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Generic elements in Zariski-dense subgroups and isospectral locally symmetric spaces

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The article contains a survey of our results on length-commensurable and isospectral locally symmetric spaces and of related problems in the theory of semisimple algebraic groups. We discuss some of the techniques involved in this work (in particular, the existence of generic tori in semisimple algebraic groups over finitely generated fields and of generic elements in finitely generated Zariski-dense subgroups) and some open problems.

1. Introduction

This article contains an exposition of recent results on isospectral and length-commensurable locally symmetric spaces associated with simple real algebraic groups [Prasad and Rapinchuk 2009; 2013] and related problems in the theory of semisimple algebraic groups [Garibaldi 2012; Garibaldi and Rapinchuk 2013; Prasad and Rapinchuk 2010a]. One of the goals of [Prasad and Rapinchuk 2009] was to study the problem beautifully formulated by Mark Kac in [1966] as "*Can one hear the shape of a drum*?" for the quotients of symmetric spaces of the groups of real points of absolutely simple real algebraic groups by cocompact arithmetic subgroups. A precise mathematical formulation of Kac's question is *whether two compact Riemannian manifolds which are isospectral* (i.e., have equal spectra — eigenvalues and multiplicities — for the Laplace–Beltrami operator) *are necessarily isometric*. In general, the answer to this question is in the negative as was shown by John Milnor already in [1964] by constructing two nonisometric isospectral flat tori of dimension 16. Later M.-F. Vignéras [1980] used arithmetic properties of quaternion algebras to produce examples of arithmetically defined isospectral, but not isometric, Riemann surfaces. On the other hand, T. Sunada [1985], inspired by a construction of nonisomorphic number fields with the same Dedekind zeta-function, proposed a general and basically purely group-theoretic method of producing nonisometric isospectral Riemannian manifolds which has since then been used in various ways. It is important to note, however, that the nonisometric isospectral manifolds constructed by Vignéras and Sunada are *commensurable*, that is, have a common finite-sheeted cover. This suggests that one should probably settle for the following weaker version of Kac's original question: *Are any two isospectral compact Riemannian manifolds necessarily commensurable?* The answer to this modified question is still negative in the general case: Lubotzky, Samuels and Vishne [Lubotzky et al. 2006], using the Langlands correspondence, have constructed examples of noncommensurable isospectral locally symmetric spaces associated with absolutely simple real groups of type A_n (see Problem 10.7). Nevertheless, it turned out that the answer is actually in the affirmative for several classes of locally symmetric spaces. Prior to our paper [Prasad and Rapinchuk 2009], this was known to be the case only for the following two classes: arithmetically defined Riemann surfaces [Reid 1992] and arithmetically defined hyperbolic 3-manifolds [Chinburg et al. 2008].

In [Prasad and Rapinchuk 2009], we used Schanuel's conjecture from transcendental number theory (for more about this conjecture, and how it comes up in our work, see below) and the results of [Garibaldi 2012; Prasad and Rapinchuk 2010a] to prove that any two compact *isospectral* arithmetically defined locally symmetric spaces associated with absolutely simple real algebraic groups of type other than A_n $(n > 1)$, D_{2n+1} $(n > 1)$, or E_6 are *necessarily commensurable*. One of the important ingredients of the proof is the connection between isospectrality and another property of Riemannian manifolds called *isolength spectrality*.

More precisely, for a Riemannian manifold M we let $L(M)$ denote the *weak length spectrum* of M, that is, the collection of the lengths of all closed geodesics in M (note that for the existence of a "nice" Laplace spectrum, M is required to be compact, but the weak length spectrum $L(M)$ can be considered for *any* M — i.e., we do not need to assume that M is compact). Then two Riemannian manifolds M_1 and M_2 are called *isolength spectral* if $L(M_1) = L(M_2)$. Any two compact isospectral locally symmetric spaces are isolength spectral; this was first proved in [Gangolli 1977] in the rank-one case, then in [Duistermaat and Guillemin 1975] and [Duistermaat et al. 1979] in the general case; see [Prasad and Rapinchuk 2009, Theorem 10.1]. So, the emphasis in this last paper is really on the analysis of isolength spectral locally symmetric spaces M_1 and M_2.

In fact, we prove our results under the much weaker assumption of *length-commensurability*, which means that $\mathbb{Q}\cdot L(M_1) = \mathbb{Q}\cdot L(M_2)$. (The set $\mathbb{Q}\cdot L(M)$ is sometimes called the *rational length spectrum* of M; its advantage, particularly in the analysis of questions involving commensurable manifolds, is that it is invariant under passing to a finite-sheeted cover — this property fails for the Laplace spectrum or the length spectrum. At the same time, $\mathbb{Q}\cdot L(M)$ can actually be computed in at least some cases, while precise computation of $L(M)$ or the Laplace spectrum is not available for any compact locally symmetric space at this point.) The notion of length-commensurability was introduced in [Prasad and Rapinchuk 2009], and the investigation of its qualitative and quantitative consequences for general locally symmetric spaces is an ongoing project. For arithmetically defined spaces, however, the main questions were answered in the same paper, and we would like to complete this introduction by showcasing the results for arithmetic hyperbolic spaces.

Let $\mathbb{H}^n$ be the real hyperbolic n-space. By an arithmetically defined real hyperbolic n-manifold we mean the quotient $\mathbb{H}^n/\Gamma$, where Γ is an arithmetic subgroup of $\mathrm{PSO}(n, 1)$ (which is the isometry group of $\mathbb{H}^n$); see Section 3 regarding the notion of arithmeticity.

Theorem 1.1 [Prasad and Rapinchuk 2009, Corollary 8.17 and Remark 8.18]. *Let M_1 and M_2 be arithmetically defined real hyperbolic n-manifolds.*

If $n \not\equiv 1 \pmod 4$, then in case M_1 and M_2 are not commensurable, after a possible interchange of M_1 and M_2, there exists $\lambda_1 \in L(M_1)$ such that for any $\lambda_2 \in L(M_2)$, the ratio λ_1/λ_2 is transcendental over $\mathbb{Q}$. (Thus, for such n the length-commensurability, and hence isospectrality, of M_1 and M_2 implies their commensurability.)

On the contrary, for any $n \equiv 1 \pmod 4$, there exist M_1 and M_2 as above that are length-commensurable, but not commensurable.

What is noteworthy is that there is no apparent geometric reason for this dramatic distinction between the length-commensurability of hyperbolic n-manifolds when $n \not\equiv 1 \pmod 4$ and $n \equiv 1 \pmod 4$ — in our argument the difference comes from considerations involving Galois cohomology; see Theorem 4.2 and subsequent comments.

Our general results for arithmetically defined length-commensurable locally symmetric spaces (Section 5) imply similar (but not identical!) assertions for complex and quaternionic hyperbolic manifolds. At the same time, one can ask about possible relations between $L(M_1)$ and $L(M_2)$ (or between $\mathbb{Q} \cdot L(M_1)$ and $\mathbb{Q} \cdot L(M_2)$) if M_1 and M_2 are *not* length-commensurable. The results we will describe in Section 8 assert that if $\mathbb{Q} \cdot L(M_1) \neq \mathbb{Q} \cdot L(M_2)$, then no polynomial-type relation between $L(M_1)$ and $L(M_2)$ can ever exist; in other words, these sets are *very* different. This is, for example, the case if M_1 and M_2 are hyperbolic manifolds of finite volume having different dimensions!

2. Length-commensurable locally symmetric spaces and weakly commensurable subgroups

2.1. *Riemann surfaces.* Our analysis of length-commensurability of locally symmetric spaces relies on a purely algebraic relation between their fundamental groups which we termed *weak commensurability*. It is easiest to motivate this notion by looking at the length-commensurability of Riemann surfaces. In this discussion we will be using the realization of $\mathbb{H}^2$ as the complex upper half-plane with the standard hyperbolic metric $ds^2 = y^{-2}(dx^2 + dy^2)$. The usual action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}^2$ by fractional linear transformations is isometric and allows us to identify $\mathbb{H}^2$ with the symmetric space $\mathrm{SO}_2(\mathbb{R})\backslash\mathrm{SL}_2(\mathbb{R})$. It is well known that any compact Riemann surface M of genus > 1 can be obtained as a quotient of $\mathbb{H}^2$ by a discrete subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ with torsion-free image in $\mathrm{PSL}_2(\mathbb{R})$. Now, given any such subgroup Γ, we let $\pi\colon \mathbb{H}^2 \to \mathbb{H}^2/\Gamma =: M$ denote the canonical projection. It is easy to see that

$$t \mapsto e^t i = i \cdot \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, \quad t \in \mathbb{R},$$

is a unit-velocity parametrization of a geodesic c in $\mathbb{H}^2$. So, if $\gamma = \mathrm{diag}(t_\gamma, t_\gamma^{-1}) \in \Gamma$, then the image $\pi(c)$ is a *closed geodesic* c_γ in M, whose length is given by the formula

$$\ell_\Gamma(c_\gamma) = \frac{2}{n_\gamma} \cdot \log t_\gamma \tag{1}$$

(assuming that $t_\gamma > 1$), where n_γ is an integer $\geqslant 1$ (winding number in case c_γ is not primitive). Generalizing this construction, one shows that every semisimple element $\gamma \in \Gamma \setminus \{\pm 1\}$ gives rise to a closed geodesic c_γ in M whose length is given by (1) where t_γ is the eigenvalue of $\pm\gamma$ which is > 1, and conversely, any closed geodesic in M is obtained this way. As a result,

$$\mathbb{Q} \cdot L(M) = \mathbb{Q} \cdot \{\log t_\gamma \mid \gamma \in \Gamma\backslash\{\pm 1\} \ \text{ semisimple}\}.$$

Now, suppose we have two quotients $M_1 = \mathbb{H}^2/\Gamma_1$ and $M_2 = \mathbb{H}^2/\Gamma_2$ as above, and let c_{γ_i} be a closed geodesic in M_i for $i = 1, 2$. Then

$$\ell_{\Gamma_1}(c_{\gamma_1})/\ell_{\Gamma_2}(c_{\gamma_2}) \in \mathbb{Q} \iff t_{\gamma_1}^m = t_{\gamma_2}^n \ \text{for some } m, n \in \mathbb{N},$$

or equivalently, the subgroups generated by the eigenvalues of γ_1 and γ_2 have nontrivial intersection. This leads us to the following.

Definition 2.2. Let $G_1 \subset \mathrm{GL}_{N_1}$ and $G_2 \subset \mathrm{GL}_{N_2}$ be two semisimple algebraic groups defined over a field F of characteristic zero.

(a) Two semisimple elements $\gamma_1 \in G_1(F)$ and $\gamma_2 \in G_2(F)$ are said to be *weakly commensurable* if the subgroups of $\overline{F}^\times$ generated by their eigenvalues intersect nontrivially.

(b) Two (Zariski-dense) subgroups $\Gamma_1 \subset G_1(F)$ and $\Gamma_2 \subset G_2(F)$ are *weakly commensurable* if every semisimple element $\gamma_1 \in \Gamma_1$ of infinite order is weakly commensurable to some semisimple element $\gamma_2 \in \Gamma_2$ of infinite order, and vice versa.

It should be noted that in [Prasad and Rapinchuk 2009] we gave a more technical, but equivalent, definition of weakly commensurable elements, viz. we required that there should exist maximal F-tori T_i of G_i for $i = 1, 2$ such that $\gamma_i \in T_i(F)$ and for some characters $\chi_i \in X(T_i)$ we have

$$\chi_1(\gamma_1) = \chi_2(\gamma_2) \neq 1.$$

This (equivalent) reformulation of (a) immediately demonstrates that the notion of weak commensurability does not depend on the choice of matrix realizations of the G_i, and more importantly, is more convenient for the proofs of our results.

The above discussion of Riemann surfaces implies that if two Riemann surfaces $M_1 = \mathbb{H}^2/\Gamma_1$ and $M_2 = \mathbb{H}^2/\Gamma_2$ are length-commensurable, then the corresponding fundamental groups Γ_1 and Γ_2 are weakly commensurable. Our next goal is to explain why this implication remains valid for general locally symmetric spaces.

2.3. ***Length-commensurability and weak commensurability*: *The general case.*** First, we need to fix some notations related to general locally symmetric spaces. Let G be a connected adjoint real semisimple algebraic group, let $\mathcal{G} = G(\mathbb{R})$ considered as a real Lie group, and let $\mathfrak{X} = \mathcal{K}\backslash\mathcal{G}$, where $\mathcal{K}$ is a maximal compact subgroup of $\mathcal{G}$, be the associated symmetric space endowed with the Riemannian metric coming from the Killing form on the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$. Furthermore, given a torsion-free discrete subgroup Γ of $\mathcal{G}$, we let $\mathfrak{X}_\Gamma = \mathfrak{X}/\Gamma$ denote the corresponding locally symmetric space. Just as in the case of Riemann surfaces, to any nontrivial semisimple element $\gamma \in \Gamma$ there corresponds a closed geodesic c_γ whose length is given by

$$\ell_\Gamma(c_\gamma) = \frac{1}{n_\gamma} \cdot \lambda_\Gamma(\gamma),$$

where n_γ is an integer $\geqslant 1$ and

$$\lambda_\Gamma(\gamma)^2 := \sum_\alpha (\log|\alpha(\gamma)|)^2, \tag{2}$$

with the summation running over all roots of G with respect to a fixed maximal $\mathbb{R}$-torus T of G whose group of $\mathbb{R}$-points contains γ. This formula looks much more intimidating than (1), so in order to make it more manageable we first make the following observation. Of course, the $\mathbb{R}$-torus T may not be $\mathbb{R}$-split, so not every root α may be defined over $\mathbb{R}$. However,

$$|\alpha(\gamma)|^2 = \chi(\gamma),$$

where $\chi = \alpha + \overline{\alpha}$ (with $\overline{\alpha}$ being the conjugate character in terms of the natural action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on $X(T)$, and, as usual, $X(T)$ is viewed as an additive group) is a character defined over $\mathbb{R}$ and which takes positive values on $T(\mathbb{R})$. Such characters will be called *positive*. So, we can now rewrite (2) in the form

$$\lambda_\Gamma(\gamma)^2 = \sum_{i=1}^{p} s_i (\log \chi_i(\gamma))^2, \tag{3}$$

where $\chi_1, \ldots, \chi_p$ are certain positive characters of T and $s_1, \ldots, s_p$ are positive rational numbers whose denominators are divisors of 4. The point to be made here is that the subgroup $P(T) \subset X(T)$ of positive characters may be rather small. More precisely, T is an almost direct product of an $\mathbb{R}$-anisotropic subtorus A and an $\mathbb{R}$-split subtorus S. Then any character of T which is defined over $\mathbb{R}$ vanishes on A. This easily implies that the restriction map yields an embedding $P(T) \hookrightarrow X(S)_\mathbb{R} = X(S)$ with finite cokernel; in particular, the rank of $P(T)$ as an abelian group coincides with the $\mathbb{R}$-rank $\mathrm{rk}_\mathbb{R}\, T$ of T.

Before formulating our results, we define the following property. Let $G \subset \mathrm{GL}_N$ be a semisimple algebraic group defined over a field F of characteristic zero. We say that a (Zariski-dense) subgroup $\Gamma \subset G(F)$ ***has property*** (***A***) if for any semisimple element $\gamma \in \Gamma$, all the eigenvalues of γ lie in the field of algebraic numbers $\overline{\mathbb{Q}}$ (note that the latter is equivalent to the fact that for any maximal F-torus T of G containing γ and any character $\chi \in X(T)$, we have $\chi(\gamma) \in \overline{\mathbb{Q}}^\times$ — this reformulation shows, in particular, that this property does not depend on the choice of a matrix realization of G). Of course, this property automatically holds if Γ is arithmetic, or more generally, if Γ can be conjugated into $\mathrm{SL}_N(K)$ for some number field K.

Let us now consider the rank-one case first.

The rank-one case. Suppose $\mathrm{rk}_\mathbb{R}\, G = 1$ (the examples include the adjoint groups of $\mathrm{SO}(n, 1)$, $\mathrm{SU}(n, 1)$ and $\mathrm{Sp}(n, 1)$; the corresponding symmetric spaces are respectively the real, complex and quaternionic hyperbolic n-spaces). Then given a nontrivial semisimple element $\gamma \in \Gamma$, for any maximal $\mathbb{R}$-torus T of G containing γ we have $\mathrm{rk}_\mathbb{R}\, T = 1$, which implies that the group $P(T)$ of positive characters is cyclic and is generated, say, by χ. Then it follows from (3) that

$$\lambda_\Gamma(\gamma) = \frac{\sqrt{m}}{2} \cdot |\log \chi(t)|, \tag{4}$$

where m is some integer $\geqslant 1$ depending only on G; note that this formula is still in the spirit of (1), but potentially involves some irrationality which can complicate the analysis of length-commensurability.

Now, suppose that G_1 and G_2 are two simple algebraic $\mathbb{R}$-groups of $\mathbb{R}$-rank one. For $i = 1, 2$, let $\Gamma_i \subset G_i(\mathbb{R}) = \mathcal{G}_i$ be a discrete torsion-free subgroup having property (A). Given a nontrivial semisimple element $\gamma_i \in \Gamma_i$, we pick a maximal $\mathbb{R}$-torus T_i of G_i whose group of $\mathbb{R}$-points contains γ_i and let χ_i be a generator of the group of positive characters $P(T_i)$. Then according to (4),

$$\lambda_{\Gamma_1}(\gamma_1) = \frac{\sqrt{m_1}}{2} \cdot |\log \chi_1(\gamma_1)| \quad \text{and} \quad \lambda_{\Gamma_2}(\gamma_2) = \frac{\sqrt{m_2}}{2} \cdot |\log \chi_2(\gamma_2)|,$$

for some integers $m_1, m_2 \geqslant 1$. By a theorem proved independently by Gelfond and Schneider in 1934 (which settled Hilbert's seventh problem; see [Baker 1990]), the ratio

$$\frac{\log \chi_1(\gamma_1)}{\log \chi_2(\gamma_2)}$$

is either rational or transcendental. This implies that the $\ell_{\Gamma_1}(\gamma_1)/\ell_{\Gamma_2}(\gamma_2)$, or equivalently the ratio $\lambda_{\Gamma_1}(\gamma_1)/\lambda_{\Gamma_2}(\gamma_2)$, can be rational only if

$$\chi_1(\gamma_1)^{n_1} = \chi_2(\gamma_2)^{n_2},$$

for some nonzero integers n_1, n_2, which makes the elements γ_1 and γ_2 weakly commensurable. (Of course, we get this conclusion without using the theorem of Gelfond–Schneider if $G_1 = G_2$, hence $m_1 = m_2$.) This argument shows that the length-commensurability of $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ implies the weak commensurability of Γ_1 and Γ_2.

Finally, we recall that if $G \subset \mathrm{GL}_N$ is an absolutely simple real algebraic group not isomorphic to PGL_2 then any lattice $\Gamma \subset G(\mathbb{R})$ can be conjugated into $\mathrm{SL}_N(K)$ for some number field K (see [Raghunathan 1972, 7.67 and 7.68]), hence possesses property (A). This implies that if $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are rank one locally symmetric spaces of finite volume then their length-commensurability always implies the weak commensurability of Γ_1 and Γ_2 except possibly in the following situation: $G_1 = \mathrm{PGL}_2$ and Γ_1 cannot be conjugated into $\mathrm{PGL}_2(K)$ for any number field $K \subset \mathbb{R}$ while $G_2 \neq \mathrm{PGL}_2$ (in [Prasad and Rapinchuk 2009] this was called the exceptional case $(\mathscr{E})$). Nevertheless, the conclusion remains valid also in this case if one assumes the truth of Schanuel's conjecture (see below) — this follows from our recent results [Prasad and Rapinchuk 2013], which we will discuss in Section 8 (see Theorem 8.1).

The general case. If $\mathrm{rk}_{\mathbb{R}}\, G > 1$ then p may be > 1 in (3), hence $\lambda_\Gamma(\gamma)$, generally speaking, is *not* a multiple of the logarithm of the value of a positive character. Consequently, the fact that the ratio $\lambda_{\Gamma_1}(\gamma_1)/\lambda_{\Gamma_2}(\gamma_2)$ is a rational number does not imply *directly* that γ_1 and γ_2 are weakly commensurable. While the implication nevertheless *is* valid (under some natural technical assumptions), it is hardly surprising now that the proof requires some nontrivial information about the logarithms of the character values. More precisely, our arguments in [Prasad and Rapinchuk 2009; 2013] assume the truth of the following famous conjecture in transcendental number theory (see [Ax 1971], for example).

Conjecture 2.4 (Schanuel's conjecture). *If $z_1, \dots, z_n \in \mathbb{C}$ are linearly independent over $\mathbb{Q}$, the transcendence degree (over $\mathbb{Q}$) of the field generated by $z_1, \dots, z_n$ and $e^{z_1}, \dots, e^{z_n}$ is at least n.*

In fact, we will only need the consequence of this conjecture that for nonzero algebraic numbers $a_1, \dots, a_n$, (any values of) their logarithms $\log a_1, \dots, \log a_n$ are algebraically independent once they are linearly independent (over $\mathbb{Q}$). In order to apply this statement in our situation, we first prove the following elementary lemma.

Lemma 2.5. *Let G_1 and G_2 be two connected semisimple real algebraic groups. For $i = 1, 2$, let T_i be a maximal $\mathbb{R}$-torus of G_i, $\gamma_i \in T_i(\mathbb{R})$ and let $\chi_1^{(i)}, \dots, \chi_{d_i}^{(i)}$ be positive characters of T_i such that the set*

$$S_i = \left\{\log \chi_1^{(i)}(\gamma_i), \dots, \log \chi_{d_i}^{(i)}(\gamma_i)\right\} \subset \mathbb{R}$$

is linearly independent over $\mathbb{Q}$. If γ_1 and γ_2 are not *weakly commensurable then the set $S_1 \cup S_2$ is also linearly independent.*

Proof. Assume the contrary. Then there exist integers $s_1, \dots, s_{d_1}, t_1, \dots, t_{d_2}$, not all zero, such that

$$s_1 \log \chi_1^{(1)}(\gamma_1) + \cdots + s_{d_1} \log \chi_{d_1}^{(1)}(\gamma_1) - t_1 \log \chi_1^{(2)}(\gamma_2) - \cdots - t_{d_2} \log \chi_{d_2}^{(2)}(\gamma_2) = 0.$$

Consider the characters

$$\psi^{(1)} := s_1 \chi_1^{(1)} + \cdots + s_{d_1} \chi_{d_1}^{(1)} \quad \text{and} \quad \psi^{(2)} := t_1 \chi_1^{(2)} + \cdots + t_{d_2} \chi_{d_2}^{(2)}$$

of T_1 and T_2 respectively. Then $\psi^{(1)}(\gamma_1) = \psi^{(2)}(\gamma_2)$, and hence

$$\psi^{(1)}(\gamma_1) = 1 = \psi^{(2)}(\gamma_2),$$

because γ_1 and γ_2 are not weakly commensurable. This means that in

$$s_1 \log \chi_1^{(1)}(\gamma_1) + \cdots + s_{d_1} \log \chi_{d_1}^{(1)}(\gamma_1) = t_1 \log \chi_1^{(2)}(\gamma_2) + \cdots + t_{d_2} \log \chi_{d_2}^{(2)}(\gamma_2)$$

both sides vanish. But the sets S_1 and S_2 are linearly independent, so all the coefficients are zero, a contradiction. □

We are now ready to connect length-commensurability with weak commensurability.

Proposition 2.6. *Let G_1 and G_2 be two connected semisimple real algebraic groups. For $i = 1, 2$, let $\Gamma_i \subset G_i(\mathbb{R})$ be a subgroup satisfying property (A). Assume that Schanuel's conjecture holds. If semisimple elements $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$ are not weakly commensurable then $\lambda_{\Gamma_1}(\gamma_1)$ and $\lambda_{\Gamma_2}(\gamma_2)$ are algebraically independent over $\mathbb{Q}$.*

Proof. It follows from (3) that

$$\lambda_{\Gamma_1}(\gamma_1)^2 = \sum_{i=1}^{p} s_i (\log \chi_i^{(1)}(\gamma_1))^2 \quad \text{and} \quad \lambda_{\Gamma_2}(\gamma_2)^2 = \sum_{i=1}^{q} t_i (\log \chi_i^{(2)}(\gamma_2))^2, \quad (5)$$

where s_i and t_i are positive rational numbers, and $\chi_i^{(1)}$ and $\chi_i^{(2)}$ are positive characters on maximal $\mathbb{R}$-tori T_1 and T_2 of G_1 and G_2 whose groups of $\mathbb{R}$-points contain the elements γ_1 and γ_2, respectively. After renumbering the characters, we can assume that

$$a_1 := \log \chi_1^{(1)}(\gamma_1), \quad \dots, \quad a_m := \log \chi_m^{(1)}(\gamma_1)$$

(resp., $b_1 := \log \chi_1^{(2)}(\gamma_2), \dots, b_n = \log \chi_n^{(2)}(\gamma_2)$) form a basis of the $\mathbb{Q}$-subspace of $\mathbb{R}$ spanned by $\log \chi_i^{(1)}(\gamma_1)$ for $i \leqslant p$ (resp., by $\log \chi_i^{(2)}(\gamma_2)$ for $i \leqslant q$). It follows from Lemma 2.5 that the numbers

$$a_1, \dots, a_m \quad \text{and} \quad b_1, \dots, b_n, \quad (6)$$

are linearly independent. By our assumption, Γ_1 and Γ_2 possess property (A), so the character values $\chi_i^{(j)}(\gamma_j)$ are all algebraic numbers. So, it follows from Schanuel's conjecture that the numbers in (6) are algebraically independent over $\mathbb{Q}$. As is seen from (5), $\lambda_{\Gamma_1}(\gamma_1)$ and $\lambda_{\Gamma_2}(\gamma_2)$ are represented by nonzero homogeneous polynomials of degree two, with rational coefficients, in $a_1, \dots, a_m$ and $b_1, \dots, b_n$, respectively, and therefore they are algebraically independent. □

This proposition leads us to the following.

Theorem 2.7. *Let G_1 and G_2 be two connected semisimple real algebraic groups. For $i = 1, 2$, let $\Gamma_i \subset G_i(\mathbb{R})$ be a discrete torsion-free subgroup having property (A). Assume that Schanuel's conjecture holds. If Γ_1 and Γ_2 are not weakly commensurable, then, possibly after reindexing, we can find $\lambda_1 \in L(\mathfrak{X}_{\Gamma_1})$ which is algebraically independent from* any *$\lambda_2 \in L(\mathfrak{X}_{\Gamma_2})$. In particular, $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are not length-commensurable.*

Combining this with the discussion above of property (A) for lattices and of the exceptional case ($\mathscr{E}$), we obtain the following.

Corollary 2.8. *Let G_1 and G_2 be two absolutely simple real algebraic groups, and for $i = 1, 2$ let Γ_i be a lattice in $G_i(\mathbb{R})$ (so that the locally symmetric space $\mathfrak{X}_{\Gamma_i}$ has finite volume). If $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are length-commensurable then Γ_1 and Γ_2 are weakly commensurable.*

The results we discussed in this section shift the focus in the analysis of length-commensurability and/or isospectrality of locally symmetric spaces to that of weak commensurability of finitely generated Zariski-dense subgroups of simple (or semisimple) algebraic groups. In Section 3, we will first present some basic results dealing with the weak commensurability of such subgroups in a completely general situation, one of which states that the mere existence of such subgroups implies that the ambient algebraic groups either are of the same type, or one of them is of type B_n and the other of type C_n for some $n \geqslant 3$ (Theorem 3.1). We then turn to much more precise results in the case where the algebraic groups are of the same type and the subgroups are S-arithmetic (see Section 4), and finally derive some geometric consequences of these results (see Section 5). Next, Section 7 contains an exposition of the recent results of [Garibaldi and Rapinchuk 2013], which completely characterize weakly commensurable S-arithmetic subgroups in the case where one of the two groups is of type B_n and the other is of type C_n $(n \geqslant 3)$. In Section 8 we discuss a more technical version of the notion of weak commensurability, which enabled us to show in [Prasad and Rapinchuk 2013] (under mild technical assumptions) that if two arithmetically defined locally symmetric spaces $M_1 = \mathfrak{X}_{\Gamma_1}$ and $M_2 = \mathfrak{X}_{\Gamma_2}$ are not length-commensurable then the sets $L(M_1)$ and $L(M_2)$ (or $\mathbb{Q} \cdot L(M_1)$ and $\mathbb{Q} \cdot (M_2)$) are *very* different. The proofs of all these results use the existence (first established in [Prasad and Rapinchuk 2003]) of special elements, which we call *generic elements*, in arbitrary finitely generated Zariski-dense subgroups; we briefly review these and more recent results in this direction in Section 9 along with the results that relate the analysis of weak commensurability with a problem of independent interest in the theory of semisimple algebraic groups of characterizing simple K-groups having the same isomorphism classes of maximal K-tori (Section 6). Finally, in Section 10 we discuss some open problems.

3. Two basic results implied by weak commensurability and the definition of arithmeticity

Our next goal is to give an account of the results from [Prasad and Rapinchuk 2009] concerning weakly commensurable subgroups of semisimple algebraic groups. We begin with the following two theorems that provide the basic results about weak commensurability of arbitrary finitely generated Zariski-dense subgroups of semisimple groups.

Theorem 3.1. *Let G_1 and G_2 be two connected absolutely almost simple algebraic groups defined over a field F of characteristic zero. Assume that there exist finitely generated Zariski-dense subgroups Γ_i of $G_i(F)$ which are weakly commensurable. Then either G_1 and G_2 are of the same Killing–Cartan type, or one of them is of type B_n and the other is of type C_n for some $n \geqslant 3$.*

The way we prove this theorem is by showing that the Weyl groups of G_1 and G_2 have the same order, as it is well known that the order of the Weyl group uniquely determines the type of the root system, except for the ambiguity between B_n and C_n. On the other hand, groups G_1 and G_2 of types B_n and C_n with $n > 2$ respectively, may indeed contain weakly commensurable subgroups. This was first shown in [Prasad and Rapinchuk 2009, Example 6.7] using a cohomological construction which we will briefly recall in Section 7. Recently in [Garibaldi and Rapinchuk 2013] another explanation was given using commutative étale subalgebras of simple algebras with involution. We refer the reader to Section 7 for this argument as well as a complete characterization of weakly commensurable S-arithmetic subgroups in the algebraic groups of types B_n and C_n (see Theorem 7.2).

Theorem 3.2. *Let G_1 and G_2 be two connected absolutely almost simple algebraic groups defined over a field F of characteristic zero. For $i = 1, 2$, let Γ_i be a finitely generated Zariski-dense subgroup of $G_i(F)$, and K_{Γ_i} be the subfield of F generated by the traces* $\mathrm{Tr}\,\mathrm{Ad}\,\gamma$, *in the adjoint representation, of $\gamma \in \Gamma_i$. If Γ_1 and Γ_2 are weakly commensurable, then $K_{\Gamma_1} = K_{\Gamma_2}$.*

We now turn to the results concerning weakly commensurable Zariski-dense S-arithmetic subgroups, which are surprisingly strong. In Section 4 we will discuss the weak commensurability of S-arithmetic subgroups in absolutely almost simple algebraic groups G_1 and G_2 of the same type, postponing the case where one of the groups is of type B_n and the other of type C_n to Section 7. Since our results rely on a specific way of describing S-arithmetic subgroups in absolutely almost simple groups, we will discuss this issue first.

3.3. ***The definition of arithmeticity.*** Let G be an algebraic group defined over a number field K, and let S be a finite subset of the set V^K of all places of K containing the set V_∞^K of archimedean places. Fix a K-embedding $G \subset \mathrm{GL}_N$, and consider the group of S-integral points

$$G(\mathcal{O}_K(S)) := G \cap \mathrm{GL}_N(\mathcal{O}_K(S)).$$

Then, for any field extension F/K, the subgroups of $G(F)$ that are commensurable[1] with $G(\mathcal{O}_K(S))$ are called *S-arithmetic*, and in the case where $S = V_\infty^K$ simply *arithmetic* (note that $\mathcal{O}_K(V_\infty^K) = \mathcal{O}_K$, the ring of algebraic integers in K). It is well known (see, for example, [Platonov and Rapinchuk 1994]) that the resulting class of S-arithmetic subgroups does not depend on the choice of K-embedding $G \subset \mathrm{GL}_N$. The question, however, is what we should mean by an arithmetic subgroup of $G(F)$ when G is an algebraic group defined over a field F of characteristic zero that is not equipped with a structure of K-group over some

[1] We recall that two subgroups $\mathcal{H}_1$ and $\mathcal{H}_2$ of an abstract group $\mathcal{G}$ are called *commensurable* if their intersection $\mathcal{H}_1 \cap \mathcal{H}_2$ is of finite index in each of the subgroups.

number field $K \subset F$. For example, what is an arithmetic subgroup of $G(\mathbb{R})$ where $G = \mathrm{SO}_3(f)$ and $f = x^2 + ey^2 - \pi z^2$? For absolutely almost simple groups the "right" concept that we will formalize below is given in terms of the forms of G over the subfields $K \subset F$ that are number fields. In our example, we can consider the following rational quadratic forms that are equivalent to f over $\mathbb{R}$:

$$f_1 = x^2 + y^2 - 3z^2 \quad \text{and} \quad f_2 = x^2 + 2y^2 - 7z^2,$$

and set $G_i = \mathrm{SO}_3(f_i)$. Then for each $i = 1, 2$, we have an $\mathbb{R}$-isomorphism $G_i \simeq G$, so the natural arithmetic subgroup $G_i(\mathbb{Z}) \subset G_i(\mathbb{R})$ can be thought of as an "arithmetic" subgroup of $G(\mathbb{R})$. Furthermore, one can consider quadratic forms over other number subfields $K \subset \mathbb{R}$ that again become equivalent to f over $\mathbb{R}$; for example,

$$K = \mathbb{Q}(\sqrt{2}) \quad \text{and} \quad f_3 = x^2 + y^2 - \sqrt{2}z^2.$$

Then for $G_3 = \mathrm{SO}_3(f_3)$, there is an $\mathbb{R}$-isomorphism $G_3 \simeq G$ which allows us to view the natural arithmetic subgroup $G_3(\mathcal{O}_K) \subset G_3(\mathbb{R})$, where $\mathcal{O}_K = \mathbb{Z}[\sqrt{2}]$, as an "arithmetic" subgroup of $G(\mathbb{R})$. One can easily generalize such constructions from arithmetic to S-arithmetic groups by replacing the rings of integers with the rings of S-integers. So, generally speaking, by an S-arithmetic subgroup of $G(\mathbb{R})$ we mean a subgroup which is commensurable to one of the subgroups obtained through this construction for some choice of a number subfield $K \subset \mathbb{R}$, a finite set S of places of K containing all the archimedean ones, and a quadratic form $\tilde{f}$ over K that becomes equivalent to f over $\mathbb{R}$. The technical definition is as follows.

Let G be a connected absolutely almost simple algebraic group defined over a field F of characteristic zero, $\overline{G}$ be its adjoint group, and $\pi\colon G \to \overline{G}$ be the natural isogeny. Suppose we are given the following data:

- a *number field* K with a *fixed* embedding $K \hookrightarrow F$;
- an F/K-form $\mathcal{G}$ of $\overline{G}$, which is an algebraic K-group such that there exists an F-isomorphism ${}_F\mathcal{G} \simeq \overline{G}$, where ${}_F\mathcal{G}$ is the group obtained from $\mathcal{G}$ by the extension of scalars from K to F;
- a finite set S of places of K containing V_∞^K but not containing any nonarchimedean places v such that $\mathcal{G}$ is K_v-anisotropic.[2]

We then have an embedding $\iota\colon \mathcal{G}(K) \hookrightarrow \overline{G}(F)$ which is well defined up to an F-automorphism of $\overline{G}$ (note that we do *not* fix an isomorphism ${}_F\mathcal{G} \simeq \overline{G}$). A subgroup Γ of $G(F)$ such that $\pi(\Gamma)$ is commensurable with $\sigma(\iota(\mathcal{G}(\mathcal{O}_K(S))))$, for

[2]We note that if $\mathcal{G}$ is K_v-anisotropic then $\mathcal{G}(\mathcal{O}_K(S))$ and $\mathcal{G}(\mathcal{O}_K(S \cup \{v\}))$ are commensurable, and therefore the classes of S- and $(S \cup \{v\})$-arithmetic subgroups coincide. Thus, this assumption on S is necessary if we want to recover it from a given S-arithmetic subgroup.

some F-automorphism σ of $\overline{G}$, will be called a $(\mathscr{G}, K, S)$*-arithmetic subgroup*,[3] or an S-arithmetic subgroup described in terms of the triple $(\mathscr{G}, K, S)$. As usual, $(\mathscr{G}, K, V_\infty^K)$-arithmetic subgroups will simply be called $(\mathscr{G}, K)$-arithmetic.

We also need to introduce a more general notion of commensurability. The point is that since weak commensurability is defined in terms of eigenvalues, a subgroup $\Gamma \subset G(F)$ is weakly commensurable with any conjugate subgroup, while the latter may not be commensurable with the former in the usual sense. So, to make theorems asserting that in certain situations *weak commensurability implies commensurability* possible (and such theorems are in fact one of the goals of our analysis) one definitely needs to modify the notion of commensurability. The following notion works well in geometric applications. Let G_i, for $i = 1, 2$, be a connected absolutely almost simple F-group, and let $\pi_i \colon G_i \to \overline{G}_i$ be the isogeny onto the corresponding adjoint group. We will say that the subgroups Γ_i of $G_i(F)$ are *commensurable up to an F-isomorphism between $\overline{G}_1$ and $\overline{G}_2$* if there exists an F-isomorphism $\sigma \colon \overline{G}_1 \to \overline{G}_2$ such that $\sigma(\pi_1(\Gamma_1))$ is commensurable with $\pi(\Gamma_2)$ in the usual sense. The key observation is that the description of S-arithmetic subgroups in terms of triples $(\mathscr{G}, K, S)$ is very convenient for determining when two such subgroups are commensurable in the new generalized sense.

Proposition 3.4. *Let G_1 and G_2 be connected absolutely almost simple algebraic groups defined over a field F of characteristic zero, and for $i = 1, 2$, let Γ_i be a Zariski-dense $(\mathscr{G}_i, K_i, S_i)$-arithmetic subgroup of $G_i(F)$. Then Γ_1 and Γ_2 are commensurable up to an F-isomorphism between $\overline{G}_1$ and $\overline{G}_2$ if and only if $K_1 = K_2 =: K$, $S_1 = S_2$, and $\mathscr{G}_1$ and $\mathscr{G}_2$ are K-isomorphic.*

It follows from the above proposition that the arithmetic subgroups Γ_1, Γ_2, and Γ_3 constructed above, of $G(\mathbb{R})$, where $G = \mathrm{SO}_3(f)$, are pairwise noncommensurable: indeed, Γ_3, being defined over $\mathbb{Q}(\sqrt{2})$, cannot possibly be commensurable to Γ_1 or Γ_2 as these two groups are defined over $\mathbb{Q}$; at the same time, the noncommensurability of Γ_1 and Γ_2 is a consequence of the fact that $\mathrm{SO}_3(f_1)$ and $\mathrm{SO}_3(f_2)$ are not $\mathbb{Q}$-isomorphic since the quadratic forms f_1 and f_2 are not equivalent over $\mathbb{Q}$.

4. Results on weakly commensurable S-arithmetic subgroups

In view of Proposition 3.4, the central question in the analysis of weak commensurability of S-arithmetic subgroups is the following: *Suppose we are given two Zariski-dense S-arithmetic subgroups that are described in terms of triples. Which components of these triples coincide given the fact that the subgroups*

[3] This notion of arithmetic subgroups coincides with that in Margulis' book [1991] for absolutely simple adjoint groups.

are weakly commensurable? As the following result demonstrates, two of these components *must* coincide.

Theorem 4.1. *Let G_1 and G_2 be two connected absolutely almost simple algebraic groups defined over a field F of characteristic zero. If Zariski-dense $(\mathscr{G}_i, K_i, S_i)$-arithmetic subgroups Γ_i of $G_i(F)$, where $i = 1, 2$, are weakly commensurable for $i = 1, 2$, then $K_1 = K_2$ and $S_1 = S_2$.*

In general, the forms $\mathscr{G}_1$ and $\mathscr{G}_2$ do not have to be K-isomorphic (see [Prasad and Rapinchuk 2009], Examples 6.5 and 6.6, as well as the general construction in Section 9). In the next theorem we list the cases where it can nevertheless be asserted that $\mathscr{G}_1$ and $\mathscr{G}_2$ are necessarily K-isomorphic, and then give a general finiteness result for the number of K-isomorphism classes.

Theorem 4.2. *Let G_1 and G_2 be two connected absolutely almost simple algebraic groups defined over a field F of characteristic zero, of the* same *type different from A_n, D_{2n+1}, with $n > 1$, or E_6. If for $i = 1, 2$, $G_i(F)$ contain Zariski-dense weakly commensurable $(\mathscr{G}_i, K, S)$-arithmetic subgroups Γ_i, then $\mathscr{G}_1 \simeq \mathscr{G}_2$ over K, and hence Γ_1 and Γ_2 are commensurable up to an F-isomorphism between $\overline{G}_1$ and $\overline{G}_2$.*

In this theorem, type D_{2n} ($n \geqslant 2$) required special consideration. The case $n > 2$ was settled in [Prasad and Rapinchuk 2010a] using the techniques of [Prasad and Rapinchuk 2009] in conjunction with results on embeddings of fields with involutive automorphisms into simple algebras with involution. The remaining case of type D_4 was treated by Skip Garibaldi [2012], whose argument actually applies to all n and explains the result from the perspective of Galois cohomology, providing thereby a cohomological insight into the difference between the types D_{2n} and D_{2n+1}. We note that the types excluded in the theorem are precisely the types for which the automorphism $\alpha \mapsto -\alpha$ of the corresponding root system is not in the Weyl group. More importantly, all these types are honest exceptions to the theorem — a general Galois-cohomological construction of weakly commensurable, but not commensurable, Zariski-dense S-arithmetic subgroups for all of these types is given in [Prasad and Rapinchuk 2009, Section 9].

Theorem 4.3. *Let G_1 and G_2 be two connected absolutely almost simple groups defined over a field F of characteristic zero. Let Γ_1 be a Zariski-dense $(\mathscr{G}_1, K, S)$-arithmetic subgroup of $G_1(F)$. Then the set of K-isomorphism classes of K-forms $\mathscr{G}_2$ of $\overline{G}_2$ such that $G_2(F)$ contains a Zariski-dense $(\mathscr{G}_2, K, S)$-arithmetic subgroup weakly commensurable to Γ_1 is finite.*

In other words, the set of all Zariski-dense (K, S)-arithmetic subgroups of $G_2(F)$ which are weakly commensurable to a given Zariski-dense (K, S)-arithmetic subgroup is a union of finitely many commensurability classes.

A noteworthy fact about weak commensurability is that it has the following implication for the existence of unipotent elements in arithmetic subgroups (even though it is formulated entirely in terms of semisimple ones). We recall that a semisimple K-group is called *K-isotropic* if $\mathrm{rk}_K\, G > 0$; in characteristic zero, this is equivalent to the existence of nontrivial unipotent elements in $G(K)$. Moreover, if K is a number field then G is K-isotropic if and only if every S-arithmetic subgroup contains unipotent elements, for any S.

Theorem 4.4. *Let G_1 and G_2 be two connected absolutely almost simple algebraic groups defined over a field F of characteristic zero. For $i = 1, 2$, let Γ_i be a Zariski-dense $(\mathscr{G}_i, K, S)$-arithmetic subgroup of $G_i(F)$. If Γ_1 and Γ_2 are weakly commensurable then* $\mathrm{rk}_K \mathscr{G}_1 = \mathrm{rk}_K \mathscr{G}_2$; *in particular, if $\mathscr{G}_1$ is K-isotropic, then so is $\mathscr{G}_2$.*

We note that in [Prasad and Rapinchuk 2009, Section 7] we prove a somewhat more precise result, viz. that if G_1 and G_2 are of the same type, then the Tits indices of $\mathscr{G}_1/K$ and $\mathscr{G}_2/K$ are isomorphic, but we will not get into these technical details here.

The following result asserts that a lattice[4] which is weakly commensurable with an S-arithmetic group is itself S-arithmetic.

Theorem 4.5. *Let G_1 and G_2 be two connected absolutely almost simple algebraic groups defined over a nondiscrete locally compact field F of characteristic zero, and for $i = 1, 2$, let Γ_i be a Zariski-dense lattice in $G_i(F)$. Assume that Γ_1 is a (K, S)-arithmetic subgroup of $G_1(F)$. If Γ_1 and Γ_2 are weakly commensurable, then Γ_2 is a (K, S)-arithmetic subgroup of $G_2(F)$.*

5. Geometric applications

We are now in a position to give the precise statements of our results on isospectral and length-commensurable locally symmetric spaces. Throughout this subsection, for $i = 1, 2$, G_i will denote an absolutely simple real algebraic group and $\mathfrak{X}_i$ the symmetric space of $\mathscr{G}_i = G_i(\mathbb{R})$. Furthermore, given a discrete torsion-free subgroup $\Gamma_i \subset \mathscr{G}_i$, we let $\mathfrak{X}_{\Gamma_i} = \mathfrak{X}_i/\Gamma_i$ denote the corresponding locally symmetric space. The geometric results are basically obtained by combining Theorem 2.7 and Corollary 2.8 with the results on weakly commensurable subgroups from the previous section. It should be emphasized that when $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are both rank.one spaces and we are not in the exceptional case $(\mathscr{E})$ (which is the case, for example, for all hyperbolic n-manifolds with $n \geqslant 4$) our

[4] A discrete subgroup Γ of a locally compact topological group $\mathscr{G}$ is said to be a lattice in $\mathscr{G}$ if $\mathscr{G}/\Gamma$ carries a *finite* $\mathscr{G}$-invariant Borel measure.

results are *unconditional*, while in all other cases they depend on the validity of Schanuel's conjecture.

Now, applying Theorems 3.1 and 3.2 we obtain the following.

Theorem 5.1. *Let G_1 and G_2 be connected absolutely simple real algebraic groups, and let $\mathfrak{X}_{\Gamma_i}$ be a locally symmetric space of finite volume, of $\mathcal{G}_i$, for $i = 1, 2$. If $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are length-commensurable, then:*

(i) *either G_1 and G_2 are of same Killing–Cartan type, or one of them is of type B_n and the other is of type C_n for some $n \geqslant 3$;*

(ii) $K_{\Gamma_1} = K_{\Gamma_2}$.

It should be pointed out that assuming Schanuel's conjecture in all cases, one can prove this theorem (in fact, a much stronger statement; see Theorem 8.1) assuming only that Γ_1 and Γ_2 are finitely generated and Zariski-dense.

Next, using Theorems 4.2 and 4.3 we obtain

Theorem 5.2. *Let G_1 and G_2 be connected absolutely simple real algebraic groups, and let $\mathcal{G}_i = G_i(\mathbb{R})$, for $i = 1, 2$. Then the set of arithmetically defined locally symmetric spaces $\mathfrak{X}_{\Gamma_2}$ of $\mathcal{G}_2$, which are length-commensurable to a given arithmetically defined locally symmetric space $\mathfrak{X}_{\Gamma_1}$ of $\mathcal{G}_1$, is a union of finitely many commensurability classes. It in fact consists of a single commensurability class if G_1 and G_2 have the same type different from A_n, D_{2n+1}, with $n > 1$, or E_6.*

Furthermore, Theorems 4.4 and 4.5 imply the following rather surprising result which has so far defied all attempts of a purely geometric proof.

Theorem 5.3. *Let G_1 and G_2 be connected absolutely simple real algebraic groups, and let $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ be length-commensurable locally symmetric spaces of $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively, of finite volume. Assume that at least one of the spaces is arithmetically defined. Then the other space is also arithmetically defined, and the compactness of one of the spaces implies the compactness of the other.*

In fact, if one of the spaces is compact and the other is not, the weak length spectra $L(\mathfrak{X}_{\Gamma_1})$ and $L(\mathfrak{X}_{\Gamma_2})$ are quite different. See Theorem 8.6 for a precise statement. (We note that the proof of this result uses Schanuel's conjecture in all cases.)

Finally, we will describe some applications to isospectral compact locally symmetric spaces. So, in the remainder of this section, the locally symmetric spaces $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ as above will be assumed to be *compact*. Then, as we discussed in Section 1, the fact that $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are isospectral implies that $L(\mathfrak{X}_{\Gamma_1}) = L(\mathfrak{X}_{\Gamma_2})$, so we can use our results on length-commensurable spaces. Thus, in particular we obtain the following.

Theorem 5.4. *Assume* $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *are isospectral. If* Γ_1 *is arithmetic, then so is* Γ_2.

Thus, the Laplace spectrum can see if the fundamental group is arithmetic or not — to our knowledge, no results of this kind, particularly for general locally symmetric spaces, were previously known in spectral theory.

The following theorem settles the question "Can one hear the shape of a drum?" for arithmetically defined compact locally symmetric spaces.

Theorem 5.5. *Let* $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *be compact locally symmetric spaces associated with absolutely simple real algebraic groups* G_1 *and* G_2, *and assume that at least one of the spaces is arithmetically defined. If* $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *are isospectral then* $G_1 = G_2 := G$. *Moreover, unless* G *is of type* A_n, D_{2n+1} $(n > 1)$, *or* E_6, *the spaces* $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *are commensurable.*

It should be noted that our methods based on length-commensurability or weak commensurability leave room for the following ambiguity in the proof of Theorem 5.5: either $G_1 = G_2$ or G_1 and G_2 are $\mathbb{R}$-split forms of types B_n and C_n for some $n \geqslant 3$ - and this ambiguity is unavoidable; see the end of Section 7. The fact that in the latter case the locally symmetric spaces cannot be isospectral was shown by Sai-Kee Yeung [2011] by comparing the traces of the heat operator (without using Schanuel's conjecture), which leads to the statement of the theorem given above.

6. Absolutely almost simple algebraic groups having the same maximal tori

The analysis of weak commensurability is related to another natural problem in the theory of algebraic groups of characterizing absolutely almost simple K-groups having the same isomorphism/isogeny classes of maximal K-tori — the exact nature of this connection will be clarified in Theorem 9.8 and the subsequent discussion. Some aspects of this problem over local and global fields were considered in [Garge 2005] and [Kariyama 1989]. Another direction of research, which has already generated a number of results — we mention [Bayer-Fluckiger 2011; Garibaldi 2012; Lee 2012; Prasad and Rapinchuk 2010a] — is the investigation of local-global principles for embedding tori into absolutely almost simple algebraic groups as maximal tori (in particular, for embedding of commutative étale algebras with involutive automorphisms into simple algebras with involution); some number-theoretic applications of these results can be found, for example, in [Fiori 2012]. A detailed discussion of these issues would be an independent undertaking, so we will limit ourselves here to the following theorem:

Theorem 6.1 (Prasad and Rapinchuk 2009, Theorem 7.5; Garibaldi and Rapinchuk 2013, Proposition 1.3). (1) *Let G_1 and G_2 be connected absolutely almost simple algebraic groups defined over a number field K, and let L_i be the smallest Galois extension of K over which G_i becomes an inner form of a split group. If G_1 and G_2 have the same K-isogeny classes of maximal K-tori then either G_1 and G_2 are of the same Killing–Cartan type, or one of them is of type B_n and the other is of type C_n, and moreover, $L_1 = L_2$.*

(2) *Fix an absolutely almost simple K-group G. Then the set of isomorphism classes of all absolutely almost simple K-groups G' having the same K-isogeny classes of maximal K-tori is finite.*

(3) *Fix an absolutely almost simple simply connected K-group G whose Killing–Cartan type is different from A_n, D_{2n+1} $(n > 1)$ or E_6. Then any K-form G' of G (in other words, any absolutely almost simple simply connected K-group G' of the* same *type as G) that has the same K-isogeny classes of maximal K-tori as G, is isomorphic to G.*

The construction described in [Prasad and Rapinchuk 2009, Section 9] shows that the types excluded in Theorem 6.1(3) are honest exceptions; that is, for each of those types one can construct nonisomorphic absolutely almost simple simply connected K-groups G_1 and G_2 of this type over a number field K that have the same isomorphism classes of maximal K-tori. Furthermore, the analysis of the situation where G_1 and G_2 are of types B_n and C_n, respectively, over a number field K and have the same isomorphism/isogeny classes of maximal K-tori is given in Theorem 7.3 below (see [Garibaldi and Rapinchuk 2013, Theorem 1.4 and 1.5]).

Of course, the question about determining absolutely almost simple algebraic K-groups by their maximal K-tori makes sense over general fields. It is particularly interesting for division algebra where it can be reformulated as the following question which is somewhat reminiscent of Amitsur's famous theorem on generic splitting fields [Amitsur 1955; Gille and Szamuely 2006]: *What can one say about two finite-dimensional central division algebras D_1 and D_2 over the same field K given the fact that they have the same isomorphism classes of maximal subfields?* For recent results on this problem, see [Chernousov et al. 2012; Garibaldi and Saltman 2010; Krashen and McKinnie 2011; Rapinchuk and Rapinchuk 2010].

7. Weakly commensurable subgroups in groups of types B and C

Let G_1 and G_2 be absolutely almost simple algebraic groups over a field K of characteristic zero. According to Theorem 3.1, finitely generated weakly commensurable Zariski-dense subgroups $\Gamma_1 \subset G_1(K)$ and $\Gamma_2 \subset G_2(K)$ can exist only if G_1 and G_2 are of the same Killing–Cartan type or one of them is of type B_n and the other is of type C_n for some $n \geqslant 3$. Moreover, the results we described in Section 4 provide virtually complete answers to the key questions about weakly commensurable S-arithmetic subgroups in the case where G_1 and G_2 are *of the same type*. In this section, we will discuss recent results [Garibaldi and Rapinchuk 2013] that determine weakly commensurable arithmetic subgroups when G_1 is *of type* B_n and G_2 is *of type* C_n $(n \geqslant 3)$.

First of all, it should be pointed out that S-arithmetic subgroups in groups of types B_n and C_n can indeed be weakly commensurable. The underlying reason is that if G_1 is a split adjoint group of type B_n and G_2 is a split simply connected group of type C_n $(n \geqslant 2)$ over any field K of characteristic $\neq 2$, then G_1 and G_2 have the *same isomorphism classes of maximal K-tori*. For the reader's convenience we briefly recall the Galois-cohomological proof of this fact given in [Prasad and Rapinchuk 2009, Example 6.7].

It is well known that for any semisimple K-group G there is a natural bijection between the set of $G(K)$-conjugacy classes of maximal K-tori of G and the set

$$\mathscr{C}_K := \mathrm{Ker}\big(H^1(K,N) \to H^1(K,G)\big),$$

where T is a maximal K-torus of G and N is the normalizer of T in G (see [Prasad and Rapinchuk 2009, Lemma 9.1]). Let $W = N/T$ be the corresponding Weyl group and introduce the following natural maps in Galois cohomology:

$$\theta_K \colon H^1(K,N) \to H^1(K,W) \quad \text{and} \quad \nu_K \colon H^1(K,W) \to H^1(K,\mathrm{Aut}\,T).$$

To apply these considerations to the groups G_1 and G_2, we will denote by T_i a fixed maximal K-split torus of G_i and let N_i, W_i, $\mathscr{C}_K^{(i)}$, $\theta_K^{(i)}$ and $\nu_K^{(i)}$ be the corresponding objects attached to G_i. It follows from an explicit description of the root systems of types B_n and C_n that there exist K-isomorphisms $\varphi\colon T_1 \to T_2$ and $\psi\colon W_1 \to W_2$ such that for the natural action of W_i on T_i we have

$$\varphi(w \cdot t) = \psi(w) \cdot \varphi(t), \quad \text{for all } t \in T_1, w \in W_1.$$

Since G_i is K-split, we have $\theta_K^{(i)}(\mathscr{C}_K^{(i)}) = H^1(K, W_i)$ (see [Gille 2004; Kottwitz 1982; Raghunathan 2004]). So, ψ induces a natural bijection between $\theta_K^{(1)}(\mathscr{C}_K^{(1)})$ and $\theta_K^{(2)}(\mathscr{C}_K^{(2)})$. Finally, we observe that if S_i is a maximal K-torus of G_i in the $G_i(K)$-conjugacy class corresponding to $c_i \in \mathscr{C}_K^{(i)}$, then the K-isomorphism class of S_i is determined by $\nu_K^{(i)}(\theta_K^{(i)}(c_i))$, and if $\psi(\theta_K^{(1)}(c_1)) = \theta_K^{(2)}(c_2)$ then S_1 and S_2 are K-isomorphic. It follows that G_1 and G_2 have the same isomorphism classes of maximal K-tori, as required.

Subsequently, in [Garibaldi and Rapinchuk 2013] a more explicit explanation of this fact was given. More precisely, let A be a central simple algebra over K with a K-linear involution τ (involution of the first kind). We recall that τ is called *orthogonal* if $\dim_K A^\tau = n(n+1)/2$ and *symplectic* if $\dim_K A^\tau = n(n-1)/2$. Furthermore, if τ is orthogonal and $n = 2m+1$ ($m \geqslant 2$) then $A = M_n(K)$ and the corresponding algebraic group $G = \mathrm{SU}(A, \tau)$ coincides with the orthogonal group $\mathrm{SO}_n(q)$ of a nondegenerate n-dimensional quadratic form $q = q_\tau$ over K, hence is a simple adjoint algebraic K-group of type B_m (note that the K-rank of G equals the Witt index of q). If τ is symplectic then necessarily $n = 2m$ and $G = \mathrm{SU}(A, \tau)$ is an almost simple simply connected K-group of type C_m; moreover G is K-split if and only if $A = M_n(K)$, in which case G is of course isomorphic to Sp_{2m}. Next, in all cases, any maximal K-torus T of G has the form $T = \mathrm{SU}(E, \sigma)$ where E is a τ-invariant n-dimensional commutative étale K-subalgebra of A such that for $\sigma = \tau | E$ we have

$$\dim E^\sigma = \left[\frac{n+1}{2}\right]. \tag{7}$$

So, the question whether $G = \mathrm{SU}(A, \tau)$, with A and τ as above, has a maximal K-torus of a specific type can be reformulated as follows: Let (E, σ) be an n-dimensional commutative étale K-algebra with an involutive K-automorphism σ satisfying (7). When does there exist an embedding $(E, \sigma) \hookrightarrow (A, \tau)$ as algebras with involution? While in the general case this question is nontrivial (see [Prasad and Rapinchuk 2010a]), the answer in the case where the group G splits over K is quite straightforward.

Proposition 7.1 [Garibaldi and Rapinchuk 2013, 2.3 and 2.5]. *Let $A = M_n(K)$ with a K-linear involution τ, and let (E, σ) be an n-dimensional commutative étale K-algebra with involution satisfying* (7). *In each of the following situations, there exists a K-embedding $(E, \sigma) \hookrightarrow (A, \tau)$:*

(1) *τ is symplectic.*

(2) *$n = 2m+1$ and τ is orthogonal such that the corresponding quadratic form q_τ has Witt index m.*

(This proposition should be viewed as an analogue for algebras with involution of the following result of Steinberg [1965]: Let G_0 be a *quasisplit* simply connected almost simple algebraic group over a field K. Then given an inner form G of G_0, *any maximal K-torus T of G admits a K-embedding into G_0.* While the proof of this result is rather technical, the proof of Proposition 7.1, as well as of the corresponding assertion for the algebras with involution arising in the description of algebraic groups of type A and D is completely elementary; see [Garibaldi and Rapinchuk 2013, Section 2].)

Now, fix $n \geqslant 2$ and let $A_1 = M_{n_1}(K)$, where $n_1 = 2n+1$, with an orthogonal involution τ_1 such that the Witt index of the corresponding quadratic form q_{τ_1} is n, and let $A_2 = M_{n_2}(K)$, where $n_2 = 2n$, with a symplectic involution τ_2. According to Proposition 7.1, for $i = 1, 2$, the maximal K-tori of $G_i = \mathrm{SU}(A_i, \tau_i)$ are of the form $T_i = \mathrm{SU}(E_i, \sigma_i)$ for a commutative étale K-algebra E_i of dimension n_i with an involution σ_i satisfying (7). On the other hand, the correspondence

$$(E_2, \sigma_2) \mapsto (E_1, \sigma_1) := (E_2 \times K\ ,\ \sigma_2 \times \mathrm{id}_K)$$

defines a natural bijection between the isomorphism classes of commutative étale K-algebras with involution satisfying (7), of dimension n_2 and n_1, respectively. Since $\mathrm{SU}(E_1, \sigma_1) = \mathrm{SU}(E_2, \sigma_2)$ in these notations, we again obtain that G_1 and G_2 have the same isomorphism classes of maximal K-tori (see [Garibaldi and Rapinchuk 2013, Remark 2.6]).

Now, let K be a number field and S be any finite set of places of K containing the set V_∞^K of archimedean places. Furthermore, let G_1 be a split adjoint K-group of type B_n, and G_2 be a split simply connected K-group of type C_n $(n \geqslant 2)$. Then the fact, discussed above, that G_1 and G_2 have the same isomorphism classes of maximal K-tori immediately implies that the S-arithmetic subgroups in G_1 and G_2 are weakly commensurable (see [Prasad and Rapinchuk 2009, Examples 6.5 and 6.7]).

A complete determination of weakly commensurable S-arithmetic subgroups in algebraic groups G_1 and G_2 of types B_n and C_n $(n \geqslant 3)$ respectively was recently obtained by Skip Garibaldi and the second-named author [Garibaldi and Rapinchuk 2013]. To formulate the result we need the following definition. Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be absolutely almost simple algebraic groups of types B_n and C_n with $n \geqslant 2$, respectively, over a number field K. We say that $\mathscr{G}_1$ and $\mathscr{G}_2$ are *twins* (over K) if for each place v of K, either both groups are split or both are anisotropic over the completion K_v. (We note that since groups of these types cannot be anisotropic over K_v when v is nonarchimedean, our condition effectively says that $\mathscr{G}_1$ and $\mathscr{G}_2$ must be K_v-split for *all* nonarchimedean v.)

Theorem 7.2 [Garibaldi and Rapinchuk 2013, Theorem 1.2]. *Let G_1 and G_2 be absolutely almost simple algebraic groups over a field F of characteristic zero having Killing–Cartan types B_n and C_n $(n \geqslant 3)$ respectively, and let Γ_i be a Zariski-dense $(\mathscr{G}_i, K, S)$-arithmetic subgroup of $G_i(F)$ for $i = 1, 2$. Then Γ_1 and Γ_2 are weakly commensurable if and only if the groups $\mathscr{G}_1$ and $\mathscr{G}_2$ are twins.*

(We recall that according to Theorem 4.1, if Zariski-dense $(\mathscr{G}_1, K_1, S_1)$- and $(\mathscr{G}_2, K_2, S_2)$-arithmetic subgroups are weakly commensurable then necessarily $K_1 = K_2$ and $S_1 = S_2$, so Theorem 7.2 in fact treats the most general situation.)

The necessity is proved using generic tori (Section 9) in conjunction with the analysis of maximal tori in real groups of types B_n and C_n (which can also be found in [Đoković and Thǎńg 1994]). The proof of sufficiency is obtained using the above description of maximal K-tori in terms of commutative étale K-subalgebras with involution and the local-global results for the existence of an embedding of commutative étale algebras with involution into simple algebras with involution established in [Prasad and Rapinchuk 2010a]; an alternative argument along the lines outlined in the beginning of this section can be given using Galois cohomology of algebraic groups (see [Garibaldi and Rapinchuk 2013, Section 9]).

As we already mentioned in Section 6, the analysis of weak commensurability involved in the proof of Theorem 7.2 leads to, and at the same time depends on, the following result describing when groups of types B_n and C_n have the same isogeny/isomorphism classes of maximal K-tori.

Theorem 7.3 [Garibaldi and Rapinchuk 2013, Theorem 1.4]. *Let G_1 and G_2 be absolutely almost simple algebraic groups over a number field K of types B_n and C_n respectively for some $n \geqslant 3$.*

(1) *The groups G_1 and G_2 have the same* isogeny *classes of maximal K-tori if and only if they are twins.*

(2) *The groups G_1 and G_2 have the same* isomorphism *classes of maximal K-tori if and only if they are twins*, *G_1 is adjoint*, *and G_2 is simply connected.*

Theorem 7.2 has the following interesting geometric applications [Prasad and Rapinchuk 2013]. Again, let G_1 and G_2 be simple real algebraic groups of types B_n and C_n respectively. For $i = 1, 2$, let Γ_i be a discrete torsion-free $(\mathcal{G}_i, K)$-arithmetic subgroup of $\mathcal{G}_i = G_i(\mathbb{R})$, and let $\mathfrak{X}_{\Gamma_i}$ be the corresponding locally symmetric space of $\mathcal{G}_i$. Then if $\mathcal{G}_1$ and $\mathcal{G}_2$ are not twins, the locally symmetric spaces $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are *not* length-commensurable. As one application of this result, we would like to point out the following assertion: *Let M_1 be an arithmetic quotient of the real hyperbolic space $\mathbb{H}^p$ ($p \geqslant 5$) and M_2 be an arithmetic quotient of the quaternionic hyperbolic space $\mathbb{H}^q_{\mathbf{H}}$ ($q \geqslant 2$). Then M_1 and M_2 are not length-commensurable.* The results of [Garibaldi and Rapinchuk 2013] are used to handle the case $p = 2n$ and $q = n - 1$ for some $n \geqslant 3$; for other values of p and q, the claim follows from Theorem 5.1.

On the other hand, suppose $G_1 = \mathrm{SO}(n+1, n)$ and $G_2 = \mathrm{Sp}_{2n}$ over $\mathbb{R}$ (i.e., G_1 and G_2 are the $\mathbb{R}$-split forms of types B_n and C_n, respectively) for some $n \geqslant 3$. Furthermore, let Γ_i be a discrete torsion-free $(\mathcal{G}_i, K)$-arithmetic subgroup of $\mathcal{G}_i$ for $i = 1, 2$, and let $M_i = \mathfrak{X}_{\Gamma_i}$. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are twins then

$$\mathbb{Q} \cdot L(M_2) = \lambda \cdot \mathbb{Q} \cdot L(M_1), \quad \text{where } \lambda = \sqrt{\frac{2n+2}{2n-1}}.$$

Thus, despite the fact that they are associated with groups of different types, the locally symmetric spaces M_1 and M_2 can be made length-commensurable by scaling the metric on one of them; this, however, will *never* make them isospectral [Yeung 2011]. What is interesting is that so far this is the *only* situation in our analysis commensurability of isospectral and length-commensurable locally symmetric spaces where isospectrality manifests itself as an essentially stronger condition.

8. On the fields generated by the lengths of closed geodesics

In Section 5 (and also at the end of Section 7) we discussed the consequences of length-commensurability of two locally symmetric spaces M_1 and M_2; our focus in this section will be on the consequences of *nonlength-commensurability* of M_1 and M_2. More precisely, we will explore how different in this case the sets $L(M_1)$ and $L(M_2)$ (or $\mathbb{Q} \cdot L(M_1)$ and $\mathbb{Q} \cdot L(M_2)$) are and whether they can in fact be related in *any* reasonable way? Of course, one can ask a number of specific questions that fit this general perspective: for example, can $L(M_1)$ and $L(M_2)$ differ only in a finite number of elements, in other words, can the symmetric difference $L(M_1) \bigtriangleup L(M_2)$ be finite? Or can it happen that $\overline{\mathbb{Q}} \cdot L(M_1) = \overline{\mathbb{Q}} \cdot L(M_2)$, where $\overline{\mathbb{Q}}$ is the field of all algebraic numbers; in other words, can the use of the field $\overline{\mathbb{Q}}$ in place of $\mathbb{Q}$ in the definition of length-commensurability essentially change this relation? One relation between $L(M_1)$ and $L(M_2)$ that would make a lot of sense geometrically is that of similarity, requiring that there be a real number $\alpha > 0$ such that

$$L(M_2) = \alpha \cdot L(M_1) \quad (\text{or } \mathbb{Q} \cdot L(M_2) = \alpha \cdot \mathbb{Q} \cdot L(M_1)),$$

which means that M_1 and M_2 can be made isolength spectral (or length-commensurable) by scaling the metric on one of them. From the algebraic standpoint, one can generalize this relation by considering arbitrary polynomial relations between $L(M_1)$ and $L(M_2)$ instead of just linear relations although this perhaps does not have a clear geometric interpretation. To formalize this general idea, we need to introduce some additional notations and definitions.

For a Riemannian manifold M, we let $\mathcal{F}(M)$ denote the subfield of $\mathbb{R}$ generated by the set $L(M)$. Given two Riemannian manifolds M_1 and M_2, we set $\mathcal{F}_i = \mathcal{F}(M_i)$ for $i \in \{1, 2\}$ and consider the following condition:

(T_i) *The compositum $\mathcal{F}_1\mathcal{F}_2$ has infinite transcendence degree over the field $\mathcal{F}_{3-i}$.*

Informally, this condition means that $L(M_i)$ contains "many" elements which are algebraically independent from all the elements of $L(M_{3-i})$, implying the nonexistence of any nontrivial polynomial dependence between $L(M_1)$ and $L(M_2)$. In particular, (T_i) implies the following condition:

(N_i) $L(M_i) \not\subset A \cdot \mathbb{Q} \cdot L(M_{3-i})$ *for any finite set* A *of real numbers.*

In [Prasad and Rapinchuk 2013], we have proved a series of results asserting that if $M_i = \mathfrak{X}_{\Gamma_i}$ for $i = 1, 2$, are the quotients of symmetric spaces $\mathfrak{X}_i$ associated with absolutely simple real algebraic groups G_i by Zariski-dense discrete torsion-free subgroups $\Gamma_i \subset G_i(\mathbb{R})$, then in many situations the fact that M_1 and M_2 are not length-commensurable implies that conditions (T_i) and (N_i) hold for at least one $i \in \{1, 2\}$. To give precise formulations, in addition to the standard notations used earlier, we let w_i denote the order of the (absolute) Weyl group of G_i. We also need to emphasize that *all geometric results in* [Prasad and Rapinchuk 2013] *assume the validity of Shanuel's conjecture.* This assumption, however, enables one to establish results that are somewhat stronger than the corresponding results in Section 5 and do not require that Γ_1 and Γ_2 have property (A) (see Theorem 2.7 and Corollary 2.8). We begin with the following result which strengthens Theorem 5.1.

Theorem 8.1. *Assume that the Zariski-dense subgroups* Γ_1 *and* Γ_2 *are finitely generated* (*which is automatically the case if these subgroups are lattices*).

(1) *If* $w_1 > w_2$ *then* (T_1) *holds.*

(2) *If* $w_1 = w_2$ *but* $K_{\Gamma_1} \not\subset K_{\Gamma_2}$ *then again* (T_1) *holds.*

Thus, unless $w_1 = w_2$ *and* $K_{\Gamma_1} = K_{\Gamma_2}$, *the condition* (T_i) *holds for at least one* $i \in \{1, 2\}$; *in particular,* M_1 *and* M_2 *are not length-commensurable.*

As follows from Theorem 8.1, we only need to consider the case where $w_1 = w_2$, which we will assume — recall that this entails that either G_1 and G_2 are of the same Killing–Cartan type, or one of them is of type B_n and the other is of type C_n ($n \geqslant 3$). Then it is convenient to divide our results for *arithmetic* subgroups Γ_1 and Γ_2 into three theorems: the first one will treat the case where G_1 and G_2 are of the *same type* which is different from A_n, D_{2n+1} ($n > 1$) and E_6, the second one — the case where both G_1 and G_2 are one of the types A_n, D_{2n+1} ($n > 1$) and E_6, and the third one – the case where G_1 is of type B_n and G_2 is of type C_n for some $n \geqslant 3$.

Theorem 8.2. *With notations as above, assume that* G_1 *and* G_2 *are of the same Killing–Cartan type which is different from* A_n, D_{2n+1} ($n > 1$) *and* E_6 *and that the subgroups* Γ_1 *and* Γ_2 *are arithmetic. Then either* $M_1 = \mathfrak{X}_{\Gamma_1}$ *and* $M_2 = \mathfrak{X}_{\Gamma_2}$ *are commensurable* (*hence also length-commensurable*), *or conditions* (T_i) *and* (N_i) *hold for at least one* $i \in \{1, 2\}$.

(This theorem strengthens part of Theorem 5.2. We also note that (T_i) and (N_i) may not hold for *both* $i = 1$ and 2; in fact, it is possible that one of $L(M_1)$ and $L(M_2)$ is contained in the other.)

Theorem 8.3. *Again, keep the above notations and assume that the common Killing–Cartan type of G_1 and G_2 is one of the following: A_n, D_{2n+1} $(n > 1)$ or E_6 and that the subgroups Γ_1 and Γ_2 are arithmetic. Assume in addition that $K_{\Gamma_i} \neq \mathbb{Q}$ for at least one $i \in \{1, 2\}$. Then either M_1 and M_2 are length-commensurable (although not necessarily commensurable), or conditions (T_i) and (N_i) hold for at least one $i \in \{1, 2\}$.*

To illustrate possible applications of these theorems, we will give in Theorem 1.1 explicit statements for real hyperbolic manifolds; similar results are available for complex and quaternionic hyperbolic spaces.

Corollary 8.4. *Let M_i $(i = 1, 2)$ be the quotients of the real hyperbolic space $\mathbb{H}^{d_i}$ with $d_i \neq 3$ by a torsion-free Zariski-dense discrete subgroup Γ_i of $G_i(\mathbb{R})$, where $G_i = \mathrm{PSO}(d_i, 1)$.*

(i) *If $d_1 > d_2$ then conditions (T_1) and (N_1) hold.*

(ii) *If $d_1 = d_2$ but $K_{\Gamma_1} \not\subset K_{\Gamma_2}$ then again conditions (T_1) and (N_1) hold.*

Thus, unless $d_1 = d_2$ and $K_{\Gamma_1} = K_{\Gamma_2}$, conditions (T_i) and (N_i) hold for at least one $i \in \{1, 2\}$.

Assume now that $d_1 = d_2 =: d$ and the subgroups Γ_1 and Γ_2 are arithmetic.

(iii) *If d is either even or is congruent to $3 \pmod 4$, then either M_1 and M_2 are commensurable, hence length-commensurable, or (T_i) and (N_i) hold for at least one $i \in \{1, 2\}$.*

(iv) *If $d \equiv 1 \pmod 4$ and in addition $K_{\Gamma_i} \neq \mathbb{Q}$ for at least one $i \in \{1, 2\}$ then either M_1 and M_2 are length-commensurable (although not necessarily commensurable), or conditions (T_i) and (N_i) hold for at least one $i \in \{1, 2\}$.*

Now, we consider the case where one of the groups is of type B_n and the other of type C_n $(n \geqslant 3)$. The theorem below strengthens the results of Section 7.

Theorem 8.5. *Notations as above, assume that G_1 is of type B_n and G_2 is of type C_n for some $n \geqslant 3$ and the subgroups Γ_1 and Γ_2 are arithmetic. Then either (T_i) and (N_i) hold for at least one $i \in \{1, 2\}$, or*

$$\mathbb{Q} \cdot L(M_2) = \lambda \cdot \mathbb{Q} \cdot L(M_1), \quad \text{where } \lambda = \sqrt{\frac{2n+2}{2n-1}}.$$

The following interesting result holds for all types (cf. Theorem 5.3):

Theorem 8.6. *For $i = 1, 2$, let $M_i = \mathfrak{X}_{\Gamma_i}$ be an arithmetically defined locally symmetric space, and assume that $w_1 = w_2$. If M_2 is compact and M_1 is not, then conditions (T_1) and (N_1) hold.*

Finally, we have the following result, which shows that the notion of "length-similarity" for arithmetically defined locally symmetric spaces is redundant if G_1 and G_2 are of the same type (but compare Theorem 8.5 for the case where G_1 and G_2 are of types B_n and C_n).

Corollary 8.7. *Let $M_i = \mathfrak{X}_{\Gamma_i}$, for $i = 1, 2$, be arithmetically defined locally symmetric spaces. Assume that there exists $\lambda \in \mathbb{R}_{>0}$ such that*

$$\mathbb{Q} \cdot L(M_1) = \lambda \cdot \mathbb{Q} \cdot L(M_2).$$

(1) *If G_1 and G_2 are of the same type which is different from A_n, D_{2n+1} $(n > 1)$ and E_6, then M_1 and M_2 are commensurable, hence length-commensurable.*

(2) *If G_1 and G_2 are of the same type which is one of the following: A_n, D_{2n+1} $(n > 1)$ or E_6, then, provided that $K_{\Gamma_i} \neq \mathbb{Q}$ for at least one $i \in \{1, 2\}$, the spaces M_1 and M_2 are length-commensurable (although not necessarily commensurable).*

The proofs of the results in this section use a generalization of the notion of weak commensurability which we termed *weak containment*. To give a precise definition, we temporarily return to the general set-up where G_1 and G_2 are semisimple algebraic groups defined over a field F of characteristic zero, and Γ_i is a Zariski-dense subgroup of $G_i(F)$ for $i = 1, 2$.

Definition 8.8. (a) Semisimple elements

$$\gamma_1^{(1)}, \ldots, \gamma_{m_1}^{(1)} \in \Gamma_1$$

are *weakly contained* in Γ_2 if there are semisimple elements $\gamma_1^{(2)}, \ldots, \gamma_{m_2}^{(2)}$ such that

$$\chi_1^{(1)}(\gamma_1^{(1)}) \cdots \chi_{m_1}^{(1)}(\gamma_{m_1}^{(1)}) = \chi_1^{(2)}(\gamma_1^{(2)}) \cdots \chi_{m_2}^{(2)}(\gamma_{m_2}^{(2)}) \neq 1, \tag{8}$$

for some maximal F-tori $T_k^{(j)}$ of G_j whose group of F-rational points contains elements $\gamma_k^{(j)}$ and some characters $\chi_k^{(j)}$ of $T_k^{(j)}$ for $j \in \{1, 2\}$ and $k \leqslant m_j$.[5]

(b) Semisimple elements $\gamma_1^{(1)}, \ldots, \gamma_{m_1}^{(1)} \in \Gamma_1$ are *multiplicatively independent* if for some (equivalently, any) choice of maximal F-tori $T_i^{(1)}$ of G_1 such that $\gamma_i^{(1)} \in T_i(F)$ for $i \leqslant m_1$, a relation of the form

$$\chi_1^{(1)}(\gamma_1^{(1)}) \cdots \chi_{m_1}^{(1)}(\gamma_{m_1}^{(1)}) = 1,$$

[5]Note that (8) means that the subgroups of $\overline{F}^{\times}$ generated by the eigenvalues of $\gamma_1^{(1)}, \ldots, \gamma_{m_1}^{(1)}$ and by those of $\gamma_1^{(2)}, \ldots, \gamma_{m_2}^{(2)}$ for some (equivalently, any) matrix realizations of $G_1 \subset \mathrm{GL}_{N_1}$ and $G_2 \subset \mathrm{GL}_{N_2}$, intersect nontrivially.

where $\chi_i \in X(T_i)$ implies that

$$\chi_1^{(1)}(\gamma_1^{(1)}) = \cdots = \chi_{m_1}^{(1)}(\gamma_{m_1}) = 1.$$

(c) We say that Γ_1 and Γ_2 as above satisfy *property* (C_i), where $i = 1$ or 2, if for any $m \geqslant 1$ there exist semisimple elements $\gamma_1^{(i)}, \ldots, \gamma_m^{(i)} \in \Gamma_i$ of infinite order that are multiplicatively independent and are *not* weakly contained in Γ_{3-i}.

Using Schanuel's conjecture, we proved in [Prasad and Rapinchuk 2013, Corollary 7.3] that *if $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are locally symmetric spaces as above with finitely generated Zariski-dense fundamental groups Γ_1 and Γ_2, then the fact that these groups satisfy property (C_i) for some $i \in \{1, 2\}$ implies that the locally symmetric spaces satisfy conditions (T_i) and (N_i) for the same i.* So, the way we prove Theorems 8.1–8.3 and 8.5–8.6 is by showing that condition (C_i) holds in the respective situations. For example, Theorem 8.1 is a consequence of the following algebraic result.

Theorem 8.9. *Assume that Γ_1 and Γ_2 are finitely generated (and Zariski-dense).*

(i) *If $w_1 > w_2$ then condition (C_1) holds.*

(ii) *If $w_1 = w_2$ but $K_{\Gamma_1} \not\subset K_{\Gamma_2}$ then again (C_1) hods.*

Thus, unless $w_1 = w_2$ and $K_{\Gamma_1} = K_{\Gamma_2}$, condition (C_i) holds for at least one $i \in \{1, 2\}$.

In [Prasad and Rapinchuk 2013] we prove much more precise results in the case where the Γ_i are arithmetic, which leads to the geometric applications described above. We refer the interested reader to [Prasad and Rapinchuk 2013] for the technical formulations of these results; a point we would like to make here, however, is that our "algebraic" results (i.e., those asserting that condition (C_i) holds in certain situations) do not depend on Schanuel's conjecture.

9. Generic elements and tori

The analysis of weak commensurability and its variations in [Garibaldi and Rapinchuk 2013; Prasad and Rapinchuk 2009; 2013] relies on the remarkable fact, first established in [Prasad and Rapinchuk 2003], that any Zariski-dense subgroup of the group of rational points of a semisimple group over a finitely generated field of characteristic zero contains special elements, to be called *generic elements* here. It is convenient to begin our discussion of these elements with the definition of *generic tori*.

Let G be a connected semisimple algebraic group defined over an infinite field K. Fix a maximal K-torus T of G, and, as usual, let $\Phi = \Phi(G, T)$ denote the corresponding root system, and let $W(G, T)$ be its Weyl group. Furthermore, we let K_T denote the (minimal) splitting field of T in a fixed separable closure $\overline{K}$

of K. Then the natural action of the Galois group $\mathrm{Gal}(K_T/K)$ on the character group $X(T)$ of T induces an injective homomorphism

$$\theta_T\colon \mathrm{Gal}(K_T/K) \to \mathrm{Aut}(\Phi(G,T)).$$

We say that T is *generic* (over K) if

$$\theta_T(\mathrm{Gal}(K_T/K)) \supset W(G,T). \tag{9}$$

For example, any maximal K-torus of $G = \mathrm{SL}_n/K$ is of the form

$$T = \mathrm{R}^{(1)}_{E/K}(\mathrm{GL}_1),$$

for some n-dimensional commutative étale K-algebra E. Then such a torus is generic over K if and only if E is a separable field extension of K and the Galois group of the normal closure L of E over K is isomorphic to the symmetric group S_n. It is well known that for each $n \geqslant 2$ one can write down a system of congruences such that any monic polynomial $f(t) \in \mathbb{Z}[t]$ satisfying this system of congruences has Galois group S_n. It turns out that one can prove a similar statement for maximal tori of an arbitrary semisimple algebraic group G over a finitely generated field K of characteristic zero (see [Prasad and Rapinchuk 2003, Theorem 3]). In order to avoid technical details, we will restrict ourselves here to the case of absolutely almost simple groups.

Theorem 9.1 [Prasad and Rapinchuk 2009, Theorem 3.1]. *Let G be a connected absolutely almost simple algebraic group over a finitely generated field K of characteristic zero, and let r be the number of nontrivial conjugacy classes of the Weyl group of G.*

(1) *There exist r inequivalent nontrivial discrete valuations $v_1, \dots, v_r$ of K such that the completion K_{v_i} is locally compact and G splits over K_{v_i} for all $i = 1, \dots, r$.*

(2) *For any choice of discrete valuations $v_1, \dots, v_r$ as in* (1), *one can find maximal K_{v_i}-tori $T(v_i)$ of G, one for each $i \in \{1, \dots, r\}$, with the property that any maximal K-torus T of G which is conjugate to $T(v_i)$ by an element of $G(K_{v_i})$, for all $i = 1, \dots, r$, is generic* (*i.e., the inclusion* (9) *holds*).

The first assertion is an immediate consequence of the following, which actually shows that we can find the v_j's so that $K_{v_j} = \mathbb{Q}_{p_j}$, where $p_1, \dots, p_r$ are distinct primes.

Proposition 9.2 [Prasad and Rapinchuk 2002; 2003]. *Let $\mathcal{K}$ be a finitely generated field of characteristic zero and $\mathcal{R} \subset \mathcal{K}$ a finitely generated subring. Then there exists an infinite set Π of primes such that for each $p \in \Pi$ there exists an embedding $\varepsilon_p\colon \mathcal{K} \hookrightarrow \mathbb{Q}_p$ with the property $\varepsilon_p(\mathcal{R}) \subset \mathbb{Z}_p$.*

To sketch a proof of the second assertion of Theorem 9.1, we fix a maximal K-torus T_0 of G. Given any other maximal torus T of G defined over an extension F of K there exists $g \in G(\bar{F})$ such that $T = \iota_g(T_0)$, where $\iota_g(x) = gxg^{-1}$. Then ι_g induces an isomorphism between the Weyl groups $W(G, T_0)$ and $W(G, T)$. A different choice of g will change this isomorphism by an inner automorphism of the Weyl group, implying that there is a *canonical bijection* between the sets $[W(G, T_0)]$ and $[W(G, T)]$ of conjugacy classes in the respective groups; we will denote this bijection by $\iota_{T_0,T}$.

Now, let v be a nontrivial discrete valuation of K such that the completion K_v is locally compact and splits T_0. Using the Frobenius automorphism of the maximal unramified extension K_v^{ur} in conjunction with the fact that $H^1(K_v, \widetilde{G})$, where $\widetilde{G}$ is the simply connected cover of G, vanishes (see [Bruhat and Tits 1987; Kneser 1965a; 1965b]),[6] one shows that given a nontrivial conjugacy class $c \in [W(G, T_0)]$, one can find a maximal K_v-torus $T(v, c)$ such that given any maximal K_v-torus T of G that is conjugate to $T(v, c)$ by an element of $G(K_v)$, for its splitting field K_{vT} we have

$$\theta_T(\mathrm{Gal}(K_{vT}/K_v)) \cap \iota_{T_0,T}(c) \neq \varnothing. \tag{10}$$

Now, if $v_1, \dots, v_r$ are as in part (1), then using the weak approximation property of the variety of maximal tori (see [Platonov and Rapinchuk 1994, Corollary 7.3]), one can pick a maximal K-torus T_0 which splits over K_{v_i} for all $i = 1, \dots, r$. Let $c_1, \dots, c_r$ be the nontrivial conjugacy classes of $W(G, T_0)$. Set $T(v_i) = T(v_i, c_i)$ for $i = 1, \dots, r$ in the above notation. Then it is not difficult to show that the tori $T(v_1), \dots, T(v_r)$ are as required.

The method described above enables one to construct generic tori with various additional properties, in particular, having prescribed local behavior.

Corollary 9.3 [Prasad and Rapinchuk 2009, Corollary 3.2]. *Let G and K be as in Theorem 9.1, and let V be a finite set of inequivalent nontrivial rank* 1 *valuations of K. Suppose that for each $v \in V$ we are given a maximal K_v-torus $T(v)$ of G. Then there exists a maximal K-torus T of G for which* (9) *holds and which is conjugate to $T(v)$ by an element of $G(K_v)$, for all $v \in V$.*

It should be noted that the method of p-adic embeddings that we used in the proof of Theorem 9.1, and which is based on Proposition 9.2, has many other applications; see [Prasad and Rapinchuk 2010b].

[6] One can alternatively use the fact that if we endow $\widetilde{G}$ with the structure of a group scheme over $\mathbb{O}_v$ as a Chevalley group, then $H^1(K_v^{\mathrm{ur}}/K_v, \widetilde{G}(\mathbb{O}_v^{\mathrm{ur}}))$, where K_v^{ur} is the maximal unramified extension of K_v with the valuation ring $\mathbb{O}_v^{\mathrm{ur}}$, vanishes, which follows from Lang's theorem [1956] or its generalization due to Steinberg [1965]; see [Platonov and Rapinchuk 1994, Theorem 6.8].

We are now prepared to discuss *generic elements* whose existence in any finitely generated Zariski-dense subgroup is the core issue in this section.

Definition 9.4. Let G be a connected semisimple algebraic group defined over a field K. A regular semisimple element $g \in G(K)$ is called *generic* (over K) if the maximal torus $T := Z_G(g)^\circ$ is generic (over K). We shall refer to T as the torus associated with g.

Before proceeding with the discussion of generic elements, we would like to point out that some authors adopt a slight variant of this definition by requiring that the extension K_g of K generated by the eigenvalues of g be "generic", which is more consistent with the notion of a "generic polynomial" in Galois theory. We note that $K_g \subset K_T$ making the Galois group $\mathrm{Gal}(K_g/K)$ a *quotient* of the group $\mathrm{Gal}(K_T/K)$. Then the requirement that the order of $\mathrm{Gal}(K_g/K)$ be divisible by $|W(G,T)|$ (which is probably one of the most natural ways to express the "genericity" of K_g/K) a priori may not imply the inclusion (9), which is most commonly used in applications (although, by Lemma 4.1 of [Prasad and Rapinchuk 2009], the former does imply the latter if G is an inner form of a split group — in particular, G is absolutely almost simple of type different from A_n $(n > 1)$, D_n $(n \geqslant 4)$ and E_6). On the other hand, even for a regular element $g \in T(K)$, where T is a maximal generic K-torus of G, the field K_g may be strictly smaller than K_T. (Example: Let $G = \mathrm{PSL}_2$ over K, and let T be a maximal K-torus of G of the form $\mathrm{R}^{(1)}_{L/K}(\mathrm{GL}_1)$ where L/K is a quadratic extension; then an element $g \in T(K)$ of order two is regular with $K_g = K$, while $K_T = L$). This problem, however, does not arise if G is absolutely simple, T is generic and $g \in T(K)$ has infinite order. Indeed, then the K-torus T is *irreducible*; that is, it does not contain any proper K-subtori since $W(G,T)$ acts irreducibly on $X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ (cf. [Prasad and Rapinchuk 2001]). It follows that every element $g \in T(K)$ of infinite order generates a Zariski-dense subgroup of T, hence $K_g = K_T$, so the order of $\mathrm{Gal}(K_g/K)$ is divisible by $|W(G,T)|$. Recall that according to a famous result of Selberg [1972, Theorem 6.11], any finitely generated subgroup Γ of $G(K)$ contains a torsion-free subgroup Γ' of finite index, which therefore is also Zariski-dense. So, the following theorem (Theorem 9.6) that asserts the existence of generic elements in an arbitrary Zariski-dense subgroup in the sense of our definition also implies the existence of generic elements in the sense of the other definition.

9.5. Let K be a field and G a connected absolutely almost simple K-group. Let $g \in G(K)$ be a generic element and let $T = Z_G(g)^\circ$. Then $g \in T(K)$ (see [Borel 1991, Corollary 11.12]). As T does not contain proper K-subtori, the cyclic group generated by any t $(\in T(K))$ of infinite order is Zariski-dense in T. So $Z_G(t) = Z_G(T) = T$; which implies that t is generic (over K) and $Z_G(t)$ is connected; in particular, if g is of infinite order, then $Z_G(g)$ $(= T)$ is connected. Moreover, if $n \in G(K)$ is such that ntn^{-1} commutes with t, then n lies in the normalizer $N_G(T)(K)$ of T in $G(K)$. If x is an element of $G(K)$ of infinite order such that for some nonzero integer a, $t := x^a$ lies in $T(K)$, then as x commutes with t, it commutes with T and hence it lies in $T(K)$.

It is also clear that the tori associated to two generic elements are equal if and only if the elements commute.

It is easily seen that the natural action of $N_G(T)(K)$ on the character group $X(T)$ commutes with the natural action of $\mathrm{Gal}(K_T/K)$. Now since T is generic, $\theta_T(\mathrm{Gal}(K_T/K)) \supset W(G,T)$, and as $W(G,T)$ acts irreducibly on $X(T) \otimes_{\mathbb{Z}} \mathbb{C}$, we see that the elements of $N_G(T)(K)$ act by $\pm I$ on $X(T)$. Therefore, for $n \in N_G(T)(K)$, n^2 commutes with T and hence lies in $T(K)$. Now if $n \in G(K)$ is such that ngn^{-1} commutes with g, then n belongs to $N_G(T)(K)$ and $n^2 \in T(K)$. So, if, moreover, n is of infinite order, then it actually lies in $T(K)$. Thus an element of $G(K)$ of infinite order which does not belong to $T(K)$ cannot normalize T.

Theorem 9.6. *Let G be a connected absolutely almost simple algebraic group over a finitely generated field K, and let Γ be a finitely generated Zariski-dense subgroup of $G(K)$. Then Γ contains a generic element (over K) of infinite order.*

(It is not difficult to show, e.g., using Burnside's characterization of absolutely irreducible linear groups, that any Zariski-dense subgroup $\Gamma \subset G(K)$ contains a *finitely generated* Zariski-dense subgroup, so the assumption in the theorem that Γ be finitely generated can actually be omitted.)

Sketch of the proof. Fix a matrix K-realization $G \subset \mathrm{GL}_N$, and pick a finitely generated subring $R \subset K$ so that $\Gamma \subset G(R) := G \cap \mathrm{GL}_N(R)$. Let r be the number of nontrivial conjugacy classes in the Weyl group $W(G,T)$. Using Proposition 9.2, we can find r distinct primes $p_1, \dots, p_r$ such that for each $i \leqslant r$ there exists an embedding $\varepsilon_i \colon K \hookrightarrow \mathbb{Q}_{p_i}$ such that $\varepsilon_i(R) \subset \mathbb{Z}_{p_i}$ and G splits over $\mathbb{Q}_{p_i}$. Let v_i be the discrete valuation of K obtained as the pullback of the p_i-adic valuation of $\mathbb{Q}_{p_i}$ so that $K_{v_i} = \mathbb{Q}_{p_i}$. Pick maximal K_{v_i}-tori $T(v_1), \dots, T(v_r)$ as in part (2) of Theorem 9.1. Let Σ_i be the Zariski-open K_{v_i}-subvariety of regular elements in $T(v_i)$. It follows from the implicit function theorem that the image Ω_i of the map

$$G(K_{v_i}) \times \Sigma_i(K_{v_i}) \to G(K_{v_i}), \quad (g,t) \mapsto gtg^{-1},$$

is open in $G(K_{v_i})$ and intersects every open subgroup of the latter. On the other hand, as explained in [Rapinchuk 2014, Section 3], the closure of the image of the diagonal embedding $\Gamma \hookrightarrow \prod_{i=1}^r G(K_{v_i})$ is open, hence contains some $U = \prod_{i=1}^r U_i$, where $U_i \subset G(K_{v_i})$ is a Zariski-open torsion-free subgroup.

Then

$$U_0 := \prod_{i=1}^{r} (U_i \cap \Omega_i)$$

is an open set that intersects Γ, and it follows from our construction that any element $g \in \Gamma \cap U_0$ is a generic element of infinite order. □

Basically, our proof shows that given a finitely generated Zariski-dense subgroup Γ of $G(K)$, one can produce a finite system of congruences (defined in terms of suitable valuations of K) such that the set of elements $\gamma \in \Gamma$ satisfying this system of congruences consists entirely of generic elements (and additionally this set is in fact a coset of a finite index subgroup in Γ, in particular, it is Zariski-dense in G). Recently, Jouve, Kowalski and Zywina [Jouve et al. 2013], Gorodnik and Nevo [2011] and Lubotzky and Rosenzweig [2012] developed different *quantitative* ways of showing that generic elements exist in abundance (in fact, these results demonstrate that "most" elements in Γ are generic). More precisely, the result of [Gorodnik and Nevo 2011] gives the asymptotics of the number of generic elements of a given height in an arithmetic group, while the results of [Lubotzky and Rosenzweig 2012], generalizing earlier results of [Jouve et al. 2013], are formulated in terms of random walks on groups and apply to arbitrary Zariski-dense subgroups in not necessarily connected semisimple groups. These papers introduce several new ideas and techniques, but at the same time employ the elements of the argument from [Prasad and Rapinchuk 2003] we outlined above.

The proofs of the results in [Prasad and Rapinchuk 2009; 2013] use not only Theorem 9.6 itself but also its different variants that provide generic elements with additional properties — e.g., having prescribed local behavior (Corollary 9.3). We refer the interested reader to these papers for precise formulations (which are rather technical), and will only indicate here the basic "multidimensional" version of Theorem 9.6.

Theorem 9.7 [Prasad and Rapinchuk 2013, Theorem 3.4]. *Let G, K and $\Gamma \subset G(K)$ be as in Theorem 9.6. Then for any $m \geqslant 1$ one can find generic semisimple elements $\gamma_1, \dots, \gamma_m \in \Gamma$ of infinite order that are multiplicatively independent.*

Finally, we would like to formulate a result that enables one to pass from the weak commensurability of two generic semisimple elements to an isogeny, and in most cases even to an isomorphism, of the ambient tori. This result relates the analysis of weak commensurability to the problem of characterizing algebraic group having the same isomorphism/isogeny classes of maximal tori.

Theorem 9.8 (isogeny theorem [Prasad and Rapinchuk 2009, Theorem 4.2]). *Let G_1 and G_2 be two connected absolutely almost simple algebraic groups defined over an infinite field K, and let L_i be the minimal Galois extension of K*

over which G_i becomes an inner form of a split group. Suppose that for $i = 1, 2$, we are given a semisimple element $\gamma_i \in G_i(K)$ contained in a maximal K-torus T_i of G_i. Assume that:

(i) *G_1 and G_2 are either of the same Killing–Cartan type*, *or one of them is of type B_n and the other is of type C_n*;

(ii) *γ_1 has infinite order*;

(iii) *T_1 is K-irreducible*; *and*

(iv) *γ_1 and γ_2 are weakly commensurable.*

Then:

(1) *There exists a K-isogeny $\pi\colon T_2 \to T_1$ which carries $\gamma_2^{m_2}$ to $\gamma_1^{m_1}$ for some integers $m_1, m_2 \geqslant 1$.*

(2) *If $L_1 = L_2 =: L$ and $\theta_{T_1}(\mathrm{Gal}(L_{T_1}/L)) \supset W(G_1, T_1)$, then*

$$\pi^*\colon X(T_1) \otimes_{\mathbb{Z}} \mathbb{Q} \to X(T_2) \otimes_{\mathbb{Z}} \mathbb{Q}$$

has the property that $\pi^*(\mathbb{Q} \cdot \Phi(G_1, T_1)) = \mathbb{Q} \cdot \Phi(G_2, T_2)$. Moreover, *if G_1 and G_2 are of the same Killing–Cartan type different from $B_2 = C_2$, F_4 or G_2*, *then a suitable rational multiple of π^* maps $\Phi(G_1, T_1)$ onto $\Phi(G_2, T_2)$*, *and if G_1 is of type B_n and G_2 is of type C_n*, *with $n > 2$*, *then a suitable rational multiple λ of π^* takes the long roots in $\Phi(G_1, T_1)$ to the short roots in $\Phi(G_2, T_2)$ while 2λ takes the short roots in $\Phi(G_1, T_1)$ to the long roots in $\Phi(G_2, T_2)$.*

It follows that in the situations where π^* can be, and has been, scaled so that $\pi^*(\Phi(G_1, T_1)) = \Phi(G_2, T_2)$, it induces K-isomorphisms $\widetilde{\pi}\colon \widetilde{T}_2 \to \widetilde{T}_1$ and $\overline{\pi}\colon \overline{T}_2 \to \overline{T}_1$ between the corresponding tori in the simply connected and adjoint groups $\widetilde{G}_i$ and $\overline{G}_i$, respectively, that extend to $\overline{K}$-isomorphisms $\widetilde{G}_2 \to \widetilde{G}_1$ and $\overline{G}_2 \to \overline{G}_1$. Thus, the fact that Zariski-dense torsion-free subgroups $\Gamma_1 \subset G_1(K)$ and $\Gamma_2 \subset G_2(K)$ are weakly commensurable implies (under some minor technical assumptions) that G_1 and G_2 have the same K-isogeny classes (and under some additional assumptions, even the same K-isomorphism classes) of generic maximal K-tori that nontrivially intersect Γ_1 and Γ_2, respectively.

For a "multidimensional" version of Theorem 9.8, which is formulated using the notion of weak containment (see Section 8) in place of weak commensurability, see [Prasad and Rapinchuk 2013, Theorem 2.3].

9.9. We conclude this section with one new observation (Theorem 9.10) which is directly related to the main theme of the workshop — thin groups. This observation was inspired by a conversation of the first-named author with Igor Rivin at the Institute for Advanced Study.

Let G be a connected absolutely almost simple algebraic group over a field K of characteristic zero, and let T be a maximal K-torus of G. We let $\Phi_{>}(G,T)$ (resp., $\Phi_{<}(G,T)$) denote the set of all long (resp., short) roots in the root system $\Phi(G,T)$; by convention,

$$\Phi_{>}(G,T) = \Phi_{<}(G,T) = \Phi(G,T)$$

if all roots have the same length. Furthermore, we let $G_T^{>}$ denote the K-subgroup of G generated by T and the one-parameter unipotent subgroups U_a for a in $\Phi_{>}(G,T)$. Then $G_T^{>}$ is a connected semisimple subgroup of G of maximal absolute rank (so, in fact, just the U_a's for $a \in \Phi_{>}(G,T)$ generate $G_T^{>}$). By direct inspection, one verifies that $G_T^{>} \neq G$ precisely when $\Phi(G,T)$ has roots of different lengths, and then $G_T^{>}$ is a semisimple group of type $(\mathsf{A}_1)^n$ if G is of type C_n, and an absolutely almost simple group of type D_n, D_4 and A_2 if G is of type B_n, F_4 and G_2, respectively. On the other hand, the subgroups U_a for $a \in \Phi_{<}(G,T)$ generate G in all cases. Finally, for any connected subgroup of G containing T there exists a subset $\Psi \subset \Phi(G,T)$ such that G is generated by T and U_a for all $a \in \Psi$.

Theorem 9.10. *Let g be a generic element of infinite order and $T := Z_G(g)$ be the associated maximal torus. Let $x \in G(K)$ be any element of infinite order not contained in $T(K)$. Furthermore, let Γ be the (abstract) subgroup of $G(K)$ generated by g and x, and let H be the identity component of the Zariski-closure of Γ. Then either $H = G$ or $H = G_T^{>}$. Consequently, g and x generate a Zariski-dense subgroup of G if all roots in the root system $\Phi(G,T)$ are of same length.*

Proof. As g is a generic element of infinite order, the cyclic group generated by it is Zariski-dense in T, and so the cyclic group generated by xgx^{-1} is Zariski-dense in the conjugate torus xTx^{-1}. Since $x \notin T(K)$ and is of infinite order, it cannot normalize T (see 9.5). Thus H contains at least two different (generic) maximal K-tori, namely T and xTx^{-1}. Assume that $H \neq G$. Since H is connected and properly contains T, it must contain a one-parameter subgroup U_a for some $a \in \Phi(G,T)$. Then being defined over K, H also contains U_b for all b of the form $b = \sigma(a)$ with $\sigma \in \mathrm{Gal}(K_T/K)$. Now since T is generic, using the fact that the Weyl group $W(G,T)$ acts transitively on the subsets of roots of same length (by [Bourbaki 1968, Chapter VI, Proposition 11]), we see that H contains U_b for all roots $b \in \Phi(G,T)$ of same length as a. If a were a short root then the above remarks would imply that $H = G$, which is not the case. Thus, a must be long, and therefore H contains $G_T^{>}$ but does not contain U_b for any short root b. This clearly implies that $H = G_T^{>}$. □

Remark 9.11. It is worth noting that the types with roots of different lengths are honest exceptions in Theorem 9.10 in the sense that for any absolutely almost simple algebraic group G of one of those types over a finitely generated field K one can find two generic elements $\gamma_1, \gamma_2 \in G(K)$ that generate $G_T^{>} \neq G$ for a generic maximal K-torus T. To see this, we first pick an arbitrary generic element $\gamma_1 \in G(K)$ of infinite order provided by Theorem 9.6, and let $T = Z_G(\gamma_1)$ be the corresponding torus. Since $H := G_T^{>}$ is semisimple, the group $H(K)$ is Zariski-dense in H (by [Borel 1991, Corollary 18.3]). So, there exists $h \in H(K)$ such that $\gamma_2 := h\gamma_1 h^{-1} \notin T(K)$. Then $\gamma_2 \in H(K)$ is also generic over K, and the Zariski-closure of the subgroup generated by γ_1 and γ_2 is contained in (actually, is equal to) H.

10. Some open problems

The analysis of weak commensurability has led to a number of interesting problems in the theory of algebraic and Lie groups (see Section 6, for example) and its applications to locally symmetric spaces, and we would like to conclude this article with a brief discussion of some of these problems.

According to Theorem 4.5, if two lattices in the groups of rational points of connected absolutely almost simple groups over a nondiscrete locally compact field are weakly commensurable and one of the lattices is arithmetic, then so is the other. At the same time, it has been shown by means of an example (see [Prasad and Rapinchuk 2009, Remark 5.5]) that a Zariski-dense subgroup weakly commensurable to a *rank-one* arithmetic subgroup need not be arithmetic. It would be interesting, however, to understand what happens with *higher-rank* S-arithmetic subgroups.

Problem 10.1. Let G_1 and G_2 be two connected absolutely almost simple algebraic groups defined over a field F of characteristic zero, and let $\Gamma_1 \subset G_1(F)$ be a Zariski-dense (K, S)-arithmetic subgroup whose S-rank[7] is at least 2. If $\Gamma_2 \subset G_2(F)$ is a Zariski-dense subgroup weakly commensurable to Γ_1, is Γ_2 necessarily S-arithmetic?

This problem appears to be very challenging; the answer is not known even in the cases where Γ_1 is $\mathrm{SL}_3(\mathbb{Z})$ or $\mathrm{SL}_2(\mathbb{Z}[1/p])$. One should probably start by considering Problem 10.1 in a more specialized situation, e.g., assuming that F is a nondiscrete locally compact field, $\Gamma_1 \subset G_1(F)$ is a *discrete* Zariski-dense (higher-rank) S-arithmetic subgroup, and $\Gamma_2 \subset G_2(F)$ is a (finitely generated) *discrete* Zariski-dense subgroup weakly commensurable to Γ_1 (these restrictions would eliminate $\mathrm{SL}_2(\mathbb{Z}[1/p])$ as a possibility for Γ_1, but many interesting groups

[7] We recall that if Γ is $(\mathscr{G}, K, S)$-arithmetic, then the S-rank of Γ is defined to be $\sum_{v \in S} \mathrm{rk}_{K_v} \mathscr{G}$, where $\mathrm{rk}_F \mathscr{G}$ denotes the rank of $\mathscr{G}$ over a field $F \supset K$.

such as $SL_3(\mathbb{Z})$ would still be included). The nature of these assumptions brings up another question of independent interest.

Problem 10.2. Let G_1 and G_2 be connected absolutely almost simple algebraic groups over a nondiscrete locally compact field F, and let Γ_i be a finitely generated Zariski-dense subgroup of $G_i(F)$ for $i = 1, 2$. Assume that Γ_1 and Γ_2 are weakly commensurable. Does the discreteness of Γ_1 imply the discreteness of Γ_2?

An affirmative answer to Problem 10.2 was given in [Prasad and Rapinchuk 2009, Propisition 5.6] for the case where F is a *nonarchimedean* local field, but the case $F = \mathbb{R}$ or $\mathbb{C}$ remains open. Another interesting question is whether weak commensurability preserves cocompactness of lattices.

Problem 10.3. Let G_1 and G_2 be connected absolutely almost simple algebraic groups over $F = \mathbb{R}$ or $\mathbb{C}$, and let $\Gamma_i \subset G_i(F)$ be a lattice for $i = 1, 2$. Assume that Γ_1 and Γ_2 are weakly commensurable. Does the compactness of $G_1(F)/\Gamma_1$ imply the compactness of $G_2(F)/\Gamma_2$?[8]

We recall that the cocompactness of a lattice in a semisimple real Lie group is equivalent to the absence of nontrivial unipotents in it; see [Raghunathan 1972, Corollary 11.13]. So, Problem 10.3 can be rephrased as the question whether for two weakly commensurable lattices Γ_1 and Γ_2, the existence of nontrivial unipotent elements in one of them implies their existence in the other; in this form the question is meaningful for arbitrary Zariski-dense subgroups (not necessarily discrete or of finite covolume). The combination of Theorems 4.4 and 4.5 implies the affirmative answer to Problem 10.3 in the case where one of the lattices is arithmetic, but no other cases have been considered so far.

From the general perspective, one important problem is to try to generalize our results on length-commensurable and/or isospectral arithmetically defined locally symmetric spaces of absolutely simple real Lie groups to arithmetically defined locally symmetric spaces of arbitrary semisimple Lie groups, or at least those of $\mathbb{R}$-simple Lie groups. To highlight the difficulty, we will make some comments about the latter case. An $\mathbb{R}$-simple adjoint group G can be written in the form $G = \mathrm{R}_{\mathbb{C}/\mathbb{R}}(H)$ (restriction of scalars) where H is an absolutely simple complex algebraic group. Arithmetic lattices in $G(\mathbb{R}) \simeq H(\mathbb{C})$ come from the forms of H over a number field admitting exactly one complex embedding. The analysis of weak commensurability of even arithmetic lattices $\Gamma_1 \subset G_1(\mathbb{R})$ and $\Gamma_2 \subset G_2(\mathbb{R})$, where $G_i = \mathrm{R}_{\mathbb{C}/\mathbb{R}}(H_i)$ for $i = 1, 2$, cannot be implemented via the

[8] It is well known that for a semisimple algebraic group G over a nondiscrete nonarchimedean locally compact field F of characteristic zero and a discrete subgroup $\Gamma \subset G(F)$, the quotient $G(F)/\Gamma$ has finite measure if and only if it is compact, so the problem in this case becomes vacuous.

study of the forms of the G_i's, forcing us to study directly the forms of the H_i's. But the relation of weak commensurability of semisimple elements $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$ in terms of G_1 and G_2 — i.e., the fact that $\chi_1(\gamma_1) = \chi_2(\gamma_2) \neq 1$ for some characters χ_i of maximal $\mathbb{R}$-tori T_i of G_i such that $\gamma_i \in T_i(\mathbb{R})$ — translates into a significantly more complicated relation in terms of H_1 and H_2. Indeed, pick maximal $\mathbb{C}$-tori S_i of H_i so that $T_i = \mathrm{R}_{\mathbb{C}/\mathbb{R}}(S_i)$, and let $\delta_i \in S_i(\mathbb{C})$ be the element corresponding to γ_i under the identification $T_i(\mathbb{R}) \simeq S_i(\mathbb{C})$. Then there exist characters χ_i', χ_i'' of S_i such that $\chi_i(\gamma_i) = \chi_i'(\delta_i)\overline{\chi_i''(\delta_i)}$. So, the relation of weak commensurability of γ_1 and γ_2 assumes the following form in terms of δ_1 and δ_2:

$$\chi_1'(\delta_1)\overline{\chi_1''(\delta_1)} = \chi_2'(\delta_2)\overline{\chi_2''(\delta_2)}.$$

It is not clear if this type of relation would lead to the results similar to those we described in this article for the weakly commensurable arithmetic subgroups of absolutely almost simple groups. So, the general problem at this stage is to formulate for general semisimple groups (or at least $\mathbb{R}$-simple groups) the "right" notion of weak commensurability and explore its consequences. We will now formulate a particular case of this general program that would be interesting for geometric applications.

Problem 10.4. Let G_1 and G_2 be almost simple complex algebraic groups. Two semisimple elements $\gamma_i \in G_i(\mathbb{C})$ are called $\mathbb{R}$-weakly commensurable if there exist complex maximal tori T_i of G_i for $i = 1, 2$ such that $\gamma_i \in T_i(\mathbb{C})$ and for suitable characters χ_i of T_i we have

$$|\chi_1(\gamma_1)| = |\chi_2(\gamma_2)| \neq 1.$$

Furthermore, Zariski-dense (discrete) subgroups $\Gamma_i \subset G_i(\mathbb{C})$ are $\mathbb{R}$-weakly commensurable if every semisimple element $\gamma_1 \in \Gamma_1$ of infinite order is $\mathbb{R}$-weakly commensurable to some semisimple element $\gamma_2 \in \Gamma_2$ of infinite order, and vice versa. Under what conditions does the $\mathbb{R}$-weak commensurability of Zariski-dense (arithmetic) lattices $\Gamma_i \subset G_i(\mathbb{C})$ $(i = 1, 2)$ imply their commensurability?

The result of [Chinburg et al. 2008] seems to imply that the $\mathbb{R}$-weak commensurability of arithmetic lattices in $\mathrm{SL}_2(\mathbb{C})$ does imply their commensurability, but no other results in this direction are available.

Turning now to the geometric aspect, we would like to reiterate that most of our results deal with the analysis of the new relation of length-commensurability, which eventually implies the results about isospectral locally symmetric spaces. At the same time, the general consequences of isospectrality and isolength spectrality are much better understood than those of length-commensurability. So, as an overarching problem, we would like to propose the following.

Problem 10.5. Understand consequences (qualitative and quantitative) of length-commensurability for locally symmetric spaces.

(Here by *quantitative consequences* we mean results stating that in certain situations a family of length-commensurable locally symmetric spaces consists either of a single commensurability class or of a certain bounded number of commensurability classes, and by *qualitative consequences* - results guaranteeing that the number of commensurability classes in a given class of length-commensurable locally symmetric spaces is finite.)

There are various concrete questions within the framework provided by Problem 10.5 that were resolved in [Prasad and Rapinchuk 2009] for arithmetically defined locally symmetric spaces but remain open for locally symmetric spaces which are not arithmetically defined. For example, according to Theorem 5.3, if $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are length-commensurable and at least one of the spaces is arithmetically defined, then the compactness of one of them implies the compactness of the other. It is natural to ask if this can be proved for locally symmetric spaces which are not arithmetically defined.

Problem 10.6 (geometric version of Problem 10.3). Let $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ be length-commensurable locally symmetric spaces of finite volume. Does the compactness of $\mathfrak{X}_{\Gamma_1}$ always imply the compactness of $\mathfrak{X}_{\Gamma_2}$?

In [Prasad and Rapinchuk 2009, Section 9], for each of the exceptional types A_n, D_{2n+1} $(n > 1)$ and E_6, we have constructed examples of length-commensurable, but not commensurable, compact arithmetically defined locally symmetric spaces associated with a simple real algebraic group of this type. It would be interesting to see if this construction can be refined to provide examples of isolength spectral or even isospectral compact arithmetically defined locally symmetric spaces that are not commensurable.

Problem 10.7. For inner and outer types A_n $(n > 1)$, D_{2n+1} $(n > 1)$ and E_6, construct examples of isospectral compact arithmetically defined locally symmetric spaces that are not commensurable.

Currently, such a construction is known only for *inner forms* of type A_n (see [Lubotzky et al. 2006]); it relies on some delicate results from the theory of automorphic forms [Harris and Taylor 2001], the analogues of which are not yet available for groups of other types.

As we already mentioned, in [Prasad and Rapinchuk 2009] we focused on the case where G_1 and G_2 are absolutely (almost) simple real algebraic groups. From the geometric perspective, however, it would be desirable to consider a more general situation where G_1 and G_2 are allowed to be either arbitrary real semisimple groups (without compact factors), or at least arbitrary $\mathbb{R}$-simple groups. This problem is intimately related to the problem, discussed above, of generalizing our results on weak commensurability from absolutely almost simple to arbitrary semisimple groups. In particular, a successful resolution of Problem 10.4 would enable us to extend our results to the (arithmetically defined) locally symmetric spaces associated with $\mathbb{R}$-simple groups providing thereby a significant generalization of the result of [Chinburg et al. 2008] where the case $G_1 = G_2 = \mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathrm{SL}_2)$ (that leads to arithmetically defined hyperbolic 3-manifolds) was considered.

Finally, the proof of the result that connects the length-commensurability of $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ to the weak commensurability of Γ_1 and Γ_2 relies (at least in the higher-rank case) on Schanuel's conjecture. It would be interesting to see if our geometric results can be made independent of Schanuel's conjecture. Some results in this direction were obtained in [Bhagwat et al. 2012].

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