

# Microlocal analysis of asymptotically hyperbolic spaces and high-energy resolvent estimates

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In this paper we describe a new method for analyzing the Laplacian on asymptotically hyperbolic spaces, which was introduced by the author in 2010. This new method in particular constructs the analytic continuation of the resolvent for even metrics (in the sense of Guillarmou), and gives high-energy estimates in strips. The key idea is an extension across the boundary for a problem obtained from the Laplacian shifted by the spectral parameter. The extended problem is nonelliptic — indeed, on the other side it is related to the Klein–Gordon equation on an asymptotically de Sitter space — but nonetheless it can be analyzed by methods of Fredholm theory. This method is a special case of a more general approach to the analysis of PDEs which includes, for instance, Kerr–de Sitter- and Minkowski-type spaces. The present paper is self-contained, and deals with asymptotically hyperbolic spaces without burdening the reader with material only needed for the analysis of the Lorentzian problems considered in earlier work by the author.

## 1. Introduction

In this paper we describe a new method for analyzing the Laplacian on asymptotically hyperbolic, or conformally compact, spaces, which was introduced in [Vasy 2010a]. This new method in particular constructs the analytic continuation of the resolvent for even metrics (in the sense of [Guillarmou 2005]), and gives high-energy estimates in strips. The key idea is an extension across the boundary for a problem obtained from the Laplacian shifted by the spectral parameter. The extended problem is nonelliptic — indeed, on the other side it is related to the Klein–Gordon equation on an asymptotically de Sitter space — but nonetheless it can be analyzed by methods of Fredholm theory. In [Vasy 2010a] these methods, with some additional ingredients, were used to analyze the wave equation on

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Kerr-de Sitter space-times; the present setting is described there as the simplest application of the tools introduced. The purpose of the present paper is to give a self-contained treatment of conformally compact spaces, without burdening the reader with the additional machinery required for the Kerr-de Sitter analysis.

We start by recalling the definition of manifolds with *even* conformally compact metrics. These are Riemannian metrics  $g_0$  on the interior of an  $n$ -dimensional compact manifold with boundary  $X_0$  such that near the boundary  $Y$ , with a product decomposition nearby and a defining function  $x$ , they are of the form

$$g_0 = \frac{dx^2 + h}{x^2},$$

where  $h$  is a family of metrics on  $Y = \partial X_0$  depending on  $x$  in an even manner, that is, only even powers of  $x$  show up in the Taylor series. (There is a much more natural way to phrase the evenness condition, see [Guillarmou 2005, Definition 1.2].) We also write  $X_{0,\text{even}}$  for the manifold  $X_0$  when the smooth structure has been changed so that  $x^2$  is a boundary defining function; thus, a smooth function on  $X_0$  is even if and only if it is smooth when regarded as a function on  $X_{0,\text{even}}$ . The analytic continuation of the resolvent in this category, but without the evenness condition, was obtained in [Mazzeo and Melrose 1987] (similar results appear in [Agmon 1986; Perry 1987; 1989] in the restricted setting of hyperbolic quotients), with the possibility of some essential singularities at pure imaginary half-integers noticed by Borthwick and Perry [2002]. Guillarmou [2005] showed that for even metrics the latter do not exist, but generically they do exist for noneven metrics, by a more careful analysis utilizing the work of Graham and Zworski [2003]. Further, if the manifold is actually asymptotic to hyperbolic space (note that hyperbolic space is of this form in view of the Poincaré model), Melrose, Sá Barreto and Vasy [Melrose et al. 2011] proved high-energy resolvent estimates in strips around the real axis via a parametrix construction; these are exactly the estimates that allow expansions for solutions of the wave equation in terms of resonances. Estimates just on the real axis were obtained in [Cardoso and Vodev 2002; Vodev 2004] for more general conformal infinities. One implication of our methods is a generalization of these results: we allow general conformal infinities, and obtain estimates in arbitrary strips.

In the sequel  $\mathcal{C}^\infty(X_0)$  denotes “Schwartz functions” on  $X_0$ , that is,  $\mathcal{C}^\infty$  functions vanishing with all derivatives at  $\partial X_0$ , and  $\mathcal{C}^{-\infty}(X_0)$  is the dual space of “tempered distributions” (these spaces are naturally identified for  $X_0$  and  $X_{0,\text{even}}$ ), while  $H^s(X_{0,\text{even}})$  is the standard Sobolev space on  $X_{0,\text{even}}$  (corresponding to extension across the boundary, see, for example, [Hörmander 1985a, Appendix B], where these are denoted by  $\bar{H}^s(X_{0,\text{even}}^\circ)$ ). For instance,

$\|u\|_{H^1(X_{0,\text{even}})}^2 = \|u\|_{L^2(X_{0,\text{even}})}^2 + \|du\|_{L^2(X_{0,\text{even}})}^2$ , with the norms taken with respect to any smooth Riemannian metric on  $X_{0,\text{even}}$  (all choices yield equivalent norms by compactness). Here we point out that while  $x^2 g_0$  is a smooth nondegenerate section of the pull-back of  $T^*X_0$  to  $X_{0,\text{even}}$  (which essentially means that it is a smooth, in  $X_{0,\text{even}}$ , nondegenerate linear combination of  $dx$  and  $dy_j$  in local coordinates), as  $\mu = x^2$  means  $d\mu = 2x dx$ , it is actually not a smooth section of  $T^*X_{0,\text{even}}$ . However,  $x^{n+1} |dg_0|$  is a smooth nondegenerate density, so  $L^2(X_{0,\text{even}})$  (up to norm equivalence) is the  $L^2$  space given by the density  $x^{n+1} |dg_0|$ , i.e., it is the space  $x^{-(n+1)/2} L^2_{g_0}(X_0)$ , which means that

$$\|x^{-(n+1)/2} u\|_{L^2(X_{0,\text{even}})} \sim \|u\|_{L^2_{g_0}(X_0)}.$$

Further, in local coordinates  $(\mu, y)$ , using  $2\partial_\mu = x^{-1}\partial_x$ , the  $H^1(X_{0,\text{even}})$  norm of  $u$  is equivalent to

$$\|u\|_{L^2(X_{0,\text{even}})}^2 + \|x^{-1}\partial_x u\|_{L^2(X_{0,\text{even}})}^2 + \sum_{j=1}^{n-1} \|\partial_{y_j} u\|_{L^2(X_{0,\text{even}})}^2.$$

We also let  $H^s_h(X_{0,\text{even}})$  be the standard semiclassical Sobolev space, that is, for  $h$  bounded away from 0 this is equipped with a norm equivalent to the standard fixed ( $h$ -independent) norm on  $H^s(X_{0,\text{even}})$ , but the uniform behavior as  $h \rightarrow 0$  is different; for example, locally the  $H^1_h(X)$  norm is given by

$$\|u\|_{H^1_h}^2 = \sum_j \|hD_j u\|_{L^2}^2 + \|u\|_{L^2}^2;$$

see [Dimassi and Sjöstrand 1999; Evans and Zworski 2010]. Thus, in (1-1), for  $s = 1$  (which is possible when  $C < \frac{1}{2}$ , that is, if one only considers the continuation into a small strip beyond the continuous spectrum),

$$\begin{aligned} s = 1 \implies \|u\|_{H^{s-1}_{|\sigma|^{-1}}(X_{0,\text{even}})} &= \|u\|_{L^2(X_{0,\text{even}})} \\ \text{and } \|u\|_{H^s_{|\sigma|^{-1}}(X_{0,\text{even}})} &= \|u\|_{L^2(X_{0,\text{even}})}^2 + |\sigma|^{-2} \|du\|_{L^2(X_{0,\text{even}})}^2, \end{aligned}$$

with the norms taken with respect to any smooth Riemannian metric on  $X_{0,\text{even}}$ .

**Theorem.** (See Theorem 5.1 for the full statement.) *Suppose that  $X_0$  is an  $n$ -dimensional manifold with boundary  $Y$  with an even Riemannian conformally compact metric  $g_0$ . Then the inverse of*

$$\Delta_{g_0} - \left(\frac{n-1}{2}\right)^2 - \sigma^2,$$

written as  $\mathcal{R}(\sigma) : L^2 \rightarrow L^2$ , has a meromorphic continuation from  $\text{Im } \sigma \gg 0$  to  $\mathbb{C}$ ,

$$\mathcal{R}(\sigma) : \mathcal{C}^\infty(X_0) \rightarrow \mathcal{C}^{-\infty}(X_0),$$

with poles with finite rank residues. If in addition  $(X_0, g_0)$  is nontrapping, then nontrapping estimates hold in every strip  $-C < \text{Im } \sigma < C_+$ ,  $|\text{Re } \sigma| \gg 0$ : for  $s > \frac{1}{2} + C$ ,

$$\begin{aligned} \|x^{-(n-1)/2+i\sigma} \mathcal{R}(\sigma) f\|_{H_{|\sigma|^{-1}}^s(X_{0,\text{even}})} \\ \leq \tilde{C} |\sigma|^{-1} \|x^{-(n+3)/2+i\sigma} f\|_{H_{|\sigma|^{-1}}^{s-1}(X_{0,\text{even}})}. \end{aligned} \quad (1-1)$$

If  $f$  has compact support in  $X_0^\circ$ , the  $s - 1$  norm on  $f$  can be replaced by the  $s - 2$  norm. For suitable  $\delta_0 > 0$ , the estimates are valid in regions  $-C < \text{Im } \sigma < \delta_0 |\text{Re } \sigma|$  if the multipliers  $x^{i\sigma}$  are slightly adjusted.

Further, as stated in Theorem 5.1, the resolvent is *semiclassically outgoing* with a loss of  $h^{-1}$ , in the sense of recent results of Datchev and Vasy [2011; 2010]. This means that for mild trapping (where, in a strip near the spectrum, one has polynomially bounded resolvent for a compactly localized version of the trapped model) one obtains resolvent bounds of the same kind as for the above-mentioned trapped models, and lossless estimates microlocally away from the trapping. In particular, one obtains logarithmic losses compared to nontrapping on the spectrum for hyperbolic trapping in the sense of [Wunsch and Zworski 2011, Section 1.2], and polynomial losses in strips, since for the compactly localized model this was shown in [Wunsch and Zworski 2011].

Our method is to change the smooth structure, replacing  $x$  by  $\mu = x^2$ , conjugate the operator by an appropriate weight as well as remove a vanishing factor of  $\mu$ , and show that the new operator continues smoothly and nondegenerately (in an appropriate sense) across  $\mu = 0$ , that is,  $Y$ , to a (nonelliptic) problem which we can analyze utilizing by now almost standard tools of microlocal analysis. These steps are reflected in the form of the estimate (1-1);  $\mu$  shows up in the use of evenness, conjugation due to the presence of  $x^{-(n+1)/2+i\sigma}$ , and the two halves of the vanishing factor of  $\mu$  being removed in  $x^{\pm 1}$  on the left and right sides.

While it might seem somewhat ad hoc, this construction in fact has origins in wave propagation in Lorentzian spaces of one dimension higher, i.e.,  $(n+1)$ -dimensional: either Minkowski space, or de Sitter space blown up at a point at future infinity. Namely in both cases the wave equation (and the Klein–Gordon equation in de Sitter space) is a totally characteristic, or b-, PDE, and after a Mellin transform this gives a PDE on the sphere at infinity in the Minkowski case, and on the front face of the blow-up in the de Sitter setting. These are exactly the PDE arising by the process described in the previous paragraph, with

the original manifold  $X_0$  lying in the interior of the light cone in Minkowski space (so there are two copies, at future and past infinity) and in the interior of the backward light cone from the blow-up point in the de Sitter case; see [Vasy 2010a] for more detail. This relationship, restricted to the  $X_0$ -region, was exploited in [Vasy 2010b, Section 7], where the work of Mazzeo and Melrose was used to construct the Poisson operator on asymptotically de Sitter spaces. Conceptually the main novelty here is that we work directly with the extended problem, which turns out to simplify the analysis of Mazzeo and Melrose in many ways and give a new explanation for Guillarmou’s results as well as yield high-energy estimates.

We briefly describe this extended operator,  $P_\sigma$ . It has radial points at the conormal bundle  $N^*Y \setminus o$  of  $Y$  in the sense of microlocal analysis, i.e., the Hamilton vector field is radial at these points — it is a multiple of the generator of dilations of the fibers of the cotangent bundle there. However, tools exist to deal with these, going back to Melrose’s geometric treatment of scattering theory on asymptotically Euclidean spaces [Melrose 1994]. Note that  $N^*Y \setminus o$  consists of two components,  $\Lambda_+$ , resp.  $\Lambda_-$ , and in  $S^*X = (T^*X \setminus o)/\mathbb{R}^+$  the images,  $L_+$ , resp.  $L_-$ , of these are sources, resp. sinks, for the Hamilton flow. At  $L_\pm$  one has choices regarding the direction one wants to propagate estimates (into or out of the radial points), which directly correspond to working with strong or weak Sobolev spaces. For the present problem, the relevant choice is propagating estimates *away from* the radial points, thus working with the “good” Sobolev spaces (which can be taken to have order as positive as one wishes; there is a minimum amount of regularity imposed by our choice of propagation direction — see the requirement  $s > \frac{1}{2} + C$  above (1-1)). All other points are either elliptic, or real principal type. It remains to either deal with the noncompactness of the “far end” of the  $n$ -dimensional de Sitter space — or instead, as is indeed more convenient when one wants to deal with more singular geometries, adding complex absorbing potentials, in the spirit of [Nonnenmacher and Zworski 2009; Wunsch and Zworski 2011]. In fact, the complex absorption could be replaced by adding a space-like boundary [Vasy 2010a], but for many microlocal purposes complex absorption is more desirable, hence we follow the latter method. However, crucially, these complex absorbing techniques (or the addition of a space-like boundary) already enter in the nonsemiclassical problem in our case, as we are in a nonelliptic setting.

One can reverse the direction of the argument and analyze the wave equation on an  $n$ -dimensional even asymptotically de Sitter space  $X'_0$  by extending it across the boundary, much like the Riemannian conformally compact space  $X_0$  is extended in this approach. Then, performing microlocal propagation in the opposite direction, which amounts to working with the adjoint operators that we

already need in order to prove existence of solutions for the Riemannian spaces, we obtain existence, uniqueness and structure results for asymptotically de Sitter spaces, recovering a large part of the results of [Vasy 2010b]. Here we only briefly indicate this method of analysis in Remark 5.3.

In other words, we establish a Riemannian–Lorentzian duality, that will have counterparts both in the pseudo-Riemannian setting of higher signature and in higher rank symmetric spaces, though in the latter the analysis might become more complicated. Note that asymptotically hyperbolic and de Sitter spaces are not connected by a “complex rotation” (in the sense of an actual deformation); they are smooth continuations of each other in the sense we just discussed.

To emphasize the simplicity of our method, we list all of the microlocal techniques (which are relevant both in the classical and in the semiclassical setting) that we use on a *compact manifold without boundary*; in all cases *only microlocal Sobolev estimates* matter (not parametrices, etc.):

- (i) Microlocal elliptic regularity.
- (ii) Real principal type propagation of singularities.
- (iii) *Rough* analysis at a Lagrangian invariant under the Hamilton flow which roughly behaves like a collection of radial points, though the internal structure does not matter, in the spirit of [Melrose 1994, Section 9].
- (iv) Complex absorbing “potentials” in the spirit of [Nonnenmacher and Zworski 2009; Wunsch and Zworski 2011].

These are almost “off the shelf” in terms of modern microlocal analysis, and thus our approach, from a microlocal perspective, is quite simple. We use these to show that on the continuation across the boundary of the conformally compact space we have a Fredholm problem, on a perhaps slightly exotic function space, which however is (perhaps apart from the complex absorption) the simplest possible coisotropic function space based on a Sobolev space, with order dictated by the radial points. Also, we propagate the estimates along bicharacteristics in different directions depending on the component  $\Sigma_{\pm}$  of the characteristic set under consideration; correspondingly the sign of the complex absorbing “potential” will vary with  $\Sigma_{\pm}$ , which is perhaps slightly unusual. However, this is completely parallel to solving the standard Cauchy, or forward, problem for the wave equation, where one propagates estimates in *opposite* directions relative to the Hamilton vector field in the two components of the characteristic set.

The complex absorption we use modifies the operator  $P_{\sigma}$  outside  $X_{0,\text{even}}$ . However, while  $(P_{\sigma} - \iota Q_{\sigma})^{-1}$  depends on  $Q_{\sigma}$ , its behavior on  $X_{0,\text{even}}$ , and even near  $X_{0,\text{even}}$ , is independent of this choice; see the proof of Section 5 for a detailed explanation. In particular, although  $(P_{\sigma} - \iota Q_{\sigma})^{-1}$  may have resonances

other than those of  $\mathcal{R}(\sigma)$ , the resonant states of these additional resonances are supported outside  $X_{0,\text{even}}$ , hence do not affect the singular behavior of the resolvent in  $X_{0,\text{even}}$ .

While the results are stated for the scalar equation, analogous results hold for operators on natural vector bundles, such as the Laplacian on differential forms. This is so because the results work if the principal symbol of the extended problem is scalar with the demanded properties, and the principal symbol of  $\frac{1}{2i}(P_\sigma - P_\sigma^*)$  is either scalar at the ‘radial sets’, or instead satisfies appropriate estimates (as an endomorphism of the pull-back of the vector bundle to the cotangent bundle) at this location; see Remark 3.1. The only change in terms of results on asymptotically hyperbolic spaces is that the threshold  $(n - 1)^2/4$  is shifted; in terms of the explicit conjugation of Section 5 this is so because of the change in the first order term in (3-2).

In Section 3 we describe in detail the setup of conformally compact spaces and the extension across the boundary. Then in Section 4 we describe in detail the necessary microlocal analysis for the extended operator. Finally, in Section 5 we translate these results back to asymptotically hyperbolic spaces.

### 2. Notation

We start by briefly recalling the basic pseudodifferential objects, in part to establish notation. As a general reference for microlocal analysis, we refer to [Hörmander 1985a; 1985b], while for semiclassical analysis, we refer to [Dimassi and Sjöstrand 1999; Evans and Zworski 2010].

First,  $S^k(\mathbb{R}^p; \mathbb{R}^\ell)$  is the set of  $\mathcal{C}^\infty$  functions on  $\mathbb{R}_z^p \times \mathbb{R}_\zeta^\ell$  satisfying uniform bounds

$$|D_z^\alpha D_\zeta^\beta a| \leq C_{\alpha\beta} \langle \zeta \rangle^{k-|\beta|}, \quad \alpha \in \mathbb{N}^p, \beta \in \mathbb{N}^\ell.$$

If  $O \subset \mathbb{R}^p$  and  $\Gamma \subset \mathbb{R}^\ell$  are open, we define  $S^k(O; \Gamma)$  by requiring these estimates to hold only for  $z \in O$  and  $\zeta \in \Gamma$ . (We could instead require uniform estimates on compact subsets; this makes no difference here.) The class of classical (or one-step polyhomogeneous) symbols is the subset  $S_{\text{cl}}^k(\mathbb{R}^p; \mathbb{R}^\ell)$  of  $S^k(\mathbb{R}^p; \mathbb{R}^\ell)$  consisting of symbols possessing an asymptotic expansion

$$a(z, r\omega) \sim \sum a_j(z, \omega)r^{k-j}, \tag{2-1}$$

where  $a_j \in \mathcal{C}^\infty(\mathbb{R}^p \times \mathbb{S}^{\ell-1})$ . Then on  $\mathbb{R}_z^n$ , pseudodifferential operators  $A \in \Psi^k(\mathbb{R}^n)$  are of the form

$$A = \text{Op}(a); (\text{Op}(a)u)(z) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(z-z') \cdot \zeta} a(z, \zeta) u(z') d\zeta dz',$$

$$u \in \mathcal{S}(\mathbb{R}^n), a \in S^k(\mathbb{R}^n; \mathbb{R}^n);$$

understood as an oscillatory integral. Classical pseudodifferential operators,  $A \in \Psi_{\text{cl}}^k(\mathbb{R}^n)$ , form the subset where  $a$  is a classical symbol. The principal symbol  $\sigma_k(A)$  of  $A \in \Psi^k(\mathbb{R}^n)$  is the equivalence class of  $a$  in  $S^k(\mathbb{R}^n; \mathbb{R}^n)/S^{k-1}(\mathbb{R}^n; \mathbb{R}^n)$ , denoted by  $[a]$ . For classical  $a$ , one can instead regard  $a_0(z, \omega)r^k$  as the principal symbol; it is a  $\mathcal{C}^\infty$  function on  $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ , which is homogeneous of degree  $k$  with respect to the  $\mathbb{R}^+$ -action given by dilations in the second factor,  $\mathbb{R}^n \setminus \{0\}$ . The principal symbol is multiplicative, that is,  $\sigma_{k+k'}(AB) = \sigma_k(A)\sigma_{k'}(B)$ . Moreover, the principal symbol of a commutator is given by the Poisson bracket (or equivalently by the Hamilton vector field):  $\sigma_{k+k'-1}(i[A, B]) = H_{\sigma_k(A)}\sigma_{k'}(B)$ , with  $H_a = \sum_{j=1}^n ((\partial_{\xi_j} a)\partial_{z_j} - (\partial_{z_j} a)\partial_{\xi_j})$ . Note that for  $a$  homogeneous of order  $k$ ,  $H_a$  is homogeneous of order  $k - 1$ .

There are two very important properties: nondegeneracy (called ellipticity) and extreme degeneracy (captured by the operator wave front set) of an operator. One says that  $A$  is elliptic at  $\alpha \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$  if there exists an open cone  $\Gamma$  (conic with respect to the  $\mathbb{R}^+$ -action on  $\mathbb{R}^n \setminus \{0\}$ ) around  $\alpha$  and  $R > 0, C > 0$  such that  $|a(x, \xi)| \geq C|\xi|^k$  for  $|\xi| > R, (x, \xi) \in \Gamma$ , where  $[a] = \sigma_k(A)$ . If  $A$  is classical, and  $a$  is taken to be homogeneous, this just amounts to  $a(\alpha) \neq 0$ .

On the other hand, for  $A = \text{Op}(a)$  and  $\alpha \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$  one says that  $\alpha \notin \text{WF}'(A)$  if there exists an open cone  $\Gamma$  around  $\alpha$  such that  $a|_\Gamma \in S^{-\infty}(\Gamma)$ , that is,  $a|_\Gamma$  is rapidly decreasing, with all derivatives, as  $|\xi| \rightarrow \infty, (x, \xi) \in \Gamma$ . Note that both the elliptic set  $\text{ell}(A)$  of  $A$  (i.e., the set of points where  $A$  is elliptic) and  $\text{WF}'(A)$  are conic.

Differential operators on  $\mathbb{R}^n$  form the subset of  $\Psi(\mathbb{R}^n)$  in which  $a$  is polynomial in the second factor,  $\mathbb{R}_\zeta^n$ , so locally

$$A = \sum_{|\alpha| \leq k} a_\alpha(z) D_z^\alpha, \quad \sigma_k(A) = \sum_{|\alpha|=k} a_\alpha(z) \zeta^\alpha.$$

If  $X$  is a manifold, one can transfer these definitions to  $X$  by localization and requiring that the Schwartz kernels are  $\mathcal{C}^\infty$  densities away from the diagonal in  $X^2 = X \times X$ ; then  $\sigma_k(A)$  is in  $S^k(T^*X)/S^{k-1}(T^*X)$ , resp.  $S_{\text{hom}}^k(T^*X \setminus o)$  when  $A \in \Psi^k(X)$ , resp.  $A \in \Psi_{\text{cl}}^k(X)$ ; here  $o$  is the zero section, and  $\text{hom}$  stands for symbols homogeneous with respect to the  $\mathbb{R}^+$  action. If  $A$  is a differential operator, then the classical (i.e., homogeneous) version of the principal symbol is a homogeneous polynomial in the fibers of the cotangent bundle of degree  $k$ . The notions of  $\text{ell}(A)$  and  $\text{WF}'(A)$  extend to give conic subsets of  $T^*X \setminus o$ ; equivalently they are subsets of the cosphere bundle  $S^*X = (T^*X \setminus o)/\mathbb{R}^+$ . We can also work with operators depending on a parameter  $\lambda \in O$  by replacing  $a \in S^k(\mathbb{R}^n; \mathbb{R}^n)$  by  $a \in S^k(\mathbb{R}^n \times O; \mathbb{R}^n)$ , with  $\text{Op}(a_\lambda) \in \Psi^k(\mathbb{R}^n)$  smoothly dependent on  $\lambda \in O$ . In the case of differential operators,  $a_\alpha$  would simply depend smoothly on the parameter  $\lambda$ .



We next consider the semiclassical operator algebra. We adopt the convention that  $\hbar$  denotes semiclassical objects, while  $h$  is the actual semiclassical parameter. This algebra,  $\Psi_{\hbar}(\mathbb{R}^n)$ , is given by

$$A_h = \text{Op}_{\hbar}(a); \text{Op}_{\hbar}(a)u(z) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{i(z-z')\cdot\xi/h} a(z, \xi, h) u(z') d\xi dz',$$

$$u \in \mathcal{S}(\mathbb{R}^n), a \in \mathcal{C}^{\infty}([0, 1]_{\hbar}; S^k(\mathbb{R}^n; \mathbb{R}_{\xi}^n));$$

its classical subalgebra,  $\Psi_{\hbar, \text{cl}}(\mathbb{R}^n)$  corresponds to  $a \in \mathcal{C}^{\infty}([0, 1]_{\hbar}; S^k_{\text{cl}}(\mathbb{R}^n; \mathbb{R}_{\xi}^n))$ . The semiclassical principal symbol is now  $\sigma_{\hbar, k}(A) = a|_{h=0} \in S^k(\mathbb{R}^n \times \mathbb{R}^n)$ . In the setting of a general manifold  $X$ ,  $\mathbb{R}^n \times \mathbb{R}^n$  is replaced by  $T^*X$ . Correspondingly,  $\text{WF}'_{\hbar}(A)$  and  $\text{ell}_{\hbar}(A)$  are subsets of  $T^*X$ . We can again add an extra parameter  $\lambda \in O$ , so  $a \in \mathcal{C}^{\infty}([0, 1]_{\hbar}; S^k(\mathbb{R}^n \times O; \mathbb{R}_{\xi}^n))$ ; then in the invariant setting the principal symbol is  $a|_{h=0} \in S^k(T^*X \times O)$ .

Differential operators now take the form

$$A_{h, \lambda} = \sum_{|\alpha| \leq k} a_{\alpha}(z, \lambda; h)(hD_z)^{\alpha}. \tag{2-2}$$

Such a family has two principal symbols, the standard one (but taking into account the semiclassical degeneration, that is, based on  $(hD_z)^{\alpha}$  rather than  $D_z^{\alpha}$ ), which depends on  $h$  and is homogeneous, and the semiclassical one, which is at  $h = 0$ , and is not homogeneous:

$$\sigma_k(A_{h, \lambda}) = \sum_{|\alpha|=k} a_{\alpha}(z, \lambda; h)\zeta^{\alpha},$$

$$\sigma_{\hbar}(A_{h, \lambda}) = \sum_{|\alpha| \leq k} a_{\alpha}(z, \lambda; 0)\zeta^{\alpha}.$$

However, the restriction of  $\sigma_k(A_{h, \lambda})$  to  $h = 0$  is the principal symbol of  $\sigma_{\hbar}(A_{h, \lambda})$ . In the special case in which  $\sigma_k(A_{h, \lambda})$  is independent of  $h$  (which is true in the setting considered below), one can simply regard the usual principal symbol as the principal part of the semiclassical symbol.

This is a convenient place to recall from [Melrose 1994] that it is often useful to consider the radial compactification of the fibers of the cotangent bundle to balls (or hemispheres, in Melrose's exposition). Thus, one adds a sphere at infinity to the fiber  $T_q^*X$  of  $T^*X$  over each  $q \in X$ . This sphere is naturally identified with  $S_q^*X$ , and we obtain compact fibers  $\bar{T}_q^*X$  with boundary  $S_q^*X$ , with the smooth structure near  $S_q^*X$  arising from reciprocal polar coordinates  $(\tilde{\rho}, \omega) = (r^{-1}, \omega)$  for  $\tilde{\rho} > 0$ , but extending to  $\tilde{\rho} = 0$ , and with  $S_q^*X$  given by

$\tilde{\rho} = 0$ . Thus, with  $X = \mathbb{R}^n$  the classical expansion (2-1) becomes

$$a(z, \tilde{\rho}, \omega) \sim \tilde{\rho}^{-k} \sum a_j(z, \omega) \tilde{\rho}^j,$$

where  $a_j \in \mathcal{C}^\infty(\mathbb{R}^p \times \mathbb{S}^{\ell-1})$ , so in particular for  $k = 0$ , this is simply the Taylor series expansion at  $S^*X$  of a function smooth up to  $S^*X = \partial\bar{T}^*X$ . In the semiclassical context then one considers  $\bar{T}^*X \times [0, 1)$ , and notes that “classical” semiclassical operators of order 0 are given locally by  $\text{Op}_\hbar(a)$  with  $a$  extending to be smooth up to the boundaries of this space, with semiclassical symbol given by restriction to  $\bar{T}^*X \times \{0\}$ , and standard symbol given by restriction to  $S^*X \times [0, 1)$ . Thus, the claim regarding the limit of the semiclassical symbol at infinity is simply a matching statement of the two symbols at the corner  $S^*X \times \{0\}$  in this compactified picture.

Finally, we recall that if  $P = \sum_{|\alpha| \leq k} a_\alpha(z) D_z^\alpha$  is an order  $k$  differential operator, then the behavior of  $P - \lambda$  as  $\lambda \rightarrow \infty$  can be converted to a semiclassical problem by considering

$$P_{\hbar, \sigma} = h^k(P - \lambda) = \sum_{|\alpha| \leq k} h^{k-|\alpha|} a_\alpha(z) (hD_z)^\alpha - \sigma,$$

where  $\sigma = h^k \lambda$ . Here there is freedom in choosing  $h$ , for example,  $h = |\lambda|^{1/k}$ , in which case  $|\sigma| = 1$ , but it is often useful to leave some flexibility in the choice so that  $h \sim |\lambda|^{1/k}$  only, and thus  $\sigma$  is in a compact subset of  $\mathbb{C}$  disjoint from 0. Note that

$$\sigma_\hbar(P_{\hbar, \sigma}) = \sum_{|\alpha|=k} a_\alpha(z) \zeta^\alpha - \sigma.$$

If we do not want to explicitly multiply by  $h^k$ , we write the full high-energy principal symbol of  $P - \lambda$  as

$$\sigma_{\text{full}}(P\lambda) = \sum_{|\alpha|=k} a_\alpha(z) \zeta^\alpha - \lambda.$$

More generally, if  $P(\lambda) = \sum_{|\alpha|+|\beta| \leq k} a_\alpha(z) \lambda^\beta D_z^\alpha$  is an order  $k$  differential operator depending on a large parameter  $\lambda$ , we let

$$\sigma_{\text{full}}(P(\lambda)) = \sum_{|\alpha|+|\beta|=k} a_\alpha(z) \lambda^\beta \zeta^\alpha$$

be the full large-parameter symbol. With  $\lambda = h^{-1} \sigma$ ,

$$P_{\hbar, \sigma} = h^k P(\lambda) = \sum_{|\alpha|+|\beta| \leq k} h^{k-|\alpha|-|\beta|} a_\alpha(z) \sigma^\beta (hD_z)^\alpha$$

is a semiclassical differential operator with semiclassical symbol

$$\sigma_h(P_{h,\sigma}) = \sum_{|\alpha|+|\beta|=k} a_\alpha(z)\sigma^\beta \zeta^\alpha.$$

Note that the full large-parameter symbol and the semiclassical symbol are “the same”, that is, they are simply related to each other.

### 3. Conformally compact spaces

**3A. From the Laplacian to the extended operator.** Suppose that  $g_0$  is an even asymptotically hyperbolic metric on  $X_0$ , with  $\dim X_0 = n$ . Then we may choose a product decomposition near the boundary such that

$$g_0 = \frac{dx^2 + h}{x^2} \tag{3-1}$$

there, where  $h$  is an even family of metrics; it is convenient to take  $x$  to be a globally defined boundary defining function. Then the dual metric is

$$G_0 = x^2(\partial_x^2 + H),$$

with  $H$  the dual metric family of  $h$  (depending on  $x$  as a parameter), and

$$|dg_0| = \sqrt{|\det g_0|} dx dy = x^{-n} \sqrt{|\det h|} dx dy,$$

so

$$\Delta_{g_0} = (xD_x)^2 + \iota(n-1+x^2\gamma)(xD_x) + x^2\Delta_h, \tag{3-2}$$

with  $\gamma$  even, and  $\Delta_h$  the  $x$ -dependent family of Laplacians of  $h$  on  $Y$ .

We show now that if we change the smooth structure on  $X_0$  by declaring that only even functions of  $x$  are smooth, that is, introducing  $\mu = x^2$  as the boundary defining function, then after a suitable conjugation and division by a vanishing factor the resulting operator smoothly and nondegenerately continues across the boundary, that is, continues to  $X_{-\delta_0} = (-\delta_0, 0)_\mu \times Y \sqcup X_{0,\text{even}}$ , where  $X_{0,\text{even}}$  is the manifold  $X_0$  with the new smooth structure.

First, changing to coordinates  $(\mu, y)$ ,  $\mu = x^2$ , we obtain

$$\Delta_{g_0} = 4(\mu D_\mu)^2 + 2\iota(n-1+\mu\gamma)(\mu D_\mu) + \mu\Delta_h, \tag{3-3}$$

Now we conjugate by  $\mu^{-\iota\sigma/2+(n+1)/4}$  to obtain

$$\begin{aligned} &\mu^{\iota\sigma/2-(n+1)/4} (\Delta_{g_0} - (n-1)^2/4 - \sigma^2) \mu^{-\iota\sigma/2+(n+1)/4} \\ &= 4(\mu D_\mu - \sigma/2 - \iota(n+1)/4)^2 + 2\iota(n-1+\mu\gamma)(\mu D_\mu - \sigma/2 - \iota(n+1)/4) \\ &\quad + \mu\Delta_h - (n-1)^2/4 - \sigma^2 \end{aligned}$$

$$= 4(\mu D_\mu)^2 - 4\sigma(\mu D_\mu) + \mu\Delta_h - 4i(\mu D_\mu) + 2i\sigma - 1 \\ + 2i\mu\gamma(\mu D_\mu - \sigma/2 - i(n+1)/4).$$

Next we multiply by  $\mu^{-1/2}$  from both sides to obtain

$$\mu^{-1/2}\mu^{i\sigma/2-(n+1)/4}(\Delta_{g_0} - (n-1)^2/4 - \sigma^2)\mu^{-i\sigma/2+(n+1)/4}\mu^{-1/2} \\ = 4\mu D_\mu^2 - \mu^{-1} - 4\sigma D_\mu - 2i\sigma\mu^{-1} + \Delta_h - 4i D_\mu + 2\mu^{-1} + 2i\sigma\mu^{-1} - \mu^{-1} \\ + 2i\gamma(\mu D_\mu - \sigma/2 - i(n-1)/4) \\ = 4\mu D_\mu^2 - 4\sigma D_\mu + \Delta_h - 4i D_\mu + 2i\gamma(\mu D_\mu - \sigma/2 - i(n-1)/4). \quad (3-4)$$

This operator is in  $\text{Diff}^2(X_{0,\text{even}})$ , and now it continues smoothly across the boundary, by extending  $h$  and  $\gamma$  in an arbitrary smooth manner. This form suffices for analyzing the problem for  $\sigma$  in a compact set, or indeed for  $\sigma$  going to infinity in a strip near the reals. However, it is convenient to modify it as we would like the resulting operator to be semiclassically elliptic when  $\sigma$  is away from the reals. We achieve this via conjugation by a smooth function, with exponent depending on  $\sigma$ . The latter would make no difference even semiclassically in the real regime as it is conjugation by an elliptic semiclassical FIO. However, in the nonreal regime (where we would like ellipticity) it does matter; the present operator is not semiclassically elliptic at the zero section. So finally we conjugate by  $(1 + \mu)^{i\sigma/4}$  to obtain

$$P_\sigma = 4(1 + a_1)\mu D_\mu^2 - 4(1 + a_2)\sigma D_\mu - (1 + a_3)\sigma^2 + \Delta_h \\ - 4i D_\mu + b_1\mu D_\mu + b_2\sigma + c_1, \quad (3-5)$$

with  $a_j$  smooth, real, vanishing at  $\mu = 0$ ,  $b_j$  and  $c_1$  smooth. In fact, we have  $a_1 \equiv 0$ , but it is sometimes convenient to have more flexibility in the form of the operator since this means that we do not need to start from the relatively rigid form (3-2).

Writing covectors as  $\xi d\mu + \eta dy$ , the principal symbol of  $P_\sigma \in \text{Diff}^2(X_{-\delta_0})$ , including in the high-energy sense ( $\sigma \rightarrow \infty$ ), is

$$p_{\text{full}} = 4(1 + a_1)\mu\xi^2 - 4(1 + a_2)\sigma\xi - (1 + a_3)\sigma^2 + |\eta|_{\mu,y}^2, \quad (3-6)$$

and is real for  $\sigma$  real. The Hamilton vector field is

$$H_{p_{\text{full}}} = 4(2(1 + a_1)\mu\xi - (1 + a_2)\sigma)\partial_\mu + \tilde{H}_{|\eta|_{\mu,y}^2} \\ - \left( 4\left(1 + a_1 + \mu\frac{\partial a_1}{\partial \mu}\right)\xi^2 - 4\frac{\partial a_2}{\partial \mu}\sigma\xi + \frac{\partial a_3}{\partial \mu}\sigma^2 + \frac{\partial |\eta|_{\mu,y}^2}{\partial \mu} \right)\partial_\xi \\ - \left( 4\frac{\partial a_1}{\partial y}\mu\xi^2 - 4\frac{\partial a_2}{\partial y}\sigma\xi - \frac{\partial a_3}{\partial y}\sigma^2 \right)\partial_\eta, \quad (3-7)$$

where  $\tilde{H}$  indicates that this is the Hamilton vector field in  $T^*Y$ , that is, with  $\mu$  considered a parameter. Correspondingly, the standard, “classical”, principal symbol is

$$p = \sigma_2(P_\sigma) = 4(1 + a_1)\mu\xi^2 + |\eta|_{\mu,y}^2, \tag{3-8}$$

which is real, independent of  $\sigma$ , while the Hamilton vector field is

$$H_p = 8(1 + a_1)\mu\xi\partial_\mu + \tilde{H}|_{\eta|_{\mu,y}^2} - \left(4\left(1 + a_1 + \mu\frac{\partial a_1}{\partial\mu}\right)\xi^2 + \frac{\partial|\eta|_{\mu,y}^2}{\partial\mu}\right)\partial_\xi - 4\frac{\partial a_1}{\partial y}\mu\xi^2\partial_\eta. \tag{3-9}$$

It is useful to keep in mind that as  $\Delta_{g_0} - \sigma^2 - (n - 1)^2/4$  is formally self-adjoint relative to the metric density  $|dg_0|$  for  $\sigma$  real, so the same holds for  $\mu^{-1/2}(\Delta_{g_0} - \sigma^2 - (n - 1)^2/4)\mu^{-1/2}$  (as  $\mu$  is real), and indeed for its conjugate by  $\mu^{-i\sigma/2}(1 + \mu)^{i\sigma/4}$  for  $\sigma$  real since this is merely unitary conjugation. As for  $f$  real,  $A$  formally self-adjoint relative to  $|dg_0|$ ,  $f^{-1}Af$  is formally self-adjoint relative to  $f^2|dg_0|$ , we then deduce that for  $\sigma$  real,  $P_\sigma$  is formally self-adjoint relative to

$$\mu^{(n+1)/2}|dg_0| = \frac{1}{2}|dh||d\mu|,$$

as  $x^{-n} dx = \frac{1}{2}\mu^{-(n+1)/2} d\mu$ . Note that  $\mu^{(n+1)/2}|dg_0|$  thus extends to a  $\mathcal{C}^\infty$  density to  $X_{-\delta_0}$ , and we deduce that with respect to the extended density,  $\sigma_1(\frac{1}{2i}(P_\sigma - P_\sigma^*))|_{\mu \geq 0}$  vanishes when  $\sigma \in \mathbb{R}$ . Since in general  $P_\sigma - P_{\text{Re}\sigma}$  differs from  $-4i(1 + a_2) \text{Im } \sigma D_\mu$  by a zeroth order operator, we conclude that

$$\sigma_1\left(\frac{1}{2i}(P_\sigma - P_\sigma^*)\right)\Big|_{\mu=0} = -4(\text{Im } \sigma)\xi. \tag{3-10}$$

We still need to check that  $\mu$  can be appropriately chosen in the interior away from the region of validity of the product decomposition (3-1) (where we had no requirements so far on  $\mu$ ). This only matters for semiclassical purposes, and (being smooth and nonzero in the interior) the factor  $\mu^{-1/2}$  multiplying from both sides does not affect any of the relevant properties (semiclassical ellipticity and possible nontrapping properties), so can be ignored — the same is true for  $\sigma$ -independent powers of  $\mu$ .

Thus, near  $\mu = 0$ , but  $\mu$  bounded away from 0, the only semiclassically nontrivial action we have done was to conjugate the operator by  $e^{-i\sigma\phi}$  where  $e^\phi = \mu^{1/2}(1 + \mu)^{-1/4}$ ; we need to extend  $\phi$  into the interior. But the semiclassical principal symbol of the conjugated operator is, with  $\sigma = z/h$ ,

$$(\zeta - z d\phi, \zeta - z d\phi)_{G_0} - z^2 = |\zeta|_{G_0}^2 - 2z(\zeta, d\phi)_{G_0} - (1 - |d\phi|_{G_0}^2)z^2. \tag{3-11}$$

For  $z$  nonreal this is elliptic if  $|d\phi|_{G_0} < 1$ . Indeed, if (3-11) vanishes then from the vanishing imaginary part we get

$$2 \operatorname{Im} z((\zeta, d\phi)_{G_0} + (1 - |d\phi|_{G_0}^2) \operatorname{Re} z) = 0, \tag{3-12}$$

and then the real part is

$$\begin{aligned} |\zeta|_{G_0}^2 - 2 \operatorname{Re} z(\zeta, d\phi)_{G_0} - (1 - |d\phi|_{G_0}^2)((\operatorname{Re} z)^2 - (\operatorname{Im} z)^2) \\ = |\zeta|_{G_0}^2 + (1 - |d\phi|_{G_0}^2)((\operatorname{Re} z)^2 + (\operatorname{Im} z)^2), \end{aligned} \tag{3-13}$$

which cannot vanish if  $|d\phi|_{G_0} < 1$ . But, reading off the dual metric from the principal symbol of (3-3),

$$\frac{1}{4} |d(\log \mu - \frac{1}{2} \log(1 + \mu))|_{G_0}^2 = \left(1 - \frac{\mu}{2(1 + \mu)}\right)^2 < 1$$

for  $\mu > 0$ , with a strict bound as long as  $\mu$  is bounded away from 0. Correspondingly,  $\mu^{1/2}(1 + \mu)^{-1/4}$  can be extended to a function  $e^\phi$  on all of  $X_0$  so that semiclassical ellipticity for  $z$  away from the reals is preserved, and we may even require that  $\phi$  is constant on a fixed (but arbitrarily large) compact subset of  $X_0^\circ$ . Then, after conjugation by  $e^{-i\sigma\phi}$ ,

$$P_{h,z} = e^{iz\phi/h} \mu^{-(n+1)/4-1/2} (h^2 \Delta_{g_0} - z) \mu^{(n+1)/4-1/2} e^{-iz\phi/h} \tag{3-14}$$

is semiclassically elliptic in  $\mu > 0$  (as well as in  $\mu \leq 0$ ,  $\mu$  near 0, where this is already guaranteed), as desired.

**Remark 3.1.** We have not considered vector bundles over  $X_0$ . However, for instance for the Laplacian on the differential form bundles it is straightforward to check that slightly changing the power of  $\mu$  in the conjugation the resulting operator extends smoothly across  $\partial X_0$ , has scalar principal symbol of the form (3-6), and the principal symbol of  $\frac{1}{2i}(P_\sigma - P_\sigma^*)$ , which plays a role below, is also as in the scalar setting, so all the results in fact go through.

**3B. Local dynamics near the radial set.** Let

$$N^*S \setminus o = \Lambda_+ \cup \Lambda_-, \quad \Lambda_\pm = N^*S \cap \{\pm\xi > 0\}, \quad S = \{\mu = 0\};$$

thus  $S \subset X_{-\delta_0}$  can be identified with  $Y = \partial X_0 (= \partial X_{0,\text{even}})$ . Note that  $p = 0$  at  $\Lambda_\pm$  and  $H_p$  is radial there since

$$N^*S = \{(\mu, y, \xi, \eta) : \mu = 0, \eta = 0\},$$

so

$$H_p|_{N^*S} = -4\xi^2 \partial_\xi.$$

This corresponds to  $dp = 4\xi^2 d\mu$  at  $N^*S$ , so the characteristic set  $\Sigma = \{p = 0\}$  is smooth at  $N^*S$ .

Let  $L_{\pm}$  be the image of  $\Lambda_{\pm}$  in  $S^*X_{-\delta_0}$ . Next we analyze the Hamilton flow at  $\Lambda_{\pm}$ . First,

$$H_p|\eta|_{\mu,y}^2 = 8(1 + a_1)\mu\xi\partial_{\mu}|\eta|_{\mu,y}^2 - 4\frac{\partial a_1}{\partial y}\mu\xi^2 \cdot_h \eta \tag{3-15}$$

and

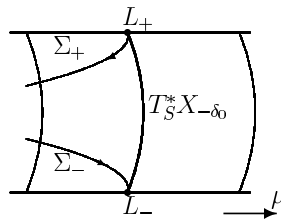
$$H_p\mu = 8(1 + a_1)\xi\mu. \tag{3-16}$$

In terms of linearizing the flow at  $N^*S$ ,  $p$  and  $\mu$  are equivalent as  $dp = 4\xi^2 d\mu$  there, so one can simply use  $\hat{p} = p/|\xi|^2$  (which is homogeneous of degree 0, like  $\mu$ ), in place of  $\mu$ . Finally,

$$H_p|\xi| = -4 \operatorname{sgn}(\xi) + b, \tag{3-17}$$

with  $b$  vanishing at  $\Lambda_{\pm}$ .

It is convenient to rehomogenize (3-15) in terms of  $\hat{\eta} = \eta/|\xi|$ . This can be phrased more invariantly by working with  $S^*X_{-\delta_0} = (T^*X_{-\delta_0} \setminus o)/\mathbb{R}^+$ , briefly discussed in Section 2. Let  $L_{\pm}$  be the image of  $\Lambda_{\pm}$  in  $S^*X_{-\delta_0}$ . Homogeneous degree zero functions on  $T^*X_{-\delta_0} \setminus o$ , such as  $\hat{p}$ , can be regarded as functions on  $S^*X_{-\delta_0}$ . For semiclassical purposes, it is best to consider  $S^*X_{-\delta_0}$  as the boundary at fiber infinity of the fiber-radial compactification  $\bar{T}^*X_{-\delta_0}$  of  $T^*X_{-\delta_0}$ , also discussed in Section 2. Then at fiber infinity near  $N^*S$ , we can take  $(|\xi|^{-1}, \hat{\eta})$  as (projective, rather than polar) coordinates on the fibers of the



**Figure 1.** The cotangent bundle of  $X_{-\delta_0}$  near  $S = \{\mu = 0\}$ . It is drawn in a fiber-radially compactified view. The boundary of the fiber compactification is the cosphere bundle  $S^*X_{-\delta_0}$ ; it is the surface of the cylinder shown.  $\Sigma_{\pm}$  are the components of the (classical) characteristic set containing  $L_{\pm}$ . They lie in  $\mu \leq 0$ , only meeting  $S^*X_{-\delta_0}$  at  $L_{\pm}$ . Semiclassically, that is, in the interior of  $\bar{T}^*X_{-\delta_0}$ , for  $z = h^{-1}\sigma > 0$ , only the component of the semiclassical characteristic set containing  $L_+$  can enter  $\mu > 0$ . This is reversed for  $z < 0$ .

cotangent bundle, with  $\tilde{\rho} = |\xi|^{-1}$  defining  $S^*X_{-\delta_0}$  in  $\bar{T}^*X_{-\delta_0}$ . Then  $W = |\xi|^{-1}H_p$  is a  $\mathcal{C}^\infty$  vector field in this region and

$$|\xi|^{-1}H_p|\hat{\eta}|_{\mu,y}^2 = 2|\hat{\eta}|_{\mu,y}^2H_p|\xi|^{-1} + |\xi|^{-3}H_p|\eta|_{\mu,y}^2 = 8(\operatorname{sgn} \xi)|\hat{\eta}|^2 + \tilde{a}, \quad (3-18)$$

where  $\tilde{a}$  vanishes cubically at  $N^*S$ . In similar notation we have

$$H_p\tilde{\rho} = 4 \operatorname{sgn}(\xi) + \tilde{a}', \quad \tilde{\rho} = |\xi|^{-1}, \quad (3-19)$$

and

$$|\xi|^{-1}H_p\mu = 8(\operatorname{sgn} \xi)\mu + \tilde{a}'', \quad (3-20)$$

with  $\tilde{a}'$  smooth (indeed, homogeneous degree zero without the compactification) vanishing at  $N^*S$ , and  $\tilde{a}''$  is also smooth, vanishing quadratically at  $N^*S$ . As the vanishing of  $\hat{\eta}$ ,  $|\xi|^{-1}$  and  $\mu$  defines  $\partial N^*S$ , we conclude that  $L_- = \partial\Lambda_-$  is a sink, while  $L_+ = \partial\Lambda_+$  is a source, in the sense that all nearby bicharacteristics (in fact, including semiclassical (null)bicharacteristics, since  $H_p|\xi|^{-1}$  contains the additional information needed; see (3-29)) converge to  $L_\pm$  as the parameter along the bicharacteristic goes to  $\mp\infty$ . In particular, the quadratic defining function of  $L_\pm$  given by

$$\rho_0 = \hat{p} + \hat{p}^2, \quad \text{where } \hat{p} = |\xi|^{-2}p, \quad \hat{p} = |\hat{\eta}|^2,$$

satisfies

$$(\operatorname{sgn} \xi)W\rho_0 \geq 8\rho_0 + \mathcal{O}(\rho_0^{3/2}). \quad (3-21)$$

We also need information on the principal symbol of  $\frac{1}{2i}(P_\sigma - P_\sigma^*)$  at the radial points. At  $L_\pm$  this is given by

$$\sigma_1\left(\frac{1}{2i}(P_\sigma - P_\sigma^*)\right)|_{N^*S} = -(4 \operatorname{sgn}(\xi)) \operatorname{Im} \sigma |\xi|; \quad (3-22)$$

here  $(4 \operatorname{sgn}(\xi))$  is pulled out due to (3-19), namely its size relative to  $H_p|\xi|^{-1}$  matters. This corresponds to the fact that  $(\mu \pm i0)^{i\sigma}$ , which are Lagrangian distributions associated to  $\Lambda_\pm$ , solve the PDE (3-5) modulo an error that is two orders lower than what one might a priori expect, that is,  $P_\sigma(\mu \pm i0)^{i\sigma} \in (\mu \pm i0)^{i\sigma}\mathcal{C}^\infty(X_{-\delta_0})$ . Note that  $P_\sigma$  is second order, so one should lose two orders a priori, that is, get an element of  $(\mu \pm i0)^{i\sigma-2}\mathcal{C}^\infty(X_{-\delta_0})$ ; the characteristic nature of  $\Lambda_\pm$  reduces the loss to 1, and the particular choice of exponent eliminates the loss. This has much in common with  $e^{i\lambda/x}x^{(n-1)/2}$  being an approximate solution in asymptotically Euclidean scattering; see [Melrose 1994].

**3C. Global behavior of the characteristic set.** By (3-8), points with  $\xi = 0$  cannot lie in the characteristic set. Thus, with

$$\Sigma_\pm = \Sigma \cap \{\pm\xi > 0\},$$



$\Sigma = \Sigma_+ \cup \Sigma_-$  and  $\Lambda_{\pm} \subset \Sigma_{\pm}$ . Further, the characteristic set lies in  $\mu \leq 0$ , and intersects  $\mu = 0$  only in  $\Lambda_{\pm}$ .

Moreover, as  $H_p \mu = 8(1 + a_1)\xi\mu$  and  $\xi \neq 0$  on  $\Sigma$ , and  $\mu$  only vanishes at  $\Lambda_+ \cup \Lambda_-$  there, for  $\epsilon_0 > 0$  sufficiently small the  $\mathcal{C}^\infty$  function  $\mu$  provides a negative global escape function on  $\mu \geq -\epsilon_0$  which is decreasing on  $\Sigma_+$ , increasing on  $\Sigma_-$ . Correspondingly, bicharacteristics in  $\Sigma_-$  travel from  $\mu = -\epsilon_0$  to  $L_-$ , while in  $\Sigma_+$  they travel from  $L_+$  to  $\mu = -\epsilon_0$ .

**3D. High energy, or semiclassical, asymptotics.** We are also interested in the high-energy behavior, as  $|\sigma| \rightarrow \infty$ . For the associated semiclassical problem one obtains a family of operators

$$P_{h,z} = h^2 P_{h^{-1}z},$$

with  $h = |\sigma|^{-1}$ , and  $z$  corresponding to  $\sigma/|\sigma|$  in the unit circle in  $\mathbb{C}$ . Then the semiclassical principal symbol  $p_{\hbar,z}$  of  $P_{h,z}$  is a function on  $T^*X_{-\delta_0}$ , whose asymptotics at fiber infinity of  $T^*X_{-\delta_0}$  is given by the classical principal symbol  $p$ . We are interested in  $\text{Im } \sigma \geq -C$ , which in semiclassical notation corresponds to  $\text{Im } z \geq -Ch$ . It is sometimes convenient to think of  $p_{\hbar,z}$ , and its rescaled Hamilton vector field, as objects on  $\bar{T}^*X_{-\delta_0}$ . Thus,

$$p_{\hbar,z} = \sigma_{2,\hbar}(P_{h,z}) = 4(1 + a_1)\mu\xi^2 - 4(1 + a_2)z\xi - (1 + a_3)z^2 + |\eta|_{\mu,y}^2, \tag{3-23}$$

so

$$\text{Im } p_{\hbar,z} = -2 \text{Im } z(2(1 + a_2)\xi + (1 + a_3) \text{Re } z). \tag{3-24}$$

In particular, for  $z$  nonreal,  $\text{Im } p_{\hbar,z} = 0$  implies  $2(1 + a_2)\xi + (1 + a_3) \text{Re } z = 0$ , so

$$\begin{aligned} \text{Re } p_{\hbar,z} = & ((1 + a_1)(1 + a_3)^2(1 + a_2)^{-2}\mu + (1 + 2a_2)(1 + a_3))(\text{Re } z)^2 \\ & + (1 + a_3)(\text{Im } z)^2 + |\eta|_{\mu,y}^2 > 0 \end{aligned} \tag{3-25}$$

near  $\mu = 0$ ; in other words,  $p_{\hbar,z}$  is semiclassically elliptic on  $T^*X_{-\delta_0}$ , but *not* at fiber infinity, i.e., at  $S^*X_{-\delta_0}$  (standard ellipticity is lost only in  $\mu \leq 0$ , of course). In  $\mu > 0$  we have semiclassical ellipticity (and automatically classical ellipticity) by our choice of  $\phi$  following (3-11). Explicitly, if we introduce for instance

$$(\mu, y, v, \hat{\eta}), \quad v = |\xi|^{-1}, \quad \hat{\eta} = \eta/|\xi|, \tag{3-26}$$

as valid projective coordinates in a (large!) neighborhood of  $L_{\pm}$  in  $\bar{T}^*X_{-\delta_0}$ , then

$$v^2 p_{\hbar,z} = 4(1 + a_1)\mu - 4(1 + a_2)(\text{sgn } \xi)zv - (1 + a_3)z^2v^2 + |\hat{\eta}|_{y,\mu}^2,$$

so

$$\nu^2 \operatorname{Im} p_{\hbar,z} = -4(1+a_2)(\operatorname{sgn} \xi)\nu \operatorname{Im} z - 2(1+a_3)\nu^2 \operatorname{Re} z \operatorname{Im} z,$$

which automatically vanishes at  $\nu = 0$ , that is, at  $S^*X_{-\delta_0}$ . Thus, for  $\sigma$  large and pure imaginary, the semiclassical problem adds no complexity to the “classical” quantum problem, but of course it does not simplify it. In fact, we need somewhat more information at the characteristic set, which is thus at  $\nu = 0$  when  $\operatorname{Im} z$  is bounded away from 0:

$$\begin{aligned} \nu \text{ small, } \operatorname{Im} z \geq 0 &\Rightarrow (\operatorname{sgn} \xi) \operatorname{Im} p_{\hbar,z} \leq 0 \Rightarrow \pm \operatorname{Im} p_{\hbar,z} \leq 0 \text{ near } \Sigma_{\hbar,\pm}, \\ \nu \text{ small, } \operatorname{Im} z \leq 0 &\Rightarrow (\operatorname{sgn} \xi) \operatorname{Im} p_{\hbar,z} \geq 0 \Rightarrow \pm \operatorname{Im} p_{\hbar,z} \geq 0 \text{ near } \Sigma_{\hbar,\pm}, \end{aligned} \quad (3-27)$$

which, as we recall in Section 4, means that for  $P_{h,z}$  with  $\operatorname{Im} z > 0$  one can propagate estimates forwards along the bicharacteristics where  $\xi > 0$  (in particular, away from  $L_+$ , as the latter is a source) and backwards where  $\xi < 0$  (in particular, away from  $L_-$ , as the latter is a sink), while for  $P_{h,z}^*$  the directions are reversed since its semiclassical symbol is  $\overline{p_{\hbar,z}}$ . The directions are also reversed if  $\operatorname{Im} z$  switches sign. This is important because it gives invertibility for  $z = \iota$  (corresponding to  $\operatorname{Im} \sigma$  large positive, that is, the physical half-plane), but does not give invertibility for  $z = -\iota$  negative.

We now return to the claim that even semiclassically, for  $z$  almost real (i.e., when  $z$  is not bounded away from the reals; we are not fixing  $z$  as we let  $h$  vary!), when the operator is not semiclassically elliptic on  $T^*X_{-\delta_0}$  as mentioned above, the characteristic set can be divided into two components  $\Sigma_{\hbar,\pm}$ , with  $L_{\pm}$  in different components. The vanishing of the factor following  $\operatorname{Im} z$  in (3-24) gives a hypersurface that separates  $\Sigma_{\hbar}$  into two parts. Indeed, this is the hypersurface given by

$$2(1+a_2)\xi + (1+a_3) \operatorname{Re} z = 0,$$

on which, by (3-25),  $\operatorname{Re} p_{\hbar,z}$  cannot vanish, so

$$\Sigma_{\hbar} = \Sigma_{\hbar,+} \cup \Sigma_{\hbar,-}, \quad \Sigma_{\hbar,\pm} = \Sigma_{\hbar} \cap \{\pm(2(1+a_2)\xi + (1+a_3) \operatorname{Re} z) > 0\}.$$

Farther in  $\mu > 0$ , the hypersurface is given, due to (3-12), by

$$(\zeta, d\phi)_{G_0} + (1 - |d\phi|_{G_0}^2) \operatorname{Re} z = 0,$$

and on it, by (3-13), the real part is  $|\zeta|_{G_0}^2 + (1 - |d\phi|_{G_0}^2)((\operatorname{Re} z)^2 + (\operatorname{Im} z)^2) > 0$ ; correspondingly

$$\Sigma_{\hbar} = \Sigma_{\hbar,+} \cup \Sigma_{\hbar,-}, \quad \Sigma_{\hbar,\pm} = \Sigma_{\hbar} \cap \{\pm((\zeta, d\phi)_{G_0} + (1 - |d\phi|_{G_0}^2) \operatorname{Re} z) > 0\}.$$

In fact, more generally, the real part is

$$\begin{aligned} & |\zeta|_{G_0}^2 - 2 \operatorname{Re} z (\zeta, d\phi)_{G_0} - (1 - |d\phi|_{G_0}^2)((\operatorname{Re} z)^2 - (\operatorname{Im} z)^2) \\ &= |\zeta|_{G_0}^2 - 2 \operatorname{Re} z ((\zeta, d\phi)_{G_0} + (1 - |d\phi|_{G_0}^2) \operatorname{Re} z) \\ &\quad + (1 - |d\phi|_{G_0}^2)((\operatorname{Re} z)^2 + (\operatorname{Im} z)^2), \end{aligned}$$

so for  $\pm \operatorname{Re} z > 0$ ,  $\mp((\zeta, d\phi)_{G_0} + (1 - |d\phi|_{G_0}^2) \operatorname{Re} z) > 0$  implies that  $p_{\hbar,z}$  does not vanish. Correspondingly, only one of the two components of  $\Sigma_{\hbar,\pm}$  enters  $\mu > 0$ : for  $\operatorname{Re} z > 0$ , it is  $\Sigma_{\hbar,+}$ , while for  $\operatorname{Re} z < 0$ , it is  $\Sigma_{\hbar,-}$ .

We finally need more information about the global semiclassical dynamics.

**Lemma 3.2.** *There exists  $\epsilon_0 > 0$  such that the following holds. All semiclassical null-bicharacteristics in  $(\Sigma_{\hbar,+} \setminus L_+) \cap \{-\epsilon_0 \leq \mu \leq \epsilon_0\}$  go to either  $L_+$  or to  $\mu = \epsilon_0$  in the backward direction and to  $\mu = \epsilon_0$  or  $\mu = -\epsilon_0$  in the forward direction, while all semiclassical null-bicharacteristics in  $(\Sigma_{\hbar,-} \setminus L_-) \cap \{-\epsilon_0 \leq \mu \leq \epsilon_0\}$  go to  $L_-$  or  $\mu = \epsilon_0$  in the forward direction and to  $\mu = \epsilon_0$  or  $\mu = -\epsilon_0$  in the backward direction.*

For  $\operatorname{Re} z > 0$ , only  $\Sigma_{\hbar,+}$  enters  $\mu > 0$ , so the  $\mu = \epsilon_0$  possibility only applies to  $\Sigma_{\hbar,+}$  then, while for  $\operatorname{Re} z < 0$ , the analogous remark applies to  $\Sigma_{\hbar,-}$ .

*Proof.* We assume that  $\operatorname{Re} z > 0$  for the sake of definiteness. Observe that the semiclassical Hamilton vector field is

$$\begin{aligned} H_{p_{\hbar,z}} &= 4(2(1 + a_1)\mu\xi - (1 + a_2)z)\partial_\mu + \tilde{H}_{|\eta|_{\mu,y}^2} \\ &\quad - \left(4(1 + a_1 + \mu \frac{\partial a_1}{\partial \mu})\xi^2 - 4\frac{\partial a_2}{\partial \mu}z\xi + \frac{\partial a_3}{\partial \mu}z^2 + \frac{\partial |\eta|_{\mu,y}^2}{\partial \mu}\right)\partial_\xi \\ &\quad - \left(4\frac{\partial a_1}{\partial y}\mu\xi^2 - 4\frac{\partial a_2}{\partial y}z\xi - \frac{\partial a_3}{\partial y}z^2\right)\partial_\eta; \end{aligned} \tag{3-28}$$

here we are concerned about  $z$  real. Near  $S^*X_{-\delta_0} = \partial\bar{T}^*X_{-\delta_0}$ , using the coordinates (3-26) (which are valid near the characteristic set), we have

$$\begin{aligned} W_{\hbar} &= \nu H_{p_{\hbar,z}} = \\ & 4(2(1 + a_1)\mu(\operatorname{sgn} \xi) - (1 + a_2)z\nu)\partial_\mu + \nu\tilde{H}_{|\eta|_{\mu,y}^2} \\ & + (\operatorname{sgn} \xi) \left(4(1 + a_1 + \mu \frac{\partial a_1}{\partial \mu}) - 4\frac{\partial a_2}{\partial \mu}z(\operatorname{sgn} \xi)\nu + \frac{\partial a_3}{\partial \mu}z^2\nu^2 + \frac{\partial |\hat{\eta}|_{\mu,y}^2}{\partial \mu}\right) (\nu\partial_\nu + \hat{\eta}\partial_{\hat{\eta}}) \\ & \quad - \left(4\frac{\partial a_1}{\partial y}\mu - 4(\operatorname{sgn} \xi)\frac{\partial a_2}{\partial y}z\nu - \frac{\partial a_3}{\partial y}z^2\nu^2\right)\partial_{\hat{\eta}}, \end{aligned} \tag{3-29}$$

with

$$\nu\tilde{H}_{|\eta|_{\mu,y}^2} = \sum_{ij} H_{ij} \hat{\eta}_i \partial_{y_j} - \sum_{ijk} \frac{\partial H_{ij}}{\partial y_k} \hat{\eta}_i \hat{\eta}_j \partial_{\hat{\eta}_k}$$

smooth. Thus,  $W_{\hbar}$  is a smooth vector field on the compactified cotangent bundle,  $\bar{T}^*X_{-\delta_0}$  which is tangent to its boundary,  $S^*X_{-\delta_0}$ , and  $W_{\hbar} - W = \nu W^\#$  (with  $W$  considered as a homogeneous degree zero vector field) with  $W^\#$  smooth and tangent to  $S^*X_{-\delta_0}$ . In particular, by (3-19) and (3-21), using that  $\tilde{\rho}^2 + \rho_0$  is a quadratic defining function of  $L_{\pm}$ ,

$$(\operatorname{sgn} \xi) W_{\hbar}(\tilde{\rho}^2 + \rho_0) \geq 8(\tilde{\rho}^2 + \rho_0) - \mathcal{O}((\tilde{\rho}^2 + \rho_0)^{3/2})$$

shows that there is  $\epsilon_1 > 0$  such that in  $\tilde{\rho}^2 + \rho_0 \leq \epsilon_1$ ,  $\xi > 0$ ,  $\tilde{\rho}^2 + \rho_0$  is strictly increasing along the Hamilton flow except at  $L_+$ , while in  $\tilde{\rho}^2 + \rho_0 \leq \epsilon_1$ ,  $\xi < 0$ ,  $\tilde{\rho}^2 + \rho_0$  is strictly decreasing along the Hamilton flow except at  $L_-$ . Indeed, all null-bicharacteristics in this neighborhood of  $L_{\pm}$  except the constant ones at  $L_{\pm}$  tend to  $L_{\pm}$  in one direction and to  $\tilde{\rho}^2 + \rho_0 = \epsilon_1$  in the other direction.

Choosing  $\epsilon'_0 > 0$  sufficiently small, the characteristic set in

$$\bar{T}^*X_{-\delta_0} \cap \{-\epsilon'_0 \leq \mu \leq \epsilon'_0\}$$

is disjoint from  $S^*X_{-\delta_0} \setminus \{\tilde{\rho}^2 + \rho_0 \leq \epsilon_1\}$ , and indeed only contains points in  $\Sigma_{\hbar,+}$  as  $\operatorname{Re} z > 0$ . Since  $H_{p_{\hbar,z}}\mu = 4(2(1 + a_1)\mu\xi - (1 + a_2)z)$ , it is negative on  $\bar{T}^*_{\{\mu=0\}}X_{-\delta_0} \setminus S^*X_{-\delta_0}$ . In particular, there is a neighborhood  $U$  of  $\mu = 0$  in  $\Sigma_{\hbar,+} \setminus S^*X_{-\delta_0}$  on which the same sign is preserved; since the characteristic set in  $\bar{T}^*X_{-\delta_0} \setminus \{\tilde{\rho}^2 + \rho_0 < \epsilon_1\}$  is compact, and is indeed a subset of  $T^*X_{-\delta_0} \setminus \{\tilde{\rho}^2 + \rho_0 < \epsilon_1\}$ , we deduce that  $|\mu|$  is bounded below on  $\Sigma \setminus (U \cup \{\tilde{\rho}^2 + \rho_0 < \epsilon_1\})$ , say  $|\mu| \geq \epsilon''_0 > 0$  there, so with  $\epsilon_0 = \min(\epsilon'_0, \epsilon''_0)$ ,  $H_{p_{\hbar,z}}\mu < 0$  on

$$\Sigma_{\hbar,+} \cap \{-\epsilon_0 \leq \mu \leq \epsilon_0\} \setminus \{\tilde{\rho}^2 + \rho_0^2 < \epsilon_1\}.$$

As  $H_{p_{\hbar,z}}\mu < 0$  at  $\mu = 0$ , bicharacteristics can only cross  $\mu = 0$  in the outward direction.

Thus, if  $\gamma$  is a bicharacteristic in  $\Sigma_{\hbar,+}$ , there are two possibilities. If  $\gamma$  is disjoint from  $\{\tilde{\rho}^2 + \rho_0 < \epsilon_1\}$ , it has to go to  $\mu = \epsilon_0$  in the backward direction and to  $\mu = -\epsilon_0$  in the forward direction. If  $\gamma$  has a point in  $\{\tilde{\rho}^2 + \rho_0 < \epsilon_1\}$ , then it has to go to  $L_+$  in the backward direction and to  $\tilde{\rho}^2 + \rho_0 = \epsilon_1$  in the forward direction; if  $|\mu| \geq \epsilon_0$  by the time  $\tilde{\rho}^2 + \rho_0 = \epsilon_1$  is reached, the result is proved, and otherwise  $H_{p_{\hbar,z}}\mu < 0$  in  $\tilde{\rho}^2 + \rho_0 \geq \epsilon_1$ ,  $|\mu| \leq \epsilon_0$ , shows that the bicharacteristic goes to  $\mu = -\epsilon_0$  in the forward direction.

If  $\gamma$  is a bicharacteristic in  $\Sigma_{\hbar,-}$ , only the second possibility exists, and the bicharacteristic cannot leave  $\{\tilde{\rho}^2 + \rho_0 < \epsilon_1\}$  in  $|\mu| \leq \epsilon_0$ , so it reaches  $\mu = -\epsilon_0$  in the backward direction (as the characteristic set is in  $\mu \leq 0$ ).  $\square$

If we assume that  $g_0$  is a nontrapping metric, that is, bicharacteristics of  $g_0$  in  $T^*X_0^\circ \setminus \circ$  tend to  $\partial X_0$  in both the forward and the backward directions, then  $\mu = \epsilon_0$  can be excluded from the statement of the lemma, and the above argument

gives the following stronger conclusion: for sufficiently small  $\epsilon_0 > 0$ , and for  $\text{Re } z > 0$ , any bicharacteristic in  $\Sigma_{\hbar,+}$  in  $-\epsilon_0 \leq \mu$  has to go to  $L_+$  in the backward direction, and to  $\mu = -\epsilon_0$  in the forward direction (with the exception of the constant bicharacteristics at  $L_+$ ), while in  $\Sigma_{\hbar,-}$ , all bicharacteristics in  $-\epsilon_0 \leq \mu$  lie in  $-\epsilon_0 \leq \mu \leq 0$ , and go to  $L_-$  in the forward direction and to  $\mu = -\epsilon_0$  in the backward direction (with the exception of the constant bicharacteristics at  $L_-$ ).

In fact, for applications, it is also useful to remark that for sufficiently small  $\epsilon_0 > 0$ , and for  $\alpha \in T^*X_0$ ,

$$0 < \mu(\alpha) < \epsilon_0, p_{\hbar,z}(\alpha) = 0 \text{ and } (H_{p_{\hbar,z}}\mu)(\alpha) = 0 \Rightarrow (H_{p_{\hbar,z}}^2\mu)(\alpha) < 0. \tag{3-30}$$

Indeed, as  $H_{p_{\hbar,z}}\mu = 4(2(1 + a_1)\mu\xi - (1 + a_2)z)$ , the hypotheses imply that  $z = 2(1 + a_1)(1 + a_2)^{-1}\mu\xi$  and

$$\begin{aligned} 0 &= p_{\hbar,z} \\ &= 4(1 + a_1)\mu\xi^2 - 8(1 + a_1)\mu\xi^2 - 4(1 + a_1)^2(1 + a_2)^{-2}(1 + a_3)\mu^2\xi^2 + |\eta|_{\mu,y}^2 \\ &= -4(1 + a_1)\mu\xi^2 - 4(1 + a_1)^2(1 + a_2)^{-2}(1 + a_3)\mu^2\xi^2 + |\eta|_{\mu,y}^2, \end{aligned}$$

so  $|\eta|_{\mu,y}^2 = 4(1 + b)\mu\xi^2$ , with  $b$  vanishing at  $\mu = 0$ . Thus, at points where  $H_{p_{\hbar,z}}\mu$  vanishes, writing  $a_j = \mu\tilde{a}_j$ ,

$$\begin{aligned} H_{p_{\hbar,z}}^2\mu &= 8(1 + a_1)\mu H_{p_{\hbar,z}}\xi + 8\mu^2\xi H_{p_{\hbar,z}}\tilde{a}_1 - 4z\mu H_{p_{\hbar,z}}\tilde{a}_2 \\ &= 8(1 + a_1)\mu H_{p_{\hbar,z}}\xi + \mathcal{O}(\mu^2\xi^2). \end{aligned} \tag{3-31}$$

Now

$$H_{p_{\hbar,z}}\xi = -\left(4\left(1 + a_1 + \mu\frac{\partial a_1}{\partial\mu}\right)\xi^2 - 4\frac{\partial a_2}{\partial\mu}z\xi + \frac{\partial a_3}{\partial\mu}z^2 + \frac{\partial|\eta|_{\mu,y}^2}{\partial\mu}\right).$$

Since  $z\xi$  is  $\mathcal{O}(\mu\xi^2)$  due to  $H_{p_{\hbar,z}}\mu = 0$ ,  $z^2$  is  $\mathcal{O}(\mu^2\xi^2)$  for the same reason, and  $|\eta|^2$  and  $\partial_\mu|\eta|^2$  are  $\mathcal{O}(\mu\xi^2)$  due to  $p_{\hbar,z} = 0$ , we deduce that  $H_{p_{\hbar,z}}\xi < 0$  for sufficiently small  $|\mu|$ , so (3-31) implies (3-30). Thus,  $\mu$  can be used for gluing constructions as in [Datchev and Vasy 2011].

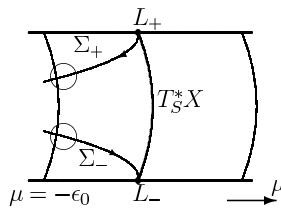
**3E. Complex absorption.** The final step of fitting  $P_\sigma$  into our general microlocal framework is moving the problem to a compact manifold, and adding a complex absorbing second order operator. We thus consider a compact manifold without boundary  $X$  for which  $X_{\mu_0} = \{\mu > \mu_0\}$ ,  $\mu_0 = -\epsilon_0 < 0$ , with  $\epsilon_0 > 0$  as above, is identified as an open subset with smooth boundary; it is convenient to take  $X$  to be the double of  $X_{\mu_0}$ , so there are two copies of  $X_{0,\text{even}}$  in  $X$ .

In the case of hyperbolic space, this doubling process can be realized from the perspective of  $(n + 1)$ -dimensional Minkowski space. Then, as mentioned in the introduction, the Poincaré model shows up in two copies, namely in the interior

of the future and past light cone inside the sphere at infinity, while de Sitter space as the “equatorial belt”, that is, the exterior of the light cone at the sphere at infinity. One can take the Minkowski equatorial plane,  $t = 0$ , as  $\mu = \mu_0$ , and place the complex absorption there, thereby decoupling the future and past hemispheres. See [Vasy 2010a] for more detail.

It is convenient to separate the “classical” (i.e., quantum!) and “semiclassical” problems, for in the former setting trapping for  $g_0$  does not matter, while in the latter it does.

We then introduce a “complex absorption” operator  $Q_\sigma \in \Psi_{cl}^2(X)$  with real principal symbol  $q$  supported in, say,  $\mu < -\epsilon_1$ , with the Schwartz kernel also supported in the corresponding region (i.e., in both factors on the product space this condition holds on the support) such that  $p \pm \iota q$  is elliptic near  $\partial X_{\mu_0}$ , that is, near  $\mu = \mu_0$ , and which satisfies that  $\pm q \geq 0$  near  $\Sigma_\pm$ . This can easily be done since  $\Sigma_\pm$  are disjoint, and away from these  $p$  is elliptic, hence so is  $p \pm \iota q$  regardless of the choice of  $q$ ; we simply need to make  $q$  to have support sufficiently close to  $\Sigma_\pm$ , elliptic on  $\Sigma_\pm$  at  $\mu = -\epsilon_0$ , with the appropriate sign near  $\Sigma_\pm$ . Having done this, we extend  $p$  and  $q$  to  $X$  in such a way that  $p \pm \iota q$  are elliptic near  $\partial X_{\mu_0}$ ; the region we added is thus irrelevant at the level of bicharacteristic dynamics (of  $p$ ) in so far as it is decoupled from the dynamics in  $X_0$ , and indeed also for analysis as we see shortly (in so far as we have two essentially decoupled copies of the same problem). This is accomplished, for instance, by using the doubling construction to define  $p$  on  $X \setminus X_{\mu_0}$  (in a smooth fashion at  $\partial X_{\mu_0}$ , as can be easily arranged; the holomorphic dependence of  $P_\sigma$  on  $\sigma$  is still easily preserved), and then, noting that the characteristic set of  $p$  still has two connected components, making  $q$  elliptic on the characteristic set of  $p$  near  $\partial X_{\mu_0}$ , with the same sign in each component as near  $\partial X_{\mu_0}$ . (An alternative would be to make  $q$  elliptic on the characteristic set of  $p$  near  $X \setminus X_{\mu_0}$ ; it is just slightly more complicated to write down such a  $q$  when the high-energy behavior



**Figure 2.** The cotangent bundle near  $S = \{\mu = 0\}$ . It is drawn in a fiber-radially compactified view, as in Figure 1. The circles on the left show the support of  $q$ ; it has opposite signs on the two disks corresponding to the opposite directions of propagation relative to the Hamilton vector field.

is taken into account. With the present choice, due to the doubling, there are essentially two copies of the problem on  $X_0$ : the original, and the one from the doubling.) Finally we take  $Q_\sigma$  be any operator with principal symbol  $q$  with Schwartz kernel satisfying the desired support conditions and which depends on  $\sigma$  holomorphically. We may choose  $Q_\sigma$  to be independent of  $\sigma$  so  $Q_\sigma$  is indeed holomorphic; in this case we may further replace it by  $\frac{1}{2}(Q_\sigma + Q_\sigma^*)$  if self-adjointness is desired.

In view of Section 3C we have arranged the following. For  $\alpha \in S^*X \cap \Sigma$ , let  $\gamma_+(\alpha)$ , resp.  $\gamma_-(\alpha)$  denote the image of the forward, resp. backward, half-bicharacteristic of  $p$  from  $\alpha$ . We write  $\gamma_\pm(\alpha) \rightarrow L_\pm$  (and say  $\gamma_\pm(\alpha)$  tends to  $L_\pm$ ) if given any neighborhood  $O$  of  $L_\pm$ ,  $\gamma_\pm(\alpha) \cap O \neq \emptyset$ ; by the source/sink property this implies that the points on the curve are in  $O$  for sufficiently large (in absolute value) parameter values. Then, with  $\text{ell}(Q_\sigma)$  denoting the elliptic set of  $Q_\sigma$ ,

$$\begin{aligned} \alpha \in \Sigma_- \setminus L_- &\Rightarrow \gamma_+(\alpha) \rightarrow L_- \text{ and } \gamma_-(\alpha) \cap \text{ell}(Q_\sigma) \neq \emptyset, \\ \alpha \in \Sigma_+ \setminus L_+ &\Rightarrow \gamma_-(\alpha) \rightarrow L_+ \text{ and } \gamma_+(\alpha) \cap \text{ell}(Q_\sigma) \neq \emptyset. \end{aligned} \tag{3-32}$$

That is, all forward and backward half-(null)bicharacteristics of  $P_\sigma$  either enter the elliptic set of  $Q_\sigma$ , or go to  $\Lambda_\pm$ , that is,  $L_\pm$  in  $S^*X$ . The point of the arrangements regarding  $Q_\sigma$  and the flow is that we are able to propagate estimates forward near where  $q \geq 0$ , backward near where  $q \leq 0$ , so by our hypotheses we can always propagate estimates for  $P_\sigma - \iota Q_\sigma$  from  $\Lambda_\pm$  towards the elliptic set of  $Q_\sigma$ . On the other hand, for  $P_\sigma^* + \iota Q_\sigma^*$ , we can propagate estimates from the elliptic set of  $Q_\sigma$  towards  $\Lambda_\pm$ . This behavior of  $P_\sigma - \iota Q_\sigma$  vs.  $P_\sigma^* + \iota Q_\sigma^*$  is important for duality reasons.

An alternative to the complex absorption would be simply adding a boundary at  $\mu = \mu_0$ ; this is easy to do since this is a space-like hypersurface, but this is slightly unpleasant from the point of view of microlocal analysis as one has to work on a manifold with boundary (though as mentioned this is easily done; see [Vasy 2010a]).

For the semiclassical problem, when  $z$  is almost real (namely when  $\text{Im } z$  is bounded away from 0 we only need to make sure we do not mess up the semiclassical ellipticity in  $T^*X_{-\delta_0}$ ) we need to increase the requirements on  $Q_\sigma$ , and what we need to do depends on whether  $g_0$  is nontrapping.

If  $g_0$  is nontrapping, we choose  $Q_\sigma$  such that  $h^2 Q_{h^{-1}z} \in \Psi_{h,\text{cl}}^2(X)$  with semiclassical principal symbol  $q_{h,z}$ , and in addition to the above requirement for the classical symbol, we need semiclassical ellipticity near  $\mu = \mu_0$ , that is, that  $p_{h,z} - \iota q_{h,z}$  and its complex conjugate are elliptic near  $\partial X_{\mu_0}$ , that is, near  $\mu = \mu_0$ , and which satisfies that for  $z$  real  $\pm q_{h,z} \geq 0$  on  $\Sigma_{h,\pm}$ . Again, we extend  $P_\sigma$  and  $Q_\sigma$  to  $X$  in such a way that  $p - \iota q$  and  $p_{h,z} - \iota q_{h,z}$  (and thus their complex

conjugates) are elliptic near  $\partial X_{\mu_0}$ ; the region we added is thus irrelevant. This is straightforward to arrange if one ignores that one wants  $Q_\sigma$  to be holomorphic: one easily constructs a function  $q_{\hbar,z}$  on  $T^*X$  (taking into account the disjointness of  $\Sigma_{\hbar,\pm}$ ), and defines  $Q_{h^{-1}z}$  to be  $h^{-2}$  times the semiclassical quantization of  $q_{\hbar,z}$  (or any other operator with the same semiclassical and standard principal symbols). Indeed, for our purposes this would suffice since we want high-energy estimates for the analytic continuation resolvent on the original space  $X_0$  (which we will know exists by the nonsemiclassical argument), and as we shall see, the resolvent is given by the same formula in terms of  $(P_\sigma - \iota Q_\sigma)^{-1}$  independently whether  $Q_\sigma$  is holomorphic in  $\sigma$  (as long as it satisfies the other properties), so there is no need to ensure the holomorphy of  $Q_\sigma$ . However, it is instructive to have an example of a holomorphic family  $Q_\sigma$  in a strip at least: in view of (3-24) we can take (with  $C > 0$ )

$$q_{h,z} = 2(2(1 + a_2)\xi + (1 + a_3)z)(\xi^2 + |\eta|^2 + z^2 + C^2 h^2)^{1/2} \chi(\mu),$$

where  $\chi \geq 0$  is supported near  $\mu_0$ ; the corresponding full symbol is

$$\sigma_{\text{full}}(Q_\sigma) = 2(2(1 + a_2)\xi + (1 + a_3)\sigma)(\xi^2 + |\eta|^2 + \sigma^2)^{1/2} \chi(\mu),$$

and  $Q_\sigma$  is taken as a quantization of this full symbol. Here the square root is defined on  $\mathbb{C} \setminus [0, -\infty)$ , with real part of the result being positive, and correspondingly  $q_{h,z}$  is defined away from  $h^{-1}z \in \pm i[C, +\infty)$ . Note that  $\xi^2 + |\eta|^2 + \sigma^2$  is an elliptic symbol in  $(\xi, \eta, \text{Re } \sigma, \text{Im } \sigma)$  as long as  $|\text{Im } \sigma| < C' |\text{Re } \sigma|$ , so the corresponding statement also holds for its square root. While  $q_{h,z}$  is only holomorphic away from  $h^{-1}z \in \pm i[C, +\infty)$ , the full (and indeed the semiclassical and standard principal) symbols are actually holomorphic in cones near infinity, and indeed, for example, via convolutions by the Fourier transform of a compactly supported function can be extended to be holomorphic in  $\mathbb{C}$ , but this is of no importance here.

If  $g_0$  is trapping, we need to add complex absorption inside  $X_0$  as well, at  $\mu = \epsilon_0$ , so we relax the requirement that  $Q_\sigma$  is supported in  $\mu < -\epsilon_0/2$  to support in  $|\mu| > \epsilon_0/2$ , but we require in addition to the other classical requirements that  $p_{\hbar,z} - \iota q_{\hbar,z}$  and its complex conjugate are elliptic near  $\mu = \pm \epsilon_0$ , and which satisfies that  $\pm q_{\hbar,z} \geq 0$  on  $\Sigma_{\hbar,\pm}$ . This can be achieved as above for  $\mu$  near  $\mu_0$ . Again, we extend  $P_\sigma$  and  $Q_\sigma$  to  $X$  in such a way that  $p - \iota q$  and  $p_{\hbar,z} - \iota q_{\hbar,z}$  (and thus their complex conjugates) are elliptic near  $\partial X_{\mu_0}$ .

In either of these semiclassical cases we have arranged that for sufficiently small  $\delta_0 > 0$ ,  $p_{\hbar,z} - \iota q_{\hbar,z}$  and its complex conjugate are *semiclassically nontrapping* for  $|\text{Im } z| < \delta_0$ , namely the bicharacteristics from any point in  $\Sigma_{\hbar} \setminus (L_+ \cup L_-)$  flow to  $\text{ell}(q_{\hbar,z}) \cup L_-$  (i.e., either enter  $\text{ell}(q_{\hbar,z})$  at some finite time, or tend to  $L_-$ ) in the forward direction, and to  $\text{ell}(q_{\hbar,z}) \cup L_+$  in the backward direction. Here



$\delta_0 > 0$  arises from the particularly simple choice of  $q_{\hbar,z}$  for which semiclassical ellipticity is easy to check for  $\text{Im } z > 0$  (bounded away from 0) and small; a more careful analysis would give a specific value of  $\delta_0$ , and a more careful choice of  $q_{\hbar,z}$  would give a better result.

#### 4. Microlocal analysis

**4A. Elliptic and real principal type points.** First, recall the basic elliptic and real principal type regularity results. Let  $\text{WF}^s(u)$  denote the  $H^s$  wave front set of a distribution  $u \in \mathcal{C}^{-\infty}(X)$ , that is,  $\alpha \notin \text{WF}^s(u)$  if there exists  $A \in \Psi^0(X)$  elliptic at  $\alpha$  such that  $Au \in H^s$ . Elliptic regularity states that

$$P_\sigma - \iota Q_\sigma \text{ elliptic at } \alpha, \alpha \notin \text{WF}^{s-2}((P_\sigma - \iota Q_\sigma)u) \Rightarrow \alpha \notin \text{WF}^s(u).$$

In particular, if  $(P_\sigma - \iota Q_\sigma)u \in H^{s-2}$  and  $p - \iota q$  is elliptic at  $\alpha$  then  $\alpha \notin \text{WF}^s(u)$ . Analogous conclusions apply to  $P_\sigma^* + \iota Q_\sigma^*$ ; since both  $p$  and  $q$  are real,  $p - \iota q$  is elliptic if and only if  $p + \iota q$  is.

We also have real principal type propagation, in the usual form valid outside  $\text{supp } q$ :

$$\text{WF}^s(u) \setminus (\text{WF}^{s-1}((P_\sigma - \iota Q_\sigma)u) \cup \text{supp } q)$$

is a union of maximally extended bicharacteristics of  $H_p$  in the characteristic set  $\Sigma = \{p = 0\}$  of  $P_\sigma$ . Putting it differently,

$$\alpha \notin \text{WF}^s(u) \cup \text{WF}^{s-1}((P_\sigma - \iota Q_\sigma)u) \cup \text{supp } q \Rightarrow \tilde{\gamma}(\alpha) \cap \text{WF}^s(u) = \emptyset,$$

where  $\tilde{\gamma}(\alpha)$  is the component of the bicharacteristic  $\gamma(\alpha)$  of  $p$  in the complement of  $\text{WF}^{s-1}((P_\sigma - \iota Q_\sigma)u) \cup \text{supp } q$ . If  $(P_\sigma - \iota Q_\sigma)u \in H^{s-1}$ , then the condition  $\text{WF}^{s-1}((P_\sigma - \iota Q_\sigma)u) = \emptyset$  can be dropped from all statements above; if  $q = 0$  one can thus replace  $\tilde{\gamma}$  by  $\gamma$ .

In general, the result does not hold for nonzero  $q$ . However, it holds in one direction (backward/forward) of propagation along  $H_p$  if  $q$  has the correct sign. Thus, let  $\tilde{\gamma}_\pm(\alpha)$  be a forward (+) or backward (−) bicharacteristic from  $\alpha$ , defined on an interval  $I$ . If  $\pm q \geq 0$  on a neighborhood of  $\tilde{\gamma}_\pm(\alpha)$  (i.e.,  $q \geq 0$  on a neighborhood of  $\tilde{\gamma}_+(\alpha)$ , or  $q \leq 0$  on a neighborhood of  $\tilde{\gamma}_-(\alpha)$ ) then (for the corresponding sign)

$$\alpha \notin \text{WF}^s(u) \text{ and } \text{WF}^{s-1}((P_\sigma - \iota Q_\sigma)u) \cap \tilde{\gamma}_\pm(\alpha) = \emptyset \Rightarrow \tilde{\gamma}_\pm(\alpha) \cap \text{WF}^s(u) = \emptyset,$$

that is, one can propagate regularity forward if  $q \geq 0$ , backward if  $q \leq 0$ . A proof of this claim that is completely analogous to Hörmander’s positive commutator proof in the real principal type setting can easily be given: see [Nonnenmacher and Zworski 2009; Datchev and Vasy 2011] in the semiclassical setting; the changes are minor in the “classical” setting. Note that at points where  $q \neq 0$ , just

$\alpha \notin \text{WF}^{s-1}((P_\sigma - iQ_\sigma)u)$  implies  $\alpha \notin \text{WF}^{s+1}(u)$  (stronger than stated above), but at points with  $q = 0$  such an elliptic estimate is unavailable (unless  $P_\sigma$  is elliptic).

As  $P_\sigma^* + iQ_\sigma^*$  has symbol  $p + iq$ , one can propagate regularity in the opposite direction as compared to  $P_\sigma - iQ_\sigma$ . Thus, if  $\mp q \geq 0$  on a neighborhood of  $\tilde{\gamma}_\pm(\alpha)$  (i.e.,  $q \leq 0$  on a neighborhood of  $\tilde{\gamma}_+(\alpha)$ , or  $q \geq 0$  on a neighborhood of  $\tilde{\gamma}_-(\alpha)$ ) then (for the corresponding sign)

$$\alpha \notin \text{WF}^s(u) \text{ and } \text{WF}^{s-1}((P_\sigma^* + iQ_\sigma^*)u) \cap \tilde{\gamma}_\pm(\alpha) = \emptyset \Rightarrow \tilde{\gamma}_\pm(\alpha) \cap \text{WF}^s(u) = \emptyset.$$

**4B. Analysis near  $\Lambda_\pm$ .** The last ingredient in the classical setting is an analogue of Melrose’s regularity result at radial sets which have the same features as ours. Although it is not stated in this generality in Melrose’s paper [1994], the proof is easily adapted. Thus, the results are:

At  $\Lambda_\pm$ , for  $s \geq m > \frac{1}{2} - \text{Im } \sigma$ , we can propagate estimates *away* from  $\Lambda_\pm$ :

**Proposition 4.1.** *Suppose  $s \geq m > \frac{1}{2} - \text{Im } \sigma$ , and  $\text{WF}^m(u) \cap \Lambda_\pm = \emptyset$ . Then*

$$\Lambda_\pm \cap \text{WF}^{s-1}(P_\sigma u) = \emptyset \Rightarrow \Lambda_\pm \cap \text{WF}^s(u) = \emptyset.$$

This is completely analogous to Melrose’s estimates in asymptotically Euclidean scattering theory at the radial sets [Melrose 1994, Section 9]. Note that the  $H^s$  regularity of  $u$  at  $\Lambda_\pm$  is “free” in the sense that we do not need to impose  $H^s$  assumptions on  $u$  anywhere; merely  $H^m$  at  $\Lambda_\pm$  does the job; of course, on  $P_\sigma u$  one must make the  $H^{s-1}$  assumption, that is, the loss of one derivative compared to the elliptic setting. At the cost of changing regularity, one can propagate estimate *towards*  $\Lambda_\pm$ . Keeping in mind that taking  $P_\sigma^*$  in place of  $P_\sigma$ , principal symbol of  $\frac{1}{2i}(P_\sigma - P_\sigma^*)$  switches sign, we have the following:

**Proposition 4.2.** *For  $s < \frac{1}{2} + \text{Im } \sigma$ , and  $O$  a neighborhood of  $\Lambda_\pm$ ,*

$$\text{WF}^s(u) \cap (O \setminus \Lambda_\pm) = \emptyset, \text{WF}^{s-1}(P_\sigma^* u) \cap \Lambda_\pm = \emptyset \Rightarrow \text{WF}^s(u) \cap \Lambda_\pm = \emptyset.$$

*Proof of Propositions 4.1 and 4.2.* The proof is a positive commutator estimate. Consider commutants  $C_\epsilon^* C_\epsilon$  with  $C_\epsilon \in \Psi^{s-1/2-\delta}(X)$  for  $\epsilon > 0$ , uniformly bounded in  $\Psi^{s-1/2}(X)$  as  $\epsilon \rightarrow 0$ ; with the  $\epsilon$ -dependence used to regularize the argument. More precisely, let

$$c = \phi(\rho_0)\tilde{\rho}^{-s+1/2}, \quad c_\epsilon = c(1 + \epsilon\tilde{\rho}^{-1})^{-\delta},$$

where  $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$  is identically 1 near 0,  $\phi' \leq 0$  and  $\phi$  is supported sufficiently close to 0 so that

$$\rho_0 \in \text{supp } d\phi \Rightarrow \pm \tilde{\rho} H_\rho \rho_0 > 0; \tag{4-1}$$

such  $\phi$  exists by (3-21). To avoid using the sharp Gårding inequality, we choose  $\phi$  so that  $\sqrt{-\phi\phi'}$  is  $\mathcal{C}^\infty$ . Note that the sign of  $H_p\tilde{\rho}^{-s+1/2}$  depends on the sign of  $-s + 1/2$  which explains the difference between  $s > 1/2$  and  $s < 1/2$  in Propositions 4.1–4.2 when there are no other contributions to the threshold value of  $s$ . The contribution of the principal symbol of  $\frac{1}{2l}(P_\sigma - P_\sigma^*)$ , however, shifts the critical value  $1/2$ .

Now let  $C \in \Psi^{s-1/2}(X)$  have principal symbol  $c$ , and have  $\text{WF}'(C) \subset \text{supp } \phi \circ \rho_0$ , and let  $C_\epsilon = CS_\epsilon$ ,  $S_\epsilon \in \Psi^{-\delta}(X)$  uniformly bounded in  $\Psi^0(X)$  for  $\epsilon > 0$ , converging to  $\text{Id}$  in  $\Psi^{\delta'}(X)$  for  $\delta' > 0$  as  $\epsilon \rightarrow 0$ , with principal symbol  $(1 + \epsilon\tilde{\rho}^{-1})^{-\delta}$ . Thus, the principal symbol of  $C_\epsilon$  is  $c_\epsilon$ .

First, consider Proposition 4.1. Then

$$\begin{aligned} \sigma_{2s}(i(P_\sigma^*C_\epsilon^*C_\epsilon - C_\epsilon^*C_\epsilon P_\sigma)) &= \sigma_1(i(P_\sigma^* - P_\sigma))c_\epsilon^2 + 2c_\epsilon H_p c_\epsilon \tag{4-2} \\ &= \pm 8\left(-\text{Im } \sigma \phi + \left(-s + \frac{1}{2}\right)\phi \pm \frac{1}{4}(\tilde{\rho}H_p\rho_0)\phi' + \delta\frac{\epsilon}{\tilde{\rho} + \epsilon}\phi\right)\phi\tilde{\rho}^{-2s}(1 + \epsilon\tilde{\rho}^{-1})^{-2\delta}, \end{aligned}$$

so

$$\begin{aligned} \pm\sigma_{2s}(i(P_\sigma^*C_\epsilon^*C_\epsilon - C_\epsilon^*C_\epsilon P_\sigma)) &\leq -8\left(s - \frac{1}{2} + \text{Im } \sigma - \delta\right)\tilde{\rho}^{-2s}(1 + \epsilon\tilde{\rho}^{-1})^{-2\delta}\phi^2 \\ &\quad + 2(\pm\tilde{\rho}H_p\rho_0)\tilde{\rho}^{-2s}(1 + \epsilon\tilde{\rho}^{-1})^{-2\delta}\phi'\phi. \tag{4-3} \end{aligned}$$

Here the first term on the right-hand side is negative if  $s - 1/2 + \text{Im } \sigma - \delta > 0$  and this is the same sign as that of  $\phi'$  term; the presence of  $\delta$  (needed for the regularization) is the reason for the appearance of  $m$  in the estimate. Thus,

$$\pm i(P_\sigma^*C_\epsilon^*C_\epsilon - C_\epsilon^*C_\epsilon P_\sigma) = -S_\epsilon^*(B^*B + B_1^*B_1 + B_{2,\epsilon}^*B_{2,\epsilon})S_\epsilon + F_\epsilon,$$

with  $B, B_1, B_{2,\epsilon} \in \Psi^s(X)$ ,  $B_{2,\epsilon}$  uniformly bounded in  $\Psi^s(X)$  as  $\epsilon \rightarrow 0$ ,  $F_\epsilon$  uniformly bounded in  $\Psi^{2s-1}(X)$ , and  $\sigma_s(B)$  an elliptic multiple of  $\phi(\rho_0)\tilde{\rho}^{-s}$ . Computing the pairing, using an extra regularization (insert a regularizer  $\Lambda_r \in \Psi^{-1}(X)$ , uniformly bounded in  $\Psi^0(X)$ , converging to  $\text{Id}$  in  $\Psi^\delta(X)$  to justify integration by parts, and use that  $[\Lambda_r, P_\sigma^*]$  is uniformly bounded in  $\Psi^1(X)$ , converging to 0 strongly, cf. [Vasy 2000, Lemma 17.1] and its use in [Vasy 2000, Lemma 17.2]) yields

$$\langle i(P_\sigma^*C_\epsilon^*C_\epsilon - C_\epsilon^*C_\epsilon P_\sigma)u, u \rangle = \langle iC_\epsilon^*C_\epsilon u, P_\sigma u \rangle - \langle iP_\sigma u, C_\epsilon^*C_\epsilon u \rangle.$$

Using Cauchy–Schwartz on the right-hand side, a standard functional analytic argument (see, for instance, Melrose [1994, Proof of Proposition 7 and Section 9]) gives an estimate for  $Bu$ , showing  $u$  is in  $H^s$  on the elliptic set of  $B$ , provided  $u$  is microlocally in  $H^{s-\delta}$ . A standard inductive argument, starting with  $s - \delta = m$  and improving regularity by  $\leq \frac{1}{2}$  in each step proves Proposition 4.1.

For Proposition 4.2, when applied to  $P_\sigma$  in place of  $P_\sigma^*$  (so the assumption is  $s < (1 - \text{Im } \sigma)/2$ ), the argument is similar, but we want to change the overall

sign of the terms in (4-2) corresponding to the first term on the right-hand side of (4-3), that is, we want it to be positive. This is satisfied if  $s - 1/2 + \text{Im } \sigma < 0$  since the regularizer now contributes the correct sign. On the other hand,  $\phi'$  now has the wrong sign, so one needs to make an assumption on  $\text{supp } d\phi$ ; one can arrange that this is in  $O \setminus \Lambda$  by making  $\phi$  have sufficiently small support, but identically 1 near 0. Since the details are standard — see [Melrose 1994, Section 9] — we leave them to the reader. When interchanging  $P_\sigma$  and  $P_\sigma^*$ , we need to take into account the switch of the sign of the principal symbol of  $\frac{1}{2i}(P_\sigma - P_\sigma^*)$ , which causes the sign change in front of  $\text{Im } \sigma$  in the statement of the proposition.  $\square$

**4C. Global estimates.** For our Fredholm results, we actually need estimates. However, these can be easily obtained from regularity results as in, for example, [Hörmander 1985b, Proof of Theorem 26.1.7] by the closed graph theorem. It should be noted that of course one really proved versions of the relevant estimates when proving regularity, but the closed graph theorem provides a particularly simple way of combining these (though it comes at the cost of using a theorem which in principle is unnecessary).

So suppose  $s \geq m > \frac{1}{2} - \text{Im } \sigma$ ,  $u \in H^m$  and  $(P_\sigma - \iota Q_\sigma)u \in H^{s-1}$ . The above results give that, first,  $\text{WF}^s(u)$  (indeed,  $\text{WF}^{s+1}(u)$ ) is disjoint from the elliptic set of  $P_\sigma - \iota Q_\sigma$ . Next  $\Lambda_\pm$  is disjoint from  $\text{WF}^s(u)$ , hence so is a neighborhood of  $\Lambda_\pm$  as the complement of the wave front set is open. Thus by propagation of singularities and (3-32), taking into account the sign of  $q$  along  $\Sigma_\pm$ ,  $\text{WF}^s(u) \cap \Sigma_\pm = \emptyset$ . Now, by the regularity result, the inclusion map

$$\mathcal{L}_s = \{u \in H^m : (P_\sigma - \iota Q_\sigma)u \in H^{s-1}\} \rightarrow H^m,$$

in fact maps to  $H^s$ .

Note that  $\mathcal{L}_s$  is complete with the norm  $\|u\|_{\mathcal{L}_s}^2 = \|u\|_{H^m}^2 + \|(P_\sigma - \iota Q_\sigma)u\|_{H^{s-1}}^2$ . Indeed, if  $\{u_j\}_{j=1}^\infty$  is Cauchy in  $\mathcal{L}_s$  we have  $u_j \rightarrow u$  in  $H^m$  and

$$(P_\sigma - \iota Q_\sigma)u_j \rightarrow v \in H^{s-1}.$$

By the first convergence,  $(P_\sigma - \iota Q_\sigma)u_j \rightarrow (P_\sigma - \iota Q_\sigma)u$  in  $H^{m-2}$ , thus, as  $s - 1 \geq m - 2$ ,  $(P_\sigma - \iota Q_\sigma)u_j \rightarrow v$  in  $H^{m-2}$  shows  $(P_\sigma - \iota Q_\sigma)u = v \in H^{s-1}$ , and thus,  $(P_\sigma - \iota Q_\sigma)u_j \rightarrow (P_\sigma - \iota Q_\sigma)u$  in  $H^{s-1}$ , so  $u_j \rightarrow u$  in  $\mathcal{L}_s$ .

The graph of the inclusion map, considered as a subset of  $\mathcal{L}_s \times H^s$  is closed, for  $(u_j, u_j) \rightarrow (u, v) \in \mathcal{L}_s \times H^s$  implies in particular  $u_j \rightarrow u$  and  $u_j \rightarrow v$  in  $H^m$ , so  $u = v \in \mathcal{L}_s \cap H^s$ . Correspondingly, by the closed graph theorem, the inclusion map is continuous, that is,

$$\|u\|_{H^s} \leq C(\|(P_\sigma - \iota Q_\sigma)u\|_{H^{s-1}} + \|u\|_{H^m}), \quad u \in \mathcal{L}_s. \tag{4-4}$$

This estimate implies that  $\text{Ker}(P_\sigma - \iota Q_\sigma)$  in  $H^s$  is finite dimensional since elements of this kernel lie in  $\mathcal{L}_s$ , and since on the unit ball of this closed subspace of  $H^s$  (for  $P_\sigma - \iota Q_\sigma : H^s \rightarrow H^{s-2}$  is continuous),  $\|u\|_{H^s} \leq C\|u\|_{H^m}$ , and the inclusion  $H^s \rightarrow H^m$  is compact. Further, elements of  $\text{Ker}(P_\sigma - \iota Q_\sigma)$  are in  $\mathcal{C}^\infty(X)$  by our regularity result, and thus this space is independent of the choice of  $s$ .

On the other hand, for the adjoint operator  $P_\sigma^* + \iota Q_\sigma^*$ , if  $s' < \frac{1}{2} + \text{Im } \sigma$  (recall that replacing  $P_\sigma$  by its adjoint switches the sign of the principal symbol of  $\frac{1}{2i}(P_\sigma - P_\sigma^*)$ ),  $u \in H^{-N}$  and  $(P_\sigma^* + \iota Q_\sigma^*)u \in H^{s'-1}$  then first  $\text{WF}^{s'}(u)$  (indeed,  $\text{WF}^{s'+1}(u)$ ) is disjoint from the elliptic set of  $P_\sigma^* + \iota Q_\sigma^*$ . Next, by propagation of singularities and (3-32), taking into account the sign of  $q$  along  $\Sigma_\pm$ , namely the sign of the imaginary part of the principal symbol switched by taking the adjoints,  $\text{WF}^{s'}(u) \cap (\Sigma_\pm \setminus \Lambda_\pm) = \emptyset$ . Finally, by the result at the radial points  $\Lambda_\pm$  is disjoint from  $\text{WF}^{s'}(u)$ . Thus, the inclusion map

$$\mathcal{W}_{s'} = \{u \in H^{-N} : (P_\sigma^* + \iota Q_\sigma^*)u \in H^{s'-1}\} \rightarrow H^{-N},$$

in fact maps to  $H^{s'}$ . We deduce, as above, by the closed graph theorem, that

$$\|u\|_{H^{s'}} \leq C(\|(P_\sigma^* + \iota Q_\sigma^*)u\|_{H^{s'-1}} + \|u\|_{H^{-N}}), \quad u \in \mathcal{W}_{s'}. \quad (4-5)$$

As above, this estimate implies that  $\text{Ker}(P_\sigma^* + \iota Q_\sigma^*)$  in  $H^{s'}$  is finite dimensional. Indeed, by our regularity results (elliptic regularity, propagation of singularities, and then regularity at the radial set) elements of  $\text{Ker}(P_\sigma^* + \iota Q_\sigma^*)$  have wave front set in  $\Lambda_+ \cup \Lambda_-$  and lie in  $\cap_{s' < 1/2 + \text{Im } \sigma} H^{s'}$ .

The dual of  $H^s$  for  $s > \frac{1}{2} - \text{Im } \sigma$ , is  $H^{-s} = H^{s'-1}$ ,  $s' = 1 - s$ , so  $s' < \frac{1}{2} + \text{Im } \sigma$  in this case, while the dual of  $H^{s-1}$ ,  $s > \frac{1}{2} - \text{Im } \sigma$ , is  $H^{1-s} = H^{s'}$ , with  $s' = 1 - s < \frac{1}{2} + \text{Im } \sigma$  again. Thus, the spaces (apart from the residual spaces  $H^m$  and  $H^{-N}$ , into which the inclusion is compact) in the left, resp. right, side of (4-5), are exactly the duals of those on the right, resp. left, side of (4-4). Thus, by a standard functional analytic argument [Hörmander 1985b, Proof of Theorem 26.1.7], namely dualization and using the compactness of the inclusion  $H^{s'} \rightarrow H^{-N}$  for  $s' > -N$ , (4-5) gives the  $H^s$ -solvability,  $s = 1 - s'$  (i.e., we demand  $u \in H^s$ ), of

$$(P_\sigma - \iota Q_\sigma)u = f, \quad s > \frac{1}{2} - \text{Im } \sigma,$$

for  $f$  in the annihilator (in  $H^{s-1} = (H^{s'})^*$  with duality induced by the  $L^2$  inner product) of the finite dimensional subspace  $\text{Ker}(P_\sigma^* + \iota Q_\sigma^*)$  of  $H^{s'}$ .

Recall from [Hörmander 1985b, Proof of Theorem 26.1.7] that this argument has two parts: first for any complementary subspace  $V$  of  $\text{Ker}(P_\sigma^* + \iota Q_\sigma^*)$  in  $H^{s'}$  (i.e.,  $V$  is closed,  $V \cap \text{Ker}(P_\sigma^* + \iota Q_\sigma^*) = \{0\}$ , and  $V + \text{Ker}(P_\sigma^* + \iota Q_\sigma^*) = H^{s'}$ ,

for example,  $V$  is the  $H^{s'}$  orthocomplement of  $\text{Ker}(P_\sigma^* + \iota Q_\sigma^*)$ , one can drop  $\|u\|_{H^{-N}}$  from the right-hand side of (4-5) when  $u \in V \cap \mathcal{W}_{s'}$  at the cost of replacing  $C$  by a larger constant  $C'$ . Indeed, if no  $C'$  existed, one would have a sequence  $u_j \in V \cap \mathcal{W}_{s'}$  such that  $\|u_j\|_{H^{s'}} = 1$  and  $\|(P_\sigma^* + \iota Q_\sigma^*)u_j\|_{H^{s'-1}} \rightarrow 0$ , so  $(P_\sigma^* + \iota Q_\sigma^*)u_j \rightarrow 0$  in  $H^{s'-1}$ . By weak compactness of the  $H^{s'}$  unit ball, there is a weakly convergent subsequence  $u_{j_\ell}$  converging to some  $u \in H^{s'}$ , by the closedness (which implies weak closedness) of  $V$ ,  $u \in V$ , so

$$(P_\sigma^* + \iota Q_\sigma^*)u_{j_\ell} \rightarrow (P_\sigma^* + \iota Q_\sigma^*)u$$

weakly in  $H^{s'-2}$ , and thus  $(P_\sigma^* + \iota Q_\sigma^*)u = 0$  so  $u \in V \cap \text{Ker}(P_\sigma^* + \iota Q_\sigma^*) = \{0\}$ . On the other hand, by compactness of the inclusion  $H^{s'} \rightarrow H^{-N}$ ,  $u_{j_\ell} \rightarrow u$  strongly in  $H^{-N}$ , so  $\{u_{j_\ell}\}$  is Cauchy in  $H^{-N}$ , hence from (4-5), it is Cauchy in  $H^{s'}$ , so it converges to  $u$  strongly in  $H^{s'}$  and hence  $\|u\|_{H^{s'}} = 1$ . This contradicts  $u = 0$ , completing the proof of

$$\|u\|_{H^{s'}} \leq C' \|(P_\sigma^* + \iota Q_\sigma^*)u\|_{H^{s'-1}}, \quad u \in V \cap \mathcal{W}_{s'}. \quad (4-6)$$

Thus, with  $s' = 1 - s$ , and for  $f$  in the annihilator (in  $H^{s-1}$ , via the  $L^2$ -pairing) of  $\text{Ker}(P_\sigma^* + \iota Q_\sigma^*) \subset H^{s'}$ , and for  $v \in V \cap \mathcal{W}_{s'}$ ,

$$|\langle f, v \rangle| \leq \|f\|_{H^{s-1}} \|v\|_{H^{s'}} \leq C' \|f\|_{H^{s-1}} \|(P_\sigma^* + \iota Q_\sigma^*)v\|_{H^{s'-1}}.$$

As adding an element of  $\text{Ker}(P_\sigma^* + \iota Q_\sigma^*)$  to  $v$  does not change either side, the inequality holds for all  $v \in \mathcal{W}_{s'} \subset H^{s'}$ . Thus, the conjugate-linear map  $(P_\sigma^* + \iota Q_\sigma^*)v \mapsto \langle f, v \rangle$ ,  $v \in \mathcal{W}_{s'}$ , which is well-defined, is continuous from  $\text{Ran}_{\mathcal{W}_{s'}}(P_\sigma^* + \iota Q_\sigma^*) \subset H^{s'-1}$  to  $\mathbb{C}$ , and by the Hahn–Banach theorem can be extended to a continuous conjugate linear functional  $\ell$  on  $H^{s'-1} = (H^s)^*$ , so there exists  $u \in H^s$  such that  $\langle u, \phi \rangle = \ell(\phi)$  for  $\phi \in H^{s'-1}$ . In particular, when  $\phi = (P_\sigma^* + \iota Q_\sigma^*)\psi$ ,  $\psi \in \mathcal{C}^\infty(X) \subset \mathcal{W}_{s'}$ ,

$$\langle u, (P_\sigma^* + \iota Q_\sigma^*)\psi \rangle = \ell(\phi) = \langle f, \psi \rangle,$$

so  $(P_\sigma - \iota Q_\sigma)u = f$  as claimed.

In order to set up Fredholm theory, let  $\tilde{P}$  be any operator with principal symbol  $p - \iota q$ ; for example,  $\tilde{P}$  is  $P_{\sigma_0} - \iota Q_{\sigma_0}$  for some  $\sigma_0$ . Then consider

$$\mathfrak{X}^s = \{u \in H^s : \tilde{P}u \in H^{s-1}\}, \quad \mathfrak{Y}^s = H^{s-1}, \quad (4-7)$$

with

$$\|u\|_{\mathfrak{X}^s}^2 = \|u\|_{H^s}^2 + \|\tilde{P}u\|_{H^{s-1}}^2.$$

Note that the  $\mathfrak{L}_s$ -norm is equivalent to the  $\mathfrak{X}^s$ -norm, and  $\mathfrak{L}_s = \mathfrak{X}^s$ , by (4-4) and the preceding discussion. Note that  $\mathfrak{X}^s$  only depends on the principal symbol of  $\tilde{P}$ . Moreover,  $\mathcal{C}^\infty(X)$  is dense in  $\mathfrak{X}^s$ ; this follows by considering  $R_\epsilon \in \Psi^{-\infty}(X)$ ,

$\epsilon > 0$ , such that  $R_\epsilon \rightarrow \text{Id}$  in  $\Psi^\delta(X)$  for  $\delta > 0$ ,  $R_\epsilon$  uniformly bounded in  $\Psi^0(X)$ ; thus  $R_\epsilon \rightarrow \text{Id}$  strongly (but not in the operator norm topology) on  $H^s$  and  $H^{s-1}$ . Then for  $u \in \mathcal{X}^s$ ,  $R_\epsilon u \in \mathcal{C}^\infty(X)$  for  $\epsilon > 0$ ,  $R_\epsilon u \rightarrow u$  in  $H^s$  and  $\tilde{P}R_\epsilon u = R_\epsilon \tilde{P}u + [\tilde{P}, R_\epsilon]u$ , so the first term on the right converges to  $\tilde{P}u$  in  $H^{s-1}$ , while  $[\tilde{P}, R_\epsilon]$  is uniformly bounded in  $\Psi^1(X)$ , converging to 0 in  $\Psi^{1+\delta}(X)$  for  $\delta > 0$ , so converging to 0 strongly as a map  $H^s \rightarrow H^{s-1}$ . Thus,  $[\tilde{P}, R_\epsilon]u \rightarrow 0$  in  $H^{s-1}$ , and we conclude that  $R_\epsilon u \rightarrow u$  in  $\mathcal{X}^s$ . (In fact,  $\mathcal{X}^s$  is a first-order coisotropic space, more general function spaces of this nature are discussed in [Melrose et al. 2009, Appendix A].)

With these preliminaries,

$$P_\sigma - \iota Q_\sigma : \mathcal{X}^s \rightarrow \mathcal{Y}^s$$

is bounded for each  $\sigma$  with  $s \geq m > \frac{1}{2} - \text{Im } \sigma$ , and is an analytic family of bounded operators in this half-plane of  $\sigma$ 's. Further, it is Fredholm for each  $\sigma$ : the kernel in  $\mathcal{X}^s$  is finite dimensional, and it surjects onto the annihilator in  $H^{s-1}$  of the (finite dimensional) kernel of  $P_\sigma^* + \iota Q_\sigma^*$  in  $H^{1-s}$ , which thus has finite codimension, and is closed, since for  $f$  in this space there exists  $u \in H^s$  with  $(P_\sigma - \iota Q_\sigma)u = f$ , and thus  $u \in \mathcal{X}^s$ . Restating this as a theorem:

**Theorem 4.3.** *Let  $P_\sigma, Q_\sigma$  be as above, and  $\mathcal{X}^s, \mathcal{Y}^s$  as in (4-7). Then*

$$P_\sigma - \iota Q_\sigma : \mathcal{X}^s \rightarrow \mathcal{Y}^s$$

*is an analytic family of Fredholm operators on*

$$\mathbb{C}_s = \{\sigma \in \mathbb{C} : \text{Im } \sigma > \frac{1}{2} - s\}. \tag{4-8}$$

Thus, analytic Fredholm theory applies, giving meromorphy of the inverse provided the inverse exists for a particular value of  $\sigma$ .

**Remark 4.4.** Note that the Fredholm property means that  $P_\sigma^* + \iota Q_\sigma^*$  is also Fredholm on the dual spaces; this can also be seen directly from the estimates. The analogue of this remark also applies to the semiclassical discussion below.

**4D. Semiclassical estimates.** There are semiclassical estimates completely analogous to those in the classical setting; we again phrase these as wave front set statements. Let  $H_h^s$  denote the semiclassical Sobolev space of order  $s$ , that is, as a function space this is the space of functions  $(u_h)_{h \in I}$ ,  $I \subset (0, 1]_h$  with values in the standard Sobolev space  $H^s$ , with  $A_h u_h$  bounded in  $L^2$  for an elliptic, semiclassically elliptic, operator  $A_h \in \Psi_h^s(X)$ . (Note that  $u_h$  need not be defined for all  $h \in (0, 1]$ ; we suppress  $I$  from the notation.) Let

$$\text{WF}_h^{s,r}(u) \subset \partial(\bar{T}^* X \times [0, 1)_h) = S^* X \times [0, 1)_h \cup T^* X \times \{0\}_h$$

denote the semiclassical wave front set of a polynomially bounded family of distributions, that is,  $u = (u_h)_{h \in I}$ ,  $I \subset (0, 1]$ , satisfying  $u_h$  is uniformly bounded in  $h^{-N} H_h^{-N}$  for some  $N$ . This is defined as follows: we say that  $\alpha \notin \text{WF}_h^{s,r}(u)$  if there exists  $A \in \Psi_h^0(X)$  elliptic at  $\alpha$  such that  $Au \in h^r H_h^s$ . Note that, in view of the description of the symbols in Section 2, ellipticity at  $\alpha$  means the ellipticity of  $\sigma_h(A_h)$  if  $\alpha \in T^*X \times \{0\}$ , that of  $\sigma_0(A_h)$  if  $\alpha \in S^*X \times (0, 1)$ , and that of either (and thus both, in view of the compatibility of these symbols) of these when  $\alpha \in S^*X \times \{0\}$ . The semiclassical wave front set captures global estimates: if  $u$  is polynomially bounded and  $\text{WF}_h^{s,r}(u) = \emptyset$ , then  $u \in h^r H_h^s$ .

Elliptic regularity states that

$$P_{h,z} - \iota Q_{h,z} \text{ elliptic at } \alpha, \alpha \notin \text{WF}_h^{s-2,0}((P_{h,z} - \iota Q_{h,z})u) \Rightarrow \alpha \notin \text{WF}_h^{s,0}(u).$$

Thus,  $(P_{h,z} - \iota Q_{h,z}) \in H_h^{s-2}$  and  $p_{h,z} - \iota q_{h,z}$  is elliptic at  $\alpha$  then  $\alpha \notin \text{WF}_h^s(u)$ .

We also have real principal type propagation:

$$\text{WF}_h^{s,-1}(u) \setminus (\text{WF}_h^{s-1,0}((P_{h,z} - \iota Q_{h,z})u) \cup \text{supp } q_{h,z})$$

is a union of maximally extended bicharacteristics of  $H_p$  in the characteristic set  $\Sigma_{h,z} = \{p_{h,z} = 0\}$  of  $P_{h,z}$ . Put differently,

$$\begin{aligned} \alpha \notin \text{WF}_h^{s,-1}(u) \cup \text{WF}_h^{s-1,0}((P_{h,z} - \iota Q_{h,z})u) \cup \text{supp } q_{h,z} \\ \Rightarrow \tilde{\gamma}(\alpha) \cap \text{WF}_h^{s,-1}(u) = \emptyset, \end{aligned}$$

where  $\tilde{\gamma}(\alpha)$  is the component of the bicharacteristic  $\gamma(\alpha)$  of  $p_{h,z}$  in the complement of  $\text{WF}_h^{s-1,0}((P_{h,z} - \iota Q_{h,z})u) \cup \text{supp } q_{h,z}$ . If  $(P_{h,z} - \iota Q_{h,z})u \in H^{s-1}$ , then  $\text{WF}_h^{s-1,0}((P_{h,z} - \iota Q_{h,z})u) = \emptyset$  can be dropped from all statements above; if  $q_{h,z} = 0$  one can thus replace  $\tilde{\gamma}$  by  $\gamma$ .

In general, the result does not hold for nonzero  $q_{h,z}$ . However, it holds in one direction (backward/forward) of propagation along  $H_{p_{h,z}}$  if  $q_{h,z}$  has the correct sign. Thus, with  $\tilde{\gamma}_\pm(\alpha)$  a forward (+) or backward (−) bicharacteristic from  $\alpha$  defined on an interval, if  $\pm q_{h,z} \geq 0$  on a neighborhood of  $\tilde{\gamma}_\pm(\alpha)$  then

$$\begin{aligned} \alpha \notin \text{WF}_h^{s,-1}(u) \text{ and } \tilde{\gamma}_\pm(\alpha) \cap \text{WF}_h^{s-1,0}((P_{h,z} - \iota Q_{h,z})u) = \emptyset \\ \Rightarrow \tilde{\gamma}_\pm(\alpha) \cap \text{WF}_h^{s,-1}(u) = \emptyset, \end{aligned}$$

that is, one can propagate regularity forward if  $q_{h,z} \geq 0$ , backward if  $q_{h,z} \leq 0$ ; see [Nonnenmacher and Zworski 2009; Datchev and Vasy 2011]. Again, for  $P_{h,z}^* + \iota Q_{h,z}^*$  the directions are reversed, that is, one can propagate regularity forward if  $q_{h,z} \leq 0$ , backward if  $q_{h,z} \geq 0$ .



A semiclassical version of Melrose’s regularity result was proved in [Vasy and Zworski 2000] in the asymptotically Euclidean setting. We need a more general result, which is an easy adaptation:

**Proposition 4.5.** *Suppose  $s \geq m > \frac{1}{2} - \text{Im } \sigma$ , and  $\text{WF}_h^{m,-N}(u) \cap L_\pm = \emptyset$  for some  $N$ . Then*

$$L_\pm \cap \text{WF}_h^{s-1,0}(P_{h,z}u) = \emptyset \Rightarrow L_\pm \cap \text{WF}_h^{s,-1}(u) = \emptyset.$$

Again, at the cost of changing regularity, one can propagate estimate *towards*  $L_\pm$ .

**Proposition 4.6.** *For  $s < \frac{1}{2} + \text{Im } \sigma$  and  $O$  a neighborhood of  $L_\pm$ ,*

$$\begin{aligned} \text{WF}_h^{s,-1}(u) \cap (O \setminus L_\pm) = \emptyset, \quad \text{WF}_h^{s-1,0}(P_{h,z}^*u) \cap L_\pm = \emptyset \\ \Rightarrow \text{WF}_h^{s,-1}(u) \cap L_\pm = \emptyset. \end{aligned}$$

*Proof.* We just need to localize in  $\tilde{\rho}$  in addition to  $\rho_0$ ; such a localization in the classical setting is implied by working on  $S^*X$  or with homogeneous symbols. We achieve this by modifying the localizer  $\phi$  in the commutant constructed in the proof of Propositions 4.1 and 4.2. As already remarked, the proof is much like at radial points in semiclassical scattering on asymptotically Euclidean spaces, studied in [Vasy and Zworski 2000], but we need to be more careful about localization in  $\rho_0$  and  $\tilde{\rho}$  as we are assuming less about the structure.

First, note that  $L_\pm$  is defined by  $\tilde{\rho} = 0, \rho_0 = 0$ , so  $\tilde{\rho}^2 + \rho_0$  is a quadratic defining function of  $L_\pm$ . Thus, let  $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$  be identically 1 near 0,  $\phi' \leq 0$  and  $\phi$  supported sufficiently close to 0 so that

$$\tilde{\rho}^2 + \rho_0 \in \text{supp } d\phi \Rightarrow \pm \tilde{\rho}(H_p \rho_0 + 2\tilde{\rho}H_p \tilde{\rho}) > 0$$

and

$$\tilde{\rho}^2 + \rho_0 \in \text{supp } \phi \Rightarrow \pm \tilde{\rho}H_p \tilde{\rho} > 0.$$

Such a  $\phi$  exists by (3-19) and (3-21) as

$$\pm \tilde{\rho}(H_p \rho_0 + 2\tilde{\rho}H_p \tilde{\rho}) \geq 8\rho_0 + 8\tilde{\rho}^2 - \mathcal{O}((\tilde{\rho}^2 + \rho_0)^{3/2}).$$

Then let  $c$  be given by

$$c = \phi(\rho_0 + \tilde{\rho}^2)\tilde{\rho}^{-s+1/2}, \quad c_\epsilon = c(1 + \epsilon\tilde{\rho}^{-1})^{-\delta}.$$

The rest of the proof proceeds exactly as for Propositions 4.1 and 4.2, except one writes  $P_{h,z} = P_{h,z,\text{Re}} + \iota P_{h,z,\text{Im}}$  with the two summands being symmetric, resp. antisymmetric, thus with principal symbol  $\text{Re } p_{h,z}$  and  $\iota \text{Im } p_{h,z}$ , one computes the commutator with  $P_{h,z,\text{Re}}$  which involves  $\text{Re } p_{h,z}$ , obtains an extra term from

$P_{h,z,\text{Im}}$  which has the correct sign as in the case of complex absorption in view of (3-27); see [Datchev and Vasy 2011, Lemma 5.1] for its treatment.  $\square$

Suppose now that  $p_{h,z}$  is semiclassically nontrapping, as discussed at the end of Section 3. Suppose again that  $s \geq m > \frac{1}{2} - \text{Im } \sigma$ ,  $h^N u_h$  is bounded in  $H_{\hbar}^m$  and  $(P_{h,z} - \iota Q_{h,z})u_h \in H_{\hbar}^{s-1}$ . The above results give that, first,  $\text{WF}_{\hbar}^{s,-1}(u)$  (indeed,  $\text{WF}_{\hbar}^{s+1,0}(u)$ ) is disjoint from the elliptic set of  $P_{h,z} - \iota Q_{h,z}$ . Next we see that  $L_{\pm}$  is disjoint from  $\text{WF}_{\hbar}^{s,-1}(u)$ , hence so is a neighborhood of  $L_{\pm}$ . Thus by propagation of singularities and the semiclassically nontrapping property, taking into account the sign of  $q$  along  $\Sigma_{\hbar,\pm}$ ,  $\text{WF}_{\hbar}^{s,-1}(u) \cap \Sigma_{\hbar,\pm} = \emptyset$ . In summary,  $\text{WF}_{\hbar}^{s,-1}(u) = \emptyset$ , i.e.,  $hu_h$  is bounded in  $H_{\hbar}^s$ , i.e.,

$$h^N u_h \text{ bounded in } H_{\hbar}^m, (P_{h,z} - \iota Q_{h,z})u_h \in H_{\hbar}^{s-1} \Rightarrow hu_h \in H_{\hbar}^s. \quad (4-9)$$

Now suppose that for a decreasing sequence  $h_j \rightarrow 0$ ,  $w_h \in \text{Ker}(P_{h,z} - \iota Q_{h,z})$  and  $\|w_h\|_{H_{\hbar}^s} = 1$ . Then for any  $N$ ,  $u_h = h^{-N} w_h$  satisfies the above hypotheses, and we deduce that  $hu_h$  is uniformly bounded in  $H_{\hbar}^s$ , that is,  $h^{-N+1} w_h$  is uniformly bounded in  $H_{\hbar}^s$ . But for  $N > 1$  this contradicts that  $\|w_h\|_{H_{\hbar}^s} = 1$ , so such a sequence  $h_j$  does not exist. Therefore  $\text{Ker}(P_{h,z} - \iota Q_{h,z}) = \{0\}$  for sufficiently small  $h$ .

Using semiclassical propagation of singularities in the reverse direction, much as we did in the previous section, we deduce that  $\text{Ker}(P_{h,z}^* + \iota Q_{h,z}^*) = \{0\}$  for  $h$  sufficiently small. Since  $P_{h,z} - \iota Q_{h,z} : \mathcal{X}^s \rightarrow \mathcal{Y}^s$  is Fredholm, we deduce immediately that there exists  $h_0$  such that it is invertible for  $h < h_0$ .

In order to obtain uniform estimates for  $(P_{h,z} - \iota Q_{h,z})^{-1}$  as  $h \rightarrow 0$ , it is convenient to “renormalize” the problem to make the function spaces (and their norms) independent of  $h$  so that one can use the uniform boundedness principle. (Again, this could have been avoided if we had just stated the estimates uniformly in  $u$  as well, much like the closed graph theorem could have been avoided in the previous section.) So for  $r \in \mathbb{R}$  let  $\Lambda_{\hbar}^r \in \Psi_{\hbar}^r$  be elliptic and invertible, and let

$$P_{h,z}^s - \iota Q_{h,z}^s = \Lambda_{\hbar}^{s-1} (P_{h,z} - \iota Q_{h,z}) \Lambda_{\hbar}^s.$$

Then, with  $\tilde{P} = P_{h_0,z_0}^s \in \Psi^1(X)$ , for instance, independent of  $h$ ,

$$\mathcal{X} = \{u \in L^2 : \tilde{P}u \in L^2\}, \quad \mathcal{Y} = L^2,$$

$P_{h,z}^s - \iota Q_{h,z}^s : \mathcal{X} \rightarrow \mathcal{Y}$  is invertible for  $h < h_0$  by the above observations. Let  $j : \mathcal{X} \rightarrow \mathcal{Z} = L^2$  be the inclusion map. Then

$$j \circ h(P_{h,z}^s - \iota Q_{h,z}^s)^{-1} : \mathcal{Y} \rightarrow \mathcal{Z}$$

is continuous for each  $h < h_0$ .

We claim that for each (nonzero)  $f \in \mathfrak{Y}$ ,  $\{ \|h(P_{h,z}^s - \iota Q_{h,z}^s)^{-1} f\|_{L^2} : h < h_0 \}$  is bounded. Indeed, let  $v_h = h(P_{h,z}^s - \iota Q_{h,z}^s)^{-1} f$ , so we need to show that  $v_h$  is bounded. Suppose first that  $h v_h$  is not bounded, so consider a sequence  $h_j$  with  $h_j \|v_{h_j}\|_{L^2} \geq 1$ . Then let  $u_h = h^{-2} v_h / \|v_h\|_{L^2}$ ,  $h \in \{h_j : j \in \mathbb{N}\}$ , so  $h^2 u_h$  is bounded in  $L^2$ , so  $u_h$  is in particular polynomially bounded in  $L^2$ . Also,  $(P_{h,z}^s - \iota Q_{h,z}^s) u_h = h^{-1} f / \|v_h\|_{L^2}$  is bounded in  $L^2$  as  $\|v_h\| \geq h^{-1}$ . Thus, by (4-9),  $h u_h$  is bounded in  $L^2$ , that is,  $h^{-1} v_h / \|v_h\|_{L^2}$  is bounded, which is a contradiction, showing that  $h v_h$  is bounded. Thus, introducing a new  $u_h$ , namely  $u_h = h^{-1} v_h$ ,  $u_h$  is polynomially bounded, and  $(P_{h,z}^s - \iota Q_{h,z}^s) u_h = f$  is bounded, so, by (4-9),  $h u_h = v_h$  is bounded as claimed.

Thus, by the uniform boundedness principle,  $j \circ h(P_{h,z}^s - \iota Q_{h,z}^s)^{-1}$  is equicontinuous. Undoing the transformation, we deduce that

$$\|(P_{h,z} - \iota Q_{h,z})^{-1} f\|_{H_h^s} \leq C h^{-1} \|f\|_{H_h^{s-1}},$$

which is exactly the high-energy estimate we were after.

Our arguments were under the assumption of semiclassical nontrapping. As discussed in Sections 3D and 3E, this always holds in sectors  $\delta |\operatorname{Re} \sigma| < \operatorname{Im} \sigma < \delta_0 |\operatorname{Re} \sigma|$  (with  $Q_\sigma$  supported in  $\mu < 0!$ ) since  $P_{h,z} - \iota Q_{h,z}$  is actually semiclassically elliptic then. In particular this gives the meromorphy of  $P_\sigma - \iota Q_\sigma$  by giving invertibility of large  $\sigma$  in such a sector. Rephrasing in the large parameter notation, using  $\sigma$  instead of  $h$ ,

**Theorem 4.7.** *Let  $P_\sigma, Q_\sigma, \mathbb{C}_s$  be as above, and  $\mathfrak{X}^s, \mathfrak{Y}^s$  as in (4-7). Then, for  $\sigma \in \mathbb{C}_s$ ,*

$$P_\sigma - \iota Q_\sigma : \mathfrak{X}^s \rightarrow \mathfrak{Y}^s$$

*has a meromorphic inverse*

$$R(\sigma) : \mathfrak{Y}^s \rightarrow \mathfrak{X}^s.$$

*Moreover, there is  $\delta_0 > 0$  such that for all  $\delta \in (0, \delta_0)$  there is  $\sigma_0 > 0$  such that  $R(\sigma)$  is invertible in*

$$\{\sigma : \delta |\operatorname{Re} \sigma| < \operatorname{Im} \sigma < \delta_0 |\operatorname{Re} \sigma|, |\operatorname{Re} \sigma| > \sigma_0\},$$

*and nontrapping estimates hold:*

$$\|R(\sigma) f\|_{H_{|\sigma|^{-1}}^s} \leq C' |\sigma|^{-1} \|f\|_{H_{|\sigma|^{-1}}^{s-1}}.$$

If the metric  $g_0$  is nontrapping then  $p_{\hbar,z} - \iota q_{\hbar,z}$  and its complex conjugate are semiclassically nontrapping by Section 3D, so the high-energy estimates are then applicable in half-planes  $\operatorname{Im} \sigma < -C$ , that is, half-planes  $\operatorname{Im} z \geq -Ch$ . The same holds for trapping  $g_0$  provided that we add a complex absorbing operator near the trapping, as discussed in Section 3E.

Translated into the classical setting, this gives:

**Theorem 4.8.** *Let  $P_\sigma, Q_\sigma, \mathbb{C}_s, \delta_0 > 0$  be as above, in particular semiclassically nontrapping, and  $\mathcal{X}^s, \mathcal{Y}^s$  as in (4-7). Let  $C > 0$ . Then there exists  $\sigma_0$  such that*

$$R(\sigma) : \mathcal{Y}^s \rightarrow \mathcal{X}^s,$$

*is holomorphic in  $\{\sigma : -C < \text{Im } \sigma < \delta_0 |\text{Re } \sigma|, |\text{Re } \sigma| > \sigma_0\}$ , assumed to be a subset of  $\mathbb{C}_s$ , and nontrapping estimates*

$$\|R(\sigma)f\|_{H^s_{|\sigma|^{-1}}} \leq C' |\sigma|^{-1} \|f\|_{H^{s-1}_{|\sigma|^{-1}}}$$

*hold. For  $s = 1$  this states that for  $|\text{Re } \sigma| > \sigma_0, \text{Im } \sigma > -C$ ,*

$$\|R(\sigma)f\|_{L^2}^2 + |\sigma|^{-2} \|dR(\sigma)\|_{L^2}^2 \leq C'' |\sigma|^{-2} \|f\|_{L^2}^2.$$

While we stated just the global results here, one also has microlocal estimates for the solution. In particular we have the following, stated in the semiclassical language, as immediate from the estimates used to derive from the Fredholm property:

**Theorem 4.9.** *Let  $P_\sigma, Q_\sigma, \mathbb{C}_s$  be as above, in particular semiclassically nontrapping, and  $\mathcal{X}^s, \mathcal{Y}^s$  as in (4-7).*

*For  $\text{Re } z > 0$  and  $s' > s$ , the resolvent  $R_{h,z}$  is semiclassically outgoing with a loss of  $h^{-1}$  in the following sense. Let  $\alpha \in \overline{T^*X} \cap \Sigma_{\hbar,\pm}$  and let  $\gamma_-$  and  $\gamma_+$  be the backward and forward bicharacteristic from  $\alpha$ , respectively.*

*If  $\text{WF}_h^{s'-1,0}(f) \cap \overline{\gamma_\mp} = \emptyset$  (where the upper sign in  $\mp$  corresponds to the upper sign in  $\pm$  in the previous paragraph), then  $\alpha \notin \text{WF}_h^{s',-1}(R_{h,z}f)$ .*

*In fact, for any  $s' \in \mathbb{R}$ , the resolvent  $R_{h,z}$  extends to  $f \in H_h^{s'}(X)$ , with nontrapping bounds, provided that  $\text{WF}_h^{s,0}(f) \cap (L_+ \cup L_-) = \emptyset$ . The semiclassically outgoing with a loss of  $h^{-1}$  result holds for such  $f$  and  $s'$  as well.*

*Proof.* The only part that is not immediate by what has been discussed is the last claim. This follows immediately, however, by microlocal solvability in arbitrary ordered Sobolev spaces away from the radial points (i.e., solvability modulo  $\mathcal{C}^\infty$ , with semiclassical estimates), combined with our preceding results to deal with this smooth remainder plus the contribution near  $L_+ \cup L_-$ , which are assumed to be in  $H_h^s(X)$ . □

This result is needed for gluing constructions as in [Datchev and Vasy 2011], namely polynomially bounded trapping with appropriate microlocal geometry can be glued to our resolvent. Furthermore, it gives nontrapping estimates microlocally away from the trapped set provided the overall (trapped) resolvent is polynomially bounded, as shown in [Datchev and Vasy 2010].

**5. Results in the conformally compact setting**

We now state our results in the original conformally compact setting. Without the nontrapping estimate, these are a special case of a result of Mazzeo and Melrose [1987], with improvements by Guillarmou [2005], with “special” meaning that evenness is assumed. If the space is asymptotic to actual hyperbolic space, the nontrapping estimate is a slightly stronger version of the estimate of [Melrose et al. 2011], where it is shown by a parametrix construction; here conformal infinity can have arbitrary geometry. The point is thus that first, we do not need the machinery of the zero calculus here, second, we do have nontrapping high-energy estimates in general (and without a parametrix construction), and third, we add the semiclassically outgoing property which is useful for resolvent gluing, including for proving nontrapping bounds microlocally away from trapping, provided the latter is mild, as shown in [Datchev and Vasy 2010; 2011].

**Theorem 5.1.** *Suppose that  $(X_0, g_0)$  is an  $n$ -dimensional manifold with boundary with an even conformally compact metric and boundary defining function  $x$ . Let  $X_{0,\text{even}}$  denote the even version of  $X_0$ , that is, with the boundary defining function replaced by its square with respect to a decomposition in which  $g_0$  is even. Then the inverse of*

$$\Delta_{g_0} - \left(\frac{n-1}{2}\right)^2 - \sigma^2,$$

written as  $\mathcal{R}(\sigma) : L^2 \rightarrow L^2$ , has a meromorphic continuation from  $\text{Im } \sigma \gg 0$  to  $\mathbb{C}$ ,

$$\mathcal{R}(\sigma) : \mathcal{C}^\infty(X_0) \rightarrow \mathcal{C}^{-\infty}(X_0),$$

with poles with finite rank residues. If in addition  $(X_0, g_0)$  is nontrapping, then, with  $\phi$  as in Section 3A, and for suitable  $\delta_0 > 0$ , nontrapping estimates hold in every region  $-C < \text{Im } \sigma < \delta_0 |\text{Re } \sigma|$ ,  $|\text{Re } \sigma| \gg 0$ : for  $s > \frac{1}{2} + C$ ,

$$\|x^{-(n-1)/2} e^{i\sigma\phi} \mathcal{R}(\sigma) f\|_{H_{|\sigma|^{-1}}^s(X_{0,\text{even}})} \leq \tilde{C} |\sigma|^{-1} \|x^{-(n+3)/2} e^{i\sigma\phi} f\|_{H_{|\sigma|^{-1}}^{s-1}(X_{0,\text{even}})}. \quad (5-1)$$

If  $f$  is supported in  $X_0^\circ$ , the  $s - 1$  norm on  $f$  can be replaced by the  $s - 2$  norm.

Furthermore, for  $\text{Re } z > 0$ ,  $\text{Im } z = \mathcal{O}(h)$ , the resolvent  $\mathcal{R}(h^{-1}z)$  is semiclassically outgoing with a loss of  $h^{-1}$  in the sense that if  $f$  has compact support in  $X_0^\circ$ ,  $\alpha \in T^*X$  is in the semiclassical characteristic set and if  $\text{WF}_h^{s-1,0}(f)$  is disjoint from the backward bicharacteristic from  $\alpha$ , then

$$\alpha \notin \text{WF}_h^{s,-1}(\mathcal{R}(h^{-1}z)f).$$

We remark that although in order to go through without changes, our methods require the evenness property, it is not hard to deduce more restricted results without this. Essentially one would have operators with coefficients that have a conormal singularity at the event horizon; as long as this is sufficiently mild relative to what is required for the analysis, it does not affect the results. The problems arise for the analytic continuation, when one needs strong function spaces ( $H^s$  with  $s$  large); these are not preserved when one multiplies by the singular coefficients.

*Proof.* All of the results of Section 4 apply.

By self-adjointness and positivity of  $\Delta_{g_0}$  and as  $\dot{\mathcal{C}}^\infty(X_0)$  is in its domain,

$$\left(\Delta_{g_0} - \sigma^2 - \left(\frac{n-1}{2}\right)^2\right)u = f \in \dot{\mathcal{C}}^\infty(X_0)$$

has a unique solution  $u = \mathcal{R}(\sigma)f \in L^2(X_0, |dg_0|)$  when  $\text{Im } \sigma \gg 0$ . On the other hand, let  $\phi$  be as in Section 3A, so  $e^\phi = \mu^{1/2}(1 + \mu)^{-1/4}$  near  $\mu = 0$  (so  $e^\phi \sim x$  there),  $\tilde{f}_0 = e^{i\sigma\phi}x^{-(n+1)/2}x^{-1}f$  in  $\mu \geq 0$ , and  $\tilde{f}_0$  still vanishes to infinite order at  $\mu = 0$ . Let  $\tilde{f}$  be an arbitrary smooth extension of  $\tilde{f}_0$  to the compact manifold  $X$  on which  $P_\sigma - iQ_\sigma$  is defined. Let  $\tilde{u} = (P_\sigma - iQ_\sigma)^{-1}\tilde{f}$ , with  $(P_\sigma - iQ_\sigma)^{-1}$  given by our results in Section 4; this satisfies  $(P_\sigma - iQ_\sigma)\tilde{u} = \tilde{f}$  and  $\tilde{u} \in \mathcal{C}^\infty(X)$ . Thus,  $u' = e^{-i\sigma\phi}x^{(n+1)/2}x^{-1}\tilde{u}|_{\mu>0}$  satisfies  $u' \in x^{(n-1)/2}e^{-i\sigma\phi}\mathcal{C}^\infty(X_0)$ , and

$$\left(\Delta_{g_0} - \sigma^2 - \left(\frac{n-1}{2}\right)^2\right)u' = f,$$

by (3-5) and (3-14) (as  $Q_\sigma$  is supported in  $\mu < 0$ ). Since  $u' \in L^2(X_0, |dg_0|)$  for  $\text{Im } \sigma > 0$ , by the aforementioned uniqueness,  $u = u'$ .

To make the extension from  $X_{0,\text{even}}$  to  $X$  more systematic, let

$$E_s : H^s(X_{0,\text{even}}) \rightarrow H^s(X)$$

be a continuous extension operator,  $R_s : H^s(X) \rightarrow H^s(X_{0,\text{even}})$  the restriction map. Then, as we have just seen, for  $f \in \dot{\mathcal{C}}^\infty(X_0)$ ,

$$\mathcal{R}(\sigma)f = e^{-i\sigma\phi}x^{(n+1)/2}x^{-1}R_s(P_\sigma - iQ_\sigma)^{-1}E_{s-1}e^{i\sigma\phi}x^{-(n+1)/2}x^{-1}f. \quad (5-2)$$

While, for the sake of simplicity,  $Q_\sigma$  is constructed in Section 3E in such a manner that it is not holomorphic in all of  $\text{Im } \sigma > -C$  due to a cut in the upper half plane, this cut can be moved outside any fixed compact subset, so taking into account that  $\mathcal{R}(\sigma)$  is independent of the choice of  $Q_\sigma$ , the theorem follows immediately from the results of Section 4.  $\square$

Our argument proves that every pole of  $\mathcal{R}(\sigma)$  is a pole of  $(P_\sigma - iQ_\sigma)^{-1}$  (for otherwise (5-2) would show  $\mathcal{R}(\sigma)$  does not have a pole either), but it is possible

for  $(P_\sigma - \iota Q_\sigma)^{-1}$  to have poles which are not poles of  $\mathcal{R}(\sigma)$ . However, in the latter case, the Laurent coefficients of  $(P_\sigma - \iota Q_\sigma)^{-1}$  would be annihilated by multiplication by  $R_S$  from the left, that is, the resonant states (which are smooth) would be supported in  $\mu \leq 0$ , in particular vanish to infinite order at  $\mu = 0$ .

In fact, a stronger statement can be made: by a calculation completely analogous to what we just performed, we can easily see that in  $\mu < 0$ ,  $P_\sigma$  is a conjugate (times a power of  $\mu$ ) of a Klein–Gordon-type operator on  $n$ -dimensional de Sitter space with  $\mu = 0$  being the boundary (i.e., where time goes to infinity). Thus, if  $\sigma$  is not a pole of  $\mathcal{R}(\sigma)$  and  $(P_\sigma - \iota Q_\sigma)\tilde{u} = 0$  then one would have a solution  $u$  of this Klein–Gordon-type equation near  $\mu = 0$ , that is, infinity, that rapidly vanishes at infinity. It is shown in [Vasy 2010b, Proposition 5.3] by a Carleman-type estimate that this cannot happen; although there  $\sigma^2 \in \mathbb{R}$  is assumed, the argument given there goes through almost verbatim in general. Thus, if  $Q_\sigma$  is supported in  $\mu < c$ , where  $c < 0$ , then  $\tilde{u}$  is also supported in  $\mu < c$ . This argument can be iterated for Laurent coefficients of higher order poles; their range (which is finite dimensional) contains only functions supported in  $\mu < c$ .

**Remark 5.2.** We now return to our previous remarks regarding the fact that our solution disallows the conormal singularities  $(\mu \pm i0)^{\iota\sigma}$  from the perspective of conformally compact spaces of dimension  $n$ . Recalling that  $\mu = x^2$ , the two indicial roots on these spaces correspond to the asymptotics  $\mu^{\pm \iota\sigma/2 + (n-1)/4}$  in  $\mu > 0$ . Thus for the operator

$$\mu^{-1/2} \mu^{\iota\sigma/2 - (n+1)/4} (\Delta_{g_0} - (n-1)^2/4 - \sigma^2) \mu^{-\iota\sigma/2 + (n+1)/4} \mu^{-1/2},$$

or indeed  $P_\sigma$ , they correspond to

$$(\mu^{-\iota\sigma/2 + (n+1)/4} \mu^{-1/2})^{-1} \mu^{\pm \iota\sigma/2 + (n-1)/4} = \mu^{\iota\sigma/2 \pm \iota\sigma/2}.$$

Here the indicial root  $\mu^0 = 1$  corresponds to the smooth solutions we construct for  $P_\sigma$ , while  $\mu^{\iota\sigma}$  corresponds to the conormal behavior we rule out. Back to the original Laplacian, thus,  $\mu^{-\iota\sigma/2 + (n-1)/4}$  is the allowed asymptotics and  $\mu^{\iota\sigma/2 + (n-1)/4}$  is the disallowed one. Notice that  $\text{Re } \iota\sigma = -\text{Im } \sigma$ , so the disallowed solution is growing at  $\mu = 0$  relative to the allowed one, as expected in the physical half plane, and the behavior reverses when  $\text{Im } \sigma < 0$ . Thus, in the original asymptotically hyperbolic picture one has to distinguish two different rates of growths, whose relative size changes. On the other hand, in our approach, we rule out the singular solution and allow the nonsingular (smooth one), so there is no change in behavior at all for the analytic continuation.

**Remark 5.3.** For *even* asymptotically de Sitter metrics on an  $n$ -dimensional manifold  $X'_0$  with boundary, the methods for asymptotically hyperbolic spaces work, except  $P_\sigma - \iota Q_\sigma$  and  $P_\sigma^* + \iota Q_\sigma^*$  switch roles, which does not affect

Fredholm properties; see Remark 4.4. Again, evenness means that we may choose a product decomposition near the boundary such that

$$g_0 = \frac{dx^2 - h}{x^2} \quad (5-3)$$

there, where  $h$  is an even family of Riemannian metrics; as above, we take  $x$  to be a globally defined boundary defining function. Then with  $\tilde{\mu} = x^2$ , so  $\tilde{\mu} > 0$  is the Lorentzian region,  $\bar{\sigma}$  in place of  $\sigma$  (recalling that our aim is to get to  $P_\sigma^* + \iota Q_\sigma^*$ ) the above calculations for  $\square_{g_0} - (n-1)^2/4 - \bar{\sigma}^2$  in place of  $\Delta_{g_0} - (n-1)^2/4 - \sigma^2$  leading to (3-4) all go through with  $\mu$  replaced by  $\tilde{\mu}$ ,  $\sigma$  replaced by  $\bar{\sigma}$  and  $\Delta_h$  replaced by  $-\Delta_h$ . Letting  $\mu = -\tilde{\mu}$ , and conjugating by  $(1 + \mu)^{\iota\bar{\sigma}/4}$  as above, yields

$$-4\mu D_\mu^2 + 4\bar{\sigma} D_\mu + \bar{\sigma}^2 - \Delta_h + 4\iota D_\mu + 2\iota\gamma(\mu D_\mu - \bar{\sigma}/2 - \iota(n-1)/4), \quad (5-4)$$

modulo terms that can be absorbed into the error terms in operators in the class (3-5), that is, this is indeed of the form  $P_\sigma^* + \iota Q_\sigma^*$  in the framework of Section 3E, at least near  $\tilde{\mu} = 0$ . If now  $X'_0$  is extended to a manifold without boundary in such a way that in  $\tilde{\mu} < 0$ , that is,  $\mu > 0$ , one has a classically elliptic, semiclassicaly nontrapping problem, then all the results of Section 4 are applicable.

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