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Navigating the MAZE

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The game of MAZE was introduced in 2006 by Albert, Nowakowski and Wolfe, and is an instance of an option-closed game and as such each position has reduced canonical form equal to a number or a switch. It was conjectured that because of the 2-dimensional structure of the board there was a bound on the denominator of the numbers which appeared as numbers or in the switches. We disprove this by constructing, for each number and each switch, a MAZE position whose reduced canonical form is that value. Surprisingly, we can also restrict the interior walls to be in one direction only, seemingly giving an advantage to one player. This also gives a linear time algorithm that determines the best move up to an infinitesimal.

1. Introduction

MAZE was introduced in [Albert et al. 2007], but apart from a few scattered observations, nothing was known about the values of the game. In the original article, MAZE is played on a rectangular grid oriented 45° to the horizontal.

The token starts at the top of the board and highlighted edges are *walls* that may not be crossed. Left is allowed to move a token any number of cells in a southwesterly direction and Right is allowed to move similarly in a southeasterly direction. However, for ease of referring to specific places in the position, we re-orient the sides parallel to the page so that Left moves downward and Right moves to the right; see Figure 1. One interesting feature is that any number of consecutive Left (Right) moves also can be accomplished in one move. This feature had been noted in several games, including HACKENBUSH strings [Berlekamp et al. 2001], and given the name of *option-closed* in [Nowakowski and Ottaway 2011], a reference we henceforth abbreviate as [NO]. Siegel [2011] notes that the partial order of option-closed games born on day n forms a planar lattice.

For a game G, G^L (G^R) is a left (right) option of G; $G^{\mathcal{L}}$ ($G^{\mathcal{R}}$) is the set of all left (right) options of G; and for a set of games S, $S^{\mathcal{L}}$ ($S^{\mathcal{R}}$) is the set of all left (right) options that can be reached from any game in S (i.e. $G^{\mathcal{L}\mathcal{L}}$ is the set

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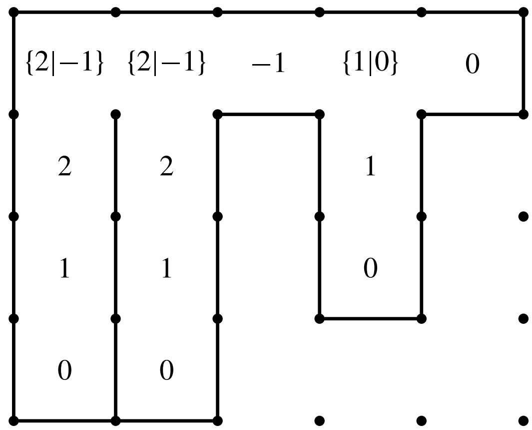

Figure 1. A MAZE board rotated so that Left and Right move down and right, respectively. Reduced canonical form of values included on board.

of all options that can be reached in two consecutive left moves from G). Most of combinatorial game theory considers the canonical forms of games, those where $G^{\mathcal{L}}$ and $G^{\mathcal{R}}$ have had dominated options removed and reversible options bypassed. We need the form of the game $G = \{G^{\mathcal{L}}|G^{\mathcal{R}}\}$ with all the options, even the bad ones. We call this the *literal* form of the game.

Definition 1 [NO]. A game G is called *option-closed* if, in the literal form, $G^{\mathcal{L}\mathcal{L}} \subset G^{\mathcal{L}}$, $G^{\mathcal{R}\mathcal{R}} \subset G^{\mathcal{R}}$ and, recursively, all the followers of G are option-closed.

For example, even though the canonical form of $G = \{0 \mid \{1 \mid 0\}, 0\}$ and $H = \{0 \mid \{1 \mid -1\}, 0\}$ are the same, G is option-closed but $H = \{0 \mid \{1 \mid -1\}, 0\}$ is not since Right can move to -1 in two moves but not in one. The fact that a game is option-closed is intrinsic to the game and is not necessarily identifiable from the canonical form [NO].

The canonical form of an option-closed game can be quite complicated. However, often the differences between many of the options in the canonical form are infinitesimals. The *reduced canonical form* (simplest game infinitesimally close, [Grossman and Siegel 2009]) should be much less complicated and indeed, in [NO], it was shown that the reduced canonical form of an option-closed game is either a number or a switch $\{a|b\}$ of numbers $a > b$.

In [NO], the authors noted that MAZE is an option-closed game and asked for an analysis of the game. It was thought that the two-dimensionality of the board would restrict the powers of 2 that could occur as a denominator of numbers that occur in reduced canonical forms. We show that this is false and give an algorithm to construct all numbers and all switches. Surprisingly, the algorithm constructs a position which is rectangular and all interior walls are vertical, that is, they prevent only Right moves.

In the next section, we present an overview of the required theory of the reduced canonical form and option-closed games before moving on to prove our main result in Section 4. Also, in Section 5 (Theorem 14), we obtain a linear time algorithm to find the reduced canonical form of any MAZE position.

We follow the combinatorial game theory notation and definitions of [Albert et al. 2007; Berlekamp et al. 2001].

2. Reduced canonical form and option-closed background.

Definition 2 [Grossman and Siegel 2009]. Two games G and H are *equalish*, written $G =_I H$, if $G - H$ is an infinitesimal. A game G is *numberish* if there is a dyadic rational x such that $G - x$ is an infinitesimal. A game G is *infinitesimally-dominated* by H if there is some integer n, such that $H + n.\uparrow - G \geq 0$; written as $H \geq_I G$. A left option, G^L, of G is *infinitesimally-reversible* if there is some $G^{LR} \leq_I G$.

Definition 3 [Grossman and Siegel 2009]. A game G is in *reduced canonical form*, denoted $\text{rcf}(G)$, if for any follower H of G, either (i) H is a number in simplest form, or (ii) H is neither a number nor numberish and H contains no infinitesimally-dominated or infinitesimally-reversible options.

Lemma 4 [Grossman and Siegel 2009]. *[Thm 4.8] If G is not numberish, then rcf(G) is obtained by (i) replacing options with simpler options infinitesimally close to the original option, (ii) eliminating infinitesimally-dominated options, and (iii) bypassing infinitesimally-reversible options.*

Lemma 5 [NO]. *If a and b are numbers with $a \geq b$, then $a \geq_I \{a \mid b\} \geq_I b$.*

To illustrate the above concepts, we sketch the proof of Lemma 5. Let a and b be as in Lemma 5. Note that a and $\{a \mid b\}$ are incomparable. Consider $a - \{a \mid b\} + 3\cdot\uparrow = a + \{-b \mid -a\} + 3\cdot\uparrow$. Left wins by moving to $a - b + 3\cdot\uparrow > 3\cdot\uparrow > 0$. By the Number Avoidance Theorem, if Right can win then she must have a good move in either $\{a \mid b\}$ or in $3\cdot\uparrow$. But $a - a + 3\cdot\uparrow > 0$ while moving in $3\cdot\uparrow$ leaves $a + \{-b \mid -a\} + \Uparrow*$, in which case Left wins by responding to $a - b + \Uparrow* \geq \Uparrow* > 0$.

We need two other relatively trivial but necessary results about the transitivity of $\geq_I$ that we will use without mention.

Lemma 6. *If $x \geq_I y$ and $y \geq_I z$ then $x \geq_I z$.*

Lemma 7. *If $x \geq_I y$ and $y > z$ with y and z numbers, then $x > z$.*

3. The structure of MAZE

The main result of [NO] is that the reduced canonical form of an option-closed game is either a number or a switch.

Lemma 8 [NO]. *If G is an option-closed game, then $rcf(G) = rcf(\{a \mid b\})$ for some numbers a and b.*

In the special case of MAZE, this result is a corollary of the results needed to prove that our construction of Section 4 works.

Let G be an option-closed game where it and all of its followers are given in their literal forms. A right option G' is a *first right option* of G if $G' \in G^{\mathcal{R}} \setminus G^{\mathcal{R}\mathcal{R}}$, i.e., can be reached in one right move but not in two right moves. A *right-option-closed sequence* of G is a sequence of right options of G, $\alpha = \langle y_1, y_2, \ldots, y_n \rangle$ where y_1 is a first right option of G, y_{i+1} is a first right option of y_i, for $0 < i < n$, and the value of y_n is a number. A *first left option* and a *left-option-closed sequence* are defined analogously. Let α be a right (left)-option-closed sequence. The *norm* of α, written $\overline{\alpha}$ is the minimum (maximum) of all numbers in the sequence. The norm of $\langle \varnothing \rangle$ is not defined.

Lemma 9. *Let H be an option-closed game $\alpha = \langle y_0, y_1, \ldots, y_n \rangle$ be a right- or left-option-closed sequence. Let i be the least index such that y_i is a number. Then $y_i = \overline{\alpha}$.*

Proof. Suppose that α is a right-option-closed sequence and let i be the least index such that y_i is a number. Suppose y_j, $j > i$ is a number then y_j is a right option of y_i and hence $y_i < y_j$. The proof for a left-option-closed sequence is similar. □

Let M be a MAZE position. Then listing the options in order, $\langle G^{\mathcal{L}} \rangle$ and $\langle G^{\mathcal{R}} \rangle$ are, respectively, a left- and a right-option-closed sequence. This is an important fact but it doesn't define MAZE. Extra conditions must hold, if, from a cell, Left can move p cells and then Right q and reversing the order gives a legal move then the two resulting positions are the same.

If M is not a number then $\langle \overline{G^{\mathcal{L}}} \rangle$ is the left stop of M and $\langle \overline{G^{\mathcal{R}}} \rangle$ is the right stop and in those terms we could invoke results from [NO]. However, we need to know more details about the order relations between the options in the special case of MAZE. The following results can be and are generalized to games where $G^{\mathcal{L}}$ and $G^{\mathcal{R}}$ contain more than one option-closed sequence.

Theorem 10. *Let G be a* MAZE *position and set $\langle G^{\mathcal{R}} \rangle = \langle x_1, x_2, \ldots, x_m \rangle$ and $\langle G^{\mathcal{L}} \rangle = \langle y_1, y_2, \ldots, y_n \rangle$. Then*

(i) $rcf(G) = rcf\big(\big\{\langle \overline{G^{\mathcal{L}}} \rangle \,\big|\, \langle \overline{G^{\mathcal{R}}} \rangle \big\}\big)$;

(ii) *If G is a number, then $G^L < G < G^R$ for all $G^L \in G^{\mathcal{L}}$ and $G^R \in G^{\mathcal{R}}$. Moreover, $G = \left\{\langle\overline{G^{\mathcal{L}}}\rangle \,\middle|\, \langle\overline{G^{\mathcal{R}}}\rangle\right\}$.*

(iii) *If G is not a number, then $\langle\overline{G^{\mathcal{L}}}\rangle \geq_I G \geq_I \langle\overline{G^{\mathcal{R}}}\rangle$.*

Proof. We induct on the size, i.e. number of cells, of the MAZE position. If the position has one cell (i.e. $G^{\mathcal{L}} = G^{\mathcal{R}} = \varnothing$), then the claims are trivially true. Suppose that the result is true for all positions with at most s cells.

Let G be a position with s cells. We may assume that G^R is not empty and that $\langle\overline{G^{\mathcal{R}}}\rangle = x_k$ for some $1 \leq k \leq m$.

(i) If $k = 1$, then x_1 is a number and by (ii), $x_1 < x_i$ for all i. If $k \neq 1$, then for $1 \leq i < k$, x_i is not a number and by induction from (iii), $x_i \geq_I x_k$. For $i > k$, $x_k < x_i$ by (ii). Combining these results gives $x_k \leq_I x_i$ for all i. Thus, $\mathrm{rcf}(G) = \mathrm{rcf}\left(\left\{\langle\overline{G^{\mathcal{L}}}\rangle \,\middle|\, x_k\right\}\right)$.

(ii) If G is a number, then $G < x_k$ since $x_k \in G^{\mathcal{R}}$. For all i we have $G < x_k \leq_I x_i$ from above, and so by Lemma 7, $G < x_i$. If $G^{\mathcal{L}} = \varnothing$, then $G = \{. \mid x_k\}$. Otherwise, $\langle\overline{G^{\mathcal{L}}}\rangle < G < x_k$, so $G = \left\{\langle\overline{G^{\mathcal{L}}}\rangle \,\middle|\, x_k\right\}$.

(iii) If G is not a number, then $G^{\mathcal{L}} \neq \varnothing$ and $\langle\overline{G^{\mathcal{L}}}\rangle = y_j$ for some $1 \leq j \leq n$, and $\mathrm{rcf}(G) = \mathrm{rcf}(\{y_j | x_k\})$. If $y_j < x_k$, then $y_i \leq_I y_j < x_k < x_l$ for all i, l, so $G = \{y_j \mid x_k\}$ is a number, which is a contradiction. Hence, $y_j \geq x_k$. Combining the above results, $G \equiv_I \{y_j \mid x_k\} \geq_I x_k = \langle\overline{G^{\mathcal{R}}}\rangle$ by Lemma 5. Thus, $G \equiv_I \{y_j \mid x_k\}$.

Thus, $y_j \geq x_k$ and $G \equiv_I \{y_j \mid x_k\}$ and so $\geq_I x_k$ by Lemma 5. □

In the case that G is not a number, then $\langle\overline{G^R}\rangle$ and $\langle\overline{G^L}\rangle$ are the right and left stops of G. In this language, Theorem 10 is an extension of [NO], applied to the specific case of Maze.

4. The construction

All positions will be rectangular mazes with vertical walls plus the horizontal walls on the lower edge of the rectangle. For brevity, we refer to such a position as a *vertical* position. An *interior* wall has its endpoints non-adjacent along the outside boundary. In any MAZE layout, each cell has a value corresponding to that position where the token is on that cell. The value of the cell at the top left of the maze the *value of the rectangle*. With a number a, we associate $m(a)$, any MAZE position with a reduced canonical form of value a. First, we must show how to adjust the height of any MAZE position without changing its value.

Lemma 11. *Let M be a MAZE position. Let M' be the MAZE position obtained by: deleting the bottom and right-hand walls of the position; adding another row at the bottom and a column on the right hand side with walls on the bottom*

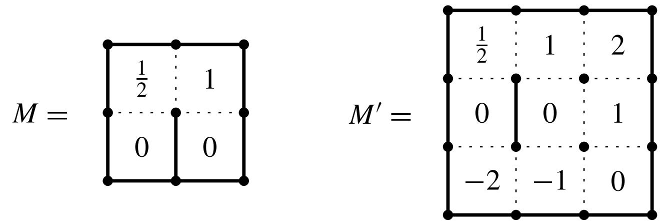

Figure 2. MAZE boards of equivalent value.

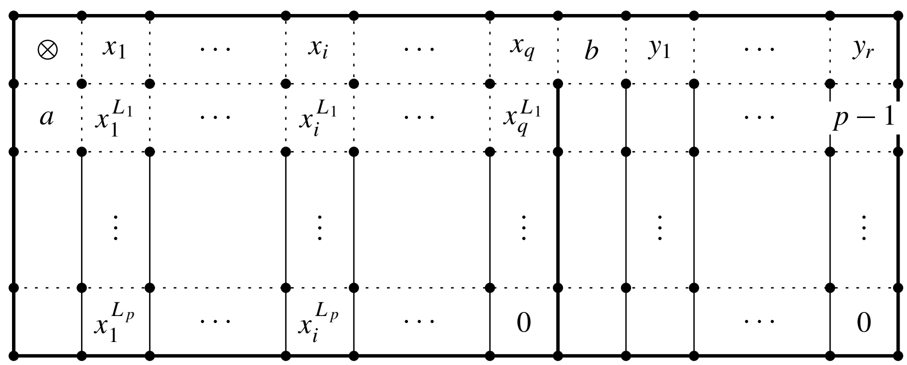

Figure 3. Outline of the Construction. Values without subscripts are actual values. Thick black lines represent existing walls, dashed lines walls that do not exist and thin lines walls that may or may not exist.

of the new row and on the right-hand side of the new column. (*Each dimension has increased by* 1.) *The values of M and M' are equal.*

Proof. Let the rows of M be indexed $1, 2 \ldots, p$ and the columns $1, 2, \ldots, q$ and let the rows and columns of M' be indexed $0, 1, 2 \ldots, p$ and $0, 1, 2 \ldots, q$ respectively, where (0,0) and (1,1) are the bottom-right corners of M' and M, respectively.

The cells on the bottom row of M' have the values $0, -1, -2, \ldots, -q$, from right to left, and the right-hand cells have values $0, 1, 2, \ldots, q$, from bottom to top.

The cell (1,1) in M' has value $\{-1 \mid 1\} = 0$, which equals the value of the (1,1) cell in M. We now proceed by induction on $i + j$. Note that the value of (i, j) is $\leq i$ since Left has at most i moves and similarly the value is $\geq -j$. Therefore, any moves from (i, j) to the zero row or to the zero column are dominated and the values of the cells then the values in the rest of M' are the same as those of M. □

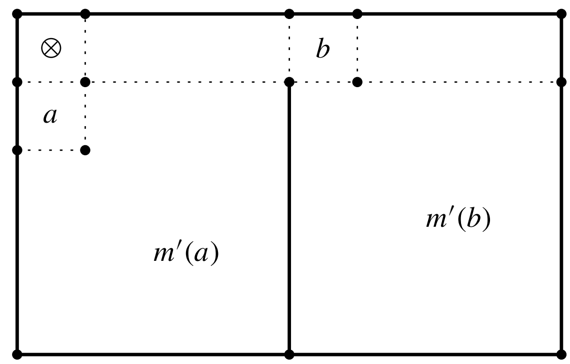

Figure 4. Outline of the Construction. Conventions are as in the previous figure.

The Construction. The idea is the following: if we wish to construct G with $\mathrm{rcf}(G) = \mathrm{rcf}(\{a \mid b\})$, a and b numbers, then take the MAZE positions $m(a)$ and $m(b)$ obtained via the Construction, adjust the heights and adjoin the two positions as in Figure 4. We need one more piece of notation. Let $x = y/2^m$ where y is odd and $m > 0$, put $\|x\| = m$. If $x = \{a \mid b\})$, the naming convention of numbers only allows the power of 2 in the denominator of x to to increase slightly over that of a and b.

Lemma 12. *If p and q, $p < q$, are dyadic rationals and $x = \{p \mid q\}$ then $\|x\| \leq \max\{\|p\|, \|q\|\} + 1$. If x is in canonical form then $\|x\| = \max\{\|p\|, \|q\|\} + 1$ and $\|x\| < \|s\|$ for any dyadic rational $p < s < q$.*

Theorem 13. *Let g be a* MAZE *position in canonical form.*

(1) *Suppose g is an integer. If $g = 0$. then let M be a single cell; if $g > 0$ then let M be a row of $g + 1$ cells; if $g < 0$ then let M be a column of $g + 1$ cells. In all cases, M has no interior walls.*

(2) *Suppose g is not an integer. Then $g = \{a \mid b\}$ where a and b are dyadic rationals. Take the* MAZE *positions obtained via the Construction for a and b. Adjust the height, if necessary, to obtain $m'(a)$, a $p \times (q + 1)$ rectangle and $m'(b)$, a $(p + 1) \times (r + 1)$ rectangle with the same values as $m(a)$ and $m(b)$ respectively. Let $m''(a)$ be formed from $m'(a)$ by removing the top outside walls and adding a new top row with no interior walls. Form the* MAZE *position M by concatenating the rows of $m''(a)$ and $m'(b)$.* (See Figure 4.)

The value of rcf(M) is g.

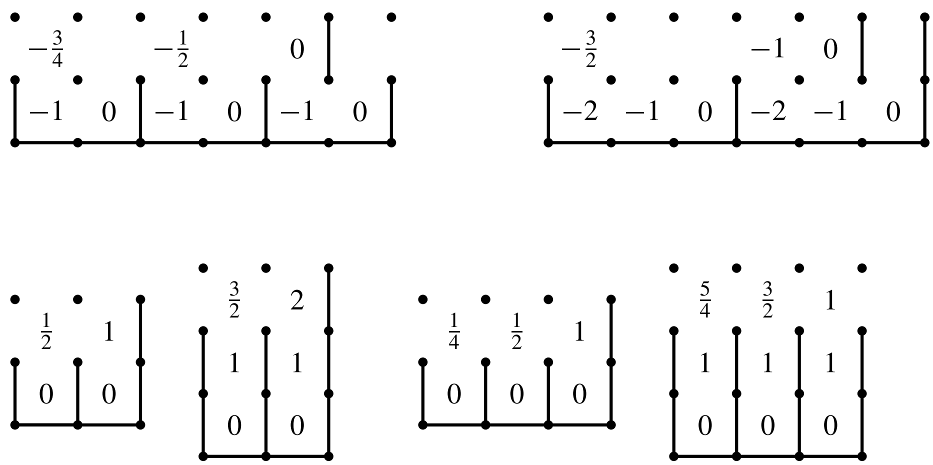

Figure 5. MAZE boards for $g = n + 1/2^k$.

Proof. (1) Suppose g is an integer then $M = \text{rcf}(M) = g$ is clear.

(2) Let $g = n + 1/2^k$, n an integer and $k \geq 1$. Then g can be constructed as follows. We only put in representative cases as the general case is clear but tedious.

Note that for any entry x of M other than the top left cell (which has value g) then $\|x\| < \|g\| = k$.

Suppose now that g is not an integer nor is $g = n + 1/2^k$ (i.e. a not an integer). Let $g = \{a \mid b\}$ in canonical form, with a and b be dyadic rationals. Let M be the MAZE position whose reduced canonical form is claimed to be g. Note that both $m''(a)$ and $m'(b)$ are obtained by the Construction and so contain no interior walls in their top rows. Moreover, if c is a cell in $m'(a)$, other than the top-left cell, then $\|c\| < \|a\|$. See Figure 3 for the naming of the values of the cells. Since there are no walls in the top row of $m(b)$, $\langle x_1, x_2, \ldots, x_q, b, y_1, y_2, \ldots, y_r = p\rangle$ is a right-option-closed sequence, and the values in the first column, a and below, form a left-option-closed sequence. Thus, $\overline{\langle \otimes^{\mathcal{L}} \rangle} = a$ by Lemma 9 and since $\langle b, y_1, y_2, \ldots, y_r\rangle$ is also a right-option-closed sequence, then $y_i > b$ by Theorem 10. Thus,

$$\text{rcf}(\otimes) = \text{rcf}\left(\{a \mid \overline{\langle x_1, x_2, \ldots, x_q, b, y_1, y_2, \ldots, y_r\rangle}\}\right) = \left(\{a \mid \overline{\langle x_1, x_2, \ldots, x_q, b\rangle}\}\right)$$

and we need to show that

$$\overline{\langle x_1, x_2, \ldots, x_q, b\rangle} = b.$$

Now, if no x_i is a number, then by Lemma 9, $\overline{\langle x_1, x_2, \ldots, x_q, b\rangle} = b$.

Therefore, we may let i be the greatest index such that x_i is a number. Necessarily then, $x_i < b$ since b is a right option of x_i. Since there are no walls in the top row of $m'(a)$, i.e. between a and any $x_i^{L_1}$, $\langle x_1^{L_1}, \dots, x_{q-1}^{L_1}, x_q^{L_1} = p-1\rangle$ is a right-option-closed sequence. Also, $x_i^{L_1}$ is a Right option of a and so by Theorem 10, $x_i^{L_1} > a$.

Now suppose $a > b$. We have the inequalities $x_i^{L_1} > a > b > x_i$ which is a contradiction since for any game G, $G^L \not> G$. Therefore, there is no i such that x_i is a number and thus $\mathrm{rcf}(\otimes) = \{a \mid b\}$.

Now suppose $a < b$ and let $g = \{a \mid b\} = \frac{y}{2^k}$ for some integers y odd, and $k \geq 1$. The case a is an integer is already covered, thus we may assume that $\|a\| \geq 1$. Repeated use of Theorem 10 gives $a < x_i^{L_1} < x_1 < b$. Since $\{a \mid b\}$ is in canonical form, then $\|x_i\| > k$ by Lemma 12. Since x_i comes from a cell in a MAZE position then, by Theorem 10,

$$x_i = \left\{\overline{\langle x_i^{\mathcal{L}}\rangle} \,\middle|\, \overline{\langle x_i^{\mathcal{R}}\rangle}\right\} = \left\{\overline{\langle x_i^{\mathcal{L}}\rangle} \,\middle|\, b\right\} = \{c \mid b\}$$

for some c in $m'(a)$ and so $\|c\| < \|a\|$ by induction. By Lemma 12, $\|a\| < \|g\|$ and so $\|c\| < \|g\| = k$. If $a < c < b$, then by Lemma 12 $\|c\| \geq \|g\|$, which is a contradiction. Thus, $c < a$.

Hence no x_i is a number and thus $\mathrm{rcf}(\otimes) = g = \{a \mid b\}$.

Note that the entries in $m'(a)$ and $m'(b)$ are the same as in $m(a)$ and $m(b)$ except for the addition of integers around the bottom and right-hand sides. Hence, the values other than a in $m'(a)$ have norms less than $\|a\|$. The values in $m''(a)$ not in $m'(a)$ are the x_i, $i = 1, \dots, p$ and none of them are numbers, so $\|\otimes\| = \max\{\|a\|, \|b\|\} + 1$ which occurs only at the top-left cell. □

5. Evaluating MAZEs

Grossman & Siegel [Grossman and Siegel 2009] note that for most situations the reduced canonical form is sufficient to evaluate a position since the infinitesimals, at most, change the parity of who gets the last move. Calculating the reduced canonical form can be done in linear time with regard to the number of cells in the position.

Theorem 14. *The reduced canonical form of an $p \times q$ MAZE position can be calculated in $O(pq)$ time.*

Note that for ease of describing the proof, the MAZE position is implicitly embedded in a rectangle and the reduced canonical form of every cell is found regardless of whether it can be reached from the top left hand cell.

Proof. The evaluation goes row by row starting at the bottom and always starting at the righthand cell. By Theorem 10, we only need look for the nearest number down a column and to the right along a row. To that end, let S_{ij} be the value of the cell (i, j) where $(1, 1)$ refers to the bottom-right corner.

(1) Define variables R_i, $i = 1, 2, \ldots, p$ and and C_j, $j = 1, 2, \ldots, q$. Initially, set $R_i = -pq$ and all $C_j = pq$ for all i and j.

(2) For j from 1 to q do
For i from 1 to p do
For cell S_{ij}, if there is a right wall from (i, j) then set $C_j = \infty$; if there is a left wall from (i, j) then set $R_i = -\infty$; set $S_{ij} = \{R_i|C_j\}$ and if $R_i < C_j$ then set $R_i = C_j = \{R_i|C_j\}$.

Note that for each pair (i, j), R_i is the norm of the left-option-closed sequence starting at $S_{i-1,j}$ the first left option of S_{ij} and C_j is the norm of the right-option-closed sequence. That $\mathrm{rcf}(C_{ij}) = \{R_i|C_j\}$ follows from Theorem 10. Each cell is looked at and there is a constant number of operations associated with each cell. (Note that evaluating $\{a \mid b\}$ for numbers a and b, $a < b$ is linear.) □

See [McKay et al. 2010] for a fanciful interpretation of this method.

6. Open questions

For any numbers a and b we have constructed, G, a MAZE position with $\mathrm{rcf}(G) = \{a \mid b\}$.

Question 15. *For numbers a and b, is there a* MAZE *position G with $G = \{a \mid b\}$?*

From [NO] we know that for any option closed game G, $\Downarrow + * < G - \mathrm{rcf}(G) < \Uparrow + *$. It is easy to see that the Construction also gives games of the form $*n$.

Question 16. *What infinitesimals occur in* MAZE?

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