$$\begin{aligned} & A_{F,MN}(y, t_1, x, \xi) \\ & = \widetilde{a}_{M,+}(y, t_1 - T_N(x, \tilde{x}), \xi) \mathscr{A}_N(x, \tilde{x}, \partial_t \alpha_{M,+}(y, t_1 - T_N(x, \tilde{x}), \xi)), \end{aligned} \tag{4-12}$$

and

$$\varphi_{MN}(y, t_1, x, \xi) = \alpha_{M,+}(y, t_1 - T_N(x, \tilde{x}), \xi) - \langle \xi, x \rangle. \tag{4-13}$$

We obtain a similar representation for $v_{h,-}$: $v_{h,-}(y, t_1) = \overline{v_{h,+}(y, t_1)}$. In the above $y \in X$.

$F_+(t_1)\chi$ is a Fourier integral operator if direct source waves are excluded. Lemma 4.1 below implies that the phase function, φ_{MN}, is nondegenerate. The canonical relation, $\Lambda_{MN}^{F_+}$, of $F_+(t_1)\chi$ is obtained as follows. The stationary point set associated with φ_{MN} contains (y, x, ξ) satisfying

$$\partial_\xi \alpha_{M,+}(y, t_1 - T_N(x, \tilde{x}), \xi) = x, \quad x \in \operatorname{supp} \chi; \tag{4-14}$$

$F_+(t_1)\chi$ propagates singularities from (x, ζ) with

$$\zeta = \xi + \partial_t \alpha_{M,+}(y, t_1 - T_N(x, \tilde{x}), \xi)\, \partial_x T_N(x, \tilde{x}), \tag{4-15}$$

to (y, t_1, η, τ) with

$$\eta = \partial_y \alpha_{M,+}(y, t_1 - T_N(x, \tilde{x}), \xi). \tag{4-16}$$

We note that

$$\partial_t \alpha_{M,+}(y, t_1 - T_N(x, \tilde{x}), \xi) = -B_M^{\text{prin}}(x, \xi).$$

Thus we can write

$$\partial_t \alpha_{M,+}(y, t_1 - T_N(x, \tilde{x}), \xi)\, \partial_x T_N(x, \tilde{x}) = -\frac{B_M^{\text{prin}}(x, \xi)}{B_N^{\text{prin}}(x, n_{\tilde{x}}(x))}\, n_{\tilde{x}}(x), \tag{4-17}$$

cf. (3-22); hence,

$$\zeta = \xi - \frac{B_M^{\text{prin}}(x, \xi)}{B_N^{\text{prin}}(x, n_{\tilde{x}}(x))}\, n_{\tilde{x}}(x). \tag{4-18}$$

For $F_-(t_1)\chi$ we get the relationship $\zeta = \xi + \dfrac{B_M^{\text{prin}}(x, \xi)}{B_N^{\text{prin}}(x, n_{\tilde{x}}(x))}\, n_{\tilde{x}}(x)$. Then

$$\begin{aligned} \Lambda_{MN}^{F_+} = \Big\{ (y, t, \eta, \tau; x, \zeta) \;\Big|\; &(y, \eta) \in (T^*X \backslash 0) \backslash V_{\tilde{x},t},\ t \in \mathbb{R},\ \tau = -B_M^{\text{prin}}(y, \eta), \\ (x, \xi) = \Phi_M^{T_N(x,\tilde{x})-t}(y, \eta),\ \zeta = \xi - &\frac{B_M^{\text{prin}}(x, \xi)}{B_N^{\text{prin}}(x, n_{\tilde{x}}(x))}\, n_{\tilde{x}}(x),\ x \in X \Big\}. \end{aligned} \tag{4-19}$$

Here, we replaced t_1 by t using that this canonical relation naturally extends to the canonical relation of $F_+(t)$ through Φ_M. In the above, $V_{\tilde{x},t}$ signifies the (conic neighborhood of a) set on which φ_{MN} is not nondegenerate.

Lemma 4.1. *The phase function φ_{MN} is nondegenerate if*

$$\partial_x T_N(x, \tilde{x}) \cdot \partial_\xi B_M^{\text{prin}}(x, \xi) \neq 1.$$

Proof. Because $\partial_\xi \partial_x \varphi_{MN} = \partial_\xi \zeta$ on the stationary point set of φ_{MN}, we need to establish whether the Jacobian, $|\partial_\xi \zeta|$, is singular. Using (4-18) we find that

$$|\partial_\xi \zeta| = \left| \det\left(I - \partial_\xi B_M^{\text{prin}}(x,\xi) \otimes \frac{1}{B_N^{\text{prin}}(x, n_{\tilde{x}}(x))}\, n_{\tilde{x}}(x) \right) \right|. \tag{4-20}$$

Hence,

$$\begin{aligned} |\partial_\xi \zeta| &= \left| 1 - n_{\tilde{x}}(x) \cdot \frac{1}{B_N^{\text{prin}}(x, n_{\tilde{x}}(x))}\, \partial_\xi B_M^{\text{prin}}(x,\xi) \right| \\ &= \left| 1 - \partial_x T_N(x,\tilde{x}) \cdot \partial_\xi B_M^{\text{prin}}(x,\xi) \right|, \end{aligned} \tag{4-21}$$

from which the statement follows. □

Hence, for $F_+(t)$ to be a Fourier integral operator, we need to invoke the assumption which excludes scattering such that $|\partial_\xi \zeta|$ is singular. The homogeneity of B_M^{prin} implies that $\xi \cdot \partial_\xi B_M^{\text{prin}}(x,\xi) = B_M^{\text{prin}}(x,\xi)$, from which it is clear that $|\partial_\xi \zeta|$ is singular if

$$\frac{1}{B_M^{\text{prin}}(x,\xi)}\, \xi = \partial_x T_N(x,\tilde{x}).$$

If $N \neq M$, the assumption is generically satisfied; if $N = M$, this excludes scattering over π, hence the reference to this assumption as the *absence of direct source waves*. In this case, $V_{\tilde{x},t}$ is a conic neighborhood of $\Xi_{\tilde{x},t}$. We introduce a t-family of pseudodifferential cutoffs, $\pi_+(t) = \pi_+(t)(y, D_y)$. For some t_c, the symbol of $\pi_+(t_c)$ vanishes on a conic neighborhood of $\Xi_{\tilde{x},t_c}$; we then set $\pi_+(t) = S_{M,+}(t - t_c)\pi_+(t_c)S_{M,+}(t_c - t)$. It follows that $\pi_+(t)F_+(t)$ is a Fourier integral operator with canonical relation given by (4-19). A similar analysis can be carried out for $F_-(t)$.

In the further analysis we will focus on the conversion where N corresponds with qP and M corresponds with qSV, in particular with a view to developing array receiver functions.

5. Reverse-time continuation from the boundary

We consider solutions to the homogeneous polarized wave equation,

$$P_M(x, D_x, D_t) w = 0.$$

We use boundary normal coordinates. We denote the restriction of w to Σ by $R_\Sigma w$, where Σ is a bounded open subset of the boundary as before. We let w_r be an anticausal solution to

$$\begin{aligned} &[\partial_t^2 + A_M(x, D_x)]\, w_r(x,t) \\ &\qquad = \delta(x_n)\, \mathcal{N}_M(x', D_{x'}, D_t)\, \Psi_{\mu,\Sigma}(x', t, D_{x'}, D_t)(R_\Sigma w)(x', t), \end{aligned} \tag{5-1}$$

where $\mathcal{N}_M(x', D_{x'}, D_t)$ was defined in (2-13), and $\Psi_{\mu,\Sigma}$ is a pseudodifferential cutoff, which removes grazing rays; that is, its symbol vanishes where

$$\frac{\partial B_M^{\text{prin}}}{\partial \xi_n}(x', 0, \tau^{-1}\xi', C_\mu(0, x', \tau^{-1}\xi', 1)) = 0.$$

In this formulation, elements in the wavefront set satisfying

$$C_\mu^{\text{prin}}(0, x', \tau^{-1}\xi', 1) = 0$$

need to be removed as well; moreover, the cutoff is designed to remove direct source waves. An alternative representation of the (principal) symbol of $\mathcal{N}_M$ is obtained using the identity

$$2\mathrm{i} B_M^{\text{prin}}(x,\xi)\frac{\partial B_M^{\text{prin}}}{\partial \xi_n}(x,\xi) = c_{nk;il}(x)\,\mathrm{i}\xi_k\, Q_{iM}^{\text{prin}}(x,\xi) Q_{lM}^{\text{prin}}(x,\xi) \tag{5-2}$$

(no sums over M) which appears in the relevant representation theorems.

We assume that X is contained in $\{x_n > 0\}$ and let $X_t = X \times \{t\} \subset \mathbb{R}^n_x \times \mathbb{R}_t$. We revisit the bicharacteristic flow

$$(x,\xi) \to ((y_M^t)'(x,\xi), t, (\eta_M^t)'(x,\xi), -B_M^{\text{prin}}(x,\xi))$$

from $T^*X_0\backslash 0 \to T^*\Sigma\backslash 0$ (cf. (3-8)), and introduce pseudodifferential illumination operators with principal symbols $\Psi_{X_s,+}$ defined by

$$\Psi_{X_s,+}(x,\xi) = \Psi_{\mu,\Sigma}\big((y_M^{t-s})'(x,\xi), t, (\eta_M^{t-s})'(x,\xi), -B_M^{\text{prin}}(x,\xi)\big)$$

if there exists t such that $y_M^{t-s}(x,\xi) \in \Sigma$, and $\Psi_{X_s,+}(x,\xi) = 0$ otherwise. Similarly, $\Psi_{X_0,-}$ is obtained by using

$$(x,\xi) \mapsto \big((y_M^{-t})'(x,\xi), -t, (\eta_M^{-t})'(x,\xi), B_M^{\text{prin}}(x,\xi)\big).$$

We assume that bicharacteristics which illuminate X intersect Σ only once, with $\mathrm{d}(y_M^t)_n/\mathrm{d}t < 0$.

Theorem 5.1. *The reverse-time continued field and the original field are related as*

$$\chi_n\, w_{r,+}(\,\cdot\,,t) = \chi_n\,[\Pi_+(t)w_+(\,\cdot\,,t) + R_{+-}(t)w_-(t)], \tag{5-3}$$

$$\chi_n\, w_{r,-}(\,\cdot\,,t) = \chi_n\,[\Pi_-(t)w_-(\,\cdot\,,t) + R_{-+}(t)w_+(t)], \tag{5-4}$$

where the $\Pi_\pm(t)$ are pseudodifferential operators of order zero with principal symbols

$$\Pi_+(t)(x,\xi) = \Psi_{X_t,+}(x,\xi), \tag{5-5}$$

$$\Pi_-(t)(x,\xi) = \Psi_{X_t,-}(x,\xi), \tag{5-6}$$

$R_{+-}(t)$ and $R_{-+}(t)$ are regularizing operators, and χ_n is a smooth cutoff supported in $x_n > 0$.

This theorem applies to the continued scattered field, $w_\pm = v_{h,\pm}$; then we write $w_{r,\pm} = v_{r,\pm}$. It also applies to the source field, $w_\pm = w_{\tilde{x};\pm}$ (replacing M by N in the above); then we write $w_{r,\pm} = w_{\tilde{x};r,\pm}$.

Proof. The anticausal solution of

$$[\partial_t + \mathrm{i} B_M(x, D_x)]\, w_+ = f_+$$

is given by

$$S_{M,+}(t, \cdot\,) f_+ = -\int_t^\infty S_{M,+}(t, s) f_+(\,\cdot\,, s)\, \mathrm{d}s.$$

The restriction operator, R_Σ, gives $R_\Sigma w(y', t) = w(y', 0, t)$ for $(y', t) \in \Sigma$, while

$$R_\Sigma^* g(y, t) = \delta(y_n)\, g(y', t),$$

for functions g defined on $\Sigma \times \mathbb{R}_t$. We use the notation

$$g_{M,\Sigma}(y', t) = \mathcal{N}_M \Psi_{\mu,\Sigma}\, (R_\Sigma w)(y', t).$$

For a given time, $t = t_c$, we study the maps $(w_+(\,\cdot\,, t_c), w_-(\,\cdot\,, t_c)) \mapsto g_{M,\Sigma}$, using that $w(\,\cdot\,, t) = S_{M,+}(t, t_c) w_+(\,\cdot\,, t_c) + S_{M,-}(t, t_c) w_-(\,\cdot\,, t_c)$ microlocally, and $g_{M,\Sigma} \mapsto \chi_n\, R_{t_c} w_{r,\pm}$, where R_{t_c} is the restriction to $t = t_c$, and their composition. For simplicity of notation we set $t_c = 0$. We proceed with the assumption that $\Psi_{\mu,\Sigma}(y', t, \eta', \tau)$ is supported in $t \in [0, t_1]$ with t_1 such that we can use the particular oscillatory integral representation (3-6)–(3-7) for the kernel of the parametrix

The solution operator $S_{M,+}(t, 0)$ has canonical relation

$$\{(y_M^t(x, \xi), t, \eta_M^t(x, \xi), -B_M(x, \xi); x, \xi)\};$$

the restriction operator R_Σ has canonical relation

$$\{(y', t, \eta', \tau; y', 0, t, \eta', \eta_n, \tau)\}.$$

The composition of these canonical relations is transversal because grazing rays have been removed. Hence, the operator $\mathcal{N}_M \Psi_{\mu,\Sigma} R_\Sigma S_{M,+}(\,\cdot\,, 0)$ is a Fourier integral operator. Its canonical relation is a subset (determined by $\Psi_{\mu,\Sigma}$) of

$$\left\{\big((y_M^t)'(x, \xi), t, (\eta_M^t)'(x, \xi), -B_M(x, \xi); x, \xi\big) \mid y_M^t(x, \xi) \in \Sigma\right\}$$

and is the graph of an invertible transformation. The kernel of this Fourier integral operator admits an oscillatory integral representation with amplitude

$$
\begin{aligned}
a^{(\mathrm{fwd})}(y', t, x, \xi) = -2\mathrm{i}\tau \frac{\partial B_M^{\mathrm{prin}}}{\partial \eta_n}(y', 0, \tau^{-1}\eta', C_\mu(0, y', \tau^{-1}\eta', 1)) \\
\times \Psi_{\mu,\Sigma}(y', t, \eta', \tau) \left| \left. \frac{\partial(y_M^t)}{\partial(x)} \right|_{\xi, x = x_M^t(y', 0, \xi)} \right|^{-1/2}
\end{aligned}
\tag{5-7}
$$

mod S^0, subject to the substitutions

$$
\begin{aligned}
\eta' &= \partial_{y'} \alpha_{M,+}(y', 0, t, \xi), \\
\tau &= \partial_t \alpha_{M,+}(y', 0, t, \xi) = -B_M(x, \xi).
\end{aligned}
\tag{5-8}
$$

Then

$$
\begin{aligned}
g_{M,\Sigma,+}(y', t) = (2\pi)^{-n} \iint_X a^{(\mathrm{fwd})}(y', t, x, \xi) \\
\times \exp\bigl(\mathrm{i}(\alpha_{M,+}(y', 0, t, \xi) - \langle \xi, x \rangle)\bigr) w_+(x, 0)\, \mathrm{d}x\, \mathrm{d}\xi.
\end{aligned}
\tag{5-9}
$$

We introduce a pseudodifferential cutoff,

$$
\widetilde{\Psi}_{\mu,\Sigma} = \widetilde{\Psi}_{\mu,\Sigma}(y', t, D_{y'}, D_t),
$$

which removes grazing rays, such that

$$
\widetilde{\Psi}_{\mu,\Sigma} \Psi_{\mu,\Sigma} = \Psi_{\mu,\Sigma}.
$$

Using the decoupling procedure, $\Lambda_M \begin{pmatrix} 0 \\ g_{M,\Sigma} \end{pmatrix}$, we find that

$$
\chi_n\, R_0 w_{r,+} = \chi_n\, S_{M,+}(0, \cdot)\, \tfrac{1}{2}\mathrm{i} B_M^{-1}\, R_\Sigma^* \widetilde{\Psi}_{\mu,\Sigma}\, g_{M,\Sigma}.
$$

The operator $\chi_n\, S_{M,+}(0, \cdot)\, \tfrac{1}{2}\mathrm{i} B_M^{-1}\, R_\Sigma^* \widetilde{\Psi}_{\mu,\Sigma}$ is a Fourier integral operator, the canonical relation of which is a subset of

$$
\bigl\{\bigl(z, \zeta; (y_M^t)'(z, \zeta), t, (\eta_M^t)'(z, \zeta), -B_M(z, \zeta)\bigr) \mid (y_M^t)_n(z, \zeta) = 0\bigr\}.
$$

The kernel of this Fourier integral operator admits an oscillatory integral representation with amplitude

$$
\begin{aligned}
&a^{(\mathrm{bkd})}(y', t, z, \zeta) \\
&\quad = \chi_n(z_n) \left| \left. \frac{\partial(y_M^t)}{\partial(x)} \right|_{\zeta, x = x_M^t(y', 0, \zeta)} \right|^{-1/2} \tfrac{1}{2}\mathrm{i}\tau^{-1} \widetilde{\Psi}_{\mu,\Sigma}(y', t, \eta', \tau)
\end{aligned}
\tag{5-10}
$$

mod S^{-2}, subject to the substitutions (5-8). Then

$$\chi_n\, R_0 w_{r,+}(z) = (2\pi)^{-n} \iiint a^{(\mathrm{bkd})}(y', t, z, \zeta) \times \exp\bigl(\mathrm{i}(-\alpha_{M,+}(y', 0, t, \zeta) + \langle \zeta, z\rangle)\bigr)\, g_{M,\Sigma}(y', t)\, \mathrm{d}y'\, \mathrm{d}t\, \mathrm{d}\zeta. \tag{5-11}$$

We now consider the composition

$$\chi_n\, S_{M,+}(0, \cdot\,)\, \tfrac{1}{2}\mathrm{i} B_M^{-1}\, R_\Sigma^* \widetilde{\Psi}_{\mu,\Sigma}\, \mathcal{N}_M \Psi_{\mu,\Sigma} R_\Sigma S_{M,+}(\,\cdot\,, 0).$$

Considering the composition of canonical relations, it follows immediately that this is a pseudodifferential operator. We construct the following representation:

$$(2\pi)^{-n} \iiint a^{(\mathrm{bkd})}(y', t, z, \zeta) a^{(\mathrm{fwd})}(y', t, x, \xi) \times \exp\bigl(\mathrm{i}(-\alpha_{M,+}(y', 0, t, \zeta) + \alpha_{M,+}(y', 0, t, \xi) + \langle \zeta, z\rangle - \langle \xi, x\rangle)\bigr)\, \mathrm{d}y'\, \mathrm{d}t\, \mathrm{d}\zeta = \sigma(z, x, \xi)\, \exp\bigl(\mathrm{i}\langle \xi, z - x\rangle\bigr). \tag{5-12}$$

We write

$$\begin{aligned} -\alpha_{M,+}(y', 0, t, \zeta) + \alpha_{M,+}(y', 0, t, \xi) &= \langle \xi - \zeta, X(y', t, \zeta, \xi)\rangle, \\ X(y', t, \zeta, \xi) &= \int_0^1 \partial_\xi \alpha_{M,+}(y', 0, t, \zeta + s(\xi - \zeta))\, \mathrm{d}s, \end{aligned} \tag{5-13}$$

and change variables of integration, $(y', t) \to X$. The phase is stationary if $y' = (y_M^t)'(x, \xi)$ and $(y_M^t)_n(x, \xi) = 0$, and $\zeta = \xi$; we have

$$X(y', t, \xi, \xi) = \partial_\xi \alpha_{M,+}(y', 0, t, \xi) = x_M^t(y', 0, \xi).$$

Using the absence of grazing rays, the relevant Jacobian can be written in the form

$$\left| \frac{\partial(X)}{\partial(y', t)} \right|_{\zeta=\xi} = \left| \frac{\partial(y_M^t)}{\partial(x)} \right|^{-1}_{\xi, x = x_M^t(y', 0, \xi)} \left| \frac{\partial(y_M^t)_n}{\partial t} \right|_{\xi, x = x_M^t(y', 0, \xi)}, \tag{5-14}$$

where

$$\frac{\partial(y_M^t)_n}{\partial t} = \frac{\partial B_M^{\mathrm{prin}}(y_M^t, \eta_M^t)}{\partial \eta_n},$$

and

$$\begin{aligned} y_M^t(x_M^t(y', 0, \xi), \xi) &= (y', 0), \\ \eta_M^t(x_M^t(y', 0, \xi), \xi) &= \partial_y \alpha_{M,+}(y', 0, t, \xi). \end{aligned}$$

Applying the method of stationary phase, we find the principal symbol of the composition under consideration:

$$\begin{aligned}
\sigma(x,x,\xi) = {} & \\
& -2\mathrm{i}\tau \frac{\partial B_M^{\mathrm{prin}}}{\partial \eta_n}(y',0,\tau^{-1}\eta',C_\mu(0,y',\eta',1))\Psi_{\mu,\Sigma}(y',t,\eta',\tau) \\
& \quad \times \left| \frac{\partial(y_M^t)}{\partial(x)}\Big|_{\xi,x=x_M^t(y',0,\xi)} \right|^{-1/2} \chi_n(x_n) \left| \frac{\partial(y_M^t)}{\partial(x)}\Big|_{\xi,x=x_M^t(y',0,\xi)} \right|^{-1/2} \\
& \quad \times \tfrac{1}{2}\mathrm{i}\tau^{-1}\widetilde{\Psi}_{\mu,\Sigma}(y',t,\eta',\tau) \left| \frac{\partial(y_M^t)}{\partial(x)}\Big|_{\xi,x=x_M^t(y',0,\xi)} \right| \\
& \quad \times \left(\frac{\partial B_M^{\mathrm{prin}}}{\partial \eta_n}(y',0,\tau^{-1}\eta',C_\mu(0,y',\tau^{-1}\eta',1)) \right)^{-1} \Bigg|_{\substack{y'=(y_M^t)'(x,\xi) \\ \eta'=(\eta_M^t)'(x,\xi) \\ \tau=-B_M(x,\xi)}} \\
& = \chi_n(x_n)\, \Psi_{X_0,+}(x,\xi), \qquad (5\text{-}15)
\end{aligned}$$

using that t is determined by $(y_M^t)_n(x,\xi) = 0$.

We extend the proof to longer times. Using a partition of unity we can decompose $\Psi_{\mu,\Sigma}$ into terms, covering time intervals $[s, s+t_1]$ $(s > 0)$, say. It is sufficient to prove the result for each term. For this, we simply change the time variable from t to $t-s$ in the above. We then use the semigroup property, microlocally, of $S_{M,+}(t,s)$. □

6. Inverse scattering: common source

Here, we develop the inverse scattering with the goal to reconstruct the singular medium perturbation given observations of the scattered field on part of the surface and the background medium. We assume that bicharacteristics which enter the region $x_n < 0$ do not return to the region $x_n \geq 0$. As mentioned before, we invoke an additional hypothesis:

Assumption 6.1 (Bolker condition). No caustics form between the source and scattering points in mode N.

Essentially, we assume the absence of multipathing in the characteristics or rays associated with the source wave field. The reflection data, d_{MN}, are modeled by $R_\Sigma v$, cf. (4-4)–(4-5). We substitute d_{MN} for $R_\Sigma w$ in (5-1) when w_r is identified with v_r, and consider the operator, H_{MN}, defined in (2-14); its canonical relation is illustrated in Figure 2.

Theorem 6.2. *Let H_{MN} be the transform defined in* (2-14) *and let w_{MN} be as defined in* (4-9)–(4-10). *With Assumption 6.1, the following holds true*:

$$H_{MN}\mathsf{P}_1 F(\,\cdot\,) = w_{MN,+}\mathcal{R}_+\, w_{MN,+}^T + w_{MN,-}\mathcal{R}_-\, w_{MN,-}^T,$$

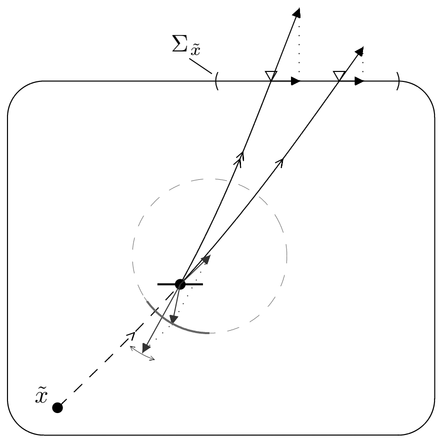

Figure 2. Illustration of the canonical relation of RTM-based inverse scattering. The receivers are contained in the set $\Sigma_{\tilde{x}}$ (the array). The ray with single arrow corresponds with the source field, which may also be observed at the boundary; the ray with double arrows corresponds with the scattered field. The covectors at the scattering point illustrate the construction of $\theta(x, x, \xi)$ (isotropic case).

where $\mathcal{R}_{\pm}$ *are pseudodifferential operators of order zero with principal symbols given by*

$$\mathcal{R}_{\pm}^{\text{prin}}(z, \zeta) = \Pi_{+}(T_N(z, \tilde{x}))(z, \xi(\pm\zeta)),$$

and $w_{MN,\pm}$ *are pseudodifferential operators with principal symbols given by*

$$w_{MN,\pm}(z, \zeta) = w_{MN}(z, T_N(z, \tilde{x}), z, \xi(\pm\zeta)).$$

Here, *the map* $\zeta \to \xi$ *is given in* (4-18).

Proof. We first carry out the analysis for symbols up to leading orders. Let w denote a wave field in $\mathcal{E}'(X \times \mathbb{R})$. We introduce the "reverse-time migration" imaging condition through the operator K, with

$$Kw(z) = w(z, T_N(z, \tilde{x})).$$

We define the pseudodifferential operator L by

$$\begin{aligned} Lw(y, t) = {} & \mathcal{A}_N(y, \tilde{x}, D_t)^{-1} 2\mathrm{i} D_t \\ & \sum_{p=0}^{n} \left(\frac{\partial T_N}{\partial y_k}(y, \tilde{x})\, Q(y, \partial_y T_N(y, \tilde{x}))_{lN}\, \Xi_p(y, -\partial_y T_N(y, \tilde{x}), 1) \right) \\ & \qquad \times \left(D_{y_j}\, Q(y, D_y)_{iM}\, \Theta_p(y, D_y, D_t) w(y, t) \right); \end{aligned}$$

hence KLv_r is an asymptotic approximation of $H_{MN}d_{MN}$.

We consider negative frequencies, identify $v_{r,+}(y,t)$ with

$$\Pi_+(t)F_+(t)\Big(\frac{\delta\rho}{\rho},-\frac{\delta c}{\rho}\Big)(y),$$

and analyze the composition $L\Pi_+(\cdot)F_+(\cdot)\chi$. The composition $\Pi_+(t)F_+(t)\chi$ is a Fourier integral operator with a phase function inherited from $F_+(t)\chi$. To highest order, its amplitude is given by

$$\Pi_+(t_1)(y,\partial_y\varphi_{MN})\,A_{F,MN}(y,t_1,x,\xi)\,w^T_{MN}(y,t_1,x,\xi);$$

see (4-12). Also, $L\Pi_+(\cdot)F_+(\cdot)\chi$ is a Fourier integral operator with leading-order amplitude

$$\begin{aligned} A_{L\Pi F,MN}(y,t_1,x,\xi) = 2\mathrm{i}\,&w_{MN}(y,t_1,y,\partial_y\alpha_{M,+})\\ &\times\sum_{p=0}^{n}\Xi_p(y,-\partial_yT_N(y,\tilde{x}),1)\Theta_p(y,\partial_y\alpha_{M,+},\partial_t\alpha_{M,+})\\ &\times\Pi_+(t_1)(y,\partial_y\alpha_{M,+})\tilde{a}_{M,+}(y,t_1-T_N(x,\tilde{x}),\xi)\\ &\times\frac{\mathcal{A}_N(x,\tilde{x},\partial_t\alpha_{M,+})}{\mathcal{A}_N(y,\tilde{x},\partial_t\alpha_{M,+})}\,w^T_{MN}(y,t_1,x,\xi), \end{aligned} \tag{6-1}$$

in which the argument of $\alpha_{M,+}$ is $(y,t_1-T_N(x,\tilde{x}),\xi)$.

The local phase function of the oscillatory integral representation of the kernel of $KL\Pi_+(\cdot)F_+(\cdot)\chi$ is obtained by setting $t_1=T_N(z,\tilde{x})$ in (4-13):

$$\alpha_{M,+}(z,T_N(z,\tilde{x})-T_N(x,\tilde{x}),\xi)-\langle\xi,x\rangle.$$

This phase is stationary at points (z,x,ξ) for which

$$\partial_\xi\alpha_{M,+}(z,T_N(z,\tilde{x})-T_N(x,\tilde{x}),\xi)=x.$$

The stationarity condition implies that the bicharacteristic with initial condition (x,ξ) arrives at z after a time lapse $T_N(z,\tilde{x})-T_N(x,\tilde{x})$. Then, however, $\tilde{x}$, x and z would lie on the same characteristic. Having excluded such (direct source) characteristics, we must have $z=x$. We note that then

$$\zeta=\partial_x\alpha_{M,+}(x,0,\xi)=\xi.$$

We write

$$\alpha_{M,+}(z,T_N(z,\tilde{x})-T_N(x,\tilde{x}),\xi)-\langle\xi,x\rangle=\langle\theta(z,x,\xi),z-x\rangle, \tag{6-2}$$

with

$$\begin{aligned}\theta(z,x,\xi) &= -\int_0^1 [\partial_x \alpha_{M,+}(z, T_N(z,\tilde{x}) - T_N(z+\mu(x-z),\tilde{x}),\xi) - \xi]\,\mathrm{d}\mu \\ &= \xi + \int_0^1 \partial_t \alpha_{M,+}\big(z, T_N(z,\tilde{x}) - T_N(z+\mu(x-z),\tilde{x}),\xi\big) \\ &\qquad\qquad \times \partial_x T_N(z+\mu(x-z),\tilde{x})\,\mathrm{d}\mu. \qquad (6\text{-}3)\end{aligned}$$

We introduce the point $\check{x} = x_M^{T_N(z,\tilde{x})-T_N(z+\mu(x-z),\tilde{x})}(z,\xi)$, that is,

$$\check{x} = \partial_\xi \alpha_{M,+}\big(z, T_N(z,\tilde{x}) - T_N(z+\mu(x-z),\tilde{x}),\xi\big),$$

with the property that the bicharacteristic with initial condition $(\check{x},\xi)$ reaches z at time $T_N(z,\tilde{x}) - T_N(z+\mu(x-z),\tilde{x})$. Then

$$\begin{aligned}&\partial_t \alpha_{M,+}\big(z, T_N(z,\tilde{x}) - T_N(z+\mu(x-z),\tilde{x}),\xi\big)\,\partial_x T_N(z+\mu(x-z),\tilde{x}) \\ &\qquad = -\frac{B_M^{\mathrm{prin}}(\check{x},\xi)}{B_N^{\mathrm{prin}}(z+\mu(x-z), n_{\tilde{x}}(z+\mu(x-z)))}\, n_{\tilde{x}}(z+\mu(x-z)) \qquad (6\text{-}4)\end{aligned}$$

cf. (4-17), and

$$\begin{aligned}\theta(z,x,\xi) &= \xi - \int_0^1 B_M^{\mathrm{prin}}(\check{x},\xi)\,\gamma_{\tilde{x}}(z,x)\,\mathrm{d}\mu, \\ \gamma_{\tilde{x}}(z,x) &= \frac{1}{B_N^{\mathrm{prin}}\big(z+\mu(x-z), n_{\tilde{x}}(z+\mu(x-z))\big)}\, n_{\tilde{x}}(z+\mu(x-z)),\end{aligned} \qquad (6\text{-}5)$$

so that

$$|\partial_\xi \theta(z,x,\xi)| = \left|\det\left(I - \int_0^1 \partial_\xi B_M^{\mathrm{prin}}(\check{x},\xi) \otimes \gamma_{\tilde{x}}(z,x)\,\mathrm{d}\mu\right)\right|; \qquad (6\text{-}6)$$

$\theta(z,x,-\xi) = -\theta(z,x,\xi)$. We note that

$$\gamma_{\tilde{x}}(z,z) = \frac{1}{B_N^{\mathrm{prin}}(z, n_{\tilde{x}}(z))}\, n_{\tilde{x}}(z) = \partial_x T_N(z,\tilde{x}),$$

while at $x = z$, $\check{x}$ can be replaced by z, and

$$\begin{aligned}\theta(z,z,\xi) &= \xi - \frac{B_M^{\mathrm{prin}}(z,\xi)}{B_N^{\mathrm{prin}}(z, n_{\tilde{x}}(z))}\, n_{\tilde{x}}(z), \\ |\partial_\xi \theta(z,z,\xi)| &= |1 - \partial_\xi B_M^{\mathrm{prin}}(z,\xi)\cdot \gamma_{\tilde{x}}(z,z)|,\end{aligned}$$

defining a mapping $\xi \to \theta(z,z,\xi)$, which is invertible; see Figure 2 and also Lemma 4.1. Thus the Schwartz kernel of $KL\Pi_+(\cdot)F_+(\cdot)\chi$ can be written in the form

$$(2\pi)^{-n} \int A_{L\Pi F,MN}(z, T_N(z,\tilde{x}), z, \xi(\theta)) \\ \times \big|\partial_\xi \theta(z,z,\xi(\theta))\big|^{-1} \exp\big(\mathrm{i}\langle \theta, z-x\rangle\big)\, \mathrm{d}\theta \;\chi(x).$$

We evaluate the principal symbol. We have

$$\sum_{p=0}^{n} \Xi_p(z, -\partial_y T_N(z,\tilde{x}), 1)\Theta_p(z, \partial_y \alpha_{M,+}, \partial_t \alpha_{M,+}) \\ = \partial_t \alpha_{M,+}\,(1 - \partial_\xi B_M^{\text{prin}}(z, \partial_y \alpha_{M,+}) \cdot \partial_y T_N(z,\tilde{x})), \tag{6-7}$$

using that the argument of $\alpha_{M,+}$ is $(z, T_N(z,\tilde{x}) - T_N(x,\tilde{x}), \xi)$; at $x = z$ we have

$$\partial_t \alpha_{M,+}(z,0,\xi) = -B_M^{\text{prin}}(z,\xi), \quad \partial_y \alpha_{M,+}(z,0,\xi) = \xi, \quad \partial_y T_N(z,\tilde{x}) = \gamma_{\tilde{x}}(z,z),$$

by (4-17), whence this expression reduces to $-B_M^{\text{prin}}(z,\xi)\,|\partial_\xi \theta(z,z,\xi)|$. Furthermore, $\widetilde{a}_{M,+}(z,0,\xi) = \frac{1}{2}\mathrm{i} B_M^{\text{prin}}(z,\xi)^{-1}$. We obtain

$$A_{L\Pi F,MN}(z, T_N(z,\tilde{x}), z, \xi(\theta))\,|\partial_\xi \theta(x,x,\xi(\theta))|^{-1} \\ = w_{MN}(z, T_N(z,\tilde{x}), z, \xi(\theta))\; \Pi_+(T_N(z,\tilde{x}))(z,\xi(\theta))\, w_{MN}^T(z, T_N(z,\tilde{x}), z, \xi(\theta)).$$

Combining the negative with the positive frequency contributions yields a point symmetry of the domain of θ integration; we obtain a pseudodifferential operator with (real-valued) principal symbol

$$w_{MN}(z, T_N(z,\tilde{x}), z, \xi(\theta))\; \Pi_+(T_N(z,\tilde{x}))(z,\xi(\theta))\, w_{MN}^T(z, T_N(z,\tilde{x}), z, \xi(\theta)) \\ + \overline{w_{MN}(z, T_N(z,\tilde{x}), z, \xi(-\theta))\; \Pi_+(T_N(z,\tilde{x}))(z,\xi(-\theta))\, w_{MN}^T(z, T_N(z,\tilde{x}), z, \xi(-\theta))},$$

from which the statement follows. □

We note that the principal symbol matrix representing the spatial resolution and contrast source radiation patterns (for a fixed source) has rank 1.

7. Array receiver functions

In this section, we assume we also observe the source field and focus on converted waves ($M \neq N$); in fact, we assume that N corresponds with qP. Thus we remove the knowledge of the source. We generalize the notion of receiver functions used in the seismological literature; in the last subsection of this section we will explain under which conditions receiver functions can be obtained from the generalization introduced here. The incident data, d_N, are modeled by $R_\Sigma w_{\tilde{x}}$; see Figure 3.

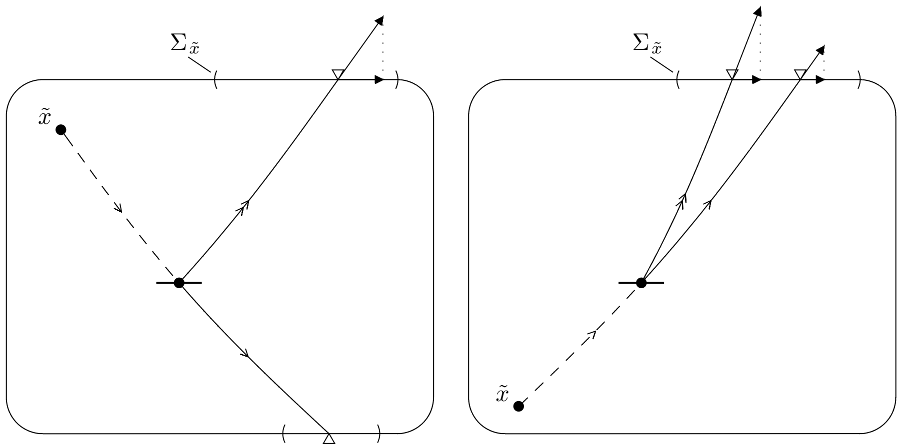

Figure 3. Array receiver functions: Detecting the incident field, and scattered field. Left: distinct arrays. Right: single array (teleseismic situation). The available set Σ may dependent on $\tilde{x}$. The ray with single arrow corresponds with the source field; the ray with double arrows corresponds with the (converted) scattered field. Knowledge of the source is eliminated.

7A. ***Inverse scattering: Reverse-time continued source wave field.*** We obtain $w_{\tilde{x};r}$ by substituting d_N for $R_\Sigma w$ in (5-1) with M replaced by N. We note that the equation which d_N satisfies is homogeneous in the relevant time interval. Applying Theorem 5.1, we obtain

$$\chi_n \, w_{\tilde{x};r,\pm}(\cdot\,, t) = \chi_n \, \Pi_{N,\pm}(t) w_{\tilde{x};\pm}(\cdot\,, t),$$

microlocally. We apply $\mathsf{P}_1 V_N$ to $w_{\tilde{x};r,\pm}$, which we use to replace G_N in the operator H_{MN} of (2-14).

In Theorem 6.2, $\mathcal{R}_\pm$ is affected by $\Pi_{N,\pm}(t)$ in a natural way. Following the propagation of singularities, it becomes clear that the singularities in the source field are recovered at x_0 only if the ray connecting $\tilde{x}$ with x_0 intersects the boundary at a point in $\Sigma_{\tilde{x}}$, see Figure 4. We reemphasize that we admit the formation of caustics between receivers and scattering points.

Without knowledge of $\tilde{x}$, the factor $1/|\widehat{G}_N(\cdot\,, \tilde{x}, \tau)|^2$ cannot be evaluated. Instead, we consider

$$-\frac{1}{2\pi} \int \frac{2\Omega(\tau)}{\mathrm{i}\tau} \sum_{p=0}^{n} \left(\frac{\partial}{\partial x_k} Q(x, D_x)_{lN} \, \Xi_p(x, D_x, \tau) \overline{\widehat{w}_{\tilde{x};r}(x, \tau)} \right) \\ \times \left(\frac{\partial}{\partial x_j} Q(x, D_x)_{iM} \, \Theta_p(x, D_x, \tau) \widehat{v}_r(x, \tau) \right) \mathrm{d}\tau.$$

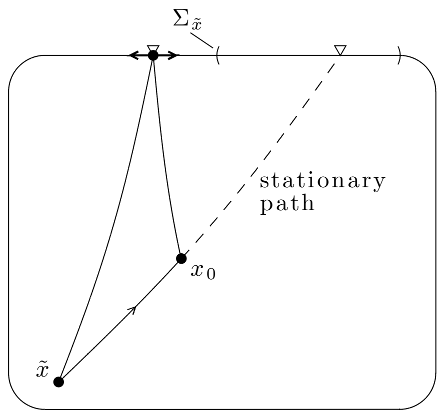

Figure 4. Source wave field: Reverse-time continuation and propagation of singularities. Reconstruction of singular perturbations can be accomplished at points (x_0) which can be connected to $\tilde{x}$ with a ray which intersects $\Sigma_{\tilde{x}}$.

We then adjust $\Omega(\tau)$ by a multiplication with $(\mathscr{A}_N(\cdot,\cdot,1)/\mathscr{A}_N(\cdot,\cdot,\tau))^2$ such that the result yields a (partial) reconstruction up to the factor

$$\left[|\mathscr{T}|^{-1}\int_{\mathscr{T}} |\widehat{w}_{\tilde{x};r}(x,\tau)|^2 \left(\frac{\mathscr{A}_N(\cdot,\cdot,1)}{\mathscr{A}_N(\cdot,\cdot,\tau)}\right)^2 \mathrm{d}\tau\right], \tag{7-1}$$

cf. (2-14). Here, $\mathscr{T}$ is the bandwidth of the data.

7B. ***Cross correlation formulation.*** We reformulate the inverse scattering procedure outlined in the previous subsection in terms of a single Fourier integral operator. To achieve this, we introduce array receiver functions; see Definition 2.1. The observational assumption is that $d_{MN} = d_M$, whence $\mathsf{R}_{MN}(d(\cdot,\cdot;\tilde{x}))$ can be obtained from the multicomponent data. (In receiver functions, correlation, or deconvolution, is considered only for $r' = r$.) We introduce an operator K_{MN} by identifying $(K_{MN}\mathsf{R}_{MN})_{ijkl}(x)$ with the transformation introduced in the previous subsection:

$$(K_{MN}\mathsf{R}_{MN})_{ijkl}(x) = -\int 2\Omega(D_t)(\mathrm{i}D_t)^{-1}\sum_{p=0}^{n}\int_{-\infty}^{t} H(-t')$$

$$\times\left[\frac{\partial}{\partial x_k}\, Q(x,D_x)_{lN}\;\Xi_p(x,D_x,D_t)(S_{N,+}(t',0)-S_{N,-}(t',0))\,\tfrac{1}{2}\mathrm{i}B_N^{-1}\,R_\Sigma^*\widetilde{\Psi}_{\nu,\Sigma}\right]_{(r')}$$

$$\times\left[\frac{\partial}{\partial x_j}\, Q(x,D_x)_{iM}\;\Theta_p(x,D_x,D_{t_0})(S_{M,+}(t',t)-S_{M,-}(t',t))\,\tfrac{1}{2}\mathrm{i}B_M^{-1}\,R_\Sigma^*\widetilde{\Psi}_{\mu,\Sigma}\right]_{(r)}\mathrm{d}t'$$

$$\times\,\mathsf{R}_{MN}(\overset{r}{\cdot},\overset{r'}{\cdot},t)\,\mathrm{d}t.$$

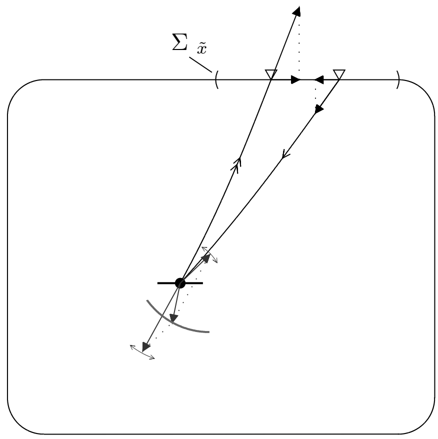

Figure 5. The canonical relation, Λ^K_{MN}, of K_{MN} (isotropic case). This is an adaption of the canonical relation illustrated in Figure 2, reflecting the cross correlation in the definition of array receiver functions.

Lemma 7.1. *Let $M \neq N$, $Z = \Sigma_{\tilde{x}} \times \Sigma_{\tilde{x}} \times (0, T)$. $K_{MN} : \mathscr{E}'(Z) \to \mathscr{D}'(X)$ is a Fourier integral operator with canonical relation*

$$\Lambda^K_{MN} = \big\{\big(x, \hat{\xi} - \tilde{\xi}; (y^{\hat{t}}_M)'(x, \hat{\xi}), \hat{t} - \tilde{t}, (y^{\tilde{t}}_N)'(x, -\tilde{\xi}), (\eta^{\hat{t}}_M)'(x, \hat{\xi}), \tau, -(\eta^{\tilde{t}}_N)'(x, -\tilde{\xi})\big) \;\big|\; B_M(x, \hat{\xi}) = B_N(x, \tilde{\xi}) = \mp\tau,\ (y^{\hat{t}}_M)_n(x, \hat{\xi}) = 0,\ (y^{\tilde{t}}_N)_n(x, -\tilde{\xi}) = 0\big\}.$$

The canonical relation Λ^K_{MN} is illustrated in Figure 5.

7C. ***Flat, translationally invariant models: propagation of singularities, receiver functions.*** In view of translational invariance, (4-3) attains the form

$$\delta G_{MN}(\hat{x}, \tilde{x}, t) = \int_{[0,Z]} \bigg(\int_0^t \int \frac{\partial}{\partial(t_0, x_{0,j})} Q(x_0, D_{x_0})_{iM} G_M(x_0, \hat{x}, t - t_0) \times \frac{\partial}{\partial(t_0, x_{0,k})} Q(x_0, D_{x_0})_{lN} G_N(x_0, \tilde{x}, t_0)\, \mathrm{d}x_0'\, \mathrm{d}t_0 \bigg) \times \left(\delta_{il} \frac{\delta\rho(z_0)}{\rho(z_0)}, -\frac{\delta c_{ijkl}(z_0)}{\rho(z_0)} \right) \mathrm{d}z_0, \quad (7\text{-}2)$$

writing $x_0 = (x_0', z_0)$ as before. Upon restriction to $\hat{x} = (r, 0)$, writing $\tilde{x} = s$, the expression in between braces on the right-hand side defines the kernel,

$\mathcal{F}^0_{MN;ijkl}(r,t;z_0)$ say, of a single scattering operator F^0_{MN}:

$$\mathcal{F}^0_{MN;ijkl}(r,t;z_0) = \iint_0^t \int \frac{\partial}{\partial(t_0, x_{0,j})} Q(x_0, D_{x_0})_{iM} G_M(x_0, r, 0, t-t_0)$$
$$\times \frac{\partial}{\partial(t_0, x_{0,k})} Q(x_0, D_{x_0})_{lN} G_N(x_0, \tilde{x}', t_0)\, \mathrm{d}x_0'\, \mathrm{d}t_0\ \mathfrak{Q}^{-1}_{Nk'}(\tilde{x}', s) e_{k'}\, \mathrm{d}\tilde{x}'. \quad (7\text{-}3)$$

The associated imaging operator, $(F^0_{MN})^*$, maps the (conversion) data to an image as a function of z_0 (and s).

We introduce so-called midpoint-offset coordinates $r' = m-h$ and $r = m+h$ (so $\mathrm{d}r\, \mathrm{d}r' = 2\, \mathrm{d}m\, \mathrm{d}h$) and find that

$$(F^0_{MN})^* d_{MN}(\cdot,\cdot;s) = K^0_{MN}(\mathsf{R}_{MN}(d(\cdot,\cdot;s))(\cdot,\cdot,\cdot)),$$

where K^0_{MN} is an operator with kernel

$$\mathcal{K}^0_{ijkl;MN}(z_0; m+h, t, m-h)$$
$$= 2\int_{-\infty}^t H(-t_0) \int \frac{\partial}{\partial x_{0,j}} Q(x_0, D_{x_0})_{iM} G_M(x_0', z_0, m+h, 0, t-t_0)$$
$$\times \frac{\partial}{\partial x_{0,k}} Q(x_0, D_{x_0})_{lN} G_N(x_0', z_0, m-h, 0, -t_0)\, \mathrm{d}x_0'\, \mathrm{d}t_0. \quad (7\text{-}4)$$

To study the propagation of singularities by this operator, we substitute the WKBJ approximations for G_M and G_N (cf. (3-33)) in this expression.

K^0_{MN}, *imaging.* We focus on the propagation of singularities and, hence, the relevant phase functions; the amplitudes follow from standard stationary phase arguments. The WKBJ phase function associated with $\mathcal{K}^0_{ijkl;MN}$ becomes

$$\tau\left[-\tau_\mu(0,z_0,\hat{p}) + \tau_\nu(0,z_0,\hat{p}') + \sum_{j=1}^{n-1}(\hat{p}-\hat{p}')_j (m-x_0')_j - \sum_{j=1}^{n-1}(\hat{p}+\hat{p}')_j h_j + t\right].$$

Carrying out the integrations over x_0' and $\hat{p}'$ leads to

$$\mathcal{K}^0_{ijkl;MN}(z_0; m+h, t, m-h) \approx \dot{\mathcal{K}}^0_{ijkl;MN}(z_0; h, t),$$

which admits an integral representation with WKBJ phase function

$$\tau\left[-\tau_\mu(0,z_0,\hat{p}) + \tau_\nu(0,z_0,\hat{p}) - 2\sum_{j=1}^{n-1} \hat{p}_j h_j + t\right].$$

We get

$$K^0_{MN}(\mathsf{R}_{MN}(d(\cdot,\cdot;s))(\cdot,\cdot,\cdot)) \approx \dot{K}^0_{MN}(\mathsf{R}^0_{MN}(d(\cdot,\cdot;s))(\cdot,\cdot,\cdot)),$$

where

$$(\mathsf{R}^0_{MN}(d(\cdot,\cdot;s)))(h,t) = \int (\mathsf{R}_{MN}(d(\cdot,\cdot;s)))(m+h,t,m-h)\,\mathrm{d}m. \tag{7-5}$$

Applying the method of stationary phase to the integral representation for $\mathcal{K}^0_{ijkl;MN}(z_0;h,t)$ in $\hat{p}$, yields stationary points $\hat{p} = \hat{p}^0(z_0,h)$ satisfying

$$-\left[\frac{\partial \tau_\mu(0,z_0,\hat{p})}{\partial \hat{p}} - \frac{\partial \tau_\nu(0,z_0,\hat{p})}{\partial \hat{p}}\right] = 2h, \tag{7-6}$$

revealing the propagation of singularities: This equation defines a pair of rays sharing the same horizontal slowness $\hat{p}$, originating at (image) depth z_0, and reaching the acquisition surface at

$$r = \frac{\partial \tau_\mu(0,z_0,\hat{p})}{\partial \hat{p}} \quad \text{and} \quad r' = \frac{\partial \tau_\nu(0,z_0,\hat{p})}{\partial \hat{p}},$$

respectively; in the imaging point of view, the rays intersect at depth z_0, whence $r' - r = 2h$. The corresponding differential travel time is given by

$$\tau_\mu(0,z_0,\hat{p}^0(z_0,h)) - \tau_\nu(0,z_0,\hat{p}^0(z_0,h)) + 2\sum_{j=1}^{n-1} \hat{p}^0_j(z_0,h)h_j$$

(we note that $\hat{p}^0_j(z_0,h)$ is the negative of the usual geometric ray parameter in view of our Fourier transform convention). The geometry is illustrated in Figure 6 (pair of solid rays).

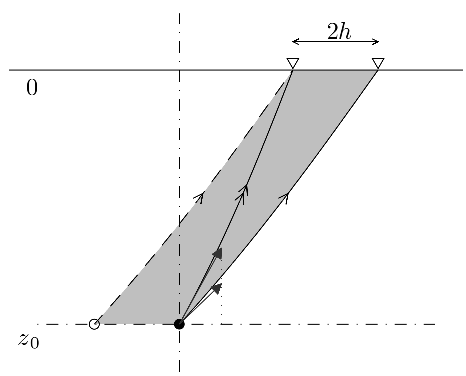

Figure 6. Propagation of singularities by K^0_{MN}. The double arrows relate to (scattered) mode μ, while the single arrow relates to (incident) mode ν. Translational invariance yields alternative ray pairs (dashed) for which the phase of $\mathcal{K}^0_{ijkl;MN}$ is stationary.

R^0_{MN}, modeling. Using (7-2), substituting the WKBJ approximations for G_M and G_N in $(\mathsf{R}^0_{MN}(d(\cdot,\cdot;s)))(h,t)$, and carrying out the integrations over x_0' and $\tilde{p}'$, we obtain a linear integral operator representation, acting on

$$\left(\delta_{il}\frac{\delta\rho(z_0')}{\rho(z_0')}, -\frac{\delta c_{ijkl}(z_0')}{\rho(z_0')}\right),$$

for the integrand in (7-5): For fixed $s = (s', z_s)$, $z_s > z_0$, the WKBJ phase function of its kernel representation is

$$\tau\Bigg[-\tau_\mu(0,z_0',\hat{p}) - \tau_\nu(z_0',z_s,\hat{p}) + \tau_\nu(0,z_s,\tilde{p}) \\ + \sum_{j=1}^{n-1}\hat{p}_j(m-h-s')_j - \sum_{j=1}^{n-1}\tilde{p}_j(m+h-s')_j + t\Bigg].$$

Carrying out the integrations over m and then $\hat{p}$ leads to

$$\int \mathsf{R}_{MN}(d(\cdot,\cdot;s))(m+h,t,m-h)\,\mathrm{d}m \approx \dot{\mathsf{R}}^0_{MN}(d(\cdot,\cdot;s))(h,t),$$

which admits an integral representation with WKBJ phase function

$$\tau\Bigg[-\tau_\mu(0,z_0',\tilde{p}) + \tau_\nu(0,z_0',\tilde{p}) - 2\sum_{j=1}^{n-1}\tilde{p}_j h_j + t\Bigg];$$

the integration over $\tilde{p}$ signifies a "plane-wave" decomposition of the source, while the explicit dependence on (s', z_s) has disappeared. Thus

$$\dot{K}^0_{MN}\big(\mathsf{R}^0_{MN}(d(\cdot,\cdot;s))(\cdot,\cdot)\big) \approx \dot{K}^0_{MN}\big(\dot{\mathsf{R}}^0_{MN}(d(\cdot,\cdot;s))(\cdot,\cdot)\big),$$

yielding a resolution analysis in depth. The stationary phase analysis in, and following (7-6) applies, leading to the introduction of $\tilde{p}^0(z_0',h)$ and associated ray geometry if there is a nonvanishing contrast (horizontal reflector) at depth z_0'.

For the singularities to appear in $(\mathsf{R}^0_{MN}(d(\cdot,\cdot;s)))(h,t)$, it is necessary that $\tilde{p}^0(z_0',h)$ coincides with the stationary value, $\tilde{p}_s$ say, of $\tilde{p}$ associated with the WKBJ approximation of the *incident* field (d_N), determined by s and $m+h$.

Receiver functions, plane-wave synthesis. Let $\tilde{d}(\cdot,\cdot;\tilde{p}_s)$ denote the frequency-domain data obtained by synthesizing a source *plane wave* with parameter $\tilde{p}_s$ (as in plane-wave Kirchhoff migration) including an appropriate amplitude weighting function derived from the WKBJ approximation (here, we depart from the single source acquisition). We correlate these data using (2-16) subjected to a Fourier transform in time, with $d(\cdot,\cdot;s)$ replaced by $\tilde{d}(\cdot,\cdot;\tilde{p}_s)$. We obtain

$$\widehat{\mathsf{R}}_{MN}(\tilde{d}(\cdot,\cdot;\tilde{p}_s))(m+h,\tau,m-h),$$

with the property

$$\dot{\mathsf{R}}^0_{MN}(d(\cdot\,,\cdot\,;s))(h,t) \sim \int \widehat{\mathsf{R}}_{MN}(\tilde{d}(\cdot\,,\cdot\,;\tilde{p}_s))(m_0,\tau,m_0)\exp\biggl(\mathrm{i}\tau\biggl[-2\sum_{j=1}^{n-1}\tilde{p}_{sj}h_j+t\biggr]\biggr)\,\mathrm{d}\tilde{p}_s\,\mathrm{d}\tau$$

for any $(m_0, 0)$ contained in Σ_s. The quantity $\widehat{\mathsf{R}}_{MN}(\tilde{d}(\cdot\,,\cdot\,;\tilde{p}_s))(m_0,\tau,m_0)$ is what seismologists call a receiver function. The phase shift and receiver function are illustrated by the dashed ray (single array) paired with the ray indicated by double arrows in Figure 6.

8. Applications in global seismology

We presented an approach to elastic-wave inverse scattering of reflection seismic data generated by an active source or a passive source, focused on mode conversions, and based on reverse-time migration (RTM). We introduced array receiver functions (ARFs), a generalization of the notion of receiver functions, which can be used for inverse scattering of passive source data with a resolution comparable to RTM. In principle, the ARFs can be used also for imaging with certain (mode-converted) multiple scattered waves. Microlocally, RTM and generalized Radon transform (GRT) based inverse scattering are the same, the key restriction being the absence of source caustics; however, typically, the GRT is obtained as the composition of the parametrix of a normal operator with the imaging (that is, adjoint of the single scattering) operator. We note that the implementation of our RTM-based inverse scattering does not involve any ray-geometrical computation.

Our RTM-based inverse scattering transform defines a Fourier integral operator the propagation of singularities of which is described by a canonical graph. Thus it directly admits expansions into wave packets or curvelets, accommodates partial reconstruction as developed in [de Hoop et al. 2009], and associated algorithms can be applied. In practice, one may carry out the addition of terms in the inverse scattering transform adaptively. Also, the polarized-wave equation formulation is well-suited for a frequency-domain implementation of the type presented in [Wang et al. 2010].

A key application of the analysis presented in this paper concerns the detection, mapping, and characterization of interfaces in Earth's upper mantle (see Figure 1). The analysis allows one to integrate contributions from different body-wave phases (for example, underside reflections beneath oceans and mode conversions, that is, ARFs beneath continents), accommodating the fact that earthquakes are highly unevenly distributed and that the data are inherently restricted to parts of the earth's surface. Concerning ARFs, we elaborate on the feasibility of

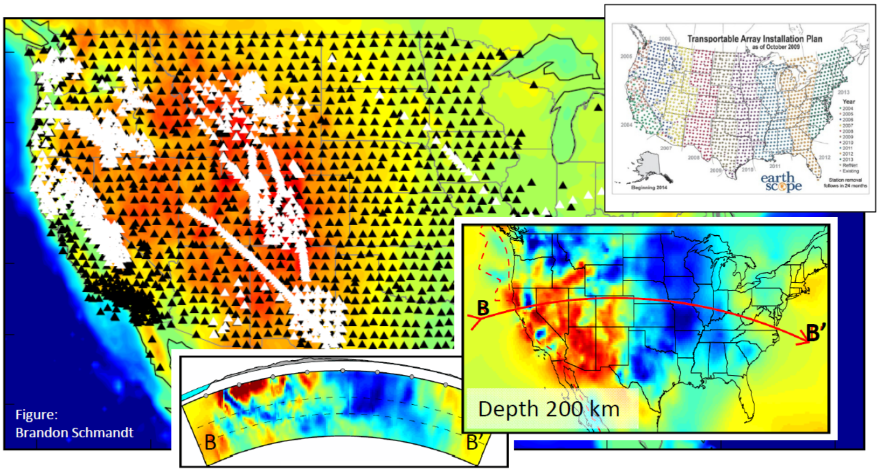

Figure 7. USArray (the seismology component of EarthScope, http://www.earthscope.org); black triangles indicate locations of permanent sensors and white triangles indicate more densely space temporary arrays. Insets: Transportable array installation plan (2004-2013), map of P-wave speed variations relative to a spherically symmetric model obtained with linearized transmission tomography at 200 km depth, and vertical mantle section to 600 km depth depicting in blue the seismically fast slab of subducted Gorda/Farallon lithosphere beneath the western States. Such a model serves as a background model in ARFs and RTM based inverse scattering. Generated by Scott Burdick.

inverse scattering beneath the North American continent using available data from USArray; see Figure 7, which also shows a recently obtained (isotropic) model, which can be used as a background model for application of the ARFs and RTM based inverse scattering presented here. The grid spacing of ~ 70 km of the three-component broadband seismograph stations that constitute the TA component of USArray is too sparse for ARF imaging of the crust mantle interface (near $35-40$ km depth) but is adequate for the imaging of upper mantle discontinuities. The images that we can produce will aid in better constraining the lateral variations in temperature and composition (including melt, volatile content) and the geological processes that produced them.

We end with a numerical example using modeled data designed for the detection of a piecewise smooth reflector. The (isotropic) model is depicted in Figure 9, top left. The finite bandwidth data, for a source indicated by an asterisk in Figure 9, bottom left, are shown in Figure 8. The smooth background model, which could be obtained from, for instance, tomography, is illustrated in Figure 9, top right. Applying the procedure outlined in Section 7A to P-to-S conversions yields the results shown in Figure 9, bottom; the bottom left figure is obtained with data generated by a single source, while the bottom right figure is obtained with data generated by a sparse set of sources.

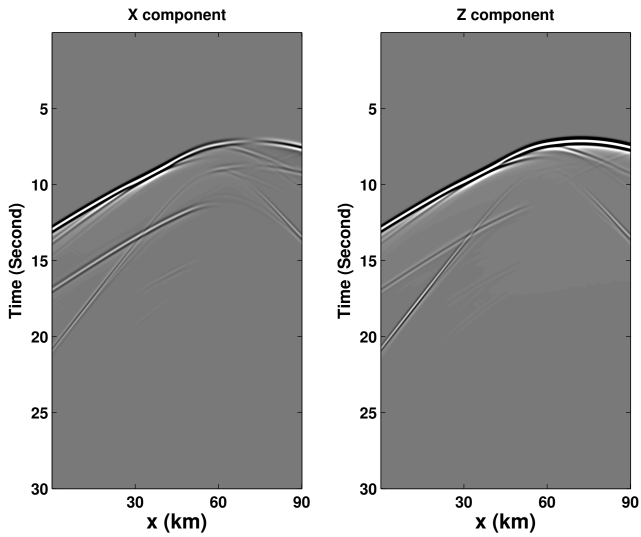

Figure 8. Array receiver functions ($n = 2$). Synthetic data generated in the model depicted in Figure 9 top, left; left: u_1; right: u_2. $\Sigma_{\tilde{x}}$ coincides with the top boundary.

Appendix: Diagonalization of A_{il} with a unitary operator

If the $A_M^{\text{prin}}(x, \xi)$ are all different, A_{il} can be diagonalized with a unitary operator, that is, $Q(x, D_x)^{-1} = Q(x, D_x)^*$. We write $\widetilde{Q}^0_{lN}(x, D_x) = Q^{\text{prin}}_{lN}(x, D_x)$; then $(\widetilde{Q}^0)^*_{Mi}(x, D_x)$ has principal symbol $(Q^{\text{prin}})^t_{Mi}(x, \xi)$. We have

$$(\widetilde{Q}^0)^*_{Mi}(x, D_x)\widetilde{Q}^0_{iN}(x, D_x) = \delta_{MN} + R^0_{MN}(x, D_x)$$

where $R^0_{MN}(x, D_x)$ is self adjoint and of order -1. Then $Q^0_{iN}(x, D_x) = (\widetilde{Q}^0(I + R^0)^{-1/2})_{iN}(x, D_x)$ is, microlocally, unitary. We write

$$A^0_1(x, D_x) = \tfrac{1}{2}[A^{\text{prin}}_1(x, D_x) + (A^{\text{prin}}_1)^*(x, D_x)],$$
$$A^0_2(x, D_x) = \operatorname{diag}(\tfrac{1}{2}[A^{\text{prin}}_M(x, D_x) + (A^{\text{prin}}_M)^*(x, D_x)]; M = 2, \dots, n),$$

so that

$$(Q^0)^* A Q^0 = \begin{pmatrix} A^0_1 & 0 \\ 0 & A^0_2 \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

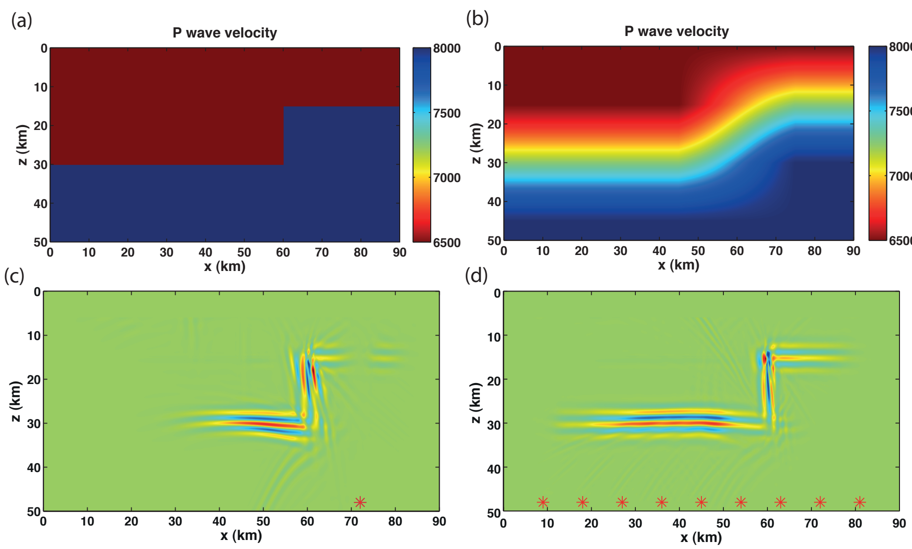

Figure 9. Array receiver functions ($n = 2$); (a) model (P-wave speed); (b) smooth background model (P-wave speed); (c) image reconstruction using data from a single source (location indicated by an asterisk); (d) image reconstruction using data from a sparse set of sources (locations indicated by asterisks). We note the effect of the illumination operators. Generated by Xuefeng Shang.

where B_{11} and the elements of B_{12} (a $1 \times (n-1)$ matrix), B_{21} (a $(n-1) \times 1$ matrix of pseudodifferential operators), and B_{22} (a $(n-1) \times (n_1)$ matrix of pseudodifferential operators), are of order 1; B must be self adjoint, whence $B_{12} = B_{21}^*$.

Next, we seek an operator, $\tilde{Q}^1_{lN}(x, D_x) = \delta_{lN} + r^1_{lN}(x, D_x)$, assuming that

$$r^1 = \begin{pmatrix} 0 & -(r^1_{21})^* \\ r^1_{21} & 0 \end{pmatrix},$$

whence $(r^1)^* = -r^1$, such that

$$\left((\tilde{Q}^1)^* \left(\begin{pmatrix} A^0_1 & 0 \\ 0 & A^0_2 \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \right) \tilde{Q}^1 \right)_{21} = 0,$$

$$\left((\tilde{Q}^1)^* \left(\begin{pmatrix} A^0_1 & 0 \\ 0 & A^0_2 \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \right) \tilde{Q}^1 \right)_{12} = 0,$$

modulo terms of order 0. This holds true if

$$r^1_{21} A^0_1 - A^0_2 r^1_{21} = B_{21};$$

r^1_{21} must be of order -1. Up to principal parts, this is a matrix equation for the symbol of r^1_{21}, given the principal symbols of A^0_1 and A^0_2; we note that the principals of A^{prin}_1 and A^0_1, and of $\mathrm{diag}(A^{\mathrm{prin}}_M; M = 2, \dots, n)$ and A^0_2, coincide. Because the eigenvalues of $A^0_2(x, \xi)$ all differ from $A^0_1(x, \xi)$, it follows that this system of algebraic equations has a solution. With the solution we form the unitary operator $Q^1_{iN}(x, D_x) = ((I + r^1)(I - (r^1)^2)^{-1/2})_{iN}(x, D_x)$. Then

$$(Q^1)^*(Q^0)^* A Q^0 Q^1 = \begin{pmatrix} A^1_1 & 0 \\ 0 & A^1_2 \end{pmatrix} + \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix},$$

where A^1_1 and A^1_2 are self adjoint. C_{11} and the elements of C_{12} (a $1 \times (n-1)$ matrix of pseudodifferential operators), C_{21} (a $(n-1) \times 1$ matrix of pseudodifferential operators), and C_{22} (a $(n-1) \times (n_1)$ matrix of pseudodifferential operators), are of order 0; C must be self adjoint, whence $C_{12} = C^*_{21}$. This procedure is continued to find $Q^0 Q^1 \cdots Q^k$ which is microlocally unitary and brings A in block diagonal form modulo terms of order $1 - k$. Next, we repeat the procedure for A^k_2.

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