Symplectic, Poisson and Noncommutative Geometry
MSRI Publications
Volume **62**, 2014

Microlocal theory of sheaves in symplectic topology

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This paper is a survey of papers by Guillermou, Kashiwara and Schapira (2012) and Guillermou and Schapira (2011) in which we expose how the microlocal theory of sheaves may be applied to symplectic topology, in particular to treat nondisplaceability problems, an idea which first appeared in Tamarkin (2008).

Introduction

D. Tamarkin [2008] gave a totally new approach for treating classical problems of nondisplaceability in symplectic geometry. His approach is based on the microlocal theory of sheaves, introduced and systematically developed in [Kashiwara and Schapira 1982; 1985; 1990]. (Note however that the use of the microlocal theory of sheaves also appeared in a related context in [Oh 1998; Nadler and Zaslow 2009].)

In these notes, we will both explain the main ideas of Tamarkin's paper, following the presentation of [Guillermou and Schapira 2011], and also the alternative approach to nondisplaceability, following [Guillermou et al. 2012]. Note that we restrict ourselves to the case where the symplectic manifold is the cotangent bundle T^*M to a real C^∞-manifold M. The case of compact symplectic manifold was announced by Tamarkin, but nothing is yet published, and this theory seems of extraordinary difficulty.

The main obstacle to applying the microlocal theory of sheaves (in the case of a cotangent bundle) to symplectic geometry is that the first theory is related to the homogeneous symplectic structure, that is the Liouville 1-form on T^*M,

MSC2000: primary 53D35; secondary 14F05.

contrarily to the second one which deal with smooth Lagrangian submanifolds, in general nonconic. There are two way to overcome this difficulty: the first is to adapt the theory of sheaves, as did Tamarkin, the second is to translate the nonhomogeneous geometrical problem to homogeneous ones, as we did in [Guillermou et al. 2012]. This last method is much easier, and we shall begin by recalling it, but the results obtained do not go as far as the first one and does not allow one to expect to construct anything which looks like the Fukaya category, contrarily to Tamarkin's approach.

1. Microlocal theory of sheaves after Kashiwara and Schapira

In this section, we recall some definitions and results from [Kashiwara and Schapira 1990], following its notations with the exception of slight modifications. We consider a real manifold M of class C^∞.

Some geometrical notions [Kashiwara and Schapira 1990, Sections 4.2 and 6.2]**.** For a locally closed subset A of M, we denote by $\mathrm{Int}(A)$ its interior and by $\overline{A}$ its closure. We denote by Δ_M or simply Δ the diagonal of $M \times M$.

We denote by $\tau\colon TM \to M$ and $\pi\colon T^*M \to M$ the tangent and cotangent bundles to M. If $L \subset M$ is a smooth submanifold, we denote by $T_L M$ its normal bundle and T_L^*M its conormal bundle. They are defined by the exact sequences

$$0 \to TL \to L \times_M TM \to T_L M \to 0,\, 0 \to T_L^*M \to L \times_M T^*M \to T^*L \to 0.$$

We identify M to T_M^*M, the zero-section of T^*M. We set

$$\dot{T}^*M := T^*M \setminus T_M^*M$$

and denote by $\dot{\pi}_M\colon \dot{T}^*M \to M$ the projection.

Let $f\colon M \to N$ be a morphism of real manifolds. To f are associated the tangent morphisms

$$\begin{array}{ccccc} TM & \xrightarrow{f'} & M \times_N TN & \xrightarrow{f_\tau} & TN \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} \\ M & = & M & \xrightarrow{f} & N. \end{array} \tag{1-1}$$

By duality, we deduce the diagram

$$\begin{array}{ccccc} T^*M & \xleftarrow{f_d} & M \times_N T^*N & \xrightarrow{f_\pi} & T^*N \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ M & = & M & \xrightarrow{f} & N. \end{array} \tag{1-2}$$

We set

$$T^*_M N := \operatorname{Ker} f_d = f_d^{-1}(T^*_M M).$$

We say that the map f is noncharacteristic with respect to a closed conic subset $\Lambda \subset T^*N$ if the map f_d is proper on $f_\pi^{-1}(\Lambda)$. This is equivalent to saying that $T^*_M N \cap f_\pi^{-1}(\Lambda) \subset M \times_N T^*_N N$.

Now consider the homogeneous symplectic manifold T^*M: it is endowed with the Liouville 1-form given in a local homogeneous symplectic coordinate system $(x;\xi)$ on T^*M by

$$\alpha_M = \langle \xi, dx \rangle.$$

The antipodal map a_M is defined by

$$a_M : T^*M \to T^*M, \quad (x;\xi) \mapsto (x;-\xi). \tag{1-3}$$

If A is a subset of T^*M, we denote by A^a instead of $a_M(A)$ its image by the antipodal map. We shall use the Hamiltonian isomorphism

$$H : T^*(T^*M) \xrightarrow{\sim} T(T^*M)$$

given in a local symplectic coordinate system $(x;\xi)$ by

$$H(\langle \lambda, dx \rangle + \langle \mu, d\xi \rangle) = -\langle \lambda, \partial_\xi \rangle + \langle \mu, \partial_x \rangle.$$

Microsupport. We consider a commutative unital ring $\boldsymbol{k}$ of finite global dimension (e.g., $\boldsymbol{k} = \mathbb{Z}$). We denote by $\mathsf{D}(\boldsymbol{k}_M)$ and $\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_M)$ the derived category and bounded derived category of sheaves of $\boldsymbol{k}$-modules on M.

Recall the definition of the microsupport (or singular support) $\mathrm{SS}(F)$ of a sheaf F.

Definition 1.1 [Kashiwara and Schapira 1990, Definition 5.1.2]. Let $F \in \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_M)$ and let $p \in T^*M$. We say that $p \notin \mathrm{SS}(F)$ if there exists an open neighborhood U of p such that for any $x_0 \in M$ and any real C^1-function φ on M defined in a neighborhood of x_0 satisfying $d\varphi(x_0) \in U$ and $\varphi(x_0) = 0$, we have $(\mathrm{R}\Gamma_{\{x;\varphi(x)\geq 0\}}(F))_{x_0} \simeq 0$.

In other words, $p \notin \mathrm{SS}(F)$ if the sheaf F has no cohomology supported by "half-spaces" whose conormals are contained in a neighborhood of p.

- By its construction, the microsupport is closed and is $\mathbb{R}^+$-conic, that is, invariant by the action of $\mathbb{R}^+$ on T^*M.
- $\mathrm{SS}(F) \cap T^*_M M = \pi_M(\mathrm{SS}(F)) = \mathrm{Supp}(F)$.
- The microsupport satisfies the triangular inequality: if $F_1 \to F_2 \to F_3 \xrightarrow{+1}$ is a distinguished triangle in $\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_M)$, then $\mathrm{SS}(F_i) \subset \mathrm{SS}(F_j) \cup \mathrm{SS}(F_k)$ for all $i, j, k \in \{1, 2, 3\}$ with $j \neq k$.

Using the notion of Whitney's normal cone, we can define the notion of being coisotropic for any locally closed subset of T^*M. We do not recall this definition here and refer to [Kashiwara and Schapira 1990, Definition 6.5.1].

Theorem 1.2 [Kashiwara and Schapira 1990, Theorem 6.5.4]. *Let* $F \in \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_M)$. *Then its microsupport* $\mathrm{SS}(F)$ *is coisotropic.*

In the sequel, for a locally closed subset Z in M, we denote by $\boldsymbol{k}_Z$ the constant sheaf with stalk $\boldsymbol{k}$ on Z, extended by 0 on $M \setminus Z$.

Example 1.3. (i) If F is a nonzero local system on a connected manifold M, then $\mathrm{SS}(F) = T^*_M M$, the zero-section.

(ii) If N is a smooth closed submanifold of M and $F = \boldsymbol{k}_N$, then $\mathrm{SS}(F) = T^*_N M$, the conormal bundle to N in M.

(iii) Let φ be C^1-function with $d\varphi(x) \neq 0$ when $\varphi(x) = 0$. Let $U = \{x \in M; \varphi(x) > 0\}$ and let $Z = \{x \in M; \varphi(x) \geq 0\}$. Then

$$\mathrm{SS}(\boldsymbol{k}_U) = U \times_M T^*_M M \cup \{(x; \lambda d\varphi(x)); \varphi(x) = 0, \lambda \leq 0\},$$
$$\mathrm{SS}(\boldsymbol{k}_Z) = Z \times_M T^*_M M \cup \{(x; \lambda d\varphi(x)); \varphi(x) = 0, \lambda \geq 0\}.$$

(iv) Assume $M = V$ is a vector space and let γ be a closed proper convex cone with vertex at 0. Then $\mathrm{SS}(\boldsymbol{k}_\gamma) \cap \pi_M^{-1}(\{0\}) = \gamma^\circ$ where $\gamma^\circ \subset V^*$ is the polar cone given by

$$\gamma_0^\circ = \{\theta \in V^*; \langle \theta, v\rangle \geq 0\} \quad \text{for all } v \in \gamma_0. \tag{1-4}$$

(v) Let $(X, \mathcal{O}_X)$ be a complex manifold and let $\mathcal{M}$ be a coherent module over the ring $\mathcal{D}_X$ of holomorphic differential operators. (Hence, $\mathcal{M}$ represents a system of linear partial differential equations on X.) Denote by $F = \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ the complex of holomorphic solutions of $\mathcal{M}$. Then $\mathrm{SS}(F) = \mathrm{char}(\mathcal{M})$, the characteristic variety of $\mathcal{M}$.

Functorial operations (proper and noncharacteristic cases). Let M and N be two real manifolds. We denote by q_i $(i = 1, 2)$ the i-th projection defined on $M \times N$ and by p_i $(i = 1, 2)$ the i-th projection defined on

$$T^*(M \times N) \simeq T^*M \times T^*N.$$

Theorem 1.4 [Kashiwara and Schapira 1990, Section 5.4]. *Let* $f : M \to N$ *be a morphism of manifolds, let* $F \in \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_M)$ *and let* $G \in \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_N)$.

(i) *Assume that* f *is proper on* $\mathrm{Supp}(F)$. *Then* $\mathrm{SS}(\mathrm{R}f_! F) \subset f_\pi f_d^{-1}\mathrm{SS}(F)$.

(ii) *Assume that* f *is noncharacteristic with respect to* $\mathrm{SS}(G)$. *Then*

$$\mathrm{SS}(f^{-1}G) \subset f_d f_\pi^{-1}\mathrm{SS}(G).$$

(iii) *Assume that f is smooth, that is, submersive. Then* $\mathrm{SS}(F) \subset M \times_N T^*N$ *if and only if, for any $j \in \mathbb{Z}$, the sheaves $H^j(F)$ are locally constant on the fibers of f.*

The next result is a particular case of the microlocal Morse lemma (see [Kashiwara and Schapira 1990, Cor. 5.4.19]) and follows immediately from Theorem 1.4 (ii). The classical theory corresponds to the constant sheaf $F = \boldsymbol{k}_M$.

Corollary 1.5. *Let $F \in \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_M)$, let $\varphi \colon M \to \mathbb{R}$ be a function of class C^1 and assume that φ is proper on* $\mathrm{supp}(F)$. *Let $a < b$ in $\mathbb{R}$ and assume that $d\varphi(x) \notin \mathrm{SS}(F)$ for $a \le \varphi(x) < b$. Then the natural morphism*

$$\mathrm{R}\Gamma(\varphi^{-1}(]-\infty, b[); F) \to \mathrm{R}\Gamma(\varphi^{-1}(]-\infty, a[); F)$$

is an isomorphism.

Corollary 1.6. *Let I be a contractible manifold and let $p \colon M \times I \to M$ be the projection. If $F \in \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_{M\times I})$ satisfies* $\mathrm{SS}(F) \subset T^*M \times T^*_I I$, *then*

$$F \simeq p^{-1}\mathrm{R}p_* F.$$

Corollary 1.7. *Let I be an open interval of $\mathbb{R}$ and let $q \colon M \times I \to I$ be the projection. Let $F \in \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_{M\times I})$ such that*

$$\mathrm{SS}(F) \cap (T^*_M M \times T^*I) \subset T^*_{M\times I}(M \times I)$$

and q is proper on $\mathrm{Supp}(F)$. *Then we have isomorphisms*

$$\mathrm{R}\Gamma(M; F_s) \simeq \mathrm{R}\Gamma(M; F_t)$$

for any $s, t \in I$, where $F_s := F|_{\{t=s\}}$.

Theorem 1.4 and its corollaries are sufficient for proving the nondisplaceability theorems obtained in [Guillermou et al. 2012]. However, Tamarkin's approach needs to consider characteristic inverse images or nonproper direct images for which we refer to [Kashiwara and Schapira 1990].

Kernels [Kashiwara and Schapira 1990, Section 3.6]**.**

Notation 1.8. Let M_i $(i = 1, 2, 3)$ be manifolds. For brevity, we write $M_{ij} := M_i \times M_j$ $(1 \le i, j \le 3)$ and $M_{123} = M_1 \times M_2 \times M_3$. We denote by q_i the projection $M_{ij} \to M_i$ or the projection $M_{123} \to M_i$ and by q_{ij} the projection $M_{123} \to M_{ij}$. Similarly, we denote by p_i the projection $T^*M_{ij} \to T^*M_i$ or the projection $T^*M_{123} \to T^*M_i$ and by p_{ij} the projection $T^*M_{123} \to T^*M_{ij}$. We also need to introduce the map p_{12^a}, the composition of p_{12} and the antipodal map on T^*M_2.

Let $A \subset T^*M_{12}$ and $B \subset T^*M_{23}$. We set

$$A \times_{T^*M_{2^a}} B = p_{12}^{-1}(A) \cap p_{2^a 3}^{-1}(B)$$

and

$$\begin{aligned} A \overset{a}{\circ} B &= p_{13}(A \times_{T^*M_{2^a}} B), \\ &= \big\{(x_1, x_3; \xi_1, \xi_3) \in T^*M_{13};\ \text{there exist } (x_2; \xi_2) \in T^*M_2, \\ &\qquad (x_1, x_2; \xi_1, \xi_2) \in A,\ (x_2, x_3; -\xi_2, \xi_3) \in B\big\}. \end{aligned} \tag{1-5}$$

We consider the operation of composition of kernels:

$$\begin{aligned} \underset{2}{\circ}\colon \mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_{M_{12}}) \times \mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_{M_{23}}) &\to \mathrm{D}^{\mathrm{b}}(\boldsymbol{k}_{M_{13}}) \\ (K_1, K_2) \mapsto K_1 \underset{2}{\circ} K_2 &:= \mathrm{R}q_{13!}(q_{12}^{-1} K_1 \overset{\mathrm{L}}{\otimes} q_{23}^{-1} K_2). \end{aligned} \tag{1-6}$$

When there is no risk of confusion, we shall write $\circ$ instead of $\underset{2}{\circ}$.

Let $A_i = \mathrm{SS}(K_i) \subset T^*M_{i,i+1}$ and assume that

(i) q_{13} is proper on $q_{12}^{-1}\operatorname{supp}(K_1) \cap q_{23}^{-1}\operatorname{supp}(K_2)$, and

(ii) the intersection $p_{12}^{-1} A_1 \cap p_{2^a3}{}^{-1} A_2 \cap (T^*_{M_1} M_1 \times T^*M_2 \times T^*_{M_3} M_3)$ is contained in $T^*_{M_1 \times M_2 \times M_3}(M_1 \times M_2 \times M_3)$.

It follows from Theorem 1.4 that under these assumptions we have

$$\mathrm{SS}(K_1 \underset{2}{\circ} K_2) \subset A_1 \overset{a}{\circ} A_2. \tag{1-7}$$

If there is no risk of confusion, we write $\circ$ instead of $\underset{2}{\circ}$.

Localization. Let $\mathcal{T}$ be a triangulated category, $\mathcal{N}$ a null system, that is a full triangulated subcategory with the property that if there is an isomorphism $F \simeq G$ in $\mathcal{T}$ with $F \in \mathcal{N}$, then $G \in \mathcal{N}$. The localization $\mathcal{T}/\mathcal{N}$ is a well defined triangulated category (we skip the problem of universes). Its objects are those of $\mathcal{T}$ and a morphism $u\colon F_1 \to F_2$ in $\mathcal{T}$ becomes an isomorphism in $\mathcal{T}/\mathcal{N}$ if, after embedding this morphism in a distinguished triangle

$$F_1 \to F_2 \to F_3 \xrightarrow{+1}, \tag{1-8}$$

we have $F_3 \in \mathcal{N}$.

Recall that the left orthogonal $\mathcal{N}^{\perp,l}$ of $\mathcal{N}$ is the full triangulated subcategory of $\mathcal{T}$ defined by

$$\mathcal{N}^{\perp,l} = \big\{F \in \mathcal{T}; \operatorname{Hom}_{\mathcal{T}}(F, G) \simeq 0 \text{ for all } G \in \mathcal{N}\big\}.$$

By classical results (see [Kashiwara and Schapira 2006, Exercise 10.15], for example), if the embedding $\mathcal{N}^{\perp,l} \hookrightarrow \mathcal{T}$ admits a left adjoint, or equivalently, if for any $F \in \mathcal{T}$, there exists a distinguished triangle

$$F' \to F \to F'' \xrightarrow{+1}$$

with $F' \in \mathcal{N}^{\perp,l}$ and $F'' \in \mathcal{N}$, then there is an equivalence $\mathcal{N}^{\perp,l} \simeq \mathcal{T}/\mathcal{N}$.

Of course, there are similar results with the right orthogonal $\mathcal{N}^{\perp,r}$.

Now let U be an open subset of T^*M and assume for simplicity that $\mathbb{R}^+ \cdot U = U$. Let $Z = T^*M \setminus U$. The full subcategory $\mathsf{D}^{\mathrm{b}}_Z(\boldsymbol{k}_M)$ of $\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_M)$ consisting of sheaves F such that $\mathrm{SS}(F) \subset Z$ is a null system. We set

$$\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_M; U) := \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_M)/\mathsf{D}^{\mathrm{b}}_Z(\boldsymbol{k}_M),$$

the localization of $\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_M)$ by $\mathsf{D}^{\mathrm{b}}_Z(\boldsymbol{k}_M)$. Hence, the objects of $\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_M; U)$ are those of $\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_M)$ but a morphism $u\colon F_1 \to F_2$ in $\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_M)$ becomes an isomorphism in $\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_M; U)$ if, after embedding this morphism in a distinguished triangle

$$F_1 \to F_2 \to F_3 \xrightarrow{+1},$$

we have $\mathrm{SS}(F_3) \cap U = \varnothing$.

For a closed subset A of U, $\mathsf{D}^{\mathrm{b}}_A(\boldsymbol{k}_M; U)$ denotes the full triangulated subcategory of $\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_M; U)$ consisting of objects whose microsupports have an intersection with U contained in A.

Quantized symplectic isomorphisms [Kashiwara and Schapira 1990, Section 7.2]. Consider two manifolds M and N, two conic open subsets $U \subset T^*M$ and $V \subset T^*N$ and a homogeneous symplectic isomorphism χ:

$$T^*N \supset V \xrightarrow[\chi]{\sim} U \subset T^*M. \tag{1-9}$$

Denote by V^a the image of V by the antipodal map a_N on T^*N and by Λ the image of the graph of χ by $\mathrm{id}_U \times a_N$. Hence Λ is a conic Lagrangian submanifold of $U \times V^a$. A *quantized contact transformation* (QCT) above χ is a kernel $K \in \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_{M\times N})$ such that $\mathrm{SS}(K) \cap (U \times V^a) \subset \Lambda$ and satisfying some technical properties that we do not recall here, so that the kernel K induces an equivalence of categories

$$K \circ \bullet : \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_N; V) \xrightarrow{\sim} \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_M; U). \tag{1-10}$$

Given χ and $q \in V$, $p = \chi(q) \in U$, there exists such a QCT after replacing U and V by sufficiently small neighborhoods of p and q.

2. Quantization of Hamiltonian isotopies after Guillermou et al.

In this section, we recall the main theorem of [Guillermou et al. 2012].

We first recall some notions of symplectic geometry. Let $\mathfrak{X}$ be a symplectic manifold with symplectic form ω. We denote by $\mathfrak{X}^a$ the same manifold endowed with the symplectic form $-\omega$. The symplectic structure induces the Hamiltonian isomorphism $\boldsymbol{h}\colon T\mathfrak{X} \xrightarrow{\sim} T^*\mathfrak{X}$ by $\boldsymbol{h}(v) = \iota_v(\omega)$, where ι_v denotes the contraction with v. For a C^∞-function $f\colon \mathfrak{X} \to \mathbb{R}$, the Hamiltonian vector field of f is by definition $H_f := -\boldsymbol{h}^{-1}(df)$.

Let I be an open interval of $\mathbb{R}$ containing the origin and let $\Phi\colon \mathfrak{X} \times I \to \mathfrak{X}$ be a map such that $\varphi_s := \Phi(\cdot\,, s)\colon \mathfrak{X} \to \mathfrak{X}$ is a symplectic isomorphism for each $s \in I$ and is the identity for $s = 0$. The map Φ induces a time dependent vector field on $\mathfrak{X}$

$$v_\Phi := \frac{\partial \Phi}{\partial s}\colon \mathfrak{X} \times I \to T\mathfrak{X}. \tag{2-1}$$

The map Φ is called a Hamiltonian isotopy if there exists some C^∞-function $f\colon \mathfrak{X} \times I \to \mathbb{R}$ such that

$$\frac{\partial \Phi}{\partial s} = H_{f_s}.$$

The fact that the isotopy Φ is Hamiltonian can be interpreted as a geometric property of its graph as follows. For a given $s \in I$ we let Λ_s be the graph of φ_s^{-1} and we let Λ' be the family formed by the Λ_s:

$$\begin{aligned} \Lambda_s &= \{(\varphi_s(v), v)\,;\, v \in \mathfrak{X}^a\} \subset \mathfrak{X} \times \mathfrak{X}^a, \\ \Lambda' &= \{(\varphi_s(v), v, s)\,;\, v \in \mathfrak{X}^a,\ s \in I\} \subset \mathfrak{X} \times \mathfrak{X}^a \times I. \end{aligned}$$

Thus Λ_s is a Lagrangian submanifold of $\mathfrak{X} \times \mathfrak{X}^a$ and Φ is a Hamiltonian isotopy if and only if there exists a Lagrangian submanifold $\Lambda \subset \mathfrak{X} \times \mathfrak{X}^a \times T^*I$ such that Λ' is the projection of Λ.

In this case Λ is written

$$\Lambda = \big\{\big(\Phi(v,s), v, s, -f(\Phi(v,s),s)\big)\,;\, v \in \mathfrak{X}, s \in I\big\}, \tag{2-2}$$

where the function $f\colon \mathfrak{X} \times I \to \mathbb{R}$ is defined up to addition of a function depending on s by $v_{\Phi,s} = H_{f_s}$.

Homogeneous case. Let us come back to the case $\mathfrak{X} = \dot{T}^*M$ and consider $\Phi\colon \dot{T}^*M \times I \to \dot{T}^*M$ such that

$$\begin{cases} \varphi_s \text{ is a homogeneous symplectic isomorphism for each } s \in I, \\ \varphi_0 = \mathrm{id}_{\dot{T}^*M}\,. \end{cases} \tag{2-3}$$

In this case Φ is a Hamiltonian isotopy and the function f associated to Φ is given by

$$f = \langle \alpha_M, v_\Phi \rangle\colon \dot{T}^*M \times I \to \mathbb{R}. \tag{2-4}$$

Since f is homogeneous of degree 1 in the fibers of $\dot{T}^*M$, the Lagrangian submanifold Λ of $\dot{T}^*M \times \dot{T}^*M \times T^*I$ associated to f in (2-2) is $\mathbb{R}^+$-conic.

The quantization theorem. We say that $F \in \mathsf{D}(\boldsymbol{k}_M)$ is locally bounded if for any relatively compact open subset $U \subset M$ we have $F|_U \in \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_U)$. We denote by $\mathsf{D}^{\mathrm{lb}}(\boldsymbol{k}_M)$ the full subcategory of $\mathsf{D}(\boldsymbol{k}_M)$ consisting of locally bounded objects.

Now, let M and N be two manifolds with the same dimension and denote by v the map $M \times N \to N \times M$, $(x, y) \mapsto (y, x)$. For $F \in \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_{M \times N})$, we set

$$F^{-1} = v^{-1} \mathrm{R}\mathscr{H}om\,(F, \omega_M \boxtimes \boldsymbol{k}_N) \in \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_{N \times M}), \tag{2-5}$$

Theorem 2.1 [Guillermou et al. 2012]. *Let* Φ *be a homogeneous Hamiltonian isotopy satisfying* (2-3). *Consider the following conditions on* $K \in \mathsf{D}^{\mathrm{lb}}(\boldsymbol{k}_{M \times M \times I})$:

(a) $\mathrm{SS}(K) \subset \Lambda \cup T^*_{M \times M \times I}(M \times M \times I)$.

(b) $K_0 \simeq \boldsymbol{k}_\Delta$.

(c) *Both projections* $\mathrm{Supp}(K) \rightrightarrows M \times I$ *are proper.*

(d) $K_s \circ K_s^{-1} \simeq K_s^{-1} \circ K_s \simeq \boldsymbol{k}_\Delta$.

Then:

(i) *Conditions* (a) *and* (b) *imply the other two conditions*, (c) *and* (d).

(ii) *There exists* K *satisfying* (a)–(d).

(iii) *Moreover such a* K *satisfying the conditions* (a)–(d) *is unique up to a unique isomorphism.*

We shall call K the *quantization* of Φ on I, or the quantization of the family $\{\varphi_s\}_{s \in I}$.

Nonhomogeneous case. Theorem 2.1 is concerned with homogeneous Hamiltonian isotopies. The next result will allow us to adapt it to nonhomogeneous cases. Let $\Phi \colon T^*M \times I \to T^*M$ be a Hamiltonian isotopy and assume that

$$\text{for some } C \subset T^*M \text{ compact, } \varphi_s|_{T^*M \setminus C} = \text{ identity for all } s \in I. \tag{2-6}$$

Proposition 2.2 [Guillermou et al. 2012]. *Assume* M *is connected and* $\dim M > 1$. *There exist a homogeneous Hamiltonian isotopy*

$$\widetilde{\Phi} \colon \dot{T}^*(M \times \mathbb{R}) \times I \to \dot{T}^*(M \times \mathbb{R})$$

and C^∞*-functions* $u \colon (T^*M) \times I \to \mathbb{R}$ *and* $v \colon I \to \mathbb{R}$ *such that the diagram*

$$\begin{array}{ccc}
\dot{T}^*(M \times \mathbb{R}) \times I_{\rho \times \mathrm{id}_I} & \xrightarrow{\ \widetilde{\Phi}\ } & \dot{T}^*(M \times \mathbb{R})_\rho \\
\downarrow & & \downarrow \\
T^*M \times I & \xrightarrow{\ \Phi\ } & T^*M
\end{array}$$

commutes and that

$$\widetilde{\Phi}((x;\xi),(t;\tau),s) = \big((x';\xi'),(t+u(x,\xi/\tau,s);\tau)\big), \tag{2-7}$$

$$\widetilde{\Phi}((x;\xi),(t;0),s) = \big((x;\xi),(t+v(s);0)\big), \tag{2-8}$$

where $(x';\xi'/\tau)=\varphi_s(x;\xi/\tau)$. *Moreover,* $u(x;\xi/\tau,s)=v(s)$ *for* $(x;\xi/\tau)\notin C$.

Remark 2.3. If $\dim M=1$, $T^*M\setminus M$ has two connected components, and we must consider two functions v_- and v_+, one for each connected component. Hence, as mentioned to us by Damien Calaque, Proposition A.6 of [Guillermou et al. 2012] should be corrected accordingly. This has no consequence for the rest of the paper.

Applications to nondisplaceability. We consider a homogeneous Hamiltonian isotopy $\Phi=\{\varphi_t\}_{t\in I}\colon \dot{T}^*M\times I\to\dot{T}^*M$ satisfying (2-3), a C^1-map $\psi\colon M\to\mathbb{R}$ such that the differential $d\psi(x)$ never vanishes and we set

$$\Lambda_\psi := \big\{(x;d\psi(x));\ x\in M\big\}\subset\dot{T}^*M.$$

Theorem 2.4 [Guillermou et al. 2012]. *Consider a homogeneous Hamiltonian isotopy* $\Phi=\{\varphi_t\}_{t\in I}$ *and a* C^1*-map* $\psi\colon M\to\mathbb{R}$ *as above. Let* $F_0\in\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_M)$ *with compact support such that* $\mathrm{R}\Gamma(M;F_0)\neq 0$. *Then for any* $t\in I$,

$$\varphi_t(\mathrm{SS}(F_0)\cap\dot{T}^*M)\cap\Lambda_\psi\neq\varnothing.$$

Proof. We let $\Lambda\subset\dot{T}^*(M\times M\times I)$ be the conic Lagrangian submanifold associated to Φ and we let $K\in\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_{M\times M\times I})$ be the quantization of Φ on I constructed in Theorem 2.1. We set

$$\begin{aligned} &F = K\circ F_0\in\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_{M\times I}),\\ &F_{t_0} = F|_{\{t=t_0\}}\simeq K_{t_0}\circ F_0\in\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_M)\quad\text{for } t_0\in I.\end{aligned} \tag{2-9}$$

Then

$$\begin{cases}\mathrm{SS}(F)\subset(\Lambda\circ\mathrm{SS}(F_0))\cup T^*_{M\times I}(M\times I),\\ \mathrm{SS}(F)\cap T^*_M M\times T^*I\subset T^*_{M\times I}(M\times I),\\ \text{the projection } \mathrm{Supp}(F)\to I \text{ is proper.}\end{cases} \tag{2-10}$$

In particular we have

$$\begin{cases}F_t \text{ has a compact support in } M,\\ \mathrm{SS}(F_t)\cap\dot{T}^*M=\varphi_t(\mathrm{SS}(F_0)\cap\dot{T}^*M).\end{cases} \tag{2-11}$$

Hence, F_t has compact support and $\mathrm{R}\Gamma(M;F_t)\neq 0$ by Corollary 1.7. Since

$$\mathrm{SS}(F_t)\subset\varphi_t(\mathrm{SS}(F_0)\cap\dot{T}^*M)\cup T^*_M M,$$

the result follows from Corollary 1.5. ☐

Corollary 2.5. *Let $\Phi = \{\varphi_t\}_{t\in I}$ and $\psi\colon M \to \mathbb{R}$ be as in Theorem 2.4. Let N be a nonempty compact submanifold of M. Then for any $t \in I$,*

$$\varphi_t(\dot{T}^*_N M) \cap \Lambda_\psi \neq \varnothing.$$

By using Proposition 2.2 one deduces nondisplaceability results in the nonhomogeneous case and in particular the Arnold's nondisplaceability conjecture (which is a theorem since long). Moreover, there exists a refined version of all these results using the Morse inequalities. We refer to [Guillermou et al. 2012].

3. Tamarkin's nondisplaceability theorem

In this section we explain Tamarkin's approach [2008], following the presentation of [Guillermou and Schapira 2011].

We consider a trivial vector bundle

$$q\colon E = M \times V \to M \tag{3-1}$$

and a trivial cone $\gamma = M \times \gamma_0 \subset E$ such that

$$\gamma_0 \text{ is a closed convex proper cone of } V \text{ containing } 0 \text{ and } \gamma_0 \neq \{0\}. \tag{3-2}$$

Recall that the polar cone $\gamma_0^\circ \subset V^*$ is defined in (1-4).

In practice we shall use these results with $V = \mathbb{R}$ and $\gamma_0 = \{t \in \mathbb{R}; t \geq 0\}$.

In the sequel, we shall say that a subset in $T^*M \times V^*$ is a cone if it is invariant by the diagonal action of $\mathbb{R}^+$.

We denote by $\widehat{\pi}_E$ or simply $\widehat{\pi}$ the projection

$$\widehat{\pi}_E\colon T^*E = T^*M \times V \times V^* \to T^*M \times V^*.$$

We set

$$\begin{aligned} U_\gamma &:= T^*M \times V \times \mathrm{Int}(\gamma_0^\circ), \\ Z_\gamma &:= T^*E \setminus U_\gamma. \end{aligned} \tag{3-3}$$

Definition 3.1 [Guillermou and Schapira 2011]. A closed cone $A \subset T^*M \times V^*$ is called a strict γ-cone if

$$A \subset (T^*M \times \mathrm{Int}\gamma_0^\circ) \cup T^*_M M \times \{0\}.$$

Definition 3.2. For $F, G \in \mathrm{D^b}(\boldsymbol{k}_E)$, we set

$$F \star G := \mathrm{R}s_!(q_1^{-1} F \overset{\mathrm{L}}{\otimes} q_2^{-1} G), \tag{3-4}$$

$$F \star_{\mathrm{np}} G := \mathrm{R}s_*(q_1^{-1} F \overset{\mathrm{L}}{\otimes} q_2^{-1} G), \tag{3-5}$$

$$\mathcal{H}om^*(G, F) := \mathrm{R}q_{1*}\mathrm{R}\mathcal{H}om\,(q_2^{-1} G, s^! F). \tag{3-6}$$

The morphism $\boldsymbol{k}_\gamma \to \boldsymbol{k}_{M\times\{0\}}$ gives the morphisms

$$F \star \boldsymbol{k}_\gamma \to F, \quad F \star_{\mathrm{np}} \boldsymbol{k}_\gamma \to F. \tag{3-7}$$

For $F_1, F_2, F_3 \in \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_E)$ one proves the isomorphisms

$$\begin{aligned} (F_1 \star F_2) \star F_3 &\simeq F_1 \star (F_2 \star F_3), \\ \mathcal{H}om^*(F_1 \star F_2, F_3) &\simeq \mathcal{H}om^*(F_1, \mathcal{H}om^*(F_2, F_3)), \\ \mathrm{RHom}\,(F_1 \star F_2, F_3) &\simeq \mathrm{RHom}\,(F_1, \mathcal{H}om^*(F_2, F_3)). \end{aligned}$$

Localization and convolution. Recall that $E = M \times V$ is a trivial vector bundle over M, γ_0 is a cone satisfying (3-2) and the sets U_γ and Z_γ are defined in (3-3). By definition $\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_E; U_\gamma)$ is a localization of $\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_E)$ and we let

$$Q_\gamma \colon \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_E) \to \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_E; U_\gamma)$$

be the functor of localization.

We introduce the kernels

$$L_\gamma := \boldsymbol{k}_\gamma \star \colon \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_E) \to \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_E), \tag{3-8}$$

$$R_\gamma := \mathcal{H}om^*(\boldsymbol{k}_\gamma, \bullet) \colon \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_E) \to \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_E). \tag{3-9}$$

Proposition 3.3. (i) *The functor L_γ defined in* (3-8) *takes its values in*

$$\mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\boldsymbol{k}_E)^{\perp,l}$$

and sends $\mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\boldsymbol{k}_E)$ to 0. *It factorizes through Q_γ and induces a functor $l_\gamma \colon \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_E; U_\gamma) \to \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_E)$ such that $L_\gamma \simeq l_\gamma \circ Q_\gamma$.*

(ii) *The functor l_γ is left adjoint to Q_γ and induces a natural equivalence $\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_E; U_\gamma) \simeq \mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\boldsymbol{k}_E)^{\perp,l}$.*

Proposition 3.3 is visualized in the diagram

$$\begin{array}{ccccc} \mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\boldsymbol{k}_E) & \hookrightarrow & \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_E) & \xrightarrow{Q_\gamma} & \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_E; U_\gamma) \\ & & & \searrow^{L_\gamma} & \sim \downarrow l_\gamma \\ & & & & \mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\boldsymbol{k}_E)^{\perp,l}. \end{array} \tag{3-10}$$

There are similar results with R_γ instead of L_γ.

Notation 3.4. Let us set for short

$$\begin{aligned} \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_M^\gamma) &:= \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_E; U_\gamma), \\ \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_M^{\gamma,l}) &:= \mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\boldsymbol{k}_E)^{\perp,l}, \\ \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_M^{\gamma,r}) &:= \mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\boldsymbol{k}_E)^{\perp,r}. \end{aligned} \tag{3-11}$$

When $M = \mathrm{pt}$, we set

$$\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}^{\gamma}) := \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}^{\gamma}_{\mathrm{pt}}), \tag{3-12}$$

and similarly with $\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}^{\gamma,l})$ and $\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}^{\gamma,r})$.

Denote by $p\colon E = M \times V \to V$ the projection. We get the diagram of categories in which the horizontal arrows are equivalences

$$\begin{array}{ccccc} \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}^{\gamma,l}_M) & \xleftarrow[\sim]{l_\gamma} & \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}^{\gamma}_M) & \xrightarrow[\sim]{r_\gamma} & \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}^{\gamma,r}_M) \\ \downarrow{\scriptstyle \mathrm{R}p_!} & & & & \downarrow{\scriptstyle \mathrm{R}p_*} \\ \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}^{\gamma,l}) & \xleftarrow[\sim]{l_\gamma} & \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}^{\gamma}) & \xrightarrow[\sim]{r_\gamma} & \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}^{\gamma,r}). \end{array} \tag{3-13}$$

Consider the functor

$$\Psi_\gamma\colon \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_M) \to \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_E), \quad F \mapsto q^{-1}F \otimes \boldsymbol{k}_\gamma.$$

We have $L_\gamma \circ \Psi_\gamma \xrightarrow{\sim} \Psi_\gamma$; in the sequel, we consider Ψ_γ as a functor

$$\Psi_\gamma\colon \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_M) \to \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}^{\gamma}_M). \tag{3-14}$$

Proposition 3.5. *The functor Ψ_γ in* (3-14) *is fully faithful.*

A separation theorem.

Lemma 3.6. *Let $F \in \mathsf{D}^{\mathrm{b}}_{Z_\gamma}(\boldsymbol{k}_E)^{\perp,l}$. We assume that there exists $A \subset T^*M \times V^*$ such that*

(i) *A is a closed strict γ-cone* (see Definition 3.1), *and*

(ii) $\mathrm{SS}(F) \cap U_\gamma \subset \widehat{\pi}_E^{-1}(A)$.

Then $\mathrm{SS}(F) \subset (\mathrm{SS}(F) \cap U_\gamma) \cup T^*_E E$.

This "cut-off" lemma is a crucial step in the proof of the theorem below, a slight generalization of [Tamarkin 2008, Theorem 3.2]. Here, we write $\widehat{\pi}$ instead of $\widehat{\pi}_E$ for short.

Theorem 3.7 (separation theorem). *Let A, B be two closed strict γ-cones in $T^*M \times V^*$. Let*

$$F \in \mathsf{D}^{\mathrm{b}}_{\widehat{\pi}^{-1}(A)}(\boldsymbol{k}_E; U_\gamma) \quad \textit{and} \quad G \in \mathsf{D}^{\mathrm{b}}_{\widehat{\pi}^{-1}(B)}(\boldsymbol{k}_E; U_\gamma).$$

Assume that $A \cap B \subset T^*_M M \times \{0\}$ and that the projection $M \times V \to V$ is proper on the set

$$\{(x, v_1 - v_2);\ (x, v_1) \in \operatorname{supp} G, (x, v_2) \in \operatorname{supp} F\}.$$

Then $\operatorname{Hom}_{\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_E; U_\gamma)}(F, G) \simeq 0$.

The Tamarkin category. We particularize the preceding results to the case where $V=\mathbb{R}$ and $\gamma_0=\{t\in\mathbb{R}; t\geq 0\}$. Hence, with the notations of (3-3), we have $U_\gamma=\{\tau>0\}$. We define the map

$$\rho\colon U_\gamma=\{\tau>0\}\to T^*M,\quad (x,t;\xi,\tau)\mapsto(x;\xi/\tau). \tag{3-15}$$

According to Notation 3.4, we set

$$\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_M^\gamma):=\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_{M\times\mathbb{R}};\{\tau>0\},\quad \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_M^{\gamma,l}):=\mathsf{D}^{\mathrm{b}}_{\{\tau\leq 0\}}(\boldsymbol{k}_{M\times\mathbb{R}})^{\perp,l}.$$

For a closed subset A of T^*M, we also set

$$\mathsf{D}^{\mathrm{b}}_A(\boldsymbol{k}_M^\gamma):=\mathsf{D}^{\mathrm{b}}_{\rho^{-1}(A)}\big(\boldsymbol{k}_{M\times\mathbb{R}};\{\tau>0\}\big).$$

Examples 3.8. (i) Let $M=\mathbb{R}$ endowed with the coordinate x and consider the set

$$Z=\{(x,t)\in M\times\mathbb{R}; -1\leq x\leq 1, 0\leq 2t<-x^2+1\}$$

Consider the sheaf $\boldsymbol{k}_Z$ and denote by $(x,t;\xi,\tau)$ the coordinates on $T^*(M\times\mathbb{R})$. The set $\mathrm{SS}(\boldsymbol{k}_Z)$ is given by

$$\begin{aligned}\{t=0,-1\leq x\leq 1,\tau>0,\xi=0\}&\cup\{2t=-x^2+1,\xi=x\tau,\tau>0\}\\ &\cup\{x=-1,t=0,0\leq-\xi\leq\tau,\tau>0\}\\ &\cup\{x=1,t=0,0\leq\xi\leq\tau,\tau>0\}\\ &\cup Z\times\{\xi=\tau=0\}.\end{aligned}$$

It follows that $\rho(\mathrm{SS}(\boldsymbol{k}_Z)\cap(T^*M\times\dot{T}^*\mathbb{R}))$ in T^*M (with coordinates $(x,u=\xi/\tau)$) is the set

$$\begin{aligned}\{u=0,-1\leq x\leq 1\}&\cup\{u=x,-1\leq x\leq 1\}\\ &\cup\{x=-1,-1\leq u\leq 0\}\\ &\cup\{x=1,0\leq u\leq 1\}.\end{aligned}$$

(ii) Let $a\in\mathbb{R}$ and consider the set $Z=\{(x,t)\in M\times\mathbb{R}; t\geq ax\}$. Then $\rho(\mathrm{SS}(\boldsymbol{k}_Z))$ in T^*M is the set $\{(x;u); u=a\}$.

(iii) If G is a sheaf on M and $F=G\boxtimes\boldsymbol{k}_{s\geq 0}$, then $\rho(\mathrm{SS}(F))=\mathrm{SS}(G)$.

Theorem 3.9 [Tamarkin 2008, Theorem 3.2]. *Let A and B be two compact subsets of T^*M and assume that $A\cap B=\varnothing$. Let $F\in\mathsf{D}^{\mathrm{b}}_A(\boldsymbol{k}_M^\gamma)$ and let $G\in\mathsf{D}^{\mathrm{b}}_B(\boldsymbol{k}_M^\gamma)$. Then* $\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_M^\gamma)}(F,G)\simeq 0$.

Torsion objects. Tamarkin [2008] introduces the notion of torsion objects, but does not study the category of such objects systematically as we did in [Guillermou and Schapira 2011].

Let us set for short $\{\tau \geq 0\}$ instead of $T^*M \times \mathbb{R} \times \{\tau \geq 0\}$, and similarly with $\{\tau \leq 0\}$ and $\{\tau > 0\}$. Recall that $\mathsf{D}^{\mathrm{b}}_{\{\tau\geq 0\}}(\boldsymbol{k}_{M\times\mathbb{R}})$ is the subcategory of $F \in \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_{M\times\mathbb{R}})$ such that $\mathrm{SS}(F) \subset \{\tau \geq 0\}$. We have $F \in \mathsf{D}^{\mathrm{b}}_{\{\tau\geq 0\}}(\boldsymbol{k}_{M\times\mathbb{R}})$ if and only if the we have the isomorphism

$$F \star_{\mathrm{np}} \boldsymbol{k}_{M\times[0,+\infty[} \xrightarrow{\sim} F. \tag{3-16}$$

Define the map

$$T_c \colon M \times \mathbb{R} \to M \times \mathbb{R}, \quad (x,t) \mapsto (x, t+c).$$

For $F \in \mathsf{D}^{\mathrm{b}}_{\{\tau\geq 0\}}(\boldsymbol{k}_{M\times\mathbb{R}})$ we have

$$F \star_{\mathrm{np}} \boldsymbol{k}_{M\times[c,+\infty[} \xrightarrow{\sim} T_{c*}F. \tag{3-17}$$

The inclusions $[d,+\infty[\, \subset [c,+\infty[$ $(c \leq d)$ induce natural morphisms of functors from $\mathsf{D}^{\mathrm{b}}_{\{\tau\geq 0\}}(\boldsymbol{k}_{M\times\mathbb{R}})$ to itself

$$\tau_{c,d} \colon T_{c*} \to T_{d*}, \quad c \leq d.$$

Definition 3.10 (Tamarkin). An object $F \in \mathsf{D}^{\mathrm{b}}_{\{\tau\geq 0\}}(\boldsymbol{k}_{M\times\mathbb{R}})$ is called a torsion object if $\tau_{0,c}(F) = 0$ for some $c \geq 0$ (and hence all $c' \geq c$).

For example, if $F \in \mathsf{D}^{\mathrm{b}}_{\{\tau\geq 0\}}(\boldsymbol{k}_{M\times\mathbb{R}})$ and F is supported by $M \times [a,b]$ for some compact interval $[a,b]$ of $\mathbb{R}$, then F is a torsion object.

We let $\mathcal{N}_{\mathrm{tor}}$ be the full subcategory of $\mathsf{D}^{\mathrm{b}}_{\{\tau\geq 0\}}(\boldsymbol{k}_{M\times\mathbb{R}})$ consisting of torsion objects.

Theorem 3.11. *The subcategory $\mathcal{N}_{\mathrm{tor}}$ is a null system in $\mathsf{D}^{\mathrm{b}}_{\{\tau\geq 0\}}(\boldsymbol{k}_{M\times\mathbb{R}})$.*

The subcategory $\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}^{\gamma,l}_M)$ of $\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_{M\times\mathbb{R}})$ is contained in $\mathsf{D}^{\mathrm{b}}_{\{\tau\geq 0\}}(\boldsymbol{k}_{M\times\mathbb{R}})$. So we can define torsion objects in $\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}^{\gamma,l}_M)$ or in the equivalent category $\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}^{\gamma}_M)$. We let $\mathcal{N}^{\gamma}_{\mathrm{tor}}$ be the subcategory of torsion objects in $\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}^{\gamma}_M)$. Then $\mathcal{N}^{\gamma}_{\mathrm{tor}}$ is a null system.

Definition 3.12. The triangulated category $\mathcal{T}(\boldsymbol{k}_M)$ is the localization of $\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}^{\gamma}_M)$ by the null system $\mathcal{N}^{\gamma}_{\mathrm{tor}}$. In other words,

$$\mathcal{T}(\boldsymbol{k}_M) = \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}^{\gamma}_M)/\mathcal{N}^{\gamma}_{\mathrm{tor}}.$$

For a closed conic subset $W \subset \{\tau > 0\}$, we let $\mathcal{T}_W(\boldsymbol{k}_M)$ be the full subcategory of $\mathcal{T}(\boldsymbol{k}_M)$ consisting of objects which are isomorphic to the image of some $F \in \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}^{\gamma}_M)$ such that $\mathrm{SS}(F) \cap \{\tau > 0\}$ is contained in W.

Invariance by Hamiltonian isotopies. The next theorem, from [Tamarkin 2008], asserts that the objects of the category $\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_M^\gamma)$ are invariant by Hamiltonian isotopies, up to torsion. This result, together with Theorem 3.9 are to compare with the main properties of the Floer homology. Note that the proof of this result proposed in [Guillermou and Schapira 2011] is much simpler that Tamarkin's original one, thanks to Theorem 2.1.

Let $\Phi\colon T^*M \times I \to T^*M$ be a Hamiltonian isotopy satisfying (2-6). We associate to it a homogeneous Hamiltonian isotopy by using Proposition 2.2 and we consider the kernel K constructed in Theorem 2.1. This kernel naturally defines a functor

$$\Psi_s\colon \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_M^\gamma) \to \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_M^\gamma),\ F \mapsto K_s \circ F.$$

Theorem 3.13. *Let $\Phi\colon T^*M \times I \to T^*M$ be a Hamiltonian isotopy satisfying* (2-6)*. Then for A a closed subset of T^*M and $F \in \mathsf{D}^{\mathrm{b}}_A(\boldsymbol{k}_M^\gamma)$ we have, for all $s \in I$:*

(i) $\Psi_s(F) \in \mathsf{D}^{\mathrm{b}}_{\varphi_s(A)}(\boldsymbol{k}_M^\gamma)$ *for any $s \in I$.*

(ii) $F \simeq \Psi_s(F)$ *in $\mathcal{T}(\boldsymbol{k}_M)$ for any $s \in I$.*

Tamarkin's nondisplaceability theorem. Recall that two compact subsets A and B of T^*M are called mutually non displaceable if, whatever be the Hamiltonian isotopy $\Phi\colon T^*M \times I \to T^*M$ satisfying (2-6), $A \cap \varphi_s(B) \neq \varnothing$ for all $s \in I$. A compact subset A is called non displaceable if A and A are mutually nondisplaceable.

Theorem 3.14. *Let A and B be two compact subsets of T^*M. Assume that there exist $F \in \mathsf{D}^{\mathrm{b}}_A(\boldsymbol{k}_M^\gamma)$ and $G \in \mathsf{D}^{\mathrm{b}}_B(\boldsymbol{k}_M^\gamma)$ such that* $\mathrm{Hom}_{\mathcal{T}(\boldsymbol{k}_M)}(F, G) \neq 0$. *Then A and B are mutually nondisplaceable in T^*M.*

In fact, Tamarkin's original result is stronger, but we restrict ourselves to this situation for sake of simplicity.

4. Exact Lagrangians and simple sheaves

We recall some results from [Guillermou and Schapira 2011; Guillermou 2012].

Simple sheaves [Kashiwara and Schapira 1990, Section 7.5]. Let $\Lambda \subset \dot{T}^*M$ be a locally closed conic Lagrangian submanifold and let $p \in \Lambda$. Simple sheaves along Λ at p are defined in [Kashiwara and Schapira 1990, Definition 7.5.4].

When Λ is the conormal bundle to a submanifold $N \subset M$, that is, when the projection $\pi_M|_\Lambda\colon \Lambda \to M$ has constant rank, then an object $F \in \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_M)$ is simple along Λ at p if $F \simeq \boldsymbol{k}_N\,[d]$ in $\mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_M; p)$ for some shift $d \in \mathbb{Z}$.

If $\mathrm{SS}(F)$ is contained in Λ on a neighborhood of Λ, Λ is connected and F is simple at some point of Λ, then F is simple at every point of Λ.

Exact Lagrangians. Recall that a (in general nonconic) Lagrangian submanifold Λ of T^*M is exact if there exists $f\colon \Lambda \to \mathbb{R}$ such that $\alpha_M|_\Lambda = df$. Assuming that Λ is a compact exact Lagrangian connected submanifold of T^*M, there exists a smooth closed conic Lagrangian submanifold $\widetilde{\Lambda}$ of $\dot{T}^*(M\times\mathbb{R})$ contained in $T^*_{\{\tau>0\}}(M\times\mathbb{R})$ such that the map ρ induces an isomorphism $\widetilde{\rho}\colon \widetilde{\Lambda}/\mathbb{R}^+ \xrightarrow{\sim} \Lambda$. Indeed, the conic submanifold

$$\widetilde{\Lambda} := \bigl\{(x,t;\xi,\tau);\ \tau>0,\ (x;\xi/\tau)\in\Lambda \text{ and } t=-f(x;\xi/\tau)\bigr\}$$

satisfies $\alpha_{M\times\mathbb{R}}|_{\widetilde{\Lambda}} = 0$ and, hence, is Lagrangian.

Theorem 4.1 [Guillermou and Schapira 2011; Guillermou 2012]. *Let $\Lambda \subset T^*M$ be a compact connected Lagrangian submanifold and let $\widetilde{\Lambda}$ be as above. Assume that there exists $F\in \mathsf{D}^{\mathrm{b}}(\boldsymbol{k}_M^{\gamma,l})$ satisfying the following properties*:

(a) *the projection $q_\mathbb{R}\colon M\times\mathbb{R}\to\mathbb{R}$ is proper on* $\operatorname{supp}(F)$.

(b) $\mathrm{SS}(F)\cap \dot{T}^*(M\times\mathbb{R}) = \widetilde{\Lambda}$.

(c) *F is simple along $\widetilde{\Lambda}$.*

(d) *there exist $a,b\in\mathbb{R}$ and $r\in\mathbb{N}$ such that $F|_{M\times]-\infty,a[}\simeq 0$ and $F|_{M\times]b,+\infty[}\simeq \boldsymbol{k}^{\oplus r}_{M\times]b,+\infty[}$.*

(e) *F is cohomologically constructible.*

Then

(i) *Λ is nondisplaceable*,

(ii) *the natural morphism $\boldsymbol{k}_M \to \mathrm{R}\pi_{M*}\boldsymbol{k}_\Lambda$ induces an isomorphism*

$$\mathrm{R}\Gamma(M;\boldsymbol{k}_M) \xrightarrow{\sim} \mathrm{R}\Gamma(\Lambda;\boldsymbol{k}_\Lambda).$$

Hence, assuming the existence of a sheaf F satisfying conditions (a)–(e) from the theorem, we recover a classical result of [Fukaya et al. 2008] and [Nadler 2009]. Claude Viterbo has informed us that he was able to construct such a sheaf F under mild hypotheses on Λ using Floer homology and Stéphane Guillermou [2012] recently obtained such a sheaf by using only sheaf theoretical methods.

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