

On a relationship between high rank cases and rank one cases of Hermitian random matrix models with external source

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We prove an identity on Hermitian random matrix models with external source relating the high rank cases to the rank 1 cases. This identity was proved and used in a previous paper of ours to study the asymptotics of the top eigenvalues. In this paper, we give an alternative, more conceptual proof of this identity based on a connection between the Hermitian matrix models with external source and the discrete KP hierarchy. This connection is obtained using the vertex operator method of Adler and van Moerbeke. The desired identity then follows from the Fay-like identity of the discrete KP τ vector.

1. Introduction

The subject of this paper is an identity between a Hermitian random matrix model with external source of rank m and m such models with external source of rank 1. This identity allows us to reduce the asymptotic study of “spiked Hermitian random matrix models” of rank higher than 1 to that of the models of rank 1. This reduction formula was used in [Baik and Wang 2013] to evaluate the limiting fluctuations of the top eigenvalue(s) of spiked models of arbitrary fixed rank with a general class of potentials. In [Baik and Wang 2013] we gave a direct proof of this identity using the formula of [Baik 2009] on the Christoffel–Darboux kernel. Here we give a different, more conceptual proof using the relation between the random matrix model with external source and discrete KP hierarchy. We show how the general results of [Adler and van Moerbeke 1999] on the construction of solutions of discrete KP hierarchy can be used on the partition functions of the Hermitian matrix model with external source.

We now introduce the model. Let $W(x)$ be a piecewise continuous function on \mathbb{R} . We assume that $W(x)$ is nonnegative, has infinite support, and vanishes sufficiently fast as $|x| \rightarrow \infty$ so that (3) converges. Let A be a $d \times d$ Hermitian

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matrix with eigenvalues a_1, \dots, a_d . We call A the external source matrix and $W(x)$ the weight function. Consider the following measure on the set \mathcal{H}_d of $d \times d$ Hermitian matrix M :

$$P(M) dM := \frac{1}{Z_d(a_1, \dots, a_d)} \det(W(M)) e^{\text{Tr}(AM)} dM, \quad (1)$$

where $W(M)$ is defined in terms of a functional calculus and

$$Z_d(a_1, \dots, a_d) := \int_{M \in \mathcal{H}_d} \det(W(M)) e^{\text{Tr}(AM)} dM \quad (2)$$

is the partition function. Note that the partition function depends only on the eigenvalues of A , but not its eigenvectors. It is also symmetric in a_1, \dots, a_d . The Harish-Chandra–Itzykson–Zuber integral formula [Harish-Chandra 1957; Itzykson and Zuber 1980] implies that

$$\begin{aligned} & Z_d(a_1, \dots, a_d) \\ &= \frac{C_d}{\prod_{1 \leq j < k \leq d} (a_k - a_j)} \int_{\mathbb{R}^d} \det[e^{a_j \lambda_k}]_{j,k=1}^d \prod_{1 \leq j < k \leq d} (\lambda_k - \lambda_j) \prod_{j=1}^d W(\lambda_j) d\lambda_j \quad (3) \end{aligned}$$

for a constant C_d that depends only on d . There is a similar formula for the density function of the eigenvalues.

If some of the eigenvalues of A are zero, we use the following short-hand: For $\mathbf{m} \leq d$,

$$\begin{aligned} Z_d(a_1, \dots, a_{\mathbf{m}}) &:= Z_d(a_1, \dots, a_{\mathbf{m}}, \underbrace{0, \dots, 0}_{d-\mathbf{m}}), \\ Z_d &:= Z_d(\underbrace{0, \dots, 0}_d). \end{aligned} \quad (4)$$

The main theorem of this paper is about the following expectations. For $E \subset \mathbb{R}$, $s \in \mathbb{C}$, and $\mathbf{m} \leq d$, we define

$$\begin{aligned} \mathfrak{E}_d(a_1, \dots, a_{\mathbf{m}}; E; s) &:= \mathbb{E} \left[\prod_{j=1}^d (1 - s \chi_E(\lambda_j)) \right] \\ &= \int_{\mathcal{H}_d} \prod_{j=1}^d (1 - s \chi_E(\lambda_j)) P(M) dM, \quad (5) \end{aligned}$$

where $\lambda_1, \dots, \lambda_d$ are eigenvalues of M and the expectation is with respect to the measure (1) when the eigenvalues of the external source matrix A are

$$a_1, \dots, a_{\mathbf{m}}, \underbrace{0, \dots, 0}_{d-\mathbf{m}}.$$

We note that the last integral is the partition function with new weight

$$(1 - s\chi_E(x))W(x)$$

divided by the original partition function with the weight $W(x)$. This observation will be used in the proof later. When $s = 1$ the above expectation is a gap probability. Set

$$\bar{\mathfrak{E}}_d(a_1, \dots, a_m; E; s) := \frac{\mathfrak{E}_d(a_1, \dots, a_m; E; s)}{\mathfrak{E}_d(E; s)}. \quad (6)$$

Let $p_j(x)$ be orthonormal polynomials with respect to $W(x) dx$. For a real number a , define the constant

$$\Gamma_j(a) := \int_{\mathbb{R}} p_j(s) e^{as} W(s) ds. \quad (7)$$

The main result is this:

Theorem 1.1. *We have*

$$\bar{\mathfrak{E}}_d(a_1, \dots, a_m; E; s) = \frac{\det[\Gamma_{d-j}(a_k) \bar{\mathfrak{E}}_{d-j+1}(a_k; E; s)]_{j,k=1}^m}{\det[\Gamma_{d-j}(a_k)]_{j,k=1}^m}. \quad (8)$$

if a_1, \dots, a_m are distinct and nonzero.

Since both sides of (8) are analytic in a_j , the above identity still holds when some of a_j are the same or equal to zero if we interpret the right-hand side using l'Hôpital's rule.

The term $\bar{\mathfrak{E}}_{d-j+1}(a_k; E; s)$ on the right-hand side of (8) is given by (6) with the rank $m = 1$ and the only nonzero eigenvalue of the external source A being a_k , and the dimension is changed to $d - j + 1$. Hence the identity (8) relates the rank m case to m rank 1 cases.

In [Baik and Wang 2011], the asymptotics $\bar{\mathfrak{E}}_{d-j+1}(a_k; E; s)$, the rank 1 cases, were obtained. The quantities $\Gamma_{d-j}(a_k)$ were also analyzed asymptotically in the same paper as an intermediate step. In [Baik and Wang 2013] the higher rank cases were then analyzed asymptotically using the identity (8). The main result was that when the potential, assuming that it is real analytic, is convex to the right of the right-end point of the support of the equilibrium measure, the phase transition behavior of the fluctuations of the top eigenvalues is same as in the Gaussian unitary ensemble. Otherwise, new types of jump transitions are possible to occur. A characterization of possible new transitions was also obtained.

2. Proof

Set $\Delta(x) := \prod_{1 \leq j < k \leq d} (x_k - x_j) = \det(x_j^{k-1})_{j,k=1}^d$, for $x = (x_1, \dots, x_d)$. Recall that, setting $a = (a_1, \dots, a_d)$ and $\lambda = (\lambda_1, \dots, \lambda_d)$,

$$\frac{\det[e^{a_j \lambda_k}]_{j,k=1}^d}{\Delta(a)\Delta(\lambda)} = \sum_{\ell(\kappa) \leq d} \frac{s_\kappa(\lambda) s_\kappa(a)}{\prod_{q=1}^d (\kappa_q + d - q)!}, \quad (9)$$

where the sum is over all partitions κ with at most d parts. Here s_κ denotes the Schur polynomial and $\ell(\kappa)$ denotes the number of parts of partition κ . This identity can be proved as follows. First, from Andréief's identity [Andréief 1886] (equivalently the Cauchy–Binnet formula),

$$\det[e^{a_j \lambda_k}]_{j,k=1}^d = \det \left[\sum_{n=0}^{\infty} \frac{a_j^n \lambda_k^n}{n!} \right] = \frac{1}{d!} \sum_{n_1, \dots, n_d=0}^{\infty} \frac{\det[a_j^{n_q}] \det[\lambda_j^{n_q}]}{\prod_{q=1}^d n_q!}. \quad (10)$$

Since the summand is symmetric in the n_q and equals to zero when two of the summation indices are the same, we can replace $(1/d!) \sum_{n_1, \dots, n_d=0}^{\infty}$ by $\sum_{0 \leq n_d < \dots < n_2 < n_1}$.

Now set $\kappa_q := n_q - d + q$. Then the summation condition becomes $\kappa_1 \geq \dots \geq \kappa_d \geq 0$, that is, all partitions with at most d parts. The identity (9) follows by recalling the classical definition of the Schur polynomial

$$s_\kappa(a) = \frac{\det(a_j^{\kappa_q + d - q})}{\Delta(\lambda)}.$$

Inserting (9) into (3), we obtain the Schur polynomial expansion of the partition function:

$$Z_d(a_1, \dots, a_d) = C_d \sum_{\ell(\kappa) \leq d} G_\kappa \frac{s_\kappa(a)}{\prod_{q=1}^d (\kappa_q + d - q)!} \quad (11)$$

where

$$G_\kappa := \int_{\mathbb{R}^d} s_\kappa(\lambda) \Delta(\lambda)^2 \prod_{j=1}^d W(\lambda_j) d\lambda_j. \quad (12)$$

Using the classical definition of the Schur function, the determinantal form of $\Delta(\lambda)$, and the Andréief's identity, we obtain

$$G_\kappa = d! \cdot \det[M_{\kappa_p + d - p + q - 1}]_{p,q=1}^d, \quad M_j := \int_{\mathbb{R}} x^j W(x) dx. \quad (13)$$

Here M_j are the moments of the measure $W(x) dx$.

We insert (13) into (11) and use the Jacobi–Trudi identity,

$$s_\kappa(a) = \det[h_{\kappa_p-p+q}(a)]_{p,q=1}^d,$$

where $h_j(a)$ denotes the complete symmetric function. The sum is over the partitions $\kappa = (\kappa_1, \dots, \kappa_d)$ where $\kappa_1 \geq \dots \geq \kappa_d \geq 0$. We set $j_p := \kappa_p + d - p$. Then $j_1 > \dots > j_d \geq 0$. Since the summand is symmetric in the j_p and vanishes when two indices are the same, we arrive at the formula

$$Z_d(a_1, \dots, a_d) = C_d \sum_{j_1, \dots, j_d=0}^{\infty} \frac{\det[M_{j_p+q-1}]_{p,q=1}^d \det[h_{j_p-d+q}(a)]_{p,q=1}^d}{\prod_{q=1}^d j_q!}. \quad (14)$$

In the below, the notation $\mathbf{t} = (t_1, t_2, \dots)$ denotes a sequence of variables. We also use the notation $[c] = (c, \frac{1}{2}c^2, \frac{1}{3}c^3, \dots)$ for the evaluation of \mathbf{t} by powers of c . The notation $[c_1] + [c_2] + \dots + [c_m]$ stands for the evaluation of \mathbf{t} obtained by substituting $t_j = \sum_{i=1}^m c_i^j / j$.

Following [Kac and Raina 1987, Definition 6.1], we define the so-called “elementary Schur polynomials” $h_j(\mathbf{t})$ by the generating function

$$\sum_{j=-\infty}^{\infty} h_j(\mathbf{t}) w^j = e^{\sum_{j=1}^{\infty} t_j w^j}. \quad (15)$$

If $\mathbf{t} = [a_1] + [a_2] + \dots + [a_d]$ (i.e., $t_j = \sum_{i=1}^d a_i^j / j$), $h_j(\mathbf{t})$ is the complete symmetric function in a_1, \dots, a_d , which we denoted earlier by $h_j(a)$. This abuse of notations is unfortunate but in the below we only use the definition $h(\mathbf{t})$ given in (15).

Now define the formal power series in $\mathbf{t} = (t_1, t_2, \dots)$

$$\widehat{Z}_d(\mathbf{t}) := \frac{1}{d! \widehat{C}_d} \sum_{j_1, \dots, j_d=0}^{\infty} \frac{\det[M_{j_p+q-1}]_{p,q=1}^d \det[h_{j_p-d+q}(\mathbf{t})]_{p,q=1}^d}{\prod_{k=1}^d j_k!}, \quad (16)$$

where $\widehat{C}_d := \frac{1}{d! C_d}$. This definition is equivalent to [Wang 2009, (26)]. Then

$$Z_d(a_1, \dots, a_d) = \widehat{Z}_d([a_1] + \dots + [a_d]). \quad (17)$$

Setting some of the parameters to zero, we also have for $m \leq d$

$$Z_d(a_1, \dots, a_m) = \widehat{Z}_d([a_1] + \dots + [a_m]). \quad (18)$$

We now show that $\widehat{Z}_d(\mathbf{t})$ solves the discrete KP hierarchy following the vertex operator construction of the general solutions due to Adler and van Moerbeke. We then show that a general property of the vertex operator solution implies an identity of which the identity (8) is a special case. Note that the definition (16)

does not require the assumption that the weight function $W(x)$ is nonnegative. In the next two subsections we drop this assumption.

Discrete KP τ vectors.

Proposition 2.1. *Let $\widehat{Z}_d(\mathbf{t})$ be defined in (16) and set $\widehat{Z}_0(\mathbf{t}) := 1$. Then the sequence*

$$(\dots, \widehat{Z}_2(-\mathbf{t}), \widehat{Z}_1(-\mathbf{t}), \widehat{Z}_0(\mathbf{t}), \widehat{Z}_1(\mathbf{t}), \widehat{Z}_2(\mathbf{t}), \dots) \quad (19)$$

constitutes a discrete KP τ vector.

Discrete KP τ vectors are solutions to a system of differential-difference equations in the discrete KP hierarchy. In [Adler and van Moerbeke 1999] several characterizations of discrete KP τ vectors were established. Here we use two of them: the vertex operator characterization (see Proposition 2.2 below) and the Hirota bilinear identity characterization (see (34) below).

Remark. Each component of a discrete KP τ vector is a KP τ function. The fact that $\widehat{Z}_d(\mathbf{t})$ is a KP τ function for each $d \in \mathbb{N}$ was proved in [Wang 2009]. In [Harnad and Orlov 2014] it was proved further that for each $d \in \mathbb{N}$, $\widehat{Z}_d(\mathbf{t})$ is a so-called 1-KP-Toda τ function and moreover these 1-KP-Toda τ functions are derived from the same Grassmannian structure. It should also be possible to prove the above proposition from this fact.

The vertex operator is a differential operator defined by

$$X(\mathbf{t}, z) := \exp\left(\sum_{k=1}^{\infty} t_k z^k\right) \exp\left(-\sum_{k=1}^{\infty} \frac{z^{-k}}{k} \frac{\partial}{\partial t_k}\right); \quad (20)$$

see [Adler and van Moerbeke 1999, (0.22)]. The vertex operator acts on a formal power series $f(\mathbf{t})$ as

$$\begin{aligned} X(\mathbf{t}, z) f(\mathbf{t}) &= \exp\left(\sum_{k=1}^{\infty} t_k z^k\right) f(\mathbf{t} - [z^{-1}]) \\ &= \left(\sum_{k=-\infty}^{\infty} h_k(\mathbf{t}) z^k\right) f(\mathbf{t} - [z^{-1}]). \end{aligned} \quad (21)$$

One can construct a discrete KP τ vector from one KP τ function and a sequence of measures by applying the vertex operator repeatedly:

Proposition 2.2 [Adler and van Moerbeke 1999, Theorem 0.3]. *Let $\tau(\mathbf{t})$ be a KP τ function. Let $(\dots, \nu_{-1}(z) dz, \nu_0(z) dz, \nu_1(z) dz, \dots)$ be a sequence of arbitrary measures. Then the infinite sequence $(\dots, \tau_{-1}(\mathbf{t}), \tau_0(\mathbf{t}), \tau_1(\mathbf{t}), \dots)$ defined by $\tau_0(\mathbf{t}) := \tau(\mathbf{t})$ and*

$$\tau_d(\mathbf{t}) := \left(\int X(\mathbf{t}, z) v_{d-1}(z) dz \right) \dots \left(\int X(\mathbf{t}, z) v_0(z) dz \right) \tau(\mathbf{t}), \quad (22)$$

$$\tau_{-d}(\mathbf{t}) := \left(\int X(-\mathbf{t}, z) v_{-d}(z) dz \right) \dots \left(\int X(-\mathbf{t}, z) v_{-1}(z) dz \right) \tau(\mathbf{t}), \quad (23)$$

for $d \in \mathbb{N}$, forms a discrete KP τ vector.

Remark. The original Theorem 0.3 of [Adler and van Moerbeke 1999] assumes that the measures $v_k(z) dz$ are defined on \mathbb{R} . However, it is easy to check that the proof to Proposition 2.2 in [Adler and van Moerbeke 1999] holds almost verbatim if we change \mathbb{R} into \mathbb{C} (or more restrictively the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$ that we are going to use in this section).

Proof of Proposition 2.1. We set $\tau(\mathbf{t}) = \hat{Z}_0(\mathbf{t}) := 1$. This is trivially a KP τ function. Hence Proposition 2.1 is proved if we can construct a sequence of measures $(\dots, v_{-1}(z) dz, v_0(z) dz, v_1(z) dz, \dots)$ such that $\hat{Z}_d(\mathbf{t})$ equals the right-hand side of (22) (resp. (23)) with $\tau(\mathbf{t}) = 1$ for $d > 0$ (resp. $d < 0$).

Define the measure on the circle $\{z \in \mathbb{C} : |z| = 1\}$ as

$$v_d(z) dz := \frac{(-1)^d \hat{C}_d}{2\pi i \hat{C}_{d+1}} \sum_{j=0}^{\infty} \frac{M_{j+d}}{j!} z^{d-j-1} dz, \quad d = 0, 1, 2, \dots, \quad (24)$$

where $\hat{C}_0 := 1$. We also define

$$v_d(z) := v_{-d-1}(z) \quad d = -1, -2, \dots \quad (25)$$

We now show that these measures satisfy the desired property.

For $d = 0$, since $\hat{Z}_0(\mathbf{t}) = 1$, (21) implies $X(\mathbf{t}, z) \hat{Z}_0(\mathbf{t}) = \sum_{k=-\infty}^{\infty} h_k(\mathbf{t}) z^k$. Hence we find from a direct evaluation of the integral using the Cauchy integral formula (the integral is over the unit circle) that

$$\oint X(\mathbf{t}, z) \hat{Z}_0(\mathbf{t}) v_0(z) dz = \frac{1}{\hat{C}_1} \sum_{j=0}^{\infty} \frac{h_j(\mathbf{t}) M_j}{j!} = \hat{Z}_1(\mathbf{t}) \quad (26)$$

from the definition (16).

We now consider $d > 0$. From (21),

$$\begin{aligned} X(\mathbf{t}, z) \hat{Z}_d(\mathbf{t}) &= \frac{1}{d! \hat{C}_d} \left(\sum_{k=-\infty}^{\infty} h_k(\mathbf{t}) z^k \right) \\ &\times \sum_{j_1, \dots, j_d=0}^{\infty} \frac{\det[M_{j_p+q-1}]_{p,q=1}^d \det[h_{j_p-d+q}(\mathbf{t} - [z^{-1}])]_{p,q=1}^d}{\prod_{k=1}^d j_k!}. \end{aligned} \quad (27)$$

Note that from (15) we have

$$\begin{aligned} \sum_{j=-\infty}^{\infty} h_j(\mathbf{t} - [z^{-1}])w^j &= \exp\left(\sum_{j=1}^{\infty} \left(t_j - \frac{z^{-j}}{j}\right)w^j\right) \\ &= \left(1 - \frac{w}{z}\right) \sum_{j=-\infty}^{\infty} h_j(\mathbf{t})w^j. \end{aligned}$$

Comparing the coefficients of w^j , we find that

$$h_j(\mathbf{t} - [z^{-1}]) = h_j(\mathbf{t}) - z^{-1}h_{j-1}(\mathbf{t}). \quad (28)$$

By (28), we have

$$\det[h_{j_p-d+q}(\mathbf{t} - [z^{-1}])]_{p,q=1}^d = \sum_{l=0}^d z^{-l} \det[h_{j_p-d+q-H[l-q]}(\mathbf{t})]_{p,q=1}^d, \quad (29)$$

where H is the discrete form of the Heaviside function such that $H[n] = 0$ for $n < 0$ and $H[n] = 1$ for $n \geq 0$. Substituting (29) into (27), we have, after the change of variable $j_0 = k - l + d$, that

$$\begin{aligned} X(\mathbf{t}, z) \hat{Z}_d(\mathbf{t}) &= \frac{1}{d! \hat{C}_d} \sum_{l=0}^d \left(\sum_{j_0=0}^{\infty} h_{j_0-d+l}(\mathbf{t}) z^{j_0-d} \right) \\ &\quad \times \sum_{j_1, \dots, j_d=0}^{\infty} \frac{\det[M_{j_p+q-1}]_{p,q=1}^d \det[h_{j_p-d+q-H[l-q]}(\mathbf{t})]_{p,q=1}^d}{\prod_{k=1}^d j_k!} \end{aligned}$$

We can reexpress this as follows, using that the right-hand side of (30) is symmetric in j_0, j_1, \dots, j_d :

$$\begin{aligned} X(\mathbf{t}, z) \hat{Z}_d(\mathbf{t}) &= \frac{(-1)^d}{(d+1)! \hat{C}_d} \sum_{j_0, j_1, \dots, j_d=0}^{\infty} \prod_{k=0}^d \frac{1}{j_k!} \\ &\quad \times \begin{vmatrix} M_{j_0} & \cdots & M_{j_0+d-1} & j_0! z^{j_0-d} \\ M_{j_1} & \cdots & M_{j_1+d-1} & j_1! z^{j_1-d} \\ \vdots & \ddots & \vdots & \vdots \\ M_{j_d} & \cdots & M_{j_d+d-1} & j_d! z^{j_d-d} \end{vmatrix} \begin{vmatrix} h_{j_0-d}(\mathbf{t}) & h_{j_0-d+1}(\mathbf{t}) & \cdots & h_{j_0}(\mathbf{t}) \\ h_{j_1-d}(\mathbf{t}) & h_{j_1-d+1}(\mathbf{t}) & \cdots & h_{j_1}(\mathbf{t}) \\ \vdots & \vdots & \ddots & \vdots \\ h_{j_d-d}(\mathbf{t}) & h_{j_d-d+1}(\mathbf{t}) & \cdots & h_{j_d}(\mathbf{t}) \end{vmatrix}. \quad (30) \end{aligned}$$

Note that in here there is one more summation index j_0 than in (27) and the determinants are of $d+1$ by $d+1$ matrices. Then from (30) and (16), and noting that the variable z appears only in the last column of the first matrix in

(30), we can check directly using the Cauchy integral formula that

$$\oint X(\mathbf{t}, z) \hat{Z}_d(\mathbf{t}) v_d(z) dz = \hat{Z}_{d+1}(\mathbf{t}), \quad d > 0. \quad (31)$$

Successive applications of the relation (31) imply that, for all $d > 0$,

$$\hat{Z}_d(\mathbf{t}) = \left(\int X(\mathbf{t}, z) v_{d-1}(z) dz \right) \dots \left(\int X(\mathbf{t}, z) v_0(z) dz \right) \hat{Z}_0(\mathbf{t}), \quad (32)$$

which is same as (22).

Finally, (25) and (32) imply that

$$\hat{Z}_d(-\mathbf{t}) = \left(\int X(-\mathbf{t}, z) v_{-d}(z) dz \right) \dots \left(\int X(-\mathbf{t}, z) v_{-1}(z) dz \right) \hat{Z}_0(\mathbf{t}). \quad (33)$$

This is same as (23). Hence the proposition is proved. \square

Fay-like identity. An importance property of discrete KP τ vector is that its components satisfy a Hirota bilinear identity (see [Adler and van Moerbeke 1999, Theorem 0.2(iii)]). (Adler and van Moerbeke, moreover, showed that the Hirota bilinear identity actually characterizes the discrete KP τ vector.) For the discrete KP τ vector (19) in our situation, replacing the notations τ_n and τ_m for components of a general KP τ vector in [Adler and van Moerbeke 1999, (0.18)] into our specific \hat{Z}_{d_1} and \hat{Z}_{d_2} , we find that the Hirota bilinear identity becomes

$$\begin{aligned} \frac{1}{2\pi i} \oint_{z=\infty} \hat{Z}_{d_1}(\tilde{\mathbf{t}} - [z^{-1}]) \hat{Z}_{d_2+1}(\mathbf{t} + [z^{-1}]) \\ \times \exp\left(\sum_{j=1}^{\infty} (\tilde{t}_j - t_j) z^j\right) z^{d_1-d_2-1} dz = 0, \end{aligned} \quad (34)$$

for all $d_1 > d_2 \geq 0$. Here the formal integral of a formal Laurent series is defined by

$$\frac{1}{2\pi i} \oint_{z=\infty} \left(\sum_{j=-\infty}^{\infty} a_j z^j \right) dz = a_{-1}. \quad (35)$$

We now show that this Hirota bilinear identity implies a Fay-like identity (41). Such a derivation of a Fay-like identity from the Hirota bilinear identity was obtained in the Toda lattice hierarchy by [Teo 2006] and we adapt this approach.

We take the special choices $d_1 = d$, $d_2 = d - 2$ and $\tilde{\mathbf{t}} = \mathbf{t} + [a] + [b]$ in (34). Then the factor $\exp\left(\sum_{j=1}^{\infty} (\tilde{t}_j - t_j) z^j\right) z^{d_1-d_2-1}$ equals

$$z \exp\left(\sum_{j=1}^{\infty} \left(\frac{a^j}{j} + \frac{b^j}{j}\right) z^j\right). \quad (36)$$

After rewriting the sum in the exponential as $-\log(1-az) - \log(1-bz)$, and using the simple identity

$$\frac{abz}{(1-az)(1-bz)} = \frac{1}{(a-b)z} \left(\frac{b}{1-az} - \frac{a}{1-bz} \right) + \frac{1}{z},$$

we find that (36) equals

$$\frac{1}{a(a-b)z} \sum_{j=0}^{\infty} a^j z^j - \frac{1}{b(a-b)z} \sum_{j=0}^{\infty} b^j z^j + \frac{1}{abz}.$$

Using this, (34) implies that

$$\begin{aligned} \frac{a}{2\pi i} \oint_{z=\infty} Q(z^{-1}) \left(\sum_{j=0}^{\infty} b^j z^j \right) \frac{dz}{z} - \frac{b}{2\pi i} \oint_{z=\infty} Q(z^{-1}) \left(\sum_{j=0}^{\infty} a^j z^j \right) \frac{dz}{z} \\ = \frac{a-b}{2\pi i} \oint_{z=\infty} Q(z^{-1}) \frac{dz}{z}, \end{aligned} \quad (37)$$

where Q is defined by

$$Q(w) := \widehat{Z}_d(\mathbf{t} + [a] + [b] - [w]) \widehat{Z}_{d-1}(\mathbf{t} + [w]). \quad (38)$$

Observe that the Laurent series of $Q(w)$ consists only of nonnegative powers of w . Hence

$$Q(w) = \sum_{n=0}^{\infty} q_n w^n, \quad (39)$$

for some q_0, q_1, \dots . Thus, from (35) we have

$$\begin{aligned} \frac{1}{2\pi i} \oint_{z=\infty} Q(z^{-1}) \left(\sum_{j=0}^{\infty} a^j z^j \right) \frac{dz}{z} &= \sum_{n=0}^{\infty} q_n a^n = Q(a) \\ &= \widehat{Z}_d(\mathbf{t} + [a]) \widehat{Z}_{d-1}(\mathbf{t} + [b]). \end{aligned} \quad (40)$$

Similar evaluations of the other integrals of (37) imply the following Fay-like identity:

$$\begin{aligned} a \widehat{Z}_d(\mathbf{t} + [a]) \widehat{Z}_{d-1}(\mathbf{t} + [b]) - b \widehat{Z}_d(\mathbf{t} + [b]) \widehat{Z}_{d-1}(\mathbf{t} + [a]) \\ = (a-b) \widehat{Z}_d(\mathbf{t} + [a] + [b]) \widehat{Z}_{d-1}(\mathbf{t}). \end{aligned} \quad (41)$$

In the remainder of this section we use identity (41) to prove the following:

Proposition 2.3. For any $d \geq m \geq 1$ and $a_1, \dots, a_m \in \mathbb{C}$,

$$\frac{\widehat{Z}_d(\mathbf{t} + [a_1] + \dots + [a_m])}{\widehat{Z}_d(\mathbf{t})} = \frac{1}{\Delta_m(a_1, \dots, a_m)} \det \left[a_k^{m-j} \frac{\widehat{Z}_{d+1-j}(\mathbf{t} + [a_k])}{\widehat{Z}_{d+1-j}(\mathbf{t})} \right]_{j,k=1}^m, \quad (42)$$

where $\Delta_m(a_1, \dots, a_m) := \prod_{1 \leq j < k \leq m} (a_j - a_k)$.

Proof. After dividing the identity (41) by $\widehat{Z}_d(\mathbf{t})\widehat{Z}_{d-1}(\mathbf{t})$, we obtain

$$\frac{\widehat{Z}_d(\mathbf{t} + [a] + [b])}{\widehat{Z}_d(\mathbf{t})} = \frac{1}{a-b} \det \begin{bmatrix} a \frac{\widehat{Z}_d(\mathbf{t} + [a])}{\widehat{Z}_d(\mathbf{t})} & b \frac{\widehat{Z}_d(\mathbf{t} + [b])}{\widehat{Z}_d(\mathbf{t})} \\ \frac{\widehat{Z}_{d-1}(\mathbf{t} + [a])}{\widehat{Z}_{d-1}(\mathbf{t})} & \frac{\widehat{Z}_{d-1}(\mathbf{t} + [b])}{\widehat{Z}_{d-1}(\mathbf{t})} \end{bmatrix}. \quad (43)$$

This is the identity (42) when $m = 2$ for all $d \geq 2$.

We now prove the general case using an induction in m . Suppose that (42) holds with $m \leq m-1$ for all $d \geq m-1$ and $a_1, \dots, a_{m-1} \in \mathbb{C}$. We are to prove that it holds with $m = m$ for all $d \geq m$ and $a_1, \dots, a_m \in \mathbb{C}$. For this purpose, we set $a = a_1$, $b = a_m$, and $\mathbf{t} \mapsto \mathbf{t} + [a_2] + \dots + [a_{m-1}]$ in (43). After pulling out the denominators of the entries of the determinant outside, we obtain

$$(a_1 - a_m) \frac{\widehat{Z}_d(\mathbf{t} + [a_1] + \dots + [a_m])}{\widehat{Z}_d(\mathbf{t})} \frac{\widehat{Z}_{d-1}(\mathbf{t} + [a_2] + \dots + [a_{m-1}])}{\widehat{Z}_{d-1}(\mathbf{t})} = \det \begin{bmatrix} a_1 \widehat{Z}_d(\mathbf{t} + [a_1] + \dots + [a_{m-1}]) & a_m \widehat{Z}_d(\mathbf{t} + [a_2] + \dots + [a_m]) \\ \widehat{Z}_{d-1}(\mathbf{t} + [a_1] + \dots + [a_{m-1}]) & \widehat{Z}_{d-1}(\mathbf{t} + [a_2] + \dots + [a_m]) \end{bmatrix}. \quad (44)$$

Let us call the entries of the last determinant A_{ij} , $i, j = 1, 2$. First we consider A_{11} on the first row. From the induction hypothesis,

$$\frac{A_{11}}{\widehat{Z}_d(\mathbf{t})} = \frac{a_1}{\Delta_{m-1}(a_1, \dots, a_{m-1})} \det \left[a_k^{m-1-j} \frac{\widehat{Z}_{d+1-j}(\mathbf{t} + [a_k])}{\widehat{Z}_{d+1-j}(\mathbf{t})} \right]_{j,k=1}^{m-1}. \quad (45)$$

If we multiply $a_2 \cdots a_{m-1}$ on both sides and bring the factor $a_1 \cdots a_{m-1}$ inside the determinant, we find that

$$a_2 \cdots a_{m-1} \frac{A_{11}}{\widehat{Z}_d(\mathbf{t})} = \frac{1}{\Delta_{m-1}(a_1, \dots, a_{m-1})} \det [B_{jk}]_{1 \leq j \leq m-1, 1 \leq k \leq m-1}, \quad (46)$$

where

$$B_{jk} := a_k^{m-j} \frac{\widehat{Z}_{d+1-j}(\mathbf{t} + [a_k])}{\widehat{Z}_{d+1-j}(\mathbf{t})}. \quad (47)$$

Note that the power of a_k has changed to $m-j$, from $m-1-j$ in (45). Similarly,

we find that

$$a_2 \cdots a_{m-1} \frac{A_{12}}{\widehat{Z}_d(\mathbf{t})} = \frac{1}{\Delta_{m-1}(a_2, \dots, a_m)} \det[B_{jk}]_{1 \leq j \leq m-1, 2 \leq k \leq m}, \quad (48)$$

with the same definition (47) of B_{jk} . Note the difference of the indices of the determinant from (46).

Now we consider A_{ij} in the second row. The induction hypothesis implies that

$$\frac{A_{21}}{\widehat{Z}_{d-1}(\mathbf{t})} = \frac{1}{\Delta_{m-1}(a_1, \dots, a_{m-1})} \det \left[a_k^{m-1-j} \frac{\widehat{Z}_{d-j}(\mathbf{t} + [a_k])}{\widehat{Z}_{d-j}(\mathbf{t})} \right]_{j,k=1}^{m-1}. \quad (49)$$

Note that d is changed to $d-1$ in the determinant from (46). If we shift the index j by $j-1$ in the determinant, we can write the above as

$$\frac{A_{21}}{\widehat{Z}_{d-1}(\mathbf{t})} = \frac{1}{\Delta_{m-1}(a_1, \dots, a_{m-1})} \det[B_{jk}]_{2 \leq j \leq m, 1 \leq k \leq m-1}. \quad (50)$$

Similarly,

$$\frac{A_{22}}{\widehat{Z}_{d-1}(\mathbf{t})} = \frac{1}{\Delta_{m-1}(a_2, \dots, a_m)} \det[B_{jk}]_{2 \leq j \leq m, 2 \leq k \leq m}. \quad (51)$$

Consider the matrix \mathcal{B} of size m whose entries are B_{jk} , $j, k = 1, \dots, m$. Let \mathcal{B}_a^b denote the matrix of size $m-1$ obtained from \mathcal{B} by deleting the row a and the column b . The determinants in (46), (48), (50), and (51) are those of the matrices \mathcal{B}_m^m , \mathcal{B}_m^1 , \mathcal{B}_1^m , and \mathcal{B}_1^1 , respectively. Hence we find that

$$\begin{aligned} \frac{a_2 \cdots a_{m-1} \Delta_{m-1}(a_1, \dots, a_{m-1}) \Delta_{m-1}(a_2, \dots, a_m)}{\widehat{Z}_{d-1}(\mathbf{t}) \widehat{Z}_d(\mathbf{t})} \det \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \\ = \det[\mathcal{B}_m^m] \det[\mathcal{B}_1^1] - \det[\mathcal{B}_m^1] \det[\mathcal{B}_1^m]. \end{aligned} \quad (52)$$

Now the Desnanot–Jacobi identity, often attributed to Charles Ludwig Dodgson, aka Lewis Carroll (see, e.g., [Krattenthaler 1999, Proposition 10] and references therein) implies that the above equals $\det[\mathcal{B}] \det[\mathcal{B}_{1,m}^{1,m}]$ where $\mathcal{B}_{1,m}^{1,m}$ is the matrix of size $m-2$ obtained by deleting the rows 1, m and the columns 1, m from \mathcal{B} . The determinant $\det[\mathcal{B}]$ is precisely the determinant in (42) with $\mathbf{m} = m$. On the other hand,

$$\begin{aligned} \frac{\det[\mathcal{B}_{1,m}^{1,m}]}{a_2 \cdots a_{m-1}} &= \det \left[a_k^{m-1-j} \frac{\widehat{Z}_{d+1-j}(\mathbf{t} + [a_k])}{\widehat{Z}_{d+1-j}(\mathbf{t})} \right]_{j,k=2}^{m-1} \\ &= \det \left[a_{k+1}^{m-2-j} \frac{\widehat{Z}_{d-j}(\mathbf{t} + [a_{k+1}])}{\widehat{Z}_{d-j}(\mathbf{t})} \right]_{j,k=1}^{m-2}. \end{aligned} \quad (53)$$

The last determinant is precisely the determinant in (42) with $\mathbf{m} = m - 1$, d replaced by $d - 1$, and the complex numbers given by a_2, \dots, a_{m-1} . The induction hypothesis implies the identity

$$\frac{\det[\mathcal{B}_{1,m}^{1,m}]}{a_2 \dots a_{m-1}} = \Delta_{m-2}(a_2, \dots, a_{m-1}) \frac{\widehat{Z}_{d-1}(\mathbf{t} + [a_2] + \dots + [a_{m-1}])}{\widehat{Z}_{d-1}(\mathbf{t})}. \quad (54)$$

Combining (44), (52), and (54), and noting that

$$\frac{\Delta_{m-2}(a_2, \dots, a_{m-1})}{(a_1 - a_m) \Delta_{m-1}(a_1, \dots, a_{m-1}) \Delta_{m-1}(a_2, \dots, a_m)} = \frac{1}{\Delta_m(a_1, \dots, a_m)}, \quad (55)$$

we obtain (42) with $\mathbf{m} = m$. Hence the induction step is established and the proposition is proved. \square

Completion of proof of Theorem 1.1. For any subset $E \in \mathbb{R}$ and $s \in \mathbb{C}$, consider the new weight function $W_{E,s}(x) := W(x)(1 - s\chi_E(x))$. Let $Z_d^{E,s}(a_1, \dots, a_d)$ be the partition function (2) with W replaced by $W_{E,s}$. We also use a similar short-hand notation as (4). Then

$$\mathfrak{E}_d(a_1, \dots, a_m; E; s) = \frac{Z_d^{E,s}(a_1, \dots, a_m)}{Z_d(a_1, \dots, a_m)}. \quad (56)$$

Taking $\mathbf{t} = (0, 0, \dots)$ in (42) and recalling (17), we find

$$\frac{Z_d(a_1, \dots, a_m)}{Z_d} = \frac{1}{\Delta_m(a_1, \dots, a_m)} \det \left[a_k^{m-j} \frac{Z_{d+1-j}(a_k)}{Z_{d+1-j}} \right]_{j,k=1}^m. \quad (57)$$

Note that this holds for any weight function W . We substitute $W \mapsto W_{E,s}$ in (57) and divide this identity by (57) with W . From this we obtain

$$\bar{\mathfrak{E}}_d(a_1, \dots, a_m; E; s) = \frac{\det \left[a_k^{m-j} Z_{d+1-j}(a_k) \bar{\mathfrak{E}}_{d+1-j}(a_k; E; s) \right]_{j,k=1}^m}{\det \left[a_k^{m-j} Z_{d+1-j}(a_k) \right]_{j,k=1}^m}. \quad (58)$$

We now consider the terms $a_k^{m-j} Z_{d+1-j}(a_k)$. For any dimension l , if $a_l = a$ and $a_1 = \dots = a_{l-1} = 0$, applying l'Hôpital's rule to (3), we have a formula of the partition function $Z_l(a)$:

$$Z_l(a) = \frac{C_l}{a^{l-1} \prod_{j=0}^{l-2} j!} \int_{\mathbb{R}^l} \det[V] \det[\lambda_i^{j-1}] \prod_{j=1}^l W(\lambda_j) d\lambda_j, \quad (59)$$

where $V = (V_{ij})_{i,j=1}^l$, with $V_{ij} = \lambda_i^{j-1}$ for $j = 1, \dots, l-1$ and $V_{il} = e^{a\lambda_i}$. Let p_j be orthonormal polynomials with respect $W(x) dx$. By using elementary row operations, we get

$$Z_l(a) = \frac{C'_l}{a^{l-1}} \int_{\mathbb{R}^l} \det[\tilde{V}] \det[p_{j-1}(\lambda_i)] \prod_{j=1}^l W(\lambda_j) d\lambda_j, \quad (60)$$

where $\tilde{V} = (\tilde{V}_{ij})_{i,j=1}^l$, with $\tilde{V}_{ij} = p_{j-1}(\lambda_i)$ for $j = 1, \dots, l-1$ and $\tilde{V}_{il} = e^{a\lambda_i}$, and C'_l is a new constant which depends only on l and W . Using the Andréief's formula and the fact that p_j are orthonormal polynomials, we obtain

$$Z_l(a) = \frac{l! C'_l}{a^{l-1}} \int_{\mathbb{R}} e^{a\lambda} p_{l-1}(\lambda) W(\lambda) d\lambda = l! C'_k a^{-l+1} \Gamma_{l-1}(a). \quad (61)$$

Inserting this into (58), we obtain Theorem 1.1. \square

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