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Universality limits via "old style" analysis

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Techniques from "old style" orthogonal polynomials have turned out to be useful in establishing universality limits for fairly general measures. We survey some of these.

1. Introduction

We focus on the classical setting of random Hermitian matrices: consider a probability distribution $P^{(n)}$ on the space of n by n Hermitian matrices $M = (m_{ij})_{1 \le i,j \le n}$:

$$\begin{aligned} P^{(n)}(M) &= cw(M)dM \\ &= cw(M)\biggl(\prod_{j=1}^{n} dm_{jj}\biggr)\biggl(\prod_{j<k} d(\operatorname{Re} m_{jk})\, d(\operatorname{Im} m_{jk})\biggr). \end{aligned}$$

Here w is some nonnegative function defined on Hermitian matrices, and c is a normalizing constant. The most important case is

$$w(M) = \exp(-2n \operatorname{tr} Q(M)),$$

for appropriate functions Q. In particular, the choice $Q(M) = M^2$, leads to the Gaussian unitary ensemble (apart from scaling) that was considered by Wigner, in the context of scattering theory for heavy nuclei. When expressed in spectral form, that is as a probability density function on the eigenvalues $x_1 \le x_2 \le \cdots \le x_n$ of M, it takes the form

$$P^{(n)}(x_1, x_2, \ldots, x_n) = c\biggl(\prod_{j=1}^{n} w(x_j)\biggr)\biggl(\prod_{i<j}(x_i - x_j)^2\biggr). \tag{1-1}$$

See [Deift 1999, p. 102 ff.]. Again, c is a normalizing constant. Note that w now can be any nonnegative measurable function.

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In most applications, we want to let $n \to \infty$, and obviously the n-fold density complicates issues. So we often integrate out most variables, forming marginal distributions. One particularly important quantity is the m-point correlation function [Deift 1999, p. 112]:

$$\widetilde{R}_m(x_1, x_2, \dots, x_m) = \frac{n!}{(n-m)!} \int \cdots \int P^{(n)}(x_1, x_2, \dots, x_n)\, dx_{m+1}\, dx_{m+2} \dots dx_n.$$

Here typically, we fix m, and study $\widetilde{R}_m$ as $n \to \infty$. $\widetilde{R}_m$ is useful in examining spacing of eigenvalues, and counting the expected number of eigenvalues in some set. For example, if B is a measurable subset of $\mathbb{R}$,

$$\int_B \dots \int_B \widetilde{R}_m(x_1, x_2, \dots, x_m)\, dx_1\, dx_2 \dots dx_m$$

counts the expected number of m-tuples $(x_1, x_2, \dots, x_m)$ of eigenvalues with each $x_j \in B$.

The *universality limit in the bulk* asserts that for fixed $m \geq 2$, and ξ in the "bulk of the spectrum" (where w above "lives") and real $a_1, a_2, \dots, a_m$, we have

$$\lim_{n\to\infty} \frac{1}{(n\omega(\xi))^m} \widetilde{R}_m\left(\xi + \frac{a_1}{n\omega(\xi)}, \xi + \frac{a_2}{n\omega(\xi)}, \dots, \xi + \frac{a_m}{n\omega(\xi)}\right) = \det(\mathbb{S}(a_i - a_j))_{1 \leq i,j \leq m}. \quad (1\text{-}2)$$

Here $\mathbb{S}$ is the *sine* or *sinc* kernel, given by

$$\mathbb{S}(t) = \frac{\sin \pi t}{\pi t}, \quad t \neq 0, \quad (1\text{-}3)$$

and $\mathbb{S}(0) = 1$. What is ω? It is basically an *equilibrium density function*, and we'll discuss this further later. It is appropriate to call the limit (1-2) universal, as it does not depend on ξ, nor on the weight function w.

One of the principal goals has been to establish the universality limit under more and more general conditions, and in this pursuit, orthogonal polynomials have turned out to be a useful tool. Throughout this paper, let μ be a finite positive Borel measure with compact support J and infinitely many points in the support. Define orthonormal polynomials

$$p_n(x) = \gamma_n x^n + \cdots, \quad \gamma_n > 0,$$

$n = 0, 1, 2, \dots$, satisfying the orthonormality conditions

$$\int_J p_j\, p_k\, d\mu = \delta_{jk}.$$

We may think of w in (1-1) as μ'. The n-th reproducing kernel for μ is

$$K_n(\mu, x, y) = \sum_{k=0}^{n-1} p_k(x) p_k(y),$$

and the normalized kernel is

$$\widetilde{K}_n(\mu, x, y) = \mu'(x)^{1/2} \mu'(y)^{1/2} K_n(\mu, x, y).$$

K_n satisfies the very useful extremal property [Freud 1971; Nevai 1979; 1986; Simon 2011]

$$K_n(\mu, \xi, \xi) = \inf_{\deg(P) \le n-1} \frac{P^2(\xi)}{\int P^2 \, d\mu}. \tag{1-4}$$

When $w = \mu'$, there are the remarkable formulae for the probability distribution $P^{(n)}$ [Deift 1999, p.112]:

$$P^{(n)}(x_1, x_2, \dots, x_n) = \frac{1}{n!} \det(\widetilde{K}_n(\mu, x_i, x_j))_{1 \le i,j \le n}, \tag{1-5}$$

and the m-point correlation function:

$$\widetilde{R}_m(x_1, x_2, \dots, x_m) = \det(\widetilde{K}_n(\mu, x_i, x_j))_{1 \le i,j \le m}. \tag{1-6}$$

Sometimes we shall find it easier to exclude the measure from the variables $x_1, x_2, \dots, x_m$, that is we consider the “stripped” m-point correlation function,

$$R_m(x_1, x_2, \dots, x_m) = \det(K_n(\mu, x_i, x_j))_{1 \le i,j \le m}. \tag{1-7}$$

Because $\widetilde{R}_m$ is the determinant of a fixed size m by m matrix, we see that (1-2) reduces to

$$\lim_{n \to \infty} \frac{\widetilde{K}_n\left(\mu, \xi + \frac{a}{n\omega(\xi)}, \xi + \frac{b}{n\omega(\xi)}\right)}{n\omega(\xi)} = \mathbb{S}(a-b), \tag{1-8}$$

for real a, b.

Let us now turn to the choice of ω. As above, suppose that μ has compact support J. Then, ν_J is the probability measure that minimizes the energy integral

$$\iint \log \frac{1}{|x-y|} \, d\nu(x) \, d\nu(y),$$

taken over all probability measures ν on J. It is called the *equilibrium measure* for the set J. It is absolutely continuous in any subinterval of J. Throughout this paper, we set $\omega(x) = \nu_J'(x), x \in J^0$, where J^0 is the interior of J. We call

ω the *equilibrium density* of J and note that $\omega > 0$ in J^0. For example, when $J = [-1, 1]$,

$$\omega(x) = \frac{1}{\pi\sqrt{1-x^2}}, \quad x \in (-1, 1).$$

Of course, the primary interest in random matrix theory is for varying measures, where at the n-th stage, $\mu'(x) = e^{-2nQ(x)}$, and there ω is an equilibrium density associated with the external field Q.

In some formulations for measures with fixed support, it is easier to prove the limit

$$\lim_{n\to\infty} \frac{\widetilde{K}_n\left(\mu, \xi + \dfrac{a}{\widetilde{K}_n(\mu,\xi,\xi)}, \xi + \dfrac{b}{\widetilde{K}_n(\mu,\xi,\xi)}\right)}{\widetilde{K}_n(\mu,\xi,\xi)} = \mathbb{S}(a-b), \tag{1-9}$$

and this is consistent with (1-8), since under quite general conditions,

$$\lim_{n\to\infty} \frac{1}{n}\widetilde{K}_n(\mu,\xi,\xi) = \lim_{n\to\infty} \frac{1}{n}\mu'(\xi) K_n(\mu,\xi,\xi) = \omega(\xi).$$

The most obvious approach to proving (1-2) is to use the Christoffel–Darboux formula,

$$K_n(\mu, u, v) = \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(u)p_{n-1}(v) - p_{n-1}(u)p_n(v)}{u-v}, \quad u \neq v, \tag{1-10}$$

and to substitute in asymptotics for p_n as $n \to \infty$. This is what effectively was done for the classical weights. Of course there are many approaches, and we cannot survey them here. We simply note that it was the Riemann–Hilbert approach that allowed dramatic breakthroughs, and refer to other papers in this proceedings, and the books [Baik et al. 2007; Baik et al. 2008; Bleher and Its 2001; Deift 1999; Deift and Gioev 2009; Forrester 2010; Mehta 2004].

In terms of "old style" orthogonal polynomials, it was Eli Levin [Levin and Lubinsky 2009] who realized that relatively weak pointwise asymptotics, such as

$$p_n(\cos\theta) = \cos n\theta + o(1), \quad n \to \infty,$$

combined with a Markov–Bernstein inequality, are sufficient for universality. However, it has since been realized that much less suffices.

In subsequent sections, we outline some approaches from classical orthogonal polynomials and complex analysis. In Section 2, it is a comparison method. In Section 3, it is a method based on the theory of entire functions of exponential type. In Section 4, we discuss a recent extremal property. This survey has a narrow focus, and we omit many important contributions and topics.

2. A comparison method

The philosophy behind the comparison method is that a lot of quantities in orthogonal polynomials have a strong local component, and a weak global one. Perhaps the primary example of this is the Christoffel function $\lambda_n(\mu, x)$, or its reciprocal, the reproducing kernel along the diagonal $K_n(\mu, x, x)$. The global component in its asymptotic is determined by the equilibrium density ω of the support of μ, often accompanied by the hypothesis of regularity: we say that a compactly supported measure μ is regular (in the sense of Stahl, Totik, Ullmann) if the leading coefficients $\{\gamma_n\}$ of the orthonormal polynomials satisfy

$$\lim_{n\to\infty} \gamma_n^{1/n} = \frac{1}{\operatorname{cap}(\operatorname{supp}[\mu])}.$$

Here cap denotes the logarithmic capacity of the support of μ (see [Ransford 1995; Saff and Totik 1997; Stahl and Totik 1992] for definitions). A simple sufficient criterion for regularity is that of Erdős–Turán: if supp$[\mu]$ consists of finitely many intervals, and $\mu' > 0$ a.e. in each of those intervals, then μ is regular. There are more general criteria in [Stahl and Totik 1992]. Note that pure jump measures and pure singularly continuous measures can be regular.

The archetypal asymptotic for K_n is due to Maté, Nevai, and Totik [Máté et al. 1991] for $[-1, 1]$, and for general support, due to Totik [2000a]:

Theorem 2.1. *Let μ have compact support J and be regular. Let ω be the equilibrium density of J.*

(a) *For a.e. $x \in J^0$, we have*

$$\liminf_{n\to\infty} \frac{1}{n} K_n(\mu, x, x) \geq \frac{\omega(x)}{\mu'(x)}.$$

(b) *If in addition, I is a subinterval of J satisfying*

$$\int_I \log \mu' > -\infty, \tag{2-1}$$

then for a.e. $x \in I$,

$$\lim_{n\to\infty} \frac{1}{n} K_n(\mu, x, x) = \frac{\omega(x)}{\mu'(x)}. \tag{2-2}$$

Why is this local in flavor? Well if two measures μ and ν have the same support, and they are equal when restricted to the interval I, then $K_n(\mu, x, x)$ and $K_n(\nu, x, x)$ have the same asymptotic in I. In fact, more is possible: using fast decreasing polynomials, and the extremal property (1-4), one can prove that the ratio $K_n(\mu, x, x)/K_n(\nu, x, x)$ has limit 1 under much weaker conditions than in (b).

What relevance does this have to universality limits? The answer lies in the following inequality: if $\mu \le \nu$, then for all real x, y,

$$\frac{|K_n(\mu,x,y)-K_n(\nu,x,y)|}{K_n(\mu,x,x)} \le \left(\frac{K_n(\mu,y,y)}{K_n(\mu,x,x)}\right)^{\frac{1}{2}}\left[1-\frac{K_n(\nu,x,x)}{K_n(\mu,x,x)}\right]^{\frac{1}{2}}. \quad (2\text{-}3)$$

In particular, if x and y vary with n, and as $n \to \infty$, $\frac{K_n(\nu,x,x)}{K_n(\mu,x,x)}$ has limit 1, while $\frac{K_n(\mu,y,y)}{K_n(\mu,x,x)}$ remains bounded, then $K_n(\mu,x,y)$ and $K_n(\nu,x,y)$ have the same asymptotic. This inequality is easily proven by using the reproducing kernel properties of K_n, and the extremal property (1-4). It enables us to use universality limits for a larger "nice" measure ν to obtain the same for a "not so nice" measure μ, which is locally the same as ν. Thus [Lubinsky 2009a, Theorem 1.1, pp. 916–917]:

Theorem 2.2. *Let μ have support* $[-1,1]$ *and be regular. Let $\xi \in (-1,1)$ and assume μ is absolutely continuous in an open set containing ξ. Assume moreover, that μ' is positive and continuous at ξ. Then uniformly for a, b in compact subsets of the real line, we have*

$$\lim_{n\to\infty} \frac{\widetilde{K}_n\left(\xi+\frac{a\pi\sqrt{1-\xi^2}}{n},\, \xi+\frac{b\pi\sqrt{1-\xi^2}}{n}\right)}{\widetilde{K}_n(\xi,\xi)} = \mathbb{S}(a-b).$$

Weaker integral forms of this limit were also established in [Lubinsky 2009a], when continuity of μ' was replaced by upper and lower bounds. However, the real potential of the inequality (2-3) was soon explored by Findley [2008], Simon [2008b] and Totik [2009]. It was Findley who replaced continuity of μ' by the Szegő condition on $[-1,1]$. Totik used the method of "polynomial pullbacks", which is based on the observation that if P is a polynomial, then $P^{[-1]}[-1,1]$ consists of finitely many intervals. This allows one to pass from asymptotics for $[-1,1]$ to finitely many intervals. In turn, one can use the latter to approximate arbitrary compact sets. Barry Simon used instead Jost functions. Here is Totik's result:

Theorem 2.3. *Let μ have compact support J and be regular. Let I be a subinterval of J in which the local Szegő condition* (2-1) *holds. Then, for a.e. $x \in I$, and all real a, b,*

$$\lim_{n\to\infty} \frac{\widetilde{K}_n\left(\mu,\xi+\frac{a}{n\omega(\xi)},\xi+\frac{b}{n\omega(\xi)}\right)}{\widetilde{K}_n(\mu,\xi,\xi)} = \frac{\sin\pi(a-b)}{\pi(a-b)}.$$

Totik actually showed that the asymptotic holds at any given ξ which is a Lebesgue point of both measure μ, and its local Szegő function. The comparison approach has also been applied to universality on the unit circle [Levin and Lubinsky 2007], to exponential weights [Levin and Lubinsky 2009], at the hard edge of the spectrum [Lubinsky 2008b], to Bergman polynomials [Lubinsky 2010], and in a generalized setting [Lubinsky 2008a].

3. A normal families approach

One pitfall of the comparison inequality, is that it needs a "starting" measure for which universality is known. For general supports, there is no such measure, unless one assumes regularity — which is a global restriction, albeit a weak one. In [Lubinsky 2008d], a method was introduced, that avoids this. It uses basic tools of complex analysis and complex approximation, such as normal families, together with some of the theory of entire functions, and reproducing kernels.

Perhaps the most fundamental idea in this approach is the notion that since K_n is a reproducing kernel for polynomials of degree $\leq n-1$, any scaled asymptotic limit of it must also be a reproducing kernel for a suitable space. It turns out that the correct limit setting is Paley–Wiener space. For given $\sigma > 0$, this is the Hilbert space of entire functions g of exponential type at most $\sigma > 0$, (so that given $\varepsilon > 0$, $|g(z)| = O(e^{(\sigma+\varepsilon)|z|})$, for large $|z|$), whose restriction to the real line is in $L_2(\mathbb{R})$, with the usual $L_2(\mathbb{R})$ inner product. Here the sinc kernel is the reproducing kernel [Stenger 1993, p. 95]:

$$g(x) = \int_{-\infty}^{\infty} g(t) \frac{\sin \sigma(x-t)}{\pi(x-t)}\, dt, \quad x \in \mathbb{R}. \tag{3-1}$$

It is not a trivial exercise to rigorously prove that reproducing kernels for polynomials turn into the reproducing kernel for Paley–Wiener space.

Assume that μ has compact support and that μ' is bounded above and below by positive constants in some open interval O containing the closed interval I. Then it is well known that for some $C_1, C_2 > 0$,

$$C_1 \leq \frac{1}{n} K_n(\mu, x, x) \leq C_2, \tag{3-2}$$

in any proper open subset O_1 of O. Indeed, this follows by comparing λ_n below to the Christoffel function of the weight 1 on a suitable subinterval of O, and comparing it above to a suitable dominating measure. Cauchy–Schwarz inequality's then gives

$$\frac{1}{n} |K_n(\mu, \xi, t)| \leq C \text{ for } \xi, \quad t \in O_1. \tag{3-3}$$

We can extend this estimate into the complex plane, by adapting Bernstein's inequality,

$$|P(z)| \leq |z + \sqrt{z^2-1}|^n \, \|P\|_{L_\infty[-1,1]},$$

which is valid for polynomials of degree $\le n$ and all complex z. The branch of $\sqrt{\ }$ is taken so that $\sqrt{z^2-1} > 0$ for $z \in (1,\infty)$. This leads to

$$\left|\frac{1}{n}K_n\left(\xi+\frac{a}{n},\xi+\frac{b}{n}\right)\right| \le C_1 e^{C_2(|\operatorname{Im} a|+|\operatorname{Im} b|)}.$$

Here C_1 and C_2 are independent of n, a and b. In view of (3-2), the same is true of $\{f_n(a,b)\}_{n=1}^{\infty}$, where

$$f_n(a,b) = \frac{K_n\left(\xi+\dfrac{a}{\widetilde{K}_n(\xi,\xi)},\xi+\dfrac{b}{\widetilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)}.$$

Thus, given $A > 0$, we have for $n \ge n_0(A)$ and $|a|, |b| \le A$, that

$$|f_n(a,b)| \le C_1 e^{C_2(|\operatorname{Im} a|+|\operatorname{Im} b|)}. \tag{3-4}$$

We emphasize that C_1 and C_2 are independent of n, A, a and b.

Let $f(a,b)$ be the limit of some subsequence $\{f_n(\cdot,\cdot)\}_{n\in\mathcal{S}}$ of $\{f_n(\cdot,\cdot)\}_{n=1}^{\infty}$. It is an entire function in a, b, but (3-4) shows even more: namely that for all complex a, b,

$$|f(a,b)| \le C_1 e^{C_2(|\operatorname{Im} a|+|\operatorname{Im} b|)}. \tag{3-5}$$

So f is bounded for $a, b \in \mathbb{R}$, and is an entire function of exponential type in each variable. Our goal is to show that

$$f(a,b) = \frac{\sin \pi(a-b)}{\pi(a-b)}. \tag{3-6}$$

So we study the properties of f. The main tool is to take elementary properties of the reproducing kernel K_n, such as properties of its zeros, and then after scaling and taking limits, to analyze the zeros of f, and related quantities. At the end, armed with a range of properties, one proves that these characterize the sinc kernel, and (3-6) follows.

The first result of this type was given in [Lubinsky 2008d]:

Theorem 3.1. *Let μ have compact support J. Let I be compact, and μ be absolutely continuous in an open set containing I. Assume that μ' is positive and continuous at each point of I. The following are equivalent*:

(I) *Uniformly for $\xi \in I$ and a in compact subsets of the real line*,

$$\lim_{n\to\infty} \frac{K_n\left(\xi+\frac{a}{n},\xi+\frac{a}{n}\right)}{K_n(\xi,\xi)} = 1. \tag{3-7}$$

(II) *Uniformly for* $\xi \in I$ *and* a, b *in compact subsets of the complex plane*, *we have*

$$\lim_{n\to\infty} \frac{K_n\left(\xi + \frac{a}{\widetilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\widetilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)} = \frac{\sin \pi(a-b)}{\pi(a-b)}. \tag{3-8}$$

One can weaken the condition of continuity of μ' to upper and lower bounds and then require ξ to be a Lebesgue point of μ, that is, we assume only

$$\lim_{h,k\to 0+} \frac{\mu([\xi-h, \xi+k])}{k+h} = \mu'(\xi).$$

The clear advantage of the theorem is that there is no global restriction on μ. The downside is that we still have to establish the ratio asymptotic (3-7) for the Christoffel functions/ reproducing kernels, and to date, these have only been established in the stronger form (2-2).

Nevertheless, the method itself has far more promise than the comparison inequality. For varying exponential weights (the “natural” setting for universality limits), it yielded [Levin and Lubinsky 2008b] universality very generally in the bulk, see below. It has also been used at the hard edge of the spectrum in [Lubinsky 2008c], at the soft edge of the spectrum [Levin and Lubinsky 2011], and to Cantor sets with positive measure by Avila, Last and Simon [Avila et al. 2010], as well as for orthogonal rational functions [Deckers and Lubinsky 2012]. Totik has observed that it yields an easier path to his Theorem 2.3 [Totik 2011].

With much more effort, and in particular a new uniqueness theorem for the sinc kernel, this set of methods also yields *universality in measure*, for arbitrary measures μ with compact support [Lubinsky 2012a]:

Theorem 3.2. *Let* μ *have compact support. Let* $\varepsilon > 0$ *and* $r > 0$. *The* (*linear Lebesgue*) *measure of the set of* ξ *satisfying* $\mu'(\xi) > 0$ *and*

$$\sup_{|u|,|v|\le r} \left| \frac{K_n\left(\xi + \frac{u}{\widetilde{K}_n(\xi,\xi)}, \xi + \frac{v}{\widetilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)} - \frac{\sin \pi(u-v)}{\pi(u-v)} \right| \ge \varepsilon$$

tends to 0 *as* $n \to \infty$.

(In the supremum, u, v are complex variables.) Because convergence in measure implies convergence a.e. of subsequences, one obtains pointwise a.e. universality for subsequences, without any local or global assumptions on μ.

Another development involves pointwise universality in the mean [Lubinsky 2012b], under some local conditions. Like all the results of the section, the essential feature is the lack of global regularity assumptions:

Theorem 3.3. *Let μ have compact support. Assume that I is an open interval in which for some $C > 0$, $\mu' \geq C$ a.e. in I. Let $\xi \in I$ be a Lebesgue point of μ. Then, for each $r > 0$,*

$$\lim_{m\to\infty} \frac{1}{m} \sum_{n=1}^{m} \sup_{|u|,|v|\leq r} \left| \frac{K_n\left(\xi + \frac{u}{\widetilde{K}_n(\xi,\xi)}, \xi + \frac{v}{\widetilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)} - \frac{\sin \pi(u-v)}{\pi(u-v)} \right| = 0.$$

In particular, this holds for a.e. $\xi \in I$.

Pointwise universality at a given point ξ seems to usually require at least something like μ' being continuous at ξ, or ξ being a Lebesgue point of μ. Indeed, when μ' has a jump discontinuity, the universality limit is different from the sine kernel [Foulquié Moreno et al. 2011], and involves de Branges spaces [Lubinsky 2009b]. It is noteworthy, though, that pure singularly continuous measures can exhibit sine kernel behavior [Breuer 2011].

From a mainstream random matrix point of view, the most impressive application of the normal families method is to exponential weights $W(x) = \exp(-Q(x))$, defined on a closed set Σ on the real line. If Σ is unbounded, we assume that

$$\lim_{|x|\to\infty, x\in\Sigma} W(x)|x| = 0. \tag{3-9}$$

Associated with Σ and Q, we may consider the extremal problem

$$\inf_{\nu}\left(\iint \log \frac{1}{|x-t|}\, d\nu(x)\, d\nu(t) + 2 \int Q\, d\nu \right),$$

where the inf is taken over all positive Borel measures ν with support in Σ and $\nu(\Sigma) = 1$. The inf is attained by a unique equilibrium measure ν_Q, characterized by the following conditions: let

$$V^{\nu_Q}(z) = \int \log \frac{1}{|z-t|}\, d\nu_Q(t)$$

denote the potential for ν_Q. Then

$$V^{\nu_Q} + Q \geq F_Q \text{ on } \Sigma;$$
$$V^{\nu_Q} + Q = F_Q \text{ in } \operatorname{supp}[\nu_Q].$$

Here the number F_Q is a constant. Using asymptotics for Christoffel functions obtained in [Totik 2000b], Eli Levin and I proved this:

Theorem 3.4 [Levin and Lubinsky 2008b, Theorem 1.1, p. 747]. *Let $W = e^{-Q}$ be a continuous nonnegative function on the set Σ, which is assumed to consist* *of at most finitely many intervals. If* Σ *is unbounded, we assume also* (3-9)*. Let* h *be a bounded positive continuous function on* Σ*, and for* $n \geq 1$*, let*

$$d\mu_n(x) = (hW^{2n})(x)\,dx. \tag{3-10}$$

Moreover, let $\widetilde{K}_n$ *denote the normalized n-th reproducing kernel for* μ_n*.*

Let I *be a closed interval lying in the interior of* $\operatorname{supp}[\nu_Q]$*. Assume that* ν_Q *is absolutely continuous in a neighborhood of* I*, and that* ν'_Q *and* Q' *are continuous in that neighborhood, while* $\nu'_Q > 0$ *there. Then uniformly for* $\xi \in I$*, and* a, b *in compact subsets of the real line, we have* (1-9).

In particular, when Q' satisfies a Lipschitz condition of some positive order in a neighborhood of I, then [Saff and Totik 1997, p. 216] ν'_Q is continuous there, and hence we obtain universality except near zeros of ν'_Q. Note too that when Q is convex in Σ, or $xQ'(x)$ is increasing there, then the support of ν_Q consists of at most finitely many intervals, with at most one interval per component of Σ [Saff and Totik 1997, p. 199].

4. A variational principle

The methods above intrinsically involve asymptotics for a single reproducing kernel, from which one can pass to the asymptotic for the general m-point correlation function. Remarkably (see [Lubinsky 2013]), there is a variational principle for the m-point correlation function R_m, for arbitrary measures μ, that generalizes the extremal property (1-4) of reproducing kernels, and allows one to investigate general m.

Its formulation involves $\mathcal{AL}_n^m$, the alternating polynomials of degree at most n in m variables. We say that $P \in \mathcal{AL}_n^m$ if

$$P(x_1, x_2, \ldots, x_m) = \sum_{0 \leq j_1, j_2, \ldots, j_m \leq n} c_{j_1 j_2 \ldots j_m} x_1^{j_1} x_2^{j_2} \ldots x_m^{j_m}, \tag{4-1}$$

so that P is a polynomial of degree $\leq n$ in each of its m variables, and in addition is *alternating*, so that for every pair (i, j) with $1 \leq i < j \leq m$,

$$P(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_m) = -P(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_m). \tag{4-2}$$

Thus swapping variables changes the sign.

Observe that if R_i is a univariate polynomial of degree $\leq n$ for each $i = 1, 2, \ldots, m$, then $P(t_1, t_2, \ldots, t_m) = \det[R_i(t_j)]_{1 \leq i,j \leq m} \in \mathcal{AL}_n^m$. Given a fixed m, we shall use the notation

$$\underline{x} = (x_1, x_2, \ldots, x_m), \quad \underline{t} = (t_1, t_2, \ldots, t_m),$$

while $\mu^{\times m}$ denotes the m-fold Cartesian product of μ, so that

$$d\mu^{\times m}(\underline{t}) = d\mu(t_1)\, d\mu(t_2) \dots d\mu(t_m).$$

Theorem 4.1 [Lubinsky 2013].

$$\det[K_n(\mu, x_i, x_j)]_{1\le i,j\le m} = m!\,\sup_{P\in\mathcal{AL}_{n-1}^m} \frac{(P(\underline{x}))^2}{\int (P(\underline{t}))^2\, d\mu^{\times m}(\underline{t})}. \tag{4-3}$$

The supremum is attained for

$$P(\underline{t}) = \det[K_n(\mu, x_i, t_j)]_{1\le i,j\le m}.$$

Here is an immediate consequence:

Corollary 4.2. $R_m^n(x_1, x_2, \dots, x_m)$ *is a monotone decreasing function of* μ, *and a monotone increasing function of* n.

The proof of Theorem 4.1 is based on multivariate orthogonal polynomials built from μ. Given $m \ge 1$, and nonnegative integers $j_1, j_2, \dots, j_m$, define

$$T_{j_1,j_2,\dots,j_m}(x_1, x_2, \dots, x_m) = \det(p_{j_i}(x_k))_{1\le i,k\le m}.$$

It is easily see that if $0 \le j_1 < j_2 < \dots < j_m$ and $0 \le k_1 < k_2 < \dots < k_m$, then

$$\int T_{j_1,j_2,\dots,j_m}(\underline{t}) T_{k_1,k_2,\dots,k_m}(\underline{t}) d\mu^{\times m}(\underline{t}) = m!\,\delta_{j_1k_1}\delta_{j_2k_2}\dots\delta_{j_mk_m}.$$

Define an associated reproducing kernel,

$$K_n^m(\underline{x}, \underline{t}) = \frac{1}{m!} \sum_{1\le j_1<j_2<\dots<j_m\le n} T_{j_1,j_2,\dots,j_m}(\underline{x}) T_{j_1,j_2,\dots,j_m}(\underline{t}).$$

Theorem 4.1 follows easily from the reproducing kernel relation

$$P(\underline{x}) = \int P(\underline{t}) K_n^m(\underline{x}, \underline{t})\, d\mu^{\times m}(\underline{t}), \quad P \in \mathcal{AL}_{n-1}^m,\ \underline{x} \in \mathbb{R}^n,$$

and the Cauchy–Schwarz inequality.

Just as the extremal property (1-4) for $K_n(\mu, x, x)$ is the main idea in proving Theorem 2.1, so we can use Theorem 4.1 to prove [Lubinsky 2013, Theorem 2.1]:

Theorem 4.3. *Let* μ *have compact support* J. *Let* $m \ge 1$.

(a) *For Lebesgue a.e.* $(x_1, x_2, \dots, x_m) \in (J^0)^m$,

$$\liminf_{n\to\infty} \frac{1}{n^m} \det[K_n(\mu, x_i, x_j)]_{1\le i,j\le m} \ge \prod_{j=1}^m \frac{\omega(x_j)}{\mu'(x_j)}.$$

The right-hand side is interpreted as ∞ *if any* $\mu'(x_j) = 0$.

(b) *Suppose that I is a compact subinterval of J, for which* (2-1) *holds. Then for Lebesgue a.e.* $(x_1, x_2, \dots, x_m) \in I^m$,

$$\limsup_{m\to\infty} \frac{1}{n^m} \det[K_n(\mu, x_i, x_j)]_{1\le i,j\le m} \le \prod_{j=1}^{m} \frac{\omega_\mu(x_j)}{\mu'(x_j)},$$

where, if ω_L denotes the equilibrium density for the compact set L,

$$\omega_\mu(x) = \inf\bigl\{\omega_L(x) : L \subset J \text{ is compact, } \mu_{|L} \text{ is regular, } x \in L\bigr\}.$$

A more impressive consequence is pointwise, almost everywhere, one-sided universality, without any local or global restrictions on μ [Lubinsky 2013, Theorem 2.2]:

Theorem 4.4. *Let μ have compact support J. Let $m \ge 1$.*

(a) *For a.e.* $x \in J^0 \cap \{\mu' > 0\}$, *and for all real* $a_1, a_2, \dots, a_m$,

$$\liminf_{n\to\infty}\left(\frac{\mu'(x)}{n\omega(x)}\right)^m R_m^n\left(x + \frac{a_1}{n\omega(x)}, \dots, x + \frac{a_m}{n\omega(x)}\right) \ge \det(\mathbb{S}(a_i - a_j))_{1\le i,j\le m}.$$

(b) *Suppose that I is a compact subinterval of J, for which* (2-1) *holds. Then for a.e.* $x \in I$, *and for all real* $a_1, a_2, \dots, a_m$,

$$\limsup_{n\to\infty}\left(\frac{\mu'(x)}{n\omega_\mu(x)}\right)^m R_m^n\left(x + \frac{a_1}{n\omega_\mu(x)}, \dots, x + \frac{a_m}{n\omega_\mu(x)}\right) \le \det(\mathbb{S}(a_i - a_j))_{1\le i,j\le m}.$$

In closing, we note that the study of universality limits has greatly enriched the asymptotics of orthogonal polynomials. A prime example of this is asymptotics for spacing of zeros [Levin and Lubinsky 2008a; 2010; Simon 2005a; 2005b; 2008a; 2011].

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