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Inverse problems in spectral geometry

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In this survey we review positive inverse spectral and inverse resonant results for the following kinds of problems: Laplacians on bounded domains, Laplace-Beltrami operators on compact manifolds, Schrödinger operators, Laplacians on exterior domains, and Laplacians on manifolds which are hyperbolic near infinity.

1. Introduction

Marc Kac [1966], in a famous paper, raised the following question: Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and let

$$0 \le \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$$

be the eigenvalues of the nonnegative Euclidean Laplacian Δ_Ω with either Dirichlet or Neumann boundary conditions. Is Ω determined up to isometries from the sequence $\lambda_0, \lambda_1, \ldots$? We can ask the same question about bounded domains in $\mathbb{R}^n$, and below we will discuss other generalizations as well. Physically, one motivation for this problem is identifying distant physical objects, such as stars or atoms, from the light or sound they emit. These inverse spectral problems, as some engineers have recently proposed in [Reuter 2007; Reuter et al. 2007; 2009; Peinecke et al. 2007], may also have interesting applications in shape-matching, copyright and medical shape analysis.

The only domains in $\mathbb{R}^n$ known to be spectrally distinguishable from all other domains are balls. It is not even known whether or not ellipses are spectrally rigid, i.e., whether or not any continuous family of domains containing an ellipse and having the same spectrum as that ellipse is necessarily trivial. We can go further and ask the same question about a compact Riemannian manifold (M, g) (with or without boundary): can we determine (M, g) up to isometries from the spectrum of the Laplace–Beltrami operator Δ_g? Or in general, what can we hear from the spectrum? For example, can we hear the area (volume in higher

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dimensions or in the case of Riemannian manifolds) or the perimeter of the domain? For the sake of brevity we only mention the historical background for the case of domains.

In 1910, Lorentz gave a series of physics lectures in Göttingen, and he conjectured that the asymptotics of the counting function of the eigenvalues are given by

$$N(\lambda) = \sharp\{\lambda_j;\ \lambda_j \le \lambda\} = \frac{\text{Area}\,(\Omega)}{2\pi}\lambda + O(\sqrt{\lambda}).$$

This asymptotic in particular implies that Area(Ω) is a spectral invariant. Hilbert thought this conjecture would not be proven in his lifetime, but less than two years later Hermann Weyl proved it using the theory of integral equations taught to him by Hilbert. Pleijel [1954] proved that one knows the perimeter of Ω, and Kac [1966] rephrased these results in terms of asymptotics of the heat trace

$$\operatorname{Tr} e^{-t\Delta_\Omega} \sim t^{-1}\sum_{j=0}^{\infty} a_j t^{j/2}, \quad t \to 0^+,$$

where the first coefficient a_0 gives the area and the second coefficient gives the perimeter. McKean and Singer [1967] proved Pleijel's conjecture that the Euler characteristic $\chi(\Omega)$ is also a spectral invariant (this is in fact given by a_2) and hence the number of holes is known. Gordon, Webb and Wolpert [1992] found examples of pairs of distinct plane domains with the same spectrum. However, their examples were nonconvex and nonsmooth, and it remains an open question to prove that convex domains are determined by the spectrum (although there are higher-dimensional counterexamples for this in [Gordon and Webb 1994]) or that smooth domains are determined by the spectrum.

In this survey we review positive inverse spectral and inverse resonant results for the following kinds of problems: Laplacians on bounded domains, Laplace–Beltrami operators on compact manifolds, Schrödinger operators, Laplacians on exterior domains, and Laplacians on manifolds which are hyperbolic near infinity. We also recommend the survey [Zelditch 2004b]. For negative results (counterexamples) we refer the reader to the surveys [Gordon 2000; Gordon, Perry, Schueth 2005].

In the next two sections of the paper we review uniqueness results for radial problems (Section 2), and for real analytic and symmetric problems (Section 3). In the first case the object to be identified satisfies very strong assumptions (radialness includes full symmetry as well as analyticity) but it is identified in a broad class of objects. In this case the first few heat invariants, together with an isoperimetric or isoperimetric-type inequality, often suffice. In the second case the assumptions on the object to be identified are somewhat weaker (only analyticity and finitely many reflection symmetries are assumed) but the identification is only within a class of objects which also satisfies the same assumptions, and generic nondegeneracy assumptions are also needed. These proofs are based on wave trace invariants corresponding to a single nondegenerate simple periodic orbit and its iterations.

In Section 4 we consider rigidity and local uniqueness results, where it is shown in the first case that isospectral deformations of a given object are necessarily trivial, and in the second case that a given object is determined by its spectrum among objects which are nearby in a suitable sense. Here the objects to be determined are more general than in the cases considered in Section 2, but less general than those in Section 3: they are ellipses, spheres, flat manifolds (which have completely integrable dynamics), and manifolds of constant negative curvature (which have chaotic dynamics). The proofs use these special features of the classical dynamics.

In Section 5 we consider compactness results, where it is shown that certain isospectral families are compact in a suitable topology. These proofs are based on heat trace invariants and on the determinant of the Laplacian, and much more general assumptions are possible than in the previous cases.

Finally, in Section 6 we review the trace invariants used for the positive results in the previous sections, and give examples of their limitations, that is to say examples of objects which have the same trace invariants but which are not isospectral. At this point we also discuss the history of these invariants, going back to the seminal paper [Selberg 1956].

We end the introduction by presenting the four basic settings we consider in this survey:

1.1. ***Dirichlet and Neumann Laplacians on bounded domains in $\mathbb{R}^n$.*** Let Ω be a bounded open set with piecewise smooth boundary. Let Δ_Ω be the nonnegative Laplacian on Ω with Dirichlet or Neumann boundary conditions. Let

$$\operatorname{spec}(\Delta_\Omega) = (\lambda_j)_{j=0}^{\infty}, \quad \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$$

be the eigenvalues included according to multiplicity, and u_j the corresponding eigenfunctions, that is to say

$$\Delta_\Omega u_j = \lambda_j u_j.$$

Recall that $\lambda_0 > 0$ in the Dirichlet case and $\lambda_0 = 0$ in the Neumann case.

1.2. ***Laplace–Beltrami operators on compact manifolds.*** Let (M, g) be a compact Riemannian manifold without boundary. Let $\Delta_g = -\operatorname{div}_g \operatorname{grad}_g$ be the nonnegative Laplace–Beltrami operator on M, which we also call the Laplacian for short. Let

$$\operatorname{spec}(\Delta_g) = (\lambda_j)_{j=0}^{\infty}, \quad 0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$$

be the eigenvalues included according to multiplicity, and u_j the corresponding eigenfunctions, that is to say

$$\Delta_g u_j = \lambda_j u_j.$$

1.3. ***Nonsemiclassical and semiclassical Schrödinger operators on*** $\mathbb{R}^n$**.** Let

$$V \in C^{\infty}(\mathbb{R}^n; \mathbb{R}), \qquad \lim_{|x|\to\infty} V(x) = \infty, \tag{1-1}$$

and let Δ be the nonnegative Laplacian on $\mathbb{R}^n$. Let

$$P_{V,h} = h^2\Delta + V, \quad h > 0,$$
$$P_V = P_{V,1}.$$

We call P_V the nonsemiclassical Schrödinger operator associated to V, and $P_{V,h}$ the semiclassical operator. For any $h > 0$, the spectrum of $P_{V,h}$ on $\mathbb{R}^n$ is discrete, and we write it as

$$\operatorname{spec}(P_{V,h}) = (\lambda_j)_{j=0}^{\infty}, \quad \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots .$$

The eigenvalues λ_j depend on h, but we do not include this in the notation. We denote by u_j the corresponding eigenfunctions (which also depend on h), so that

$$P_{V,h} u_j = \lambda_j u_j.$$

1.4. ***Resonance problems for obstacle and potential scattering.*** In this section we discuss problems where the spectrum consists of a half line of essential spectrum, together with possibly finitely many eigenvalues. In such settings the spectrum contains limited information, but one can often define resonances, which supplement the discrete spectral data and contain more information.

Obstacle scattering in $\mathbb{R}^n$. Let $O \subset \mathbb{R}^n$ be a bounded open set with smooth boundary, let $\Omega = \mathbb{R}^n \setminus \overline{O}$, and suppose that Ω is connected. Let Δ_Ω be the nonnegative Dirichlet or Neumann Laplacian on Ω. Then the spectrum of Δ_Ω is continuous and equal to $[0, \infty)$, and so it contains no (further) information about Ω. One way to reformulate the inverse spectral problem in this case is in terms of *resonances*, which are defined as follows. Introduce a new spectral parameter $z = \sqrt{\lambda}$, with $\sqrt{\ }$ taken so as to map $\mathbb{C} \setminus [0, \infty)$ to the upper half-plane. As $\operatorname{Im} z \to 0^+$, z^2 approaches $[0, \infty)$ and the resolvent $(\Delta_\Omega - z^2)^{-1}$ has no limit as a map $L^2(\Omega) \to L^2(\Omega)$. However, if we restrict the domain of the resolvent and expand the range it is possible not only to take the limit but also to take a meromorphic continuation to a larger set. More precisely the resolvent

$$(\Delta_\Omega - z^2)^{-1} \colon L^2_{\mathrm{comp}} \to L^2_{\mathrm{loc}},$$

(where L^2_{comp} denotes compactly supported L^2 functions and L^2_{loc} denotes functions which are locally L^2) continues meromorphically as an operator-valued function of z from $\{\operatorname{Im} z > 0\}$ to $\mathbb{C}$ when n is odd and to the Riemann surface of $\log z$ when n is even. Resonances are defined to be the poles of this continuation of the resolvent. Let $\operatorname{res}(\Delta_\Omega)$ denote the set of resonances, included according to multiplicity. See for example [Melrose 1995; Sjöstrand 2002; Zworski 2011] for more information.

Potential scattering in $\mathbb{R}^n$. Let $P_{V,h}$ be as before, but instead of (1-1) assume $V \in C_0^\infty(\mathbb{R}^n)$. Then the continuous spectrum of $P_{V,h}$ is equal to $[0,\infty)$, but if V is not everywhere nonnegative then $P_{V,h}$ may have finitely many negative eigenvalues. In either case, the resolvent

$$(P_{V,h} - z^2)^{-1} \colon L^2_{\mathrm{comp}} \to L^2_{\mathrm{loc}},$$

has a meromorphic continuation from $\{\operatorname{Im} z > 0\}$ to $\mathbb{C}$ when n is odd and to the Riemann surface of $\log z$ when n is even, and resonances are defined to be the poles of this continuation. Let $\operatorname{res}(P_{V,h})$ denote the set of resonances, included according to multiplicity. Again, see for example [Melrose 1995; Sjöstrand 2002; Zworski 2011] for more information.

Scattering on asymptotically hyperbolic manifolds. The problem of determining a noncompact manifold from the scattering resonances of the associated Laplace–Beltrami is in general a much more difficult one, but some progress has been made in the asymptotically hyperbolic setting. Meromorphic continuation of the resolvent was established in [Mazzeo and Melrose 1987], and a wave trace formula in the case of surfaces with exact hyperbolic ends was found by Guillopé and Zworski [1997], which has led to some compactness results: see Section 5.4.

2. The radial case

In this case one makes a strong assumption (radial symmetry) on the object to be spectrally determined (whether it is an open set in $\mathbb{R}^n$, a compact manifold, or a potential) but makes almost no assumption on the class of objects within which it is determined. The methods involved use the first few heat invariants, and in many cases the isoperimetric inequality or an isoperimetric-type inequality.

2.1. ***Bounded domains in*** $\mathbb{R}^n$**.** The oldest inverse spectral results are for radial problems. If $\Omega \subset \mathbb{R}^n$ is a bounded open set with smooth boundary, then the spectrum of the Dirichlet (or Neumann) Laplacian on Ω agrees with the spectrum on the unit ball if and only if Ω is a translation of this ball. This can be proved in many ways; one way is to use heat trace invariants. These are defined to be the coefficients of the asymptotic expansion of the heat trace as $t \to 0^+$:

$$\sum_{j=0}^{\infty} e^{-t\lambda_j} = \operatorname{Tr} e^{-t\Delta_\Omega} \sim t^{-n/2} \sum_{j=0}^{\infty} a_j t^{j/2}, \tag{2-1}$$

where in both the Dirichlet and the Neumann case a_0 is a universal constant times $\operatorname{vol}(\Omega)$, and a_1 is a universal constant times $\operatorname{vol}(\partial\Omega)$. The left-hand side is clearly determined by the spectrum, and so the conclusion follows from the isoperimetric inequality.

2.2. ***Compact manifolds.*** Let (M, g) be a smooth Riemannian manifold of dimension n without boundary. If $n \le 6$, then the spectrum of the Laplacian on M agrees with the spectrum on S^n (equipped with the round metric) if and only if M is isometric to S^n. This was proved in [Tanno 1973; 1980] using the first four coefficients, a_0, a_1, a_2, a_3, of the heat trace expansion, which in this case takes the form

$$\sum_{j=0}^{\infty} e^{-t\lambda_j} = \operatorname{Tr} e^{-t\Delta_g} \sim t^{-n/2} \sum_{j=0}^{\infty} a_j t^{j}.$$

In higher dimensions the analogous result is not known. Zelditch [1996] proved that if the multiplicities m_k of the *distinct* eigenvalues $0 = E_0 < E_1 < E_2 < \cdots$ of the Laplacian on M obey the asymptotic $m_k = ak^{n-1} + O(k^{n-2})$, for some $a > 0$ as $k \to \infty$ (this is the asymptotic behavior for the multiplicities of the eigenvalues of the sphere), then (M, g) is a Zoll manifold, that is to say a manifold on which all geodesics are periodic with the same period.

2.3. ***Schrödinger operators.*** In general it is impossible to determine a potential V from the spectrum of the nonsemiclassical Schrödinger operator $\Delta + V$. For example, McKean and Trubowitz [1981] found an infinite-dimensional family of potentials in $C^\infty(\mathbb{R})$ which are isospectral with the harmonic oscillator $V(x) = x^2$.

However, analogous uniqueness results to those above were proved in [Datchev, Hezari, and Ventura 2011], where it is shown that radial, monotonic potentials in $\mathbb{R}^n$ (such as for example the harmonic oscillator) are determined by the spectrum of the associated *semiclassical* Schrödinger operator among all potentials with discrete spectrum. The approach is based in part on that of [Colin de Verdière 2011] and [Guillemin and Wang 2009] (see also [Guillemin and Sternberg 2010, §10.6]), where a one-dimensional version of the result is proved. Colin de Verdière and Guillemin–Wang show that an even function (or a suitable noneven function) is determined by its spectrum within the class of functions monotonic away from 0.

The method of proof is similar to that used to prove spectral uniqueness of balls in $\mathbb{R}^n$ as discussed in Section 2.1 above. Namely, we use the first two trace invariants, this time of the semiclassical trace formula of Helffer and Robert [1983], together with the isoperimetric inequality. We show that if V, V_0 are as in (1-1), if $V_0(x) = R(|x|)$ where $R(0) = 0$ and $R'(r) > 0$ for $r > 0$, and if $\mathrm{spec}(P_{V,h}) = \mathrm{spec}(P_{V_0,h})$ up to order[1] $o(h^2)$ for $h \in \{h_j\}_{j=0}^{\infty}$ with $h_j \to 0^+$, then $V(x) = V_0(x - x_0)$ for some $x_0 \in \mathbb{R}^n$.

The semiclassical trace formula we use is

$$\mathrm{Tr}(f(P_{V,h})) = \frac{1}{(2\pi h)^n}\biggl(\int_{\mathbb{R}^{2n}} f(|\xi|^2+V)\,dx\,d\xi + \frac{h^2}{12}\int_{\mathbb{R}^{2n}} |\nabla V|^2 f^{(3)}(|\xi|^2+V)\,dx\,d\xi + \mathbb{O}(h^4)\biggr), \tag{2-2}$$

where $f \in C_0^\infty(\mathbb{R})$.

Because the spectrum of $P_{V,h}$ is known up to order $o(h^2)$ we obtain from (2-2) the two trace invariants

$$\int_{\{|\xi|^2+V(x)<\lambda\}} dx\,d\xi, \quad \int_{\{|\xi|^2+V(x)<\lambda\}} |\nabla V(x)|^2 dx\,d\xi, \tag{2-3}$$

for each λ. It follows in particular that V is nonnegative. By integrating in the ξ variable, we rewrite these invariants as follows:

$$\int_{\{V(x)<\lambda\}} (\lambda - V)^{n/2} dx, \quad \int_{\{V(x)<\lambda\}} |\nabla V(x)|^2 (\lambda - V)^{n/2} dx. \tag{2-4}$$

Using the coarea formula we rewrite the invariants in (2-4) as

$$\int_0^\lambda \biggl(\int_{\{V=s,\nabla V\neq 0\}} \frac{(\lambda - V)^{n/2}}{|\nabla V|} dS\biggr) ds,$$
$$\int_0^\lambda \biggl(\int_{\{V=s\}} |\nabla V|\,(\lambda - V)^{n/2} dS\biggr) ds.$$

Using the fact that $V = s$ in the inner integrand, the factor of $(\lambda - V)^{n/2} = (\lambda - s)^{n/2}$ can be taken out of the surface integral, leaving

$$\int_0^\lambda (\lambda - s)^{n/2} I_1(s)\,ds, \quad \int_0^\lambda (\lambda - s)^{n/2} I_2(s)\,ds, \tag{2-5}$$

[1] The implicit rate of convergence here must be uniform on $[0, \lambda_0]$ for each $\lambda_0 > 0$.

where

$$I_1(s) = \int_{\{V=s, \nabla V \neq 0\}} \frac{1}{|\nabla V|} dS, \quad I_2(s) = \int_{\{V=s\}} |\nabla V| dS. \tag{2-6}$$

We denote the integrals (2-5) by $A_{1+n/2}(I_1)(\lambda)$ and $A_{1+n/2}(I_1)(\lambda)$. These are Abel fractional integrals of I_1 and I_2 (see for example [Zelditch 1998a, §5.2] and [Guillemin and Sternberg 2010, (10.45)]), and they can be inverted by applying $A_{1+n/2}$, using the formula

$$\frac{1}{\Gamma(\alpha)} A_\alpha \circ \frac{1}{\Gamma(\beta)} A_\beta = \frac{1}{\Gamma(\alpha+\beta)} A_{\alpha+\beta}, \tag{2-7}$$

and differentiating $n+1$ times. From this we conclude that the functions I_1 and I_2 in (2-6) are spectral invariants for every $s > 0$.

Integrating I_1 and using the coarea formula again we find that the volumes of the sets $\{V < s\}$ are spectral invariants:

$$\int_0^s I_1(s')\, ds' = \int_0^s \int_{\{V=s', \nabla V \neq 0\}} \frac{1}{|\nabla V|} dS\, ds' = \int_{\{V<s\}} 1\, dx. \tag{2-8}$$

From Cauchy–Schwarz and the fact that I_1 and I_2 are spectral invariants we obtain

$$\begin{aligned} \left(\int_{\{V=s\}} 1\, dS \right)^2 &\le \int_{\{V=s\}} \frac{1}{|\nabla V|} dS \int_{\{V=s\}} |\nabla V|\, dS \\ &= \int_{\{R=s\}} \frac{1}{R'} dS \int_{\{R=s\}} R' dS, \end{aligned} \tag{2-9}$$

when s is not a critical value of V, and thus, by Sard's theorem, for almost every $s \in (0, \lambda_0)$. On the other hand, using the invariants obtained in (2-8) and the fact that the sets $\{R < s\}$ are balls, by the isoperimetric inequality we find

$$\int_{\{R=s\}} 1\, dS \le \int_{\{V=s\}} 1 dS. \tag{2-10}$$

However,

$$\left(\int_{\{R=s\}} 1\, dS \right)^2 = \int_{\{R=s\}} \frac{1}{R'} dS \int_{\{R=s\}} R' dS,$$

because $1/R'$ and R' are constant on $\{R = s\}$. Consequently

$$\int_{\{R=s\}} 1\, dS = \int_{\{V=s\}} 1\, dS,$$

and so $\{V = s\}$ is a sphere for almost every s, because only spheres extremize the isoperimetric inequality. Moreover,

$$\left(\int_{\{V=s\}} 1\, dS\right)^2 = \int_{\{V=s\}} \frac{1}{|\nabla V|} dS \int_{\{V=s\}} |\nabla V|\, dS,$$

and so $|\nabla V|^{-1}$ and $|\nabla V|$ are proportional on the surface $\{V = s\}$ for almost every s, again by Cauchy–Schwarz. Using (2-9) to determine the constant of proportionality, we find that

$$|\nabla V|^2 = R'(R^{-1}(s))^2 = (R^{-1})'(s)^{-2} \stackrel{\text{def}}{=} F(s)$$

on $\{V = s\}$. In other words

$$|\nabla V|^2 = F(V), \tag{2-11}$$

for all $x \in V^{-1}(s)$ for almost all s. However, because $F(V) \neq 0$ when $V \neq 0$, it follows by continuity that this equation holds for all $x \in V^{-1}((0, \infty))$.

We solve this equation by restricting it to flowlines of ∇V, with initial conditions taken on a fixed level set $\{V = s_0\}$, and conclude that, the level surfaces are not only spheres (as follows from (2-10)) but are moreover spheres with a common center. Hence, up to a translation, V is radial. Since the volumes (2-8) are spectral invariants, it follows that $V(x) = R(|x|)$.

2.4. *Resonance problems.* We first mention briefly some results for inverse problems for resonances for the nonsemiclassical Schrödinger problem when $n = 1$. Zworski [2001] proved that a compactly supported even potential $V \in L^1(\mathbb{R})$ is determined from the resonances of P_V among other such potentials, and Korotyaev [2005] showed that a potential which is not necessarily even is determined by some additional scattering data.

Analogous results to those discussed in Section 2.1 hold in the case of obstacle scattering. Hassell and Zworski [1999] showed that a ball is determined by its Dirichlet resonances among all compact obstacles in $\mathbb{R}^3$. Christiansen [2008] extended this result to multiple balls, to higher odd dimensions, and to Neumann resonances. As in the other results discussed above, the proofs use two trace invariants and isoperimetric-type inequalities, although the invariants and inequalities are different here. There is also a large literature of inverse scattering results where data other than the resonances are used. A typical datum here is the *scattering phase*: see for example [Melrose 1995, §4.1].

In [Datchev and Hezari 2012] we prove the analogue for resonances of the result in the previous section for semiclassical Schrödinger operators with discrete spectrum. Let $n \geq 1$ be odd, and let $V_0, V \in C_0^\infty(\mathbb{R}^n; [0, \infty))$. Suppose $V_0(x) = R(|x|)$, and $R'(r)$ vanishes only at $r = 0$ and whenever $R(r) = 0$, and suppose that $\operatorname{res}(P_{V_0,h}) = \operatorname{res}(P_{V,h})$, up to order[2] $o(h^2)$, for $h \in \{h_j\}_{j=1}^{\infty}$ for some sequence $h_j \to 0$. Then there exists $x_0 \in \mathbb{R}^n$ such that $V(x) = V_0(x - x_0)$.

Our proof is, as before, based on recovering and analyzing first two integral invariants of the Helffer–Robert semiclassical trace formula [1983, Proposition 5.3] (see also [Guillemin and Sternberg 2010, §10.5]):

$$\begin{aligned}\operatorname{Tr}(f(P_{V,h}) - f(P_{0,h}))&\\ = \frac{1}{(2\pi h)^n}\biggl(\int_{\mathbb{R}^{2n}} f(|\xi|^2 + V) - f(|\xi|^2)\,dx\,d\xi&\\ + \frac{h^2}{12}\int_{\mathbb{R}^{2n}} |\nabla V|^2 f^{(3)}(|\xi|^2 + V)\,dx\,d\xi + \mathbb{O}(h^4)\biggr).&\qquad (2\text{-}12)\end{aligned}$$

To express the left-hand side of (2-12) in terms of the resonances of $P_{V,h}$, we use Melrose's Poisson formula [Melrose 1982], an extension of the formula of Bardos, Guillot, and Ralston [1982]:

$$2\operatorname{Tr}\bigl(\cos(t\sqrt{P_{V,h}}) - \cos(t\sqrt{P_{0,h}})\bigr) = \sum_{\lambda \in \operatorname{res}(P_{V,h})} e^{-i|t|\lambda}, \quad t \neq 0, \qquad (2\text{-}13)$$

where equality is in the sense of distributions on $\mathbb{R} \setminus 0$.

From (2-13), it follows that if

$$\hat{g} \in C_0^\infty(\mathbb{R} \setminus 0) \text{ is even,} \qquad (2\text{-}14)$$

then

$$\operatorname{Tr}\bigl(g(\sqrt{-h^2\Delta + V}) - g(\sqrt{-h^2\Delta})\bigr) = \frac{1}{4\pi}\sum_{\lambda \in \operatorname{res}(P_{V,h})} \int_{\mathbb{R}} e^{-i|t|\lambda}\hat{g}(t)\,dt. \qquad (2\text{-}15)$$

Now setting the right-hand sides of (2-15) and (2-12) equal and taking $h \to 0$, we find that

$$\int_{\mathbb{R}^{2n}} f(|\xi|^2 + V) - f(|\xi|^2)\,dx\,d\xi, \quad \int_{\mathbb{R}^{2n}} |\nabla V|^2 f^{(3)}(|\xi|^2 + V)\,dx\,d\xi \qquad (2\text{-}16)$$

are resonant invariants (i.e., are determined by knowledge of the resonances up to $o(h^2)$) provided that $f(\tau^2) = g(\tau)$ for all τ and for some g as in (2-14). Taylor expanding, we write the first invariant as

$$\sum_{k=1}^{m} \frac{1}{k!}\int_{\mathbb{R}^n} f^{(k)}(|\xi|^2)\,d\xi \int_{\mathbb{R}^n} V(x)^k\,dx + \int_{\mathbb{R}^{2n}} \frac{V(x)^{m+1}}{m!}\int_0^1 (1-t)^m f^{(m+1)}(|\xi|^2 + tV(x))\,dt\,dx\,d\xi.$$

[2]The implicit rate of convergence here must be uniform on the disk of radius λ_0 for each $\lambda_0 > 0$.

Replacing f by f_λ, where $f_\lambda(\tau) = f(\tau/\lambda)$ (note that $g_\lambda(\tau) = f_\lambda(\tau^2)$ satisfies (2-14)) gives

$$\sum_{k=1}^{m} \lambda^{n/2-k} \frac{1}{k!} \int_{\mathbb{R}^n} f^{(k)}(|\xi|^2)\, d\xi \int_{\mathbb{R}^n} V(x)^k\, dx + \mathbb{O}(\lambda^{n/2-m-1})$$

Taking $\lambda \to \infty$ and $m \to \infty$ we obtain the invariants

$$\int_{\mathbb{R}^n} f^{(k)}(|\xi|^2)\, d\xi \int_{\mathbb{R}^n} V(x)^k\, dx,$$

for every $k \geq 1$.

In [Datchev and Hezari 2012, Lemma 2.1] it is shown that there exists g satisfying (2-14) such that if $f(\tau^2) = g(\tau)$, then $\int_{\mathbb{R}^n} f^{(k)}(|\xi|^2)\, d\xi \neq 0$, provided $k \geq n$.

This shows that

$$\int_{\mathbb{R}^n} V(x)^k\, dx = \int_{\mathbb{R}^n} V_0(x)^k\, dx \tag{2-17}$$

for every $k \geq n$, and a similar analysis of the second invariant of (2-16) proves that

$$\int_{\mathbb{R}^n} V(x)^k |\nabla V(x)|^2 dx = \int_{\mathbb{R}^n} V_0(x)^k |\nabla V_0(x)|^2 dx \tag{2-18}$$

for every $k \geq n$.

We rewrite the invariant (2-17) using $V_* dx$, the pushforward of Lebesgue measure by V, as

$$\int_{\mathbb{R}^n} V(x)^k\, dx = \int_{\mathbb{R}} s^k (V_* dx)_s = i^k\, \widehat{V_* dx}^{(k)}(0). \tag{2-19}$$

Since V and V_0 are both bounded functions, the pushforward measures are compactly supported and hence have entire Fourier transforms, and we conclude that

$$V_* dx = V_{0*} dx + \sum_{k=0}^{n-1} c_k \delta_0^{(k)} = V_{0*} dx + c_0 \delta_0.$$

For the first equality we used the invariants (2-19), and for the second the fact that $V_* dx$ is a measure. In other words

$$\operatorname{vol}(\{V > \lambda\}) = \operatorname{vol}(\{V_0 > \lambda\})$$

whenever $\lambda > 0$. Moreover, this shows that $V_* dx$ is absolutely continuous on $(0, \infty)$, and so by Sard's lemma the critical set of V is Lebesgue-null on

$V^{-1}((0,\infty))$. As a result we may use the coarea formula[3] to write

$$V_* dx = \int_{\{V=s\}} |\nabla V|^{-1} dS\, ds \quad \text{on } (0,\infty)$$

and to conclude that

$$\int_{\{V=s\}} |\nabla V|^{-1} dS = \int_{\{V_0=s\}} |\nabla V_0|^{-1} dS$$

for almost every $s > 0$. Similarly, rewriting the invariants (2-18) as

$$\int_{\mathbb{R}^n} V(x)^k |\nabla V(x)|^2 dx = \int_{\mathbb{R}} s^k \int_{\{V=s\}} |\nabla V|\, dS\, ds,$$

we find that

$$\int_{\{V=s\}} |\nabla V|\, dS = \int_{\{V_0=s\}} |\nabla V_0|\, dS \quad s > 0.$$

From this point on the proof proceeds as in the previous section.

To our knowledge it is not known whether such results hold in even dimensions. The higher-dimensional results discussed above all rely on the Poisson formula (2-13) which is only valid for odd dimensions. A similar formula is also true in the obstacle case [Melrose 1983b], although slightly more care is needed in the definition of $\cos(t\sqrt{\Delta_\Omega}) - \cos(t\sqrt{\Delta_{\mathbb{R}^n}})$ because the two operators act on different spaces. poles of R_V. When $n = 1$ a stronger trace formula, valid for all $t \in \mathbb{R}$, is known: see for example [Zworski 1997, page 3]. When n is even, because the meromorphic continuation of the resolvent is not to $\mathbb{C}$ but to the Riemann surface of the logarithm, Poisson formulæ for resonances are more complicated and contain error terms: see [Sjöstrand 1997; Zworski 1998]. A proof based on Sjöstrand's local trace formula [1997] would be of particular interest, firstly because this formula applies in all dimensions and to a very general class of operators, and also because it uses only resonances in a sector (and in certain versions, as in [Bony 2002], resonances in a strip) around the real axis. This would strengthen the known results in odd dimensions as well as proving results in even dimensions, as one would only have to assume that these resonances agreed and not that all resonances do.

3. The real analytic and symmetric case

In this case uniqueness results about nonradial objects are obtained, so the assumptions on the object to be determined are weaker. However, the assumptions on the class of objects within which it is determined are much stronger – in fact

[3]If $n = 1$ we put $\int_{\{V=s\}} |\nabla V|^{-1} dS = \sum_{x \in V^{-1}(s)} |V'(x)|^{-1}$.

they are the same as the assumptions on the object to be determined. The two main assumptions are analyticity and symmetry. In each case wave invariants are used which are microlocalized near certain periodic orbits, as opposed to the nonmicrolocal heat invariants of the previous section.

3.1. ***Bounded domains in*** $\mathbb{R}^n$**.** Here the main tool is the following result from [Guillemin and Melrose 1979b]. When $\Omega \subset \mathbb{R}^n$ is a bounded, open set with smooth boundary, they prove that $\operatorname{Tr}(\cos(t\sqrt{\Delta_\Omega}))$ is a tempered distribution in $\mathbb{R}$ with the property

$$\operatorname{sing\,supp}\operatorname{Tr}(\cos(t\sqrt{\Delta_\Omega})) \subset \{0\} \cup \overline{\operatorname{Lsp}(\Omega)},$$

where Lsp denotes the length spectrum, that is to say the lengths of periodic billiard orbits in Ω. Moreover, they show that if $T \in \operatorname{Lsp}(\Omega)$ is of simple length[4] and γ_T is nondegenerate,[5] then for t sufficiently near T we have

$$\begin{aligned}\operatorname{Tr}\cos(t\sqrt{\Delta_\Omega}) \\ = \operatorname{Re}\Bigg[i^{\sigma_T} \frac{T^\sharp}{\sqrt{|\det(I-P_T)|}} (t-T+i0)^{-1} \\ \times \Bigg(1+\sum_{j=1}^{\infty} a_j (t-T)^j \log(t-T+i0)\Bigg)\Bigg] + S(t), \quad \text{(3-1)}\end{aligned}$$

where S is smooth near T. Here $T^\sharp$ is the primitive length of γ_T, which is the length of γ_T without retracing, and σ_T is the Maslov index of γ_T (which can be defined geometrically but which appears here as the signature of the Hessian in the stationary phase expansion of the wave trace). The coefficients a_j are known as *wave invariants*. Viewing the boundary locally as the graph of a function f, they are polynomials in the Taylor coefficients of f at the reflection points of γ_T. In general there is no explicit formula, but they were computed in [Zelditch 2009] in the special case discussed below.

We now assume that $n = 2$, with coordinates (x, y), and make these further assumptions:

(1) Ω is simply connected, symmetric about the x-axis, and $\partial\Omega$ is analytic on $\{y \neq 0\}$.
(2) There is a nondegenerate vertical bouncing ball orbit γ of length T such that both T and $2T$ are simple lengths in $\operatorname{Lsp}(\Omega)$.
(3) The endpoints of γ are not critical points of the curvature of $\partial\Omega$.

[4] This means that only one periodic orbit (up to time reversal), γ_T, has length T.

[5] This means that γ_T is transversal to the boundary and P_T the linearized Poincaré map of γ_T, which is the derivative of the first return map, does not have eigenvalue 1.

We recall that a bouncing ball orbit is a 2-link periodic trajectory of the billiard flow, i.e., a reversible periodic billiard trajectory that bounces back and forth along a line segment orthogonal to the boundary at both endpoints. Without loss of generality we may assume that the bouncing ball orbit in assumption (2) is on the y-axis.

Zelditch [2009] proved that if Ω and Ω' both satisfy these assumptions, and if $\operatorname{spec}(\Delta_\Omega) = \operatorname{spec}(\Delta_{\Omega'})$ (for either Dirichlet or Neumann boundary conditions), then $\Omega = \Omega'$ up to a reflection about the y-axis. This improves a previous result in [Zelditch 2000], where an additional symmetry assumption is needed, which in turn improves upon [Colin de Verdière 1984], where rigidity is proved in the class of analytic domains with two reflection symmetries. Under the above assumptions, for $\varepsilon > 0$ sufficiently small, there exists a real analytic function $f : (-\varepsilon, \varepsilon) \to \mathbb{R}$ such that

$$\Omega \cap \{|x| < \varepsilon\} = \{(x, y) : |x| < \varepsilon, |y| < f(x)\}.$$

To prove the theorem it is enough to show that the Taylor coefficients of f at 0 are determined by $\operatorname{spec}(\Omega)$ (up to possibly replacing $f(x)$ by $f(-x)$). Zelditch does this by writing a formula for the coefficients a_j of (3-1), which are determined by $\operatorname{spec}(\Omega)$, applied to γ and to γ^2 (the iteration of γ):

$$a_j(\gamma^r) = A_j(r) f^{(2j+2)}(0) + B_j(r) f^{(2j+1)}(0) f^{(3)}(0) \\ + \big[\text{terms containing } f^{(k)}(0) \text{ only for } k \le j\big]. \tag{3-2}$$

Here $A_j(r)$ and $B_j(r)$ are spectral invariants which are determined by the first term of (3-1). One can show that $(A_j(1), B_j(1))$ as a vector is linearly independent from $(A_j(2), B_j(2))$. Hence, by an inductive argument, if $f^{(3)}(0) \neq 0$, all the coefficients are determined (up to a sign ambiguity for $f^{(3)}(0)$, which corresponds to reflection about the y-axis). The condition $f^{(3)}(0) \neq 0$ is equivalent to assumption (3) above, and Zelditch [2009, §6.9] outlined a possible proof in the case where $f^{(3)}(0) = 0$.

In [Hezari and Zelditch 2010], it is proved that bounded analytic domains $\Omega \subset \mathbb{R}^n$ with $\pm$ reflection symmetries across all coordinate axes, and with one axis height fixed (and also satisfying some generic nondegeneracy conditions) are spectrally determined among other such domains. This inverse result gives a higher-dimensional analogue of the result discussed above from [Zelditch 2009], but with n axes of symmetry rather than $n - 1$. To our knowledge, it is the first positive higher-dimensional inverse spectral result for Euclidean domains which is not restricted to balls. The proof is based as before on (3-1) and on formulas for the $a_j(\gamma^r)$, but there are additional algebraic and combinatorial complications coming from the fact that Taylor coefficients must be recovered corresponding to all possible combinations of partial derivatives. These complications are very similar to those that arise for higher-dimensional semiclassical Schrödinger operators discussed below in Section 3.3.

3.2. ***Compact manifolds.*** To our knowledge all uniqueness results in this category are about surfaces of revolution. Bérard [1976] and Gurarie [1995] have shown that the joint spectrum of Δ_g and $\partial/\partial\theta$ (the generator of rotations) of a smooth surface of revolution determines the metric among smooth surfaces of revolution, by reducing the problem to a semiclassical Schrödinger operator in one dimension.

Brüning and Heintze [1984] showed that the spectrum of Δ_g alone determines the metric of a smooth surface of revolution with an up-down symmetry among such surfaces. They proved that the spectrum of Δ_g determines the S^1-invariant spectrum (but not necessarily the full joint spectrum), allowing them to apply [Marchenko 1952, Theorem 2.3.2] for one-dimensional Schrödinger operators.

Zelditch [1998a] proved that a convex analytic surface of revolution satisfying a nondegeneracy condition and a simplicity condition is determined uniquely by the spectrum among all such surfaces. He used analyticity and convexity to show that the spectrum determines the full joint spectrum of Δ_g and $\partial/\partial\theta$, reducing the problem to a semiclassical Schrödinger operator in one dimension.

3.3. ***Schrödinger operators.*** When $n = 1$, Marchenko [1952] showed that an even potential is determined by the spectrum of the associated nonsemiclassical Schrödinger operator among all even potentials. More specifically, Marchenko's Theorem 2.3.2 states that a Schrödinger operator on $[0,\infty)$ is determined by knowledge of both the Dirichlet and the Neumann spectrum. The result for even potentials on $\mathbb{R}$ follows from the result on $[0,\infty)$ as follows: Let $V \in C^\infty(\mathbb{R})$ obey $\lim_{|x|\to\infty} V(x) = \infty$. If u_j is the eigenfunction of $-\frac{d^2}{dx^2} + V$ corresponding to the eigenvalue λ_j, then u_j has exactly j zeros and they are all simple; see [Berezin and Shubin 1991, Chapter 2, Theorem 3.5]. If V is even then every eigenfunction is either odd or even and this result shows that the parity of u_j is the same as the parity of j. In particular $(\lambda'_j)_{j=0}^\infty$ with $\lambda'_j = \lambda_{2j}$ is the spectrum of

$$-\frac{d^2}{dx^2} + V \text{ on } L^2([0,\infty)) \text{ with Neumann boundary condition,}$$

and $(\lambda''_j)_{j=0}^\infty$ with $\lambda''_j = \lambda_{2j+1}$ is the spectrum of

$$-\frac{d^2}{dx^2} + V \text{ on } L^2([0,\infty)) \text{ with Dirichlet boundary condition.}$$

This reduces the problem on $\mathbb{R}$ to the result of Marchenko.

However, noneven potentials may have the same spectrum: indeed, McKean and Trubowitz [1981] constructed an infinite-dimensional family of potentials having the same spectrum as the one-dimensional harmonic oscillator $V(x) = x^2$.

Guillemin and Uribe [2007] considered potentials V in $\mathbb{R}^n$ which are analytic and even in all variables, which have a unique global minimum $V(0) = 0$, which obey $\liminf_{|x|\to\infty} V(x) > 0$, and such that the square roots of the eigenvalues of Hess $V(0)$ are linearly independent over $\mathbb{Q}$. They showed that such potentials are determined by their low lying semiclassical eigenvalues, that is to say by $\operatorname{spec}(P_{V,h}) \cap [0, \varepsilon]$ for any $\varepsilon > 0$. In [Hezari 2009], the second author removed the symmetry assumption in the case $n = 1$ but assumed $V'''(0) \neq 0$, and for $n \geq 2$ he replaced the symmetry assumption by the assumption that $V(x) = f(x_1^2, \dots x_n^2) + x_n^3 g(x_1^2, \dots, x_n^2)$. Another proof of this result is given in [Colin de Verdière and Guillemin 2011; Colin de Verdière 2011] for the case $n = 1$, and in [Guillemin and Uribe 2011] in the higher-dimensional case.

The proofs in these last three works and in [Guillemin and Uribe 2007] are based on quantum Birkhoff normal forms, a quantum version of the Birkhoff normal forms of classical mechanics. In the classical case, one constructs a symplectomorphism which puts a Hamiltonian function into a canonical form in a neighborhood of a periodic orbit. In the quantum case, one constructs a Fourier integral operator associated to this symplectomorphism which puts a pseudodifferential operator which is a quantization of this Hamiltonian into a canonical form, microlocally near the periodic orbit. Quantum Birkhoff normal forms were developed by Sjöstrand [1992] for semiclassical Schrödinger operators near a global minimum of the potential. Guillemin [1996] and Zelditch [1997; 1998b] put the Laplace–Beltrami operator on a compact Riemannian manifold into a quantum Birkhoff normal form. General semiclassical Schrödinger operators on a manifold at nondegenerate energy levels were studied in [Sjöstrand and Zworski 2002; Iantchenko, Sjöstrand, and Zworski 2002].

The proof in [Hezari 2009] (that the Taylor coefficients of the potential at the bottom of the well are determined by the low-lying eigenvalues) is based on Schrödinger trace invariants. These are coefficients of the expansion

$$\operatorname{Tr}\big(e^{-itP_{V,h}/h}\chi(P_{V,h})\big) = \sum_{j=0}^{\infty} a_j(t)h^j, \quad h \to 0^+,$$

where $\chi \in C_0^\infty(\mathbb{R})$ is 1 near 0 and is supported in a sufficiently small neighborhood of 0. The coefficients a_j in dimension $n = 1$ have exactly the form (3-2) (and in higher dimensions they have the same form as the higher-dimensional coefficients of the wave trace on a bounded domain) and hence, once this fact is established, the remainder of the uniqueness proof is the same for both problems.

3.4. ***Resonance problems.*** The case of an analytic obstacle with two mutually symmetric connected components is treated in [Zelditch 2004a] by using the singularities of the wave trace generated by the bouncing ball between the two components. Zworski [2007] gave a general method for reducing inverse problems for resonances on a noncompact space to corresponding inverse problems for spectra on a compact space.

Iantchenko [2008] considered potentials V in $\mathbb{R}^n$ which are analytic and even in all variables, which have a unique global maximum at $V(0) = E$, which extend holomorphically to a sector around the real axis and obey $\liminf_{|x|\to\infty} V(x) = 0$ in that sector, and such that the square roots of the eigenvalues of Hess $V(0)$ are linearly independent over $\mathbb{Q}$. He used the quantum Birkhoff normal form method of [Guillemin and Uribe 2007] to recover the Taylor coefficients of the potential at the maximum and to show that potentials V in this class are determined by the resonances in a small neighborhood of E.

4. Rigidity and local uniqueness results

In this section we consider results which show nonexistence of nontrivial isospectral deformations.

4.1. ***Bounded domains in*** $\mathbb{R}^n$***.*** Marvizi and Melrose [1982] introduced new invariants for strictly convex bounded domains $\Omega \subset \mathbb{R}^2$ based on the length spectrum, associated with the boundary. They show that, for $m \in \mathbb{N}$ fixed,

$$\sup\{L(\gamma)\colon \gamma \text{ is a periodic billiard orbit with } m \text{ rotations and } n \text{ reflections}\} \sim mL(\partial\Omega) + \sum_{k=1}^{\infty} c_{k,m} n^{-2k}, \quad n \to \infty, \tag{4-1}$$

where L denotes the length. Then they introduce the following *noncoincidence condition* on Ω, which holds for a dense open family (in the C^∞ topology) of strictly convex domains: suppose there exists $\varepsilon > 0$ such that if γ is a closed orbit with $L(\partial\Omega) - \varepsilon < L(\gamma) < L(\partial\Omega)$, then γ consists of one rotation. They show that under this condition, the coefficients $c_{k,m}$ are spectral invariants, and they use the invariants $c_{1,1}$ and $c_{2,1}$ to construct a two-parameter family of planar domains which are locally spectrally unique (meaning that each domain has a neighborhood in the C^∞ topology within which it is determined by its spectrum). The two-parameter family consists of domains defined by elliptic integrals, and which resemble, but are not, ellipses.

Guillemin and Melrose [1979a] considered the Laplacian on an ellipse Ω given by $x^2/a + y^2/b = 1$, with $a > b > 0$, and with boundary condition

$$\partial u/\partial n = Ku \quad \text{on } \partial\Omega, \tag{4-2}$$

where $K \in C^\infty(\partial\Omega)$ and is even in both x and y. They showed K is determined by $\operatorname{spec}(\Delta_{\Omega,K})$, where $\Delta_{\Omega,K}$ is the Laplacian on Ω with boundary condition (4-2).

To explain their method, let us introduce some terminology. For $T > 0$ the length of a periodic orbit, the fixed point set of T, denoted by Y_T, is the set of $(q,\eta) \in B^*\partial\Omega$, the coball bundle of $\partial\Omega$, such that the billiard orbit corresponding to the initial condition (q,η) is periodic and has length T. For a more general domain there will often be only one periodic orbit of length T (up to time reversal), but an ellipse, because of the complete integrability of its billiard flow, always has one or several one-parameter families of such orbits. Guillemin and Melrose proved that, in the case of the ellipse, for any T which is the length of a periodic orbit such that $L(\partial\Omega) - T > 0$ is sufficiently small, Y_T has one connected component Γ (up to time reversal). This connected component is necessarily a curve which is invariant under the billiard map. Moreover, they showed that the asymptotic expansion of

$$\operatorname{Tr}\big(\cos(t\sqrt{\Delta_{\Omega,K}})\big) - \operatorname{Tr}\big(\cos(t\sqrt{\Delta_{\Omega,0}})\big)$$

in fractional powers of $t - T$ has leading coefficient

$$\int_\Gamma \frac{K}{\sqrt{1-\eta^2}} d\mu_\Gamma. \tag{4-3}$$

Here μ_Γ is the Leray measure on Γ. Under the symmetry assumptions, K is determined from a sequence of such integrals for T_j with T_j tending to $L(\partial\Omega)$ from below.

In [Hezari and Zelditch 2010], it is proved that an ellipse is infinitesimally spectrally rigid among C^∞ domains with the symmetries of the ellipse. This means that if Ω_0 is an ellipse, and if ρ_ϵ is a smooth one-parameter family of smooth functions on $\partial\Omega_0$ which are even in x and y, and if Ω_ϵ is a domain whose boundary is defined by

$$\partial\Omega_\epsilon = \{z + \rho_\epsilon(z) n_z : z \in \partial\Omega_0\},$$

and if $\operatorname{spec}(\Omega_0) = \operatorname{spec}(\Omega_\epsilon)$ for $\epsilon \in [0, \epsilon_0)$, then the Taylor expansion of ρ_ϵ vanishes at $\epsilon = 0$. In particular, if ρ depends on ϵ analytically, the deformation is constant. The proof uses Hadamard's variational formula for the wave trace:

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \operatorname{Tr}(\cos(t\sqrt{\Delta_{\Omega_\epsilon}})) = \frac{t}{2}\int_{\partial\Omega_0} \partial_{n_1}\partial_{n_2} S_{\Omega_0}(t,z,z)\left(\frac{d}{d\epsilon}\Big|_{\epsilon=0}\rho_\epsilon(z)\right) dz, \tag{4-4}$$

where ∂_{n_1} and ∂_{n_2} denote normal derivatives in the first and second variables respectively, S_{Ω_0} is the kernel of $\sin(t\sqrt{\Delta_{\Omega_0}})/\sqrt{\Delta_{\Omega_0}}$. They then use (4-4) to prove that for any T in the length spectrum of Ω_0, the leading order singularity of the wave trace variation is,

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \operatorname{Tr}(\cos(t\sqrt{\Delta_{\Omega_\epsilon}})) \sim \frac{t}{2}\operatorname{Re}\Big\{\Big(\sum_{\Gamma\subset Y_T} C_\Gamma \int_\Gamma \Big(\frac{d}{d\epsilon}\Big|_{\epsilon=0}\rho_\epsilon\Big)\sqrt{1-|\eta|^2}\,d\mu_\Gamma\Big)(t-T+i0)^{-\frac{5}{2}}\Big\}, \tag{4-5}$$

modulo lower order singularities, where the sum is over the connected components Γ of the set Y_T of periodic points of the billiard map on $B^*\partial\Omega_0$ (and its powers) of length T, and where $d\mu_\Gamma$ is as in (4-3). As before, if $L(\partial\Omega_0)-T>0$ is sufficiently small, there is only one connected component and the sum has only one term. For an isospectral deformation, the left-hand side of (4-5) vanishes, and hence the integrals

$$\int_\Gamma \Big(\frac{d}{d\epsilon}\Big|_{\epsilon=0}\rho_\epsilon\Big)\sqrt{1-|\eta|^2}\,d\mu_\Gamma$$

vanish when $L(\partial\Omega_0)-T>0$ sufficiently small. From this point on proceeding as in [Guillemin and Melrose 1979a] above one can show that

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0}\rho_\epsilon = 0,$$

and reparametrizing the variation one can show that all Taylor coefficients of the variation are 0. In [Hezari and Zelditch 2010] it is shown that expansions of the form (4-4) and (4-5) hold more generally and in higher dimensions; indeed (4-4) holds for any C^1 variation of any bounded domain, and a version of (4-5) holds whenever the fixed point sets Y_T are clean. These formulas may be useful for example in a possible proof of spectral rigidity of ellipsoids.

4.2. ***Compact manifolds.*** Tanno [1980] used heat trace invariants to show local spectral uniqueness of spheres in all dimensions. This means that there is a C^∞ neighborhood of the round metric on the sphere within which this metric is spectrally determined. Kuwabara [1980] did this for compact flat manifolds and Sharafutdinov [2009] for compact manifolds of constant negative curvature.

Guillemin and Kazhdan [1980a; 1980b] proved that a negatively curved compact manifold (M,g) with simple length spectrum is spectrally rigid if its sectional curvatures satisfy the pinching condition that for every $x\in M$ there is $A(x)>0$ such that $|K/A+1|<1/n$, where K is any sectional curvature at x (note that the pinching condition is satisfied for all negatively curved surfaces because in that case there is only one sectional curvature at each point x and we may take $A(x)=-K$). Spectrally rigid here means if g_ϵ is a smooth family of metrics on M with $g_0 = g$ and with $\mathrm{spec}(\Delta_{g_\epsilon}) = \mathrm{spec}(\Delta_g)$, then (M, g_ϵ) is isometric to (M, g) for every ϵ. They further used a similar method of proof to establish a spectral uniqueness result for Schrödinger operators on these manifolds. The pinching condition was relaxed in [Maung 1986] and removed in [Croke and Sharafutdinov 1998], and the result was extended to Anosov surfaces with no focal points in [Sharafutdinov and Uhlmann 2000].

4.3. ***Schrödinger operators.*** In work in progress, the second author considers anisotropic harmonic oscillators: $V(x) = a_1^2 x_1^2 + \cdots + a_n^2 x_n^2$, where the a_j are linearly independent over $\mathbb{Q}$. It is shown that if $V_\epsilon(x)$ is a smooth deformation of $V(x)$ within the class of C^∞ functions which are even in each x_j, and if $\mathrm{spec}(P_{V,1}) = \mathrm{spec}(P_{V_\epsilon,1})$ for all $\epsilon \in [0, \epsilon)$, then the deformation is flat at $\epsilon = 0$, just as in the infinitesimal rigidity result for the ellipse in Section 4.1.

5. Compactness results

5.1. ***Bounded domains in $\mathbb{R}^n$.*** Melrose [1983a] used heat trace invariants and Sobolev embedding to prove compactness of isospectral sets of domains $\Omega \subset \mathbb{R}^2$ in the sense of the C^∞ topology on the curvature functions in $C^\infty(\partial\Omega)$. This result allows the possibility of a sequence of isospectral domains whose curvatures converge but which "pinch off" in such a way that the limit object is not a domain, but Melrose [1996] showed that this possibility can be ruled out using the fact that the singularity of the wave trace at $t = 0$ is isolated.

Osgood, Phillips, and Sarnak [1989] gave another approach to this problem based on the determinant of the Laplacian. This is defined via the analytic continuation of the zeta function

$$Z(s) = \sum_{j=1}^{\infty} \lambda_j^{-s}, \quad \det \Delta_\Omega = e^{-Z'(0)}.$$

They consider the domain Ω as the image of the unit disk D under a conformal map F, with $e^{2\phi} g_0$ the induced metric on D, where $\phi = \log |F'|$ is a harmonic function. Thus ϕ is determined by its boundary values, and the topology of [Osgood, Phillips, and Sarnak 1989] is the C^∞ topology on $\phi|_{\partial D}$, and in this case pinching degenerations are ruled out automatically. Hassell and Zelditch [1999] gave a nice review of these results and an application of these methods to the compactness problem for isophasal obstacles in $\mathbb{R}^2$.

To our knowledge there is no compactness result in higher dimensions.

5.2. ***Compact manifolds.*** Osgood, Phillips, and Sarnak [1988a; 1988b] extended their determinant methods to the case of surfaces and prove that the set of isospectral metrics on a given Riemannian surface is sequentially compact in the C^∞ topology, up to isometry. Further compactness results for isospectral metrics in a given conformal class on a three-dimensional manifold appeared in [Chang and Yang 1989; Brooks, Perry, and Yang 1989]. Brooks, Perry, and Petersen [1992] proved compactness for isospectral families of Riemannian manifolds provided that either the sectional curvatures are all negative or there is a uniform lower bound on the Ricci curvatures. Zhou [1997] showed that on a given manifold, the family of isospectral Riemmanian metrics with uniformly bounded curvature is compact, with no restriction on the dimension.

5.3. *Schrödinger operators.* Brüning [1984] considered Schrödinger operators $\Delta_g + V$ on a compact Riemannian manifold (M, g), where $V \in C^\infty(M)$, and proves that if the dimension $n \leq 3$, then any set of isospectral potentials is compact. In higher dimensions he proved the same result under the additional condition that the H^s norm of V for some $s > 3(n/2)-2$ is known to be bounded by some constant C. Donnelly [2005] improved this condition to $s > (n/2) - 2$, and derived alternative compactness criteria: he shows that isospectral families of nonnegative potentials are compact in dimensions $n \leq 9$. If one considers instead $\Delta_g + \gamma V$, he shows that a family of potentials which is isospectral for more than $(n/2) - 1$ different values of γ is compact. In particular, this implies compactness of families which are isospectral for the semiclassical problem $h^2 \Delta_g + V$.

5.4. *Resonance problems.* Let (X_0, g_0) be a conformally compact surface that is hyperbolic (has constant curvature) outside a given compact set $K_0 \subset X_0$. This means that, if K_0 is taken sufficiently large, then $X_0 \setminus K_0$ is a finite disjoint union of funnel ends, which is to say ends of the form

$$(0, \infty)_r \times S^1_\theta, \quad dr^2 + \ell^2 \cosh^2(r)\, d\theta^2, \tag{5-1}$$

where $\ell \neq 0$ may vary between the funnels. Then the continuous spectrum of Δ_{g_0} is given by $[1/4, \infty)$, and the point spectrum is either empty or finite and contained in $(0, 1/4)$ (and there is no other spectrum). If we introduce the spectral parameter $z = \sqrt{\lambda - 1/4}$, where $\sqrt{\ }$ is taken to map $\mathbb{C} \setminus [0, \infty)$ to the upper half-plane, then the resolvent $(\Delta_g - 1/4 - z^2)^{-1}$ continues meromorphically from $\{\operatorname{Im} z > 0\}$ to $\mathbb{C}$ as an operator $L^2_{\text{comp}} \to L^2_{\text{loc}}$. This meromorphic continuation can be proved by writing a parametrix in terms of the resolvent of the Laplacian on the ends (5-1), which in this case can be written explicitly in terms of special functions: see [Mazzeo and Melrose 1987] for the general construction, and [Guillopé and Zworski 1995, §5] for a simpler version in this case.

Borthwick and Perry [2011] used a Poisson formula for resonances due to Guillopé and Zworski [1997] and a heat trace expansion to show that the set of surfaces which are isoresonant with (X_0, g_0) and for which there is a compact set $K \subset X$ such that $(X_0 \setminus K_0, g_0)$ is isometric to $(X \setminus K, g)$ is compact in the C^∞ topology, improving a previous result of Borthwick, Judge, and Perry [2003]. They also proved related but weaker results in higher dimensions.

6. Trace invariants and their limitations

6.1. ***Bounded domains in*** $\mathbb{R}^n$**.** For Δ_Ω with $\Omega \subset \mathbb{R}^n$ a bounded smooth domain we have seen two kinds of trace invariants. The first are heat trace invariants, which are the coefficients a_j of the expansion

$$\operatorname{Tr} e^{-t\Delta_\Omega} \sim t^{-n/2} \sum_{j=0}^{\infty} a_j t^{j/2}, \quad t \to 0^+,$$

are given by integrals along the boundary of polynomials in the curvature and its derivatives. These are equivalent to the invariants obtained from coefficients of the expansion of the wave trace $\operatorname{Tr} \cos(t\sqrt{\Delta_\Omega})$ at $t = 0$.

The other kind are wave trace invariants obtained from coefficients of the expansion of the wave trace at the length of a periodic billiard orbit, always assumed to be nondegenerate and usually assumed to be simple. In this case the formula, as already mentioned in (3-1), is

$$\begin{aligned} \operatorname{Tr}\cos(t\sqrt{\Delta_\Omega}) &= \operatorname{Re}\Bigg[i^{\sigma_T} \frac{T^\sharp}{\sqrt{\det(I - P_T)}} (t - T + i0)^{-1} \\ &\qquad \times \Bigg(1 + \sum_{j=1}^{\infty} b_j (t-T)^j \log(t - T + i0)\Bigg)\Bigg] + S(t), \end{aligned} \tag{6-1}$$

where γ_T is the simple periodic orbit of length T, and where the coefficients b_j are polynomials in the Taylor coefficients at the reflection points of γ_T of the function of which the boundary is a graph. Because of this requirement on the periodic orbit, positive inverse results of the kind described above, which are based on the wave trace, always require generic assumptions such as nondegeneracy and simple length spectrum. Although there has been some work on the degenerate case, such as [Popov 1998], it does not seem to have led yet to uniqueness, rigidity, or compactness results. However, Marvizi and Melrose [1982] obtained information from invariants at lengths approaching the length of $\partial\Omega$ (see Section 4.1 above for more information).

Another limitation comes from the fact that domains can have the same trace invariants without being isospectral. That is to say, we can construct Ω and Ω' such that $\operatorname{Tr}(\cos(t\sqrt{\Delta_\Omega})) - \operatorname{Tr}(\cos(t\sqrt{\Delta_{\Omega'}})) \in C^\infty(\mathbb{R})$ (recall that the wave trace invariants are the coefficients in the expansion of the wave trace near a singularity, as in (6-1)), but $\mathrm{spec}(\Delta_\Omega) \neq \mathrm{spec}(\Delta_{\Omega'})$. This was done by Fulling and Kuchment [2005], following a conjecture of Zelditch [2004b], where the following types of domains are considered (these were first introduced by Penrose to study the illumination problem, and then shown by Lifshits to be examples of nonisometric domains with the same length spectrum):

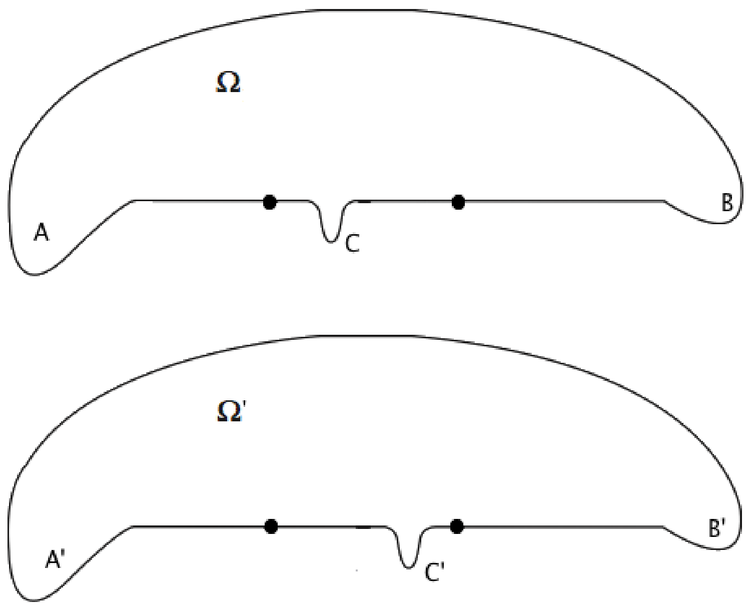

Figure 1. Two domains Ω, Ω' with $\mathrm{Tr}(\cos(t\sqrt{\Delta_\Omega}))-\mathrm{Tr}(\cos(t\sqrt{\Delta_{\Omega'}}))$ in $C^\infty(\mathbb{R})$ but $\mathrm{spec}(\Delta_\Omega) \neq \mathrm{spec}(\Delta_{\Omega'})$.

These two domains are obtained by taking a semiellipse and adding two asymmetric bumps A, B and A', B', with $A = A'$ and $B = B'$, such that the foci are left unperturbed (as in the figure). Then one adds bumps C and C', the small bumps in the middle which are in between the foci, such that $C \neq C'$ but C and C' are reflections of one another. These two domains are not isometric but have the same heat invariants, because heat invariants are given by integrals along the boundary of polynomials in the curvature and its derivatives – indeed we have freedom to 'slide' C back and forth along the boundary without changing any heat invariants, although this is not the case for the wave trace invariants.

We now show that $\mathrm{Tr}(\cos(t\sqrt{\Delta_\Omega})) - \mathrm{Tr}(\cos(t\sqrt{\Delta_{\Omega'}})) \in C^\infty(\mathbb{R})$. This is because of the following separation of the phase spaces[6] $B^*\partial\Omega$ and $B^*\partial\Omega'$ into

[6] Recall that $B^*\partial\Omega$ is the ball bundle, the fibers of which are intervals $[-1, 1]$.

two disconnected rooms each, which are invariant under the billiard maps of the domains, and which we denote R_1, R_2, R'_1, and R'_2, and which have the property that $\overline{R_1 \cup R_2} = B^*\partial\Omega$ and $\overline{R'_1 \cup R'_2} = B^*\partial\Omega'$. These are defined as follows: R_1 is the set of points in $B^*\partial\Omega$ whose billiard flowout intersects the part of the boundary strictly in between the two foci, R_2 is the set of points in $B^*\partial\Omega$ whose billiard flowout intersects the part of the boundary which is strictly outside the two foci but on the axis of the ellipse or below (and similarly for R'_1 and R'_2). These two sets are disjoint because billiards in an ellipse which intersect the major axis in between the two foci once do so always. Now we make the generic assumption that no trajectory which passes through the two foci in the initial semiellipse is periodic. Because R_1 is isometric to R'_1, and R_2 is isometric to R'_2, we have $\operatorname{Tr}(\cos(t\sqrt{\Delta_\Omega})) - \operatorname{Tr}(\cos(t\sqrt{\Delta_{\Omega'}})) \in C^\infty(\mathbb{R})$. This is because the singularities of $\operatorname{Tr}(\cos(t\sqrt{\Delta_\Omega}))$ occur at $t = T$, where T is the length of a periodic orbit, and only depend on the structure of $B^*\partial\Omega$ in an arbitrarily small neighborhood of the orbits of length T.

To show that $\operatorname{spec}(\Delta_\Omega) \neq \operatorname{spec}(\Delta_{\Omega'})$, Fulling and Kuchment use a perturbation argument based on Hadamard's variational formula for the ground state to show that, for suitably chosen small C, the ground states are not the same.

6.2. ***Compact manifolds.*** As we have already mentioned, heat trace invariants can be defined for compact manifolds (M, g) as well. In the boundaryless case the expansion takes the form

$$\operatorname{Tr} e^{-t\Delta_g} \sim t^{-n/2} \sum_{j=0}^{\infty} a_j t^j, \quad t \to 0^+,$$

where the a_j are given by integrals on M of polynomials in the curvature and its derivatives. Half powers of t appear only when there is a boundary, as in the case of domains considered above. Once again, these invariants are equivalent to the invariants obtained from coefficients of the expansion of the wave trace $\operatorname{Tr}\cos(t\sqrt{\Delta_g})$ at $t = 0$. In analogy with the example given in the previous section, we can construct manifolds which are not isometric but which have the same heat invariants by taking a sphere, adding two disjoint bumps, and moving them around. For suitable choices of bumps it should be possible to make the length spectra nonequal, as a result of which the manifolds will be nonisospectral. It seems to be an open problem, however, to find an example of two manifolds (M, g) and (M', g') which are nonisospectral but which have $\operatorname{Tr}(\cos(t\sqrt{\Delta_g})) - \operatorname{Tr}(\cos(t\sqrt{\Delta_{g'}})) \in C^\infty(\mathbb{R})$, that is to say which have identical wave trace invariants.

In this setting the wave trace expansion was established by Duistermaat and Guillemin [1975], building off of previous work by Colin de Verdière [1973] and Chazarain [1974]. It is a generalization of Selberg's Poisson formula [1956] to an arbitrary compact boundaryless Riemannian manifold. For T the length of a simple nondegenerate periodic geodesic γ_T, it takes the form

$$\operatorname{Tr} e^{it\sqrt{\Delta_g}} = i^{\sigma_T} \frac{T^\sharp}{\sqrt{|\det(I-P_T)|}} (t-T+i0)^{-1} \Bigl(1+\sum_{j=1}^{\infty} b_j (t-T)^j \log(t-T+i0) \Bigr) + S(t),$$

where $S(t)$ is smooth near T. Using quantum Birkhoff normal forms, Zelditch [1998b] showed these coefficients b_j to be integrals of polynomials in the metric and its derivatives along γ_T. See also [Zelditch 1999] for a more detailed survey on wave invariants. Because of this very local nature of these invariants, to prove uniqueness results one must either assume analyticity (as is done in the results discussed above) or find a way to combine information from many different orbits (no one seems to have been able to do this so far).

6.3. ***Schrödinger operators.*** In the setting of semiclassical Schrödinger operators the analogue of the Duistermaat–Guillemin wave trace is the Gutzwiller trace formula near the length T of a periodic trajectory of the Hamiltonian vector field H_p in $p^{-1}(E)$, where $p(x,\xi) = |\xi|^2 + V(x)$:

$$\operatorname{Tr} e^{-it(P_{V,h}-E)/h} \chi(P_{V,h}) \sim \sum_\gamma i^{\sigma_\gamma} \frac{e^{iS_\gamma/h}}{\sqrt{|\det(I-P_\gamma)|}} \sum_{j=0}^{\infty} a_{j,\gamma} h^j, \qquad a_{0,\gamma} = \delta_0(t-T),$$

for t near T, where $\chi \in C_0^\infty(\mathbb{R})$ has $\chi = 1$ near E. Here the sum in γ is over periodic trajectories in $p^{-1}(0)$ of length T and the $a_{j,\gamma}$ are distributions whose singular support is contained in $\{T\}$. This formula goes back to [Gutzwiller 1971], and was proved in various degrees of generality and with various methods from [Guillemin and Uribe 1989] and [Combescure, Ralston, and Robert 1999] (see also this last paper for further history and references). This formula is also valid for more general pseudodifferential operators of real principal type, so long as E is a regular value of the principal symbol p and so long as the periodic trajectories in $p^{-1}(E)$ are nondegenerate (so that the determinants in the denominator are nonzero). In particular it also applies on manifolds. Iantchenko, Sjöstrand, and Zworski used it in [Iantchenko, Sjöstrand, and Zworski 2002] to recover quantum and classical Birkhoff normal forms of semiclassical classical Schrödinger operators, at nondegenerate periodic orbits. When the energy level is degenerate, the Gutzwiller trace formula becomes more complicated: see Section 3.3 for a discussion of the case where p has a unique global minimum at

E, and see for example [Brummelhuis, Paul, and Uribe 1995] and [Khuat-Duy 1997] for other cases.

Colin de Verdière [2011] gave an example of a pair of potentials $V \not\equiv V' \in C^\infty(\mathbb{R})$ such that $\operatorname{spec}(P_{V,h}) = \operatorname{spec}(P_{V',h})$ up to $\mathbb{O}(h^\infty)$, so that in particular all semiclassical trace invariants for these two potentials agree. He conjectured, however, that the spectra are not equal. In [Guillemin and Hezari 2012], two potentials are constructed that have different ground states and hence different spectra, although the spectra still agree up to $\mathbb{O}(h^\infty)$. These potentials are perturbations of the harmonic oscillator analogous to the perturbations of the semiellipse discussed in Section 6.1.

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