Random Matrices
MSRI Publications
Volume **65**, 2014

Universality conjecture for all Airy, sine and Bessel kernels in the complex plane

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We address the question of how the celebrated universality of local correlations for the real eigenvalues of Hermitian random matrices of size $N \times N$ can be extended to complex eigenvalues in the case of random matrices without symmetry. Depending on the location in the spectrum, particular large-N limits (the so-called weakly non-Hermitian limits) lead to one-parameter deformations of the Airy, sine and Bessel kernels into the complex plane. This makes their universality highly suggestive for all symmetry classes. We compare all the known limiting real kernels and their deformations into the complex plane for all three Dyson indices $\beta = 1, 2, 4$, corresponding to real, complex and quaternion real matrix elements. This includes new results for Airy kernels in the complex plane for $\beta = 1, 4$. For the Gaussian ensembles of elliptic Ginibre and non-Hermitian Wishart matrices we give all kernels for finite N, built from orthogonal and skew-orthogonal polynomials in the complex plane. Finally we comment on how much is known to date regarding the universality of these kernels in the complex plane, and discuss some open problems.

1. Introduction

The topic of universality in Hermitian random matrix theory (RMT) has attracted a lot of attention in the mathematics community recently, particularly in the context of matrices with elements that are independent random variables, as reviewed in [Tao and Vu 2012; Erdős and Yau 2012]. The question that one tries to answer is this: under what conditions are the statistics of eigenvalues of $N \times N$ matrices with independent Gaussian variables the same (for large matrices) as for more general RMT where matrix elements may become coupled? This has been answered under very general assumptions, and we refer to some recent reviews on invariant [Kuijlaars 2011; Deift and Gioev 2009] and noninvariant [Tao and Vu 2012; Erdős and Yau 2012] ensembles.

In this short note we would like to advocate the idea that non-Hermitian RMT with eigenvalues in the complex plane also warrants the investigation of universality. Apart from the interest in its own right, these models have important applications in physics and other sciences (see, e.g., [Akemann et al. 2011a]). We will focus here on RMT that is close to Hermitian, a regime that is particularly important for applications in quantum chaotic scattering (see [Fyodorov and Sommers 2003] for a review) and quantum chromodynamics (QCD), for example. In the latter case, the non-Hermiticity may arise from describing the effect of quark chemical potential (as reviewed in [Verbaarschot 2011; Akemann 2007]), or from finite lattice spacing effects of the Wilson–Dirac operator (see [Damgaard et al. 2010; Akemann et al. 2011b] as well as [Kieburg 2012] for the solution of this non-Hermitian RMT).

Being a system of N coupled eigenvalues, Hermitian RMT already offers a rich variety of large-N limits, where one has to distinguish the bulk and (soft) edge of the spectrum for Wigner–Dyson (WD) ensembles, and in addition the origin (hard edge) for Wishart–Laguerre (WL, or chiral) RMT. Not surprisingly complex eigenvalues offer even more possibilities. The limit we will investigate is known as the weakly non-Hermitian regime; it *connects* Hermitian and (strongly) non-Hermitian RMT, and was first introduced in [Fyodorov et al. 1997a; 1997b] in the bulk of the spectrum. For strong non-Hermiticity — which includes the well-known circular law and the corresponding universality results — we refer to [Khoruzhenko and Sommers 2011] and references therein, although the picture there is also far from being complete; an important breakthrough was published recently in [Tao and Vu 2014].

In the next section we give a brief list of the six non-Hermitian WD and WL ensembles, and indicate where they were first solved in the weak limit. There are three principal reasons why we believe that universality may hold. First, in some cases two different Gaussian RMT both give the same answers. Second, there are heuristic arguments available for i.i.d. matrix elements using supersymmetry [Fyodorov et al. 1998], as well as for invariant non-Gaussian ensembles using large-N factorisation and orthogonal polynomials (OP) [Akemann 2002]. Third, the resulting limiting kernels of (skew-) OP look very similar to the corresponding kernels of real eigenvalues, being merely one-parameter deformations of them. One of the main goals of this paper is to illustrate this fact. For this purpose we give a complete list of all the known Airy, sine, and Bessel kernels for real eigenvalues, side-by-side with their deformed kernels in the complex plane, where some of our results are new.

2. Random matrices and their limiting kernels

In this section we briefly introduce the Gaussian random matrix ensembles that we consider, and give a list of the limiting kernels they lead to, for both real and complex eigenvalues. For simplicity we have restricted ourselves to Gaussian ensembles in the Hermitian cases, in order to highlight the parallels to their non-Hermitian counterparts.

We begin with the classical WD and Ginibre ensembles in Section 2.1, displaying Airy (Section 2.2) and sine (Section 2.3) behaviour at the (soft) edge and in the bulk of the spectrum respectively, as well as their deformations. We then introduce the WL ensembles and their non-Hermitian counterparts in Section 2.4, in order to access the Bessel behaviour (Section 2.5) at the origin (or hard edge). The corresponding orthogonal and skew-orthogonal Hermite and Laguerre polynomials are given in Appendix A, and precise statements of the limits that lead to the microscopic kernels can be found in Appendix B.

2.1. ***Gaussian ensembles with eigenvalues on $\mathbb{R}$ and $\mathbb{C}$.*** The three classical Gaussian Wigner–Dyson ensembles (the GOE, GUE and GSE) are defined as [Mehta 2004]

$$\begin{aligned}\mathcal{Z}_N^{\mathrm{G}\beta\mathrm{E}} &= \int dH \exp[-\beta \operatorname{Tr} H^2/4] \\ &= c_{N,\beta} \prod_{j=1}^{N} \int_{\mathbb{R}} dx_j\, w_\beta(x_j)\, |\Delta_N(\{x\})|^\beta. \end{aligned} \tag{2.1}$$

The random matrix elements H_{kl} are real, complex, or quaternion real numbers for $\beta = 1, 2, 4$ respectively, with the condition that the $N \times N$ matrix H (N is taken to be even for simplicity) is real symmetric, complex Hermitian or complex Hermitian and self-dual for $\beta = 1, 2, 4$. In the first equation we integrate over all independent matrix elements denoted by dH. The Gaussian weight completely factorises and thus the independent elements are normal random variables; for $\beta = 1$, for example, the real elements are distributed $\mathcal{N}(0, 1)$ for off-diagonal elements, and $\mathcal{N}(0, 2)$ for diagonal elements.

In the second equality of (2.1), we diagonalised the matrix

$$H = U \operatorname{diag}(x_1, \dots, x_N)\, U^{-1},$$

where U is an orthogonal, unitary or unitary-symplectic matrix for $\beta = 1, 2, 4$. The integral over U factorises and leads to the known constants $c_{N,\beta}$. We obtain a Gaussian weight $w_\beta(x)$ and the Vandermonde determinant $\Delta_N(\{x\})$ from the Jacobian of the diagonalisation,

$$w_\beta(x) = \exp[-\beta x^2/4], \quad \Delta_N(\{x\}) = \prod_{1 \le l < k \le N} (x_k - x_l). \tag{2.2}$$

The integrand on the right-hand side of (2.1) times $c_{N,\beta}/\mathcal{Z}_N^{\mathrm{G}\beta\mathrm{E}}$ defines the normalised joint probability distribution function (jpdf) of all eigenvalues. The k-point correlation function R_k^β, which is proportional to the jpdf integrated over

$N-k$ eigenvalues, can be expressed through a single kernel $K_N^{\beta=2}$ of orthogonal polynomials (OP) for $\beta=2$, or through a 2×2 matrix-valued kernel involving skew-OP for $\beta=1,4$:

$$\begin{aligned} R_k^{\beta=2}(x_1,\ldots,x_k) &= \det_{i,j=1,\ldots,k}[K_N^{\beta=2}(x_i,x_j)],\\ R_k^{\beta=1,4}(x_1,\ldots,x_k) &= \operatorname*{Pf}_{i,j=1,\ldots,k}\left[\begin{pmatrix} K_N^{\beta=1,4}(x_i,x_j) & -G_N^{\beta=1,4}(x_i,x_j)\\ G_N^{\beta=1,4}(x_j,x_i) & -W_N^{\beta=1,4}(x_i,x_j)\end{pmatrix}\right]. \end{aligned} \tag{2.3}$$

The matrix kernel elements K_N and W_N are not independent of G_N but are related by differentiation and integration, respectively. These relations will be given later for the limiting kernels.

The three parameter-dependent Ginibre (i.e., elliptic or Ginibre–Girko) ensembles, denoted by GinOE, GinUE, and GinSE, can be written as

$$\begin{aligned} \mathfrak{X}_N^{\mathrm{Gin}\beta\mathrm{E}}(\tau) &= \int dJ\,\exp\left[\frac{-\gamma_\beta}{1-\tau^2}\operatorname{Tr}\left(JJ^\dagger-\frac{\tau}{2}(J^2+J^{\dagger\,2})\right)\right]\\ &= \int dH_1\,dH_2\,\exp\left[-\frac{\gamma_\beta\operatorname{Tr}H_1^2}{1+\tau}-\frac{\gamma_\beta\operatorname{Tr}H_2^2}{1-\tau}\right], \end{aligned} \tag{2.4}$$

with $\tau\in[0,1)$. We use the parametrisation of [Khoruzhenko and Sommers 2011], with $\gamma_{\beta=2}=1$ and $\gamma_{\beta=1,4}=\frac{1}{2}$. The matrix elements of J are of the same types as for H for all three values of β, but without any further symmetry constraint. Decomposing $J=H_1+iH_2$ into its Hermitian and anti-Hermitian parts, these ensembles can be viewed as Gaussian two-matrix models. For $\tau=0$ (maximal non-Hermiticity) the distribution for all matrix elements again factorises. In the opposite, that is, Hermitian, limit ($\tau\to 1$), the parameter-dependent Ginibre ensembles become the Wigner–Dyson ensembles. The jpdf of complex (and real) eigenvalues can be computed by transforming J into the following form, $J=U(Z+T)U^{-1}$. For $\beta=2$ this is the Schur decomposition, with $Z=\operatorname{diag}(z_1,\ldots,z_N)$ containing the complex eigenvalues, and T being upper triangular:[1]

$$\begin{aligned} \mathfrak{X}_N^{\mathrm{GinUE}}(\tau) &= c_{N,\mathbb{C}}^{\beta=2}\prod_{j=1}^N\int_{\mathbb{C}} d^2z_j\,w_{\beta=2}^{\mathbb{C}}(z_j)\,|\Delta_N(\{z\})|^2,\\ w_{\beta=2}^{\mathbb{C}}(z) &= \exp\left[\frac{-1}{1-\tau^2}\left(|z|^2-\frac{\tau}{2}(z^2+z^{*\,2})\right)\right]. \end{aligned} \tag{2.5}$$

For $\beta=1,4$ we follow [Khoruzhenko and Sommers 2011] where the two ensembles have been cast into a unifying framework. For simplicity we choose

[1] The resulting jpdf of complex eigenvalues for normal matrices with $T\equiv 0$ at $\beta=2$ is the same.

N to be even. Here the matrix Z can be chosen to be 2×2 block diagonal and T to be upper block triangular. The calculation of the jpdf reduces to a 2×2 calculation, yielding

$$\mathcal{Z}_N^{\mathrm{GinO/SE}}(\tau)=c_{N,\mathbb{C}}^{\beta=1,4}\prod_{j=1}^{N}\int_{\mathbb{C}}d^2z_j\,\prod_{k=1}^{N/2}\mathcal{F}_{\beta=1,4}^{\mathbb{C}}(z_{2k-1},z_{2k})\,\Delta_N(\{z\}),\tag{2.6}$$

where we have introduced an antisymmetric bivariate weight function. For $\beta=1$, this is given by

$$\begin{aligned}\mathcal{F}_{\beta=1}^{\mathbb{C}}(z_1,z_2)&=w_{\beta=1}^{\mathbb{C}}(z_1)w_{\beta=1}^{\mathbb{C}}(z_2)\\&\qquad\times\big(2i\delta^2(z_1-z_2^*)\,\mathrm{sign}(y_1)+\delta^1(y_1)\delta^1(y_2)\,\mathrm{sign}(x_2-x_1)\big),\\(w_{\beta=1}^{\mathbb{C}}(z))^2&=\mathrm{erfc}\left(\frac{|z-z^*|}{\sqrt{2(1-\tau^2)}}\right)\exp\left[\frac{-1}{2(1+\tau)}(z^2+z^{*2})\right],\end{aligned}\tag{2.7}$$

and for $\beta=4$ by

$$\begin{aligned}\mathcal{F}_{\beta=4}^{\mathbb{C}}(z_1,z_2)&=w_{\beta=4}^{\mathbb{C}}(z_1)w_{\beta=4}^{\mathbb{C}}(z_2)\,(z_1-z_2)\,\delta(z_1-z_2^*),\\(w_{\beta=4}^{\mathbb{C}}(z))^2&=w_{\beta=2}^{\mathbb{C}}(z).\end{aligned}\tag{2.8}$$

For $\beta=1$, it should be noted that the integrand in (2.6) is not always positive, and so a symmetrisation must be applied when determining the correlation functions below.[2] For $\beta=4$, the parameter N in (2.6) should — in our convention — be taken to be the size of the complex-valued matrix that is equivalent to the original quaternion real matrix.

The correlation functions can be written in a similar form as for the real eigenvalues

$$\begin{aligned}R_{k,\mathbb{C}}^{\beta=2}(z_1,\dots,z_k)&=\det_{i,j=1,\dots,k}[K_{N,\mathbb{C}}^{\beta=2}(z_i,z_j^*)],\\R_{k,\mathbb{C}}^{\beta=1,4}(z_1,\dots,z_k)&=\mathop{\mathrm{Pf}}_{i,j=1,\dots,k}\left[\begin{pmatrix}K_{N,\mathbb{C}}^{\beta=1,4}(z_i,z_j)&-G_{N,\mathbb{C}}^{\beta=1,4}(z_i,z_j)\\G_{N,\mathbb{C}}^{\beta=1,4}(z_j,z_i)&-W_{N,\mathbb{C}}^{\beta=1,4}(z_i,z_j)\end{pmatrix}\right],\end{aligned}\tag{2.9}$$

where the elements of the matrix kernels are related through

$$\begin{aligned}G_{N,\mathbb{C}}^{\beta=1,4}(z_i,z_j)&=-\int_{\mathbb{C}}d^2z\,K_{N,\mathbb{C}}^{\beta=1,4}(z_i,z)\mathcal{F}_{\beta=1,4}^{\mathbb{C}}(z,z_j),\\W_{N,\mathbb{C}}^{\beta=1,4}(z_i,z_j)&=\int_{\mathbb{C}^2}d^2z\,d^2z'\,\mathcal{F}_{\beta=1,4}^{\mathbb{C}}(z_i,z)K_{N,\mathbb{C}}^{\beta=1,4}(z,z')\mathcal{F}_{\beta=1,4}^{\mathbb{C}}(z',z_j)\\&\qquad-\mathcal{F}_{\beta=1,4}^{\mathbb{C}}(z_i,z_j).\end{aligned}\tag{2.10}$$

[2]It is, however, possible to write the partition function $\mathcal{Z}_N^{\mathrm{GinOE}}$ as an integral over a true (i.e., positive) jpdf, by, for example, appropriately ordering the eigenvalues; however, such a representation is technically more difficult to work with.

The kernels $K_{N,\mathbb{C}}^{\beta}(z, z')$ are given explicitly in Appendix A.

For $\beta = 1$, we can write

$$
\begin{aligned}
G_{N,\mathbb{C}}^{\beta=1}(z_1, z_2) &= \delta^1(y_2) G_{N,\mathbb{C},\mathrm{real}}^{\beta=1}(z_1, x_2) + G_{N,\mathbb{C},\mathrm{com}}^{\beta=1}(z_1, z_2),\\
W_{N,\mathbb{C}}^{\beta=1}(z_1, z_2) &= \delta^1(y_1)\delta^1(y_2) W_{N,\mathbb{C},\mathrm{real},\mathrm{real}}^{\beta=1}(x_1, x_2)\\
&\quad + \delta^1(y_1) W_{N,\mathbb{C},\mathrm{real},\mathrm{com}}^{\beta=1}(x_1, z_2) + \delta^1(y_2) W_{N,\mathbb{C},\mathrm{com},\mathrm{real}}^{\beta=1}(z_1, x_2)\\
&\quad + W_{N,\mathbb{C},\mathrm{com},\mathrm{com}}^{\beta=1}(z_1, z_2) - \mathscr{F}_{\beta=1}^{\mathbb{C}}(z_1, z_2),
\end{aligned}
\tag{2.11}
$$

whereas, for $\beta = 4$, (2.8) implies the following relations:

$$
\begin{aligned}
G_{N,\mathbb{C}}^{\beta=4}(z_1, z_2) &= (z_2 - z_2^*) w_{\beta=2}^{\mathbb{C}}(z_2) K_{N,\mathbb{C}}^{\beta=4}(z_1, z_2^*),\\
W_{N,\mathbb{C}}^{\beta=4}(z_1, z_2) &= -(z_1 - z_1^*)(z_2 - z_2^*) w_{\beta=2}^{\mathbb{C}}(z_1) w_{\beta=2}^{\mathbb{C}}(z_2) K_{N,\mathbb{C}}^{\beta=4}(z_1^*, z_2^*),
\end{aligned}
\tag{2.12}
$$

where in the final expression we have dropped the term representing the perfect correlation between an eigenvalue z and its complex conjugate z^*. For this reason, for $\beta = 4$ we will only give one of the matrix kernel elements in the following.

Note that $\beta = 1$ is special as the eigenvalues of a real asymmetric matrix are either real or come in complex conjugate pairs. Therefore we will have to distinguish kernels (and k-point densities) of real, complex or mixed arguments.

In order to specify the limiting kernels we first need the behaviour of the mean (or macroscopic) spectral density. At large N, and for all three values of β, the (real) eigenvalues in the Hermitian ensembles are predominantly concentrated within the Wigner semicircle $\rho_{\mathrm{sc}}(x) = (2\pi N)^{-1}\sqrt{4N - x^2}$ on $[-2\sqrt{N}, 2\sqrt{N}]$, whereas in the non-Hermitian ensemble, the complex eigenvalues lie mostly within an ellipse with half-axes of lengths $(1 + \tau)\sqrt{N}$ and $(1 - \tau)\sqrt{N}$, with constant density $\rho_{\mathrm{el}}(z) = (N\pi(1 - \tau^2))^{-1}$. Depending on where (and how) we magnify the spectrum locally, we obtain different asymptotic Airy or sine kernels for each $\beta = 1, 2, 4$. In the following we will give all of the known real kernels; see [Kuijlaars 2011], for example, for a complete list and references, together with their deformations into the complex plane. For the Bessel kernels which will be introduced later we need to consider different matrix ensembles, see Section 2.4 below.

2.2. ***Limiting Airy kernels on*** $\mathbb{R}$ ***and*** $\mathbb{C}$. When appropriately zooming into the "square root" edge of the semicircle, the three well-known Airy kernels (matrix-valued for $\beta = 1, 4$) are obtained for real eigenvalues. For complex eigenvalues we have to consider the vicinity of the eigenvalues on a thin ellipse which have the largest real parts, and where the weakly non-Hermitian limit introduced in [Bender 2010] is defined such that

$$
\sigma = N^{\frac{1}{6}}\sqrt{1 - \tau} \tag{2.13}
$$

remains fixed (see Appendix B for the precise details of the scaling of the eigenvalues). This leads to one-parameter deformations of the Airy kernels in the complex plane. Whilst the results for $\beta = 2$ are already known [Bender 2010; Akemann and Bender 2010], our results for $\beta = 1, 4$, stated below, are new [Akemann and Phillips 2014]:

$\underline{\beta = 2}$:

$$K_{\mathrm{Ai}}^{\beta=2}(x_1, x_2) = \frac{\mathrm{Ai}(x_1)\,\mathrm{Ai}'(x_2) - \mathrm{Ai}'(x_1)\,\mathrm{Ai}(x_2)}{x_1 - x_2} = \int_0^\infty dt\,\mathrm{Ai}(x_1+t)\,\mathrm{Ai}(x_2+t), \tag{2.14}$$

$$K_{\mathrm{Ai},\mathbb{C}}^{\beta=2}(z_1, z_2) = \frac{1}{\sigma\sqrt{\pi}}\,\exp\Bigl(-\frac{y_1^2+y_2^2}{2\sigma^2} + \frac{\sigma^6}{6} + \frac{\sigma^2(z_1+z_2)}{2}\Bigr) \times \int_0^\infty dt\,\mathrm{e}^{\sigma^2 t}\,\mathrm{Ai}\Bigl(z_1+t+\frac{\sigma^4}{4}\Bigr)\,\mathrm{Ai}\Bigl(z_2+t+\frac{\sigma^4}{4}\Bigr). \tag{2.15}$$

In the Hermitian limit $\sigma \to 0$ we obtain

$$K_{\mathrm{Ai},\mathbb{C}}^{\beta=2}(z_1, z_2) \to \sqrt{\delta^1(y_1)\delta^1(y_2)}\,K_{\mathrm{Ai}}^{\beta=2}(x_1, x_2),$$

with the factor in front of the integral in (2.15) projecting the imaginary parts of the eigenvalues to zero. For the integral itself — which is obtained from the limit of the sum of the OP on $\mathbb{C}$ given in (A.4) — the deformation in σ is very smooth. The same deformed Airy kernel can be obtained from the corresponding WL ensemble equation (2.29) [Akemann and Bender 2010] with kernel equation (A.5), and is thus universal.

$\underline{\beta = 4}$:

$$\begin{aligned}
G_{\mathrm{Ai}}^{\beta=4}(x_1,x_2) &= -\tfrac{1}{2}K_{\mathrm{Ai}}^{\beta=2}(x_1, x_2) + \tfrac{1}{4}\,\mathrm{Ai}(x_1)\int_{x_2}^\infty dt\,\mathrm{Ai}(t),\\
K_{\mathrm{Ai}}^{\beta=4}(x_1, x_2) &= \frac{\partial}{\partial x_2}G_{\mathrm{Ai}}^{\beta=4}(x_1, x_2),\\
W_{\mathrm{Ai}}^{\beta=4}(x_1, x_2) &= -\int_{x_1}^\infty ds\,G_{\mathrm{Ai}}^{\beta=4}(s, x_2) \qquad (2.16)\\
&= -\tfrac{1}{4}\int_0^\infty ds\int_0^s dt\bigl(\mathrm{Ai}(x_2{+}t)\,\mathrm{Ai}(x_1{+}s) - \mathrm{Ai}(x_2{+}s)\,\mathrm{Ai}(x_1{+}t)\bigr),
\end{aligned}$$

$$
\begin{aligned}
G^{\beta=4}_{\mathrm{Ai},\mathbb{C}}(z_1,z_2) = {} & \frac{i y_2}{4\sigma^3\sqrt{\pi}} \exp\left(-\frac{y_1^2+y_2^2}{2\sigma^2}+\frac{\sigma^6}{6}+\frac{\sigma^2(z_1+z_2^*)}{2}\right) \\
& \times \int_0^\infty ds \int_0^s dt\, e^{\frac{1}{2}\sigma^2(s+t)} \\
& \times \left(\mathrm{Ai}\Big(z_2^*+s+\frac{\sigma^4}{4}\Big)\mathrm{Ai}\Big(z_1+t+\frac{\sigma^4}{4}\Big)-(z_1\leftrightarrow z_2^*)\right). \quad (2.17)
\end{aligned}
$$

The integral in (2.17) which is also present in the other two kernel elements — see (2.12) — clearly reduces to that in (2.16) in the Hermitian limit, whereas the prefactors provide the appropriate Dirac delta functions. When analysing the Hermitian limit in detail, the real kernel elements $G^{\beta=4}_{\mathrm{Ai}}$ and $K^{\beta=4}_{\mathrm{Ai}}$ follow from a Taylor expansion of $W^{\beta=4}_{\mathrm{Ai},\mathbb{C}}$; see [Akemann and Basile 2007] for a discussion of the analogous Hermitian limit of the Bessel kernel.

$\underline{\beta=1}$:

$$
G^{\beta=1}_{\mathrm{Ai}}(x_1,x_2) = -\int_0^\infty dt\, \mathrm{Ai}(x_1+t)\,\mathrm{Ai}(x_2+t) - \tfrac{1}{2}\,\mathrm{Ai}(x_1)\left(1-\int_{x_2}^\infty dt\, \mathrm{Ai}(t)\right),
$$

$$
K^{\beta=1}_{\mathrm{Ai}}(x_1,x_2) = \frac{\partial}{\partial x_2} G^{\beta=1}_{\mathrm{Ai}}(x_1,x_2),
$$

$$
\begin{aligned}
W^{\beta=1}_{\mathrm{Ai}}(x_1,x_2) = {} & -\int_{x_1}^\infty ds\; G^{\beta=1}_{\mathrm{Ai}}(s,x_2) - \frac{1}{2}\int_{x_1}^{x_2} dt\, \mathrm{Ai}(t) \\
& + \frac{1}{2}\int_{x_1}^\infty ds\, \mathrm{Ai}(s)\int_{x_2}^\infty dt\, \mathrm{Ai}(t) - \tfrac{1}{2}\,\mathrm{sign}(x_1-x_2), \quad (2.18)
\end{aligned}
$$

$$
\begin{aligned}
G^{\beta=1}_{\mathrm{Ai},\mathbb{C},\mathrm{real}}(x_1,x_2) = {} & -\exp\left(\frac{\sigma^6}{6}+\frac{\sigma^2(x_1+x_2)}{2}\right) \\
& \times \int_0^\infty dt\; e^{\sigma^2 t}\, \mathrm{Ai}\Big(x_1+t+\frac{\sigma^4}{4}\Big)\,\mathrm{Ai}\Big(x_2+t+\frac{\sigma^4}{4}\Big) \\
& - \tfrac{1}{2}\exp\left(\frac{\sigma^6}{12}+\frac{\sigma^2 x_1}{2}\right)\mathrm{Ai}\Big(x_1+\frac{\sigma^4}{4}\Big) \\
& \times \left(1-e^{\sigma^6/12}\int_{x_2}^\infty dt\, e^{\sigma^2 t/2}\,\mathrm{Ai}\Big(t+\frac{\sigma^4}{4}\Big)\right),
\end{aligned}
$$

$$
\begin{aligned}
G^{\beta=1}_{\mathrm{Ai},\mathbb{C},\mathrm{com}}(z_1,z_2) = {} & -\frac{i}{2\sigma^2}\,\mathrm{sign}(y_2)(z_1-z_2^*)\,\exp\left(\frac{\sigma^6}{6}+\frac{\sigma^2(x_1+x_2)}{2}\right) \\
& \times \sqrt{\mathrm{erfc}\big(|y_1|/\sigma\big)\,\mathrm{erfc}\big(|y_2|/\sigma\big)} \\
& \times \int_0^\infty dt\,(e^{\sigma^2 t}-1)\,\mathrm{Ai}\Big(z_1+t+\frac{\sigma^4}{4}\Big)\,\mathrm{Ai}\Big(z_2^*+t+\frac{\sigma^4}{4}\Big),
\end{aligned}
$$

$$K^{\beta=1}_{\mathrm{Ai},\mathbb{C}}(z_1, z_2) = \frac{i}{2}\,\mathrm{sign}(y_2) G^{\beta=1}_{\mathrm{Ai},\mathbb{C},\mathrm{com}}(z_1, z_2^*),$$

$$\begin{aligned} W^{\beta=1}_{\mathrm{Ai},\mathbb{C}}(z_1, z_2) = \big(&-2A(x_1, x_2) + B(x_1)B(x_2) + B(x_2) - B(x_1)\big)\delta^1(y_1)\delta^1(y_2)\\ &+ 2i\,\mathrm{sign}(y_2) G^{\beta=1}_{\mathrm{Ai},\mathbb{C},\mathrm{real}}(z_2^*, x_1)\delta^1(y_1)\\ &- 2i\,\mathrm{sign}(y_1) G^{\beta=1}_{\mathrm{Ai},\mathbb{C},\mathrm{real}}(z_1^*, x_2)\delta^1(y_2)\\ &- 2i\,\mathrm{sign}(y_1) G^{\beta=1}_{\mathrm{Ai},\mathbb{C},\mathrm{com}}(z_1^*, z_2)\\ &- 2i\delta^2(z_1 - z_2^*)\,\mathrm{sign}(y_1) - \delta^1(y_1)\delta^1(y_2)\,\mathrm{sign}(x_2 - x_1), \end{aligned}$$

$$\begin{aligned} A(x_1, x_2) = {}&\exp\left(\frac{\sigma^6}{6} + \frac{\sigma^2(x_1 + x_2)}{2}\right)\\ &\times \int_0^\infty ds \int_0^s dt\, \mathrm{e}^{\frac{1}{2}\sigma^2(s+t)}\,\mathrm{Ai}\Big(x_1 + s + \frac{\sigma^4}{4}\Big)\,\mathrm{Ai}\Big(x_2 + t + \frac{\sigma^4}{4}\Big), \end{aligned}$$

$$B(x) = \exp\left(\frac{\sigma^6}{12} + \frac{\sigma^2 x}{2}\right)\int_0^\infty dt\, \mathrm{e}^{\frac{1}{2}\sigma^2 t}\,\mathrm{Ai}\Big(x + t + \frac{\sigma^4}{4}\Big). \tag{2.19}$$

Clearly $G^{\beta=1}_{\mathrm{Ai},\mathbb{C},\mathrm{real}}(x_1, x_2) \to G^{\beta=1}_{\mathrm{Ai}}(x_1, x_2)$ as $\sigma \to 0$, whereas the complex part vanishes in this Hermitian limit: $G^{\beta=1}_{\mathrm{Ai},\mathbb{C},\mathrm{com}}(z_1, z_2) \to 0$. We have also explicitly verified the corresponding limits for $K^{\beta=1}_{\mathrm{Ai},\mathbb{C}}(z_1, z_2)$ and $W^{\beta=1}_{\mathrm{Ai},\mathbb{C}}(z_1, z_2)$.

2.3. *Limiting sine kernels on $\mathbb{R}$ and $\mathbb{C}$.* For real eigenvalues the sine kernels are obtained by zooming into the bulk of the spectrum, sufficiently far away from the edges. The weakly non-Hermitian limit of the complex eigenvalues introduced in [Fyodorov et al. 1997a; 1997b] is taken such that

$$\sigma = N^{1/2}\sqrt{1-\tau} \tag{2.20}$$

remains finite (see Appendix B for further details). In this limit the macroscopic support of the spectral density on an ellipse shrinks to the semicircle distribution on the real axis, whereas microscopically we still have correlations of the eigenvalues in the complex plane.

The list of the known one-parameter deformations of the sine kernels for $\beta = 2$ is as follows:

$\underline{\beta = 2}$:

$$K^{\beta=2}_{\sin}(x_1, x_2) = \frac{\sin(x_1 - x_2)}{\pi(x_1 - x_2)} = \frac{1}{\pi}\int_0^1 dt\,\cos[(x_1 - x_2)t], \tag{2.21}$$

$$K^{\beta=2}_{\sin,\mathbb{C}}(z_1, z_2) = \frac{1}{\sigma\pi^{3/2}}\,\mathrm{e}^{-(y_1^2+y_2^2)/(2\sigma^2)}\int_0^1 dt\,\mathrm{e}^{-\sigma^2 t^2}\cos[(z_1 - z_2)t]. \tag{2.22}$$

The corresponding spectral density of complex eigenvalues was first derived in [Fyodorov et al. 1997a] using supersymmetry, and the kernel with all correlation functions in [Fyodorov et al. 1997b; 1998] using OP; see (A.4). In the Hermitian limit $\sigma \to 0$, we have

$$K^{\beta=2}_{\sin,\mathbb{C}}(z_1,z_2) \to \sqrt{\delta^1(y_1)\delta^1(y_2)}\,K^{\beta=2}_{\sin}(x_1,x_2).$$

In [Fyodorov et al. 1998] it was shown using supersymmetric techniques that the same result holds for the microscopic density of random matrices with i.i.d. matrix elements for $\beta = 1, 2$. Further arguments in favour of universality were added in [Akemann 2002] for the kernel for $\beta = 2$ using large-N factorisation and asymptotic OP. The universal parameter is the mean macroscopic spectral density $\rho(x_0)$.

$\underline{\beta = 4}$:

$$G^{\beta=4}_{\sin}(x_1,x_2) = -\frac{\sin[2(x_1-x_2)]}{2\pi(x_1-x_2)},$$

$$K^{\beta=4}_{\sin}(x_1,x_2) = \frac{\partial}{\partial x_1} G^{\beta=4}_{\sin}(x_1,x_2),$$

$$W^{\beta=4}_{\sin}(x_1,x_2) = \int_0^{x_1-x_2} dt\ G^{\beta=4}_{\sin}(t,0) = \frac{1}{2\pi}\int_0^1 \frac{dt}{t}\sin[2(x_1-x_2)t], \tag{2.23}$$

$$G^{\beta=4}_{\sin,\mathbb{C}}(z_1,z_2) = \frac{i2\sqrt{2}\,y_2}{\pi^{3/2}\sigma^3} e^{-2y_2^2/\sigma^2}\int_0^1 \frac{dt}{t}\, e^{-2\sigma^2t^2}\sin[2(z_1-z_2^*)t]. \tag{2.24}$$

The corresponding spectral density of complex eigenvalues was derived in [Kolesnikov and Efetov 1999] using supersymmetry, and the kernel with all correlation functions was derived in [Kanzieper 2002] using skew-OP leading to (A.9).

$\underline{\beta = 1}$:

$$\begin{aligned}
G^{\beta=1}_{\sin}(x_1,x_2) &= -K^{\beta=2}_{\sin}(x_1,x_2),\\
K^{\beta=1}_{\sin}(x_1,x_2) &= \frac{\partial}{\partial x_1} G^{\beta=1}_{\sin}(x_1,x_2) = \frac{1}{\pi}\int_0^1 dt\ t\sin[(x_2-x_1)t],\\
W^{\beta=1}_{\sin}(x_1,x_2) &= \int_0^{x_1-x_2} dt\ G^{\beta=1}_{\sin}(t,0) + \tfrac{1}{2}\,\mathrm{sign}(x_1-x_2),
\end{aligned} \tag{2.25}$$

$$G^{\beta=1}_{\sin,\mathbb{C},\mathrm{real}}(z_1,x_2) = -\frac{1}{\pi}\int_0^1 dt\, e^{-\sigma^2t^2}\cos[(z_1-x_2)t],$$

$$G^{\beta=1}_{\sin,\mathbb{C},\mathrm{com}}(z_1,z_2) = -2i\,\mathrm{sign}(y_2)\,\mathrm{erfc}\left|\frac{y_1}{\sigma}\right| K^{\beta=1}_{\sin,\mathbb{C}}(z_1,z_2^*),$$

$$K^{\beta=1}_{\sin,\mathbb{C}}(z_1,z_2)=\frac{1}{\pi}\int_0^1 dt\, t\, \mathrm{e}^{-\sigma^2t^2}\sin[(z_2-z_1)t],$$

$$\begin{aligned}W^{\beta=1}_{\sin,\mathbb{C}}(z_1,z_2)=&-\bigg(\frac{1}{\pi}\int_0^1\frac{dt}{t}\,\mathrm{e}^{-\sigma^2t^2}\sin[(x_2-x_1)t]\bigg)\delta^1(y_1)\delta^1(y_2)\\&-2i\,\mathrm{sign}(y_2)\,\mathrm{erfc}\Big|\frac{y_2}{\sigma}\Big|G^{\beta=1}_{\sin,\mathbb{C},\mathrm{real}}(z_2^*,x_1)\delta^1(y_1)\\&+2i\,\mathrm{sign}(y_1)\,\mathrm{erfc}\Big|\frac{y_1}{\sigma}\Big|G^{\beta=1}_{\sin,\mathbb{C},\mathrm{real}}(z_1^*,x_2)\delta^1(y_2)\\&+4\,\mathrm{sign}(y_1)\,\mathrm{sign}(y_2)\,\mathrm{erfc}\Big|\frac{y_1}{\sigma}\Big|\,\mathrm{erfc}\Big|\frac{y_2}{\sigma}\Big|K^{\beta=1}_{\sin,\mathbb{C}}(z_1^*,z_2^*)\\&-\sqrt{\mathrm{erfc}\Big|\frac{y_1}{\sigma}\Big|\,\mathrm{erfc}\Big|\frac{y_2}{\sigma}\Big|}\big(2i\,\delta^2(z_1-z_2^*)\,\mathrm{sign}(y_1)\\&\qquad+\delta^1(y_1)\delta^1(y_2)\,\mathrm{sign}(x_2-x_1)\big).\end{aligned}\tag{2.26}$$

The kernel elements $G^{\beta=1}_{\sin,\mathbb{C}}(z_1,z_2)$ and $K^{\beta=1}_{\sin,\mathbb{C}}(z_1,z_2)$ were derived in [Forrester and Nagao 2008] using skew-OP; see also (A.11). The same resulting spectral densities of complex and real eigenvalues were derived previously in [Efetov 1997a; 1997b] using a sigma-model calculation, which again indicates universality. It can easily be verified that the Hermitian limit $\sigma\to 0$ of $G^{\beta=1}_{\sin,\mathbb{C},\mathrm{real}}(x_1,x_2)$ is indeed $G^{\beta=1}_{\sin}(x_1,x_2)$, and that $G^{\beta=1}_{\sin,\mathbb{C},\mathrm{com}}(z_1,z_2)$ vanishes in this limit.

2.4. ***Wishart–Laguerre ensembles with eigenvalues on $\mathbb{R}$ and $\mathbb{C}$.*** In order to be able to access the Bessel kernels for real and complex eigenvalues as well, we briefly introduce the Wishart–Laguerre (or chiral) ensembles (LβE) and their non-Hermitian counterparts ($\mathbb{C}$LβE). We begin with the former which are defined as

$$\begin{aligned}\mathscr{Z}_N^{\mathrm{L}\beta\mathrm{E}}&=\int dW\exp[-\beta\,\mathrm{Tr}\,WW^\dagger/2]\\&=c_{N,\beta,\nu}\prod_{j=1}^N\int_{\mathbb{R}_+}dx_j\,w_\beta^\nu(x_j)\,|\Delta_N(\{x\})|^\beta.\end{aligned}\tag{2.27}$$

The elements of the rectangular $N\times(N+\nu)$ matrix W are again real, complex, or quaternion real for $\beta=1,2,4$, without further symmetry constraints. The integration denoted by dW runs over all the independent matrix elements. Because we want to access the so-called hard edge of the spectrum we will only consider fixed $\nu=O(1)$ in the following. The distribution of the (positive definite) eigenvalues x_j of $WW^\dagger$ in the Wishart picture (or equivalently the distribution of the singular values of W in the Dirac picture used in QCD) is of the same form as (2.1), but with different weight functions

$$w_\beta^\nu(x)=x^{\frac{1}{2}\beta(\nu+1)-1}\exp[-\beta x/2],\tag{2.28}$$

that now depend on β in a nontrivial way. Consequently the k-point correlation functions take the same form as in (2.3), with the corresponding kernels.

In analogy to the Ginibre ensembles we define a parameter-dependent family of non-Hermitian Wishart–Laguerre (also called complex chiral) ensembles as the following two-matrix model

$$\mathscr{Z}_N^{\mathbb{C}\mathrm{L}\beta\mathrm{E}}(\tau) = \int dW\, dV \exp\Big[-\frac{1}{1-\tau}\operatorname{Tr}\big(WW^\dagger + V^\dagger V - \tau(WV + V^\dagger W^\dagger)\big)\Big], \quad (2.29)$$

with W and $V^\dagger$ being two rectangular $N \times (N+\nu)$ matrices. Here we follow the notation of [Akemann and Bender 2010]. This two-matrix model was first introduced and solved for $\beta = 1$ [Akemann et al. 2009; 2010b], $\beta = 2$ [Osborn 2004] and $\beta = 4$ in [Akemann 2005]. For $\tau = 0$ the jpdf of all the matrix elements again factorises, and in the opposite limit we have

$$\mathscr{Z}_N^{\mathbb{C}\mathrm{L}\beta\mathrm{E}}(\tau) \to \mathscr{Z}_N^{\mathrm{L}\beta\mathrm{E}}$$

as $\tau \to 1$. Here we are seeking the complex (and real) eigenvalues of the product matrix WV (W and V are the off-diagonal blocks of the Dirac matrix that we diagonalise). Its jpdf takes the same form as in (2.5) and (2.6), but with different weight functions that are no longer Gaussian

$$\begin{aligned} w_{\beta=2}^{\nu,\mathbb{C}}(z) &= |z|^\nu \exp\left[\frac{\tau(z+z^*)}{1-\tau^2}\right] K_\nu\left(\frac{2|z|}{1-\tau^2}\right),\\ w_{\beta=4}^{\nu,\mathbb{C}}(z) &= \sqrt{w_{\beta=2}^{2\nu,\mathbb{C}}(z)}. \end{aligned} \quad (2.30)$$

The function $K_\nu(z)$ here is the modified Bessel function. For $\beta = 4$ the antisymmetric weight function is defined as in (2.8). For $\beta = 1$ we explicitly specify two functions in the antisymmetric weight function

$$\begin{aligned} \mathscr{F}_{\beta=1}^{\nu,\mathbb{C}}(z_1, z_2) &= i g_\nu(z_1, z_2)\operatorname{sign}(y_1)\,\delta^2(z_1 - z_2^*)\\ &\qquad + \tfrac{1}{2} h_\nu(x_1) h_\nu(x_2)\delta(y_1)\delta(y_2)\operatorname{sign}(x_2 - x_1),\\ h_\nu(x) &= 2|x|^{\frac{\nu}{2}} \exp\left[\frac{\tau x}{1-\tau^2}\right] K_{\frac{\nu}{2}}\left(\frac{|x|}{1-\tau^2}\right), \end{aligned} \quad (2.31)$$

$$\begin{aligned} g_\nu(z_1, z_2) &= 2|z_1 z_2|^{\frac{\nu}{2}} \exp\left[\frac{\tau(z_1+z_2)}{1-\tau^2}\right]\\ &\qquad \times \int_0^\infty \frac{dt}{t} \exp\left[-\frac{(z_1^2+z_2^2)t}{(1-\tau^2)^2} - \frac{1}{4t}\right]\\ &\qquad \times K_{\frac{\nu}{2}}\left(\frac{2z_1 z_2 t}{(1-\tau^2)^2}\right) \operatorname{erfc}\left(\frac{|z_2 - z_1|\sqrt{t}}{1-\tau^2}\right), \end{aligned}$$

which are related by $g_\nu(z, z^*) \to h_\nu(x)^2$ as $y \to 0$. We give the corresponding kernels in Appendix A.

In the large-N limit (with $\nu = O(1)$ fixed), for all three β the real positive Wishart eigenvalues of $WW^\dagger$ are concentrated on the interval $(0, 4N]$, with a density $\rho(x) = (2\pi N)^{-1}\sqrt{(4N-x)/x}$. This is a special case of the Marchenko–Pastur density. After mapping to Dirac eigenvalues $\lambda = \sqrt{x}$, this becomes the same semicircle distribution as for WD, but with eigenvalues coming in $\pm\lambda$ pairs. The density of the Wishart eigenvalues has a singularity at the origin; however, after mapping to the Dirac picture, we obtain a macroscopic density function that is flat on an ellipse, just as in the Ginibre ensemble[3].

We now give a list of all the known Bessel kernels. For real eigenvalues we follow [Deift et al. 2007; Nagao and Forrester 1995] where a most comprehensive list and references can be found. In some cases the parallel between kernels of real and complex eigenvalues is more transparent after using some identities for Bessel functions.

2.5. ***Limiting Bessel kernels on*** $\mathbb{R}$ ***and*** $\mathbb{C}$**.** The hard-edge limit is defined by zooming into the origin (see Appendix B), where for the complex eigenvalues we have to keep $\sigma = \sqrt{N(1-\tau)}$ fixed as in the weakly non-Hermitian bulk limit equation (2.20). The corresponding limiting kernels are given as follows:

$\underline{\beta = 2}$:

$$K_{\mathrm{Bes}}^{\beta=2}(x_1, x_2) = \frac{J_\nu(\sqrt{x_1})\sqrt{x_2}\, J_{\nu-1}(\sqrt{x_2}) - (x_1 \leftrightarrow x_2)}{2(x_1 - x_2)}$$
$$= \frac{1}{2}\int_0^1 dt\, t\, J_\nu(\sqrt{x_1}\, t) J_\nu(\sqrt{x_2}\, t), \tag{2.32}$$

$$K_{\mathrm{Bes},\mathbb{C}}^{\beta=2}(z_1, z_2) = \frac{1}{8\pi\sigma^2} K_\nu\left(\frac{|z_1|}{4\sigma^2}\right)^{1/2} K_\nu\left(\frac{|z_2|}{4\sigma^2}\right)^{1/2} \exp\left(\frac{x_1 + x_2}{8\sigma^2}\right)$$
$$\times \int_0^1 dt\, t\, \mathrm{e}^{-2\sigma^2 t^2} J_\nu(t\sqrt{z_1}) J_\nu(t\sqrt{z_2}). \tag{2.33}$$

It can be shown that

$$K_{\mathrm{Bes},\mathbb{C}}^{\beta=2}(z_1, z_2) \to \sqrt{\delta^1(y_1)\delta^1(y_2)}\,\Theta(x_1)\Theta(x_2) K_{\mathrm{Bes}}^{\beta=2}(x_1, x_2)$$

as $\sigma \to 0$, where $\Theta(x)$ is the Heaviside step function. The kernel of complex eigenvalues was derived in [Osborn 2004]. The same density following from this kernel was obtained from a different Gaussian non-Hermitian one-matrix model [Splittorff and Verbaarschot 2004] using replicas, and is in that sense universal.

[3]Note that the ν exact eigenvalues do not contribute to the macroscopic spectral density.

$\underline{\beta = 4}$:

$$K_{\mathrm{Bes}}^{\beta=4}(x_1, x_2) = -\frac{\partial}{\partial x_2} G_{\mathrm{Bes}}^{\beta=4}(x_1, x_2),$$

$$\begin{aligned} G_{\mathrm{Bes}}^{\beta=4}(x_1, x_2) = -2\sqrt{x_1} \int_0^1 dt \int_0^1 ds\, s^2 \big(& J_{2\nu}(2\sqrt{x_1}\, st) J_{2\nu+1}(2\sqrt{x_2}\, s) \\ & - t\, J_{2\nu}(2\sqrt{x_1}\, s) J_{2\nu+1}(2\sqrt{x_2}\, st) \big), \end{aligned}$$

$$\begin{aligned} W_{\mathrm{Bes}}^{\beta=4}(x_1, x_2) &= \sqrt{x_1 x_2} \int_0^1 dt \int_0^1 ds\, s \big(J_{2\nu}(2\sqrt{x_1}\, st) J_{2\nu}(2\sqrt{x_2}\, s) - (x_1 \leftrightarrow x_2) \big) \\ &= \int_{x_1}^{x_2} dx'\, G_{\mathrm{Bes}}^{\beta=4}(x_1, x'), \end{aligned} \tag{2.34}$$

$$\begin{aligned} K_{\mathrm{Bes},\mathbb{C}}^{\beta=4}(z_1, z_2) = \frac{1}{\sigma^4} \int_0^1 dt \int_0^1 & ds\, s\, \mathrm{e}^{-2\sigma^2 s^2 (1+t^2)} \\ & \times \big(J_{2\nu}(2\sqrt{z_1}\, st) J_{2\nu}(2\sqrt{z_2}\, s) - (z_1 \leftrightarrow z_2) \big). \end{aligned} \tag{2.35}$$

The complex kernel was first derived in [Akemann 2005], whereas the matching in the Hermitian limit — which can best be seen when comparing the kernels $K_{\mathrm{Bes}}^{\beta=4}$ and $K_{\mathrm{Bes},(\mathbb{C})}^{\beta=4}$ — is discussed in detail in [Akemann and Basile 2007].

$\underline{\beta = 1}$:

$$\begin{aligned} K_{\mathrm{Bes}}^{\beta=1}(x_1, x_2) &= \frac{-1}{8\sqrt{x_1 x_2}} \int_0^1 ds\, s^2 \left(\sqrt{x_1}\, J_{\nu+1}(s\sqrt{x_1}) J_\nu(s\sqrt{x_2}) - (x_1 \leftrightarrow x_2) \right) \\ &= -\frac{\partial}{\partial x_2} G_{\mathrm{Bes}}^{\beta=1}(x_1, x_2), \end{aligned}$$

$$\begin{aligned} G_{\mathrm{Bes}}^{\beta=1}(x_1, x_2) = &-\frac{1}{2} \int_0^1 dt\, t\, J_{\nu-1}(\sqrt{x_1}\, t) J_{\nu-1}(\sqrt{x_2}\, t) \\ & - \frac{1}{4\sqrt{x_1}} J_\nu(\sqrt{x_1}) \int_{\sqrt{x_2}}^\infty ds\, J_{\nu-2}(s), \end{aligned}$$

$$W_{\mathrm{Bes}}^{\beta=1}(x_1, x_2) = -\int_{x_1}^{x_2} ds\, G_{\mathrm{Bes}}^{\beta=1}(s, x_2) - \tfrac{1}{2}\, \mathrm{sign}(x_1 - x_2), \tag{2.36}$$

$$\begin{aligned} K_{\mathrm{Bes},\mathbb{C}}^{\beta=1}(z_1, z_2) = \frac{1}{256\pi\sigma^2} & \int_0^1 ds\, s^2\, \mathrm{e}^{-2\sigma^2 s^2} \\ & \times \left(\sqrt{z_1}\, J_{\nu+1}(s\sqrt{z_1}) J_\nu(s\sqrt{z_2}) - (z_1 \leftrightarrow z_2) \right), \end{aligned}$$

$$\begin{aligned} G_{\mathrm{Bes},\mathbb{C},\mathrm{com}}^{\beta=1}(z_1, z_2) = -2i\, \mathrm{sign}(y_2)\, \mathrm{e}^{x_2/(4\sigma^2)} & \int_0^\infty \frac{dt}{t} \exp\left(-\frac{t(z_2^2 + z_2^{*\,2})}{64\sigma^4} - \frac{1}{4t} \right) \\ & \times K_{\frac{\nu}{2}}\left(\frac{t}{32\sigma^4} |z_2|^2 \right) \mathrm{erfc}\left(\frac{\sqrt{t}\, |y_2|}{4\sigma^2} \right) K_{\mathrm{Bes},\mathbb{C}}^{\beta=1}(z_1, z_2^*), \end{aligned}$$

$$
\begin{aligned}
&G^{\beta=1}_{\mathrm{Bes},\mathbb{C},\mathrm{real}}(x_1,x_2) \qquad (2.37)\\
&= -\frac{2\,\mathrm{e}^{x_2/(8\sigma^2)}K_{\frac{\nu}{2}}\big(|x_2|/8\sigma^2\big)}{[\mathrm{sign}(x_2)]^{\nu/2}}\\
&\quad\times\Bigg\{\bigg((-i)^{\nu}\int_{-\infty}^{0}dy+\frac{2}{[\mathrm{sign}(x_2)]^{\frac{\nu}{2}}}\int_0^{x_2}dy\bigg)K^{\beta=1}_{\mathrm{Bes},\mathbb{C}}(x_1,y)2\,\mathrm{e}^{y/(8\sigma^2)}K_{\frac{\nu}{2}}\bigg(\frac{|y|}{8\sigma^2}\bigg)\\
&\qquad-\frac{1}{32\sqrt{\pi}}\bigg[-\frac{1}{\sigma}\,\mathrm{e}^{-\sigma^2}\,J_\nu(\sqrt{x_1})\\
&\qquad\qquad+\frac{2\sigma^{\nu}}{\Gamma\big(\frac{\nu+1}{2}\big)}\int_0^1 ds\,\mathrm{e}^{-\sigma^2s^2}s^{\nu+2}\\
&\qquad\qquad\qquad\times\bigg(\frac{\sqrt{x_1}}{2}E_{\frac{1-\nu}{2}}(\sigma^2s^2)\,J_{\nu+1}(s\sqrt{x_1})\\
&\qquad\qquad\qquad\qquad-\sigma^2s\big(E_{\frac{-1-\nu}{2}}(\sigma^2s^2)-E_{\frac{1-\nu}{2}}(\sigma^2s^2)\big)J_\nu(s\sqrt{x_1})\bigg)\bigg]\Bigg\}.
\end{aligned}
$$

In the final equation above,

$$
E_n(x)=\int_1^{\infty}dt\,\frac{\mathrm{e}^{-xt}}{t^n} \qquad (2.38)
$$

is the exponential integral. The kernels $K^{\beta=1}_{\mathrm{Bes},\mathbb{C}}(z_1,z_2)$ and $G^{\beta=1}_{\mathrm{Bes},\mathbb{C},\mathrm{com}}(z_1,z_2)$ were derived in [Akemann et al. 2010b], and $G^{\beta=1}_{\mathrm{Bes},\mathbb{C},\mathrm{real}}(x_1,x_2)$ in [Akemann et al. 2011c; Phillips 2011]; the latter case is somewhat subtle, since we cannot simply commute the (weakly non-Hermitian) large-N limit operation with the integral that appears in the finite-N expression, to give an integral over the limit of the integrand. We also note that there is numerical evidence from studying some examples of non-Gaussian RMT that the density of the real eigenvalues resulting from $G^{\beta=1}_{\mathrm{Bes},\mathbb{C},\mathrm{real}}(x_1,x_2)$ and the corresponding distribution of the smallest eigenvalues may be universal [Phillips 2011].

The Hermitian limit is much more involved here; but compare $K^{\beta=1}_{\mathrm{Bes},\mathbb{C}}(z_1,z_2)$ and $K^{\beta=1}_{\mathrm{Bes}}(z_1,z_2)$ — in particular, as $N\to\infty$ and $\sigma\to0$, we find that

$$
G^{\beta=1}_{\mathrm{Bes},\mathbb{C},\mathrm{com}}(z_1,z_2)\to 0 \quad\text{and}\quad G^{\beta=1}_{\mathrm{Bes},\mathbb{C},\mathrm{real}}(x_1,x_2)\to G^{\beta=1}_{\mathrm{Bes}}(x_1,x_2),
$$

and we refer to [Akemann et al. 2011c; Phillips 2011] for more details.

3. Discussion and open problems

In this short article we have collected together all the known kernels for RMT with real eigenvalues along with all the known and new kernels for RMT with complex eigenvalues at weak non-Hermiticity. This comprises the Airy, sine and Bessel kernels of the three Wigner–Dyson and the three Wishart–Laguerre ensembles, as well as their non-Hermitian counterparts. In order to highlight the nature of this deformation we have used real integral representations for the kernels of real eigenvalues, rather than the asymptotic forms resulting from the Christoffel–Darboux identity for $\beta = 2$ or from the rewriting à la Tracy–Widom for $\beta = 1, 4$. The extra exponential factor (and shift for the Airy case) in the integral representation of the kernels on $\mathbb{C}$ is a very smooth deformation. This makes it very plausible that the universality which is very well studied for real eigenvalues extends to the weakly non-Hermitian limit for all ensembles, beyond what is already known for $\beta = 2$. The universality of the factor in front of the integral which contains special functions such as the complementary error function or modified Bessel function will be more difficult to establish. However, the presence of these factors is crucial when taking the Hermitian limit, in projecting the imaginary parts of the eigenvalues to zero.

Whilst we have already mentioned what is known about universality in the weak limit so far, let us give some more open problems. To date, a mathematically rigorous derivation of most of the limiting kernels on $\mathbb{C}$ is lacking, apart from the complex Airy kernel for $\beta = 2$ [Bender 2010]. There is no doubt that the kernels we have listed and which have been derived using different techniques such as asymptotic OP, supersymmetry or replicas are correct. This is based not only on numerical evidence but also, and more importantly, on a comparison with complex eigenvalue spectra in physics; see, for example, [Verbaarschot 2011; Akemann 2007] and references therein, where the complex Bessel kernels for $\beta = 2$ and 4 were successfully compared with complex spectra from QCD and QCD-like theories. Because the latter are field theories and not Gaussian RMT this gives a further indication that universality holds in this regime. Preliminary numerical investigations with non-Gaussian, non-Hermitian RMT appear to confirm this [Phillips 2011] for $\beta = 1$.

A much more challenging problem will be to show the universality of these kernels on $\mathbb{C}$, either by going to non-Gaussian potentials of polynomial or harmonic form, or by considering non-Hermitian Wigner matrix ensembles, with elements being independent random variables.

A further reason why we believe that this universality question is important is that some of the kernels on $\mathbb{C}$ reappear in the same integral form (with real arguments) when looking at symmetry transitions between two different *Hermitian* RMTs, say from one GUE to another GUE, in a corresponding "weak" limit. Their eigenvalue correlations are also called parametric. For $\beta = 2$ this fact can be observed for the Bessel, sine [Akemann et al. 2007; Forrester et al. 1999] and Airy [Macêdo 1994; Forrester et al. 1999] kernels.

Appendix A: finite-N (skew-) orthogonal polynomials and kernels on $\mathbb{C}$

In this appendix we specify the orthogonal polynomials (OP) and skew-OP as well as their (skew-) symmetric scalar products. These may be used to construct the kernels — which we will list for all the above matrix ensembles with complex eigenvalues — in terms of which all k-point correlation functions can be expressed; see (2.3) and (2.9). We also highlight the relations between expectation values of characteristic polynomials on the one hand and (skew-) OP and their kernels on the other, valid on both $\mathbb{R}$ and $\mathbb{C}$.

Starting with $\beta = 2$ we define the monic OP on $\mathbb{R}$ (and $\mathbb{C}$) by

$$\int_{\mathbb{R}(\mathbb{C})} d^{(2)}z\, w_{\beta=2}^{(\nu,\mathbb{C})}(z)\, P_k(z)\, P_l(z)^{(*)} = h_{k(\nu,\mathbb{C})}^{\beta=2}\delta_{kl}, \tag{A.1}$$

with squared norms $h_{k(\nu,\mathbb{C})}^{\beta=2}$. The symbol ν labels the rectangular $N \times (N+\nu)$ matrices of the Wishart–Laguerre ensembles considered in Section 2.4. Because all moments exist in our examples these OP can be constructed via the Gram–Schmidt procedure. Alternatively, they can be written as

$$P_k(z) = \langle \det[z-H]\rangle_k = \frac{1}{\mathcal{Z}_k^{\mathrm{GUE}}}\int dH\, \det[z-H]\exp[-\operatorname{Tr} H^2/2], \tag{A.2}$$

and similarly for the GinUE and ($\mathbb{C}$)LUE, replacing the $k \times k$ matrix H inside the determinant with J, or with the Wishart matrices $WW^\dagger$ and WV, respectively. In fact, this relation holds for general weight functions. For the Gaussian ensembles we obtain Hermite, and for the WL ensembles Laguerre polynomials on $\mathbb{R}$ and on $\mathbb{C}$. The corresponding kernels are then obtained by summing over the *normalised* OP (multiplied by the weights). Most conveniently, a second relation to characteristic polynomials exists [Akemann and Vernizzi 2003],

$$K_{N,\mathbb{C}}^{\beta=2}(u,v) = w_{\beta=2}^{\mathbb{C}}(u)^{\frac{1}{2}}\, w_{\beta=2}^{\mathbb{C}}(v)^{\frac{1}{2}}\, \frac{1}{h_{N-1,\mathbb{C}}^{\beta=2}}\langle \det[u-J]\det[v-J^\dagger]\rangle_{N-1}, \tag{A.3}$$

which we state here for the GinUE. Correspondingly it holds for the GUE and $\beta = 2$ WL ensembles, and, indeed, for arbitrary weights. We can now give the two $\beta = 2$ kernels in the complex plane, following [Fyodorov et al. 1998] and [Osborn 2004] respectively:

$\underline{\beta = 2}$:

$$K_{N,\mathbb{C}}^{\beta=2}(u,v) = w_{\beta=2}^{\mathbb{C}}(u)^{\frac{1}{2}} w_{\beta=2}^{\mathbb{C}}(v)^{\frac{1}{2}} \frac{1}{\pi\sqrt{1-\tau^2}} \sum_{j=0}^{N-1} \frac{\tau^j}{2^j j!} H_j\left(\frac{u}{\sqrt{2\tau}}\right) H_j\left(\frac{v}{\sqrt{2\tau}}\right), \tag{A.4}$$

$$K_{N,\nu,\mathbb{C}}^{\beta=2}(u,v) = w_{\beta=2}^{\nu,\mathbb{C}}(u)^{\frac{1}{2}} w_{\beta=2}^{\nu,\mathbb{C}}(v)^{\frac{1}{2}} \frac{2}{\pi(1-\tau^2)} \sum_{j=0}^{N-1} \frac{\tau^{2j} j!}{(j+\nu)!} L_j^{\nu}\left(\frac{u}{\tau}\right) L_j^{\nu}\left(\frac{v}{\tau}\right). \tag{A.5}$$

For the skew-OP related to $\beta = 1, 4$, we have to distinguish between the skew products for complex and real eigenvalues. Because the latter are very well known (see, e.g., [Mehta 2004]) we will focus on the former, which can be written in a unified way [Akemann et al. 2010a] as

$$\int_{\mathbb{C}^2} d^2z_1 d^2z_2\, \mathcal{F}_{\beta=1,4}^{(\nu)\mathbb{C}}(z_1,z_2) \det\begin{bmatrix} Q_{2k}^{\beta=1,4}(z_1) & Q_{2l+1}^{\beta=1,4}(z_1) \\ Q_{2k}^{\beta=1,4}(z_2) & Q_{2l+1}^{\beta=1,4}(z_2)\end{bmatrix} = h_{k,(\nu)\mathbb{C}}^{\beta=1,4}\delta_{kl}, \tag{A.6}$$

for skew-OP of even-odd degree, and which is vanishing for even-even and odd-odd degree. Here and in the following we again choose N even. Once more, the skew-OP satisfying this can be written as follows for $\beta = 4$ [Kanzieper 2002] and $\beta = 1$ [Akemann et al. 2010a]:

$$Q_{2k}^{\beta=1,4}(z) = \langle \det[z-J]\rangle_{2k}, \quad Q_{2k+1}^{\beta=1,4}(z) = \langle \det[z-J](z+c+\operatorname{Tr} J)\rangle_{2k}. \tag{A.7}$$

Note that, for $\beta = 4$, the matrix J here should be taken as the complex-valued matrix of size $2k \times 2k$. The odd skew-OP in (A.7) are defined only up to a constant c times the even skew-OP. The same relation holds for real eigenvalues and arbitrary weights, see [Akemann et al. 2010a] for references. Moreover, the antisymmetric kernel matrix element $K_{N,\mathbb{C}}^{\beta=1,4}$ (sometimes called the prekernel) enjoys a similar relation to that in (A.3), as was observed for $\beta = 4$ [Akemann and Basile 2007] and $\beta = 1$ [Akemann et al. 2009]:

$$K_{N,\mathbb{C}}^{\beta=1,4}(u,v) = (u-v)\frac{1}{h_{\frac{N}{2}-1,\mathbb{C}}^{\beta=1,4}} \langle \det[u-J]\det[v-J^{\dagger}]\rangle_{N-2}. \tag{A.8}$$

An analogous relation holds for the Wishart–Laguerre ensembles (see, e.g., in [Akemann et al. 2010a]). We list the corresponding kernel matrix elements. From [Kanzieper 2002] and [Akemann 2005], respectively, we have:

$\underline{\beta=4}$:

$$K_{N,\mathbb{C}}^{\beta=4}(u,v)=\frac{1}{\pi(1-\tau)\sqrt{1-\tau^2}}\sum_{k=0}^{N/2-1}\sum_{l=0}^{k}\frac{1}{(2k+1)!!(2l)!!}\left(\frac{\tau}{2}\right)^{k+l+\frac{1}{2}}\times\left(H_{2k+1}\left(\frac{u}{\sqrt{2\tau}}\right)H_{2l}\left(\frac{v}{\sqrt{2\tau}}\right)-(u\leftrightarrow v)\right),\tag{A.9}$$

$$K_{N,\nu,\mathbb{C}}^{\beta=4}(u,v)=-\frac{2}{\pi(1-\tau^2)^2}\sum_{k=0}^{N/2-1}\sum_{j=0}^{k}\frac{2^{2k-2j}k!(k+\nu)!(2j)!}{(2k+2\nu+1)!j!(j+\nu)!}\tau^{2k+2j+1}\times\left(L_{2k+1}^{2\nu}\left(\frac{u}{\tau}\right)L_{2j}^{2\nu}\left(\frac{v}{\tau}\right)-(u\leftrightarrow v)\right),\tag{A.10}$$

recalling that N here is the size of the complex-valued matrix that is equivalent to the original quaternion real matrix. From [Forrester and Nagao 2008] and [Akemann et al. 2009] we have

$\underline{\beta=1}$:

$$K_{N,\mathbb{C}}^{\beta=1}(u,v)=\frac{1}{2\sqrt{2\pi}(1+\tau)}\sum_{l=0}^{N-2}\frac{1}{l!}\times\left(\frac{\tau}{2}\right)^{l+\frac{1}{2}}\left(H_{l+1}\left(\frac{u}{\sqrt{2\tau}}\right)H_{l}\left(\frac{v}{\sqrt{2\tau}}\right)-(u\leftrightarrow v)\right),\tag{A.11}$$

$$K_{N,\nu,\mathbb{C}}^{\beta=1}(u,v)=-\frac{1}{8\pi(1-\tau^2)}\sum_{l=0}^{N-2}\frac{(l+1)!}{(l+\nu)!}\tau^{2l+1}\times\left(L_{l+1}^{\nu}\left(\frac{u}{\tau}\right)L_{l}^{\nu}\left(\frac{v}{\tau}\right)-(u\leftrightarrow v)\right).\tag{A.12}$$

The other elements of the matrix-valued kernel follow by integration.

Appendix B: large-N limits at weak non-Hermiticity

In this appendix we will specify the different large-N limits that lead to the limiting kernels listed in Section 2. Let us emphasise that these are not all of the possible large-N limits of the above finite-N kernels that one can take. We will give only those limits where $(1-\tau)N^{\delta}=\sigma$ is kept fixed for some $\delta>0$, limits where the degree of non-Hermiticity is weak. The reason is that it is only these particular limiting kernels that relate closely to the known universal kernels on $\mathbb{R}$. However, many of the results at strong non-Hermiticity (i.e., where τ is N-independent) can be recovered from the weak limit by taking $\sigma\to\infty$ and rescaling the complex eigenvalues accordingly.

Soft edge limit. We consider fluctuations around the right end-point of the long half-axis of the supporting ellipse, to obtain from (A.4) [Bender 2010]

$$z = (1+\tau)\sqrt{N} + \frac{X}{N^{1/6}} + i\,\frac{Y}{N^{1/6}}, \quad \sigma = N^{1/6}\sqrt{1-\tau}, \tag{B.1}$$

$$K^{\beta=2}_{\mathrm{Ai},\mathbb{C}}(X_1+iY_1, X_2+iY_2) \equiv \lim_{\substack{N\to\infty\\ \tau\to 1}} \frac{1}{N^{1/3}} K^{\beta=2}_{N,\mathbb{C}}(z_1, z_2). \tag{B.2}$$

The same limit applies to the WL kernel on $\mathbb{C}$ given in (A.5) [Akemann and Bender 2010]. By symmetry we expect the same limiting behaviour around the left end-point $-(1+\tau)\sqrt{N}$ as well as for $\nu = O(N)$. The limiting kernels for $\beta = 1, 4$ are defined in the same way. Note that the eigenvalues in the bulk of the spectrum are at strong non-Hermiticity in this limit, since $\sigma_{\mathrm{sine}} = N^{1/3}\sigma_{\mathrm{Airy}} \to \infty$ as $N \to \infty$. In fact, in order to reach weak non-Hermiticity in the bulk we need to consider the following scaling limit.

Bulk limit. Without loss of generality we consider fluctuations around the origin, being representative of the Gaussian ensembles equation (2.4). On rescaling, we obtain from (A.4) [Fyodorov et al. 1997a; 1997b]

$$z = \frac{X}{N^{1/2}} + i\,\frac{Y}{N^{1/2}}, \quad \sigma = N^{1/2}\sqrt{1-\tau}, \tag{B.3}$$

$$K^{\beta=2}_{\sin,\mathbb{C}}(X_1+iY_1, X_2+iY_2) \equiv \lim_{\substack{N\to\infty\\ \tau\to 1}} \frac{1}{N} K^{\beta=2}_{N,\mathbb{C}}(z_1, z_2), \tag{B.4}$$

and likewise for $\beta = 1, 4$. The macroscopic spectral density collapses onto the real axis, and becomes the semicircle for our Gaussian ensembles. The functions $R_{k,\mathbb{C}}$ given by the determinant or Pfaffian of the rescaled kernels describe the microscopic correlations in the complex plane.

If we magnify around any other point $|x_0| < 2\sqrt{N}$ inside the bulk, then we rescale the fluctuations $z - x_0$ as in (B.3). The correlations are then universal when measured in units of the local mean density $\pi\rho_{\mathrm{sc}}(x_0)$ [Fyodorov et al. 1998; Akemann 2002].

Hard edge limit. Whilst we expect that in the bulk of the spectrum the WL and Gaussian ensembles (Equation (2.29) and (2.4) respectively) show the same behaviour, the origin is singled out in the latter case. The rescaling here is given by [Osborn 2004]

$$z = \frac{X}{4N} + i\,\frac{Y}{4N}, \quad \sigma = N^{1/2}\sqrt{1-\tau}, \tag{B.5}$$

$$K^{\beta=2}_{\mathrm{Bes},\mathbb{C}}(X_1+iY_1, X_2+iY_2) \equiv \lim_{\substack{N\to\infty\\ \tau\to 1}} \frac{1}{(4N)^2} K^{\beta=2}_{N,\nu,\mathbb{C}}(z_1, z_2), \tag{B.6}$$

with the Laguerre polynomials in (A.5) displaying a Bessel function asymptotic.

In all three scaling limits the asymptotic kernels are obtained by replacing the sums with integrals (the Christoffel–Darboux identity does not hold for OP in the complex plane), and the Hermite and Laguerre polynomials by their corresponding Plancherel–Rotach asymptotics (proven for real arguments) in the corresponding region.

An additional problem arises from the integrations with the antisymmetric weight function $\mathcal{F}$ used to obtain the limiting kernel elements G and W. For $\beta = 1$ these integrals are not absolutely convergent, and hence the limit $N \to \infty$ and the integration cannot be interchanged. For a detailed discussion we refer to [Akemann et al. 2011c; Phillips 2011].

Acknowledgements

The organisers and participants of the workshop "Random matrix theory and its applications" at MSRI Berkeley, 13–17 September 2010, are thanked for many inspiring talks and discussions.

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