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Whittaker functions and related stochastic processes

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We review some recent results on connections between Brownian motion, Whittaker functions, random matrices and representation theory.

1. Harish-Chandra formula, Duistermaat–Heckman measure and Gelfand–Tsetlin patterns

Define $J_\lambda(x) = h(\lambda)^{-1}\det(e^{\lambda_i x_j})$, where $h(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j)$. For each x, $J_\lambda(x)$ is an analytic function of λ; in particular,

$$J_0(x) = \left(\prod_{j=1}^{n-1} j!\right)^{-1} h(x).$$

The functions $J_\lambda(x)$ play a central role in random matrix theory. For example, if Λ and X are Hermitian matrices with eigenvalues given by λ and x, respectively, then

$$\int_{U(n)} e^{\operatorname{tr}\Lambda U X U^*} dU = \frac{J_\lambda(x)}{J_0(x)}, \tag{1}$$

where the integral is with respect to normalised Haar measure on the unitary group. This is known as the Harish-Chandra, or Itzykson–Zuber, formula.

Let $\beta = (\beta_t,\ t \geq 0)$ be a standard Brownian motion in $\mathbb{R}^n$ with drift λ. Denote by $\mathbb{P}_x$ the law of β started at x and by $\mathbb{E}_x$ the corresponding expectation. Set

$$\Omega = \{x \in \mathbb{R}^n :\ x_1 > x_2 > \cdots > x_n\}, \qquad T = \inf\{t > 0 :\ \beta_t \notin \Omega\}.$$

For $\lambda, x \in \mathbb{R}^n$, write $\lambda(x) = \sum_i \lambda_i x_i$.

Proposition 1. *For $x, \lambda \in \Omega$, $J_\lambda(x) = h(\lambda)^{-1} e^{\lambda(x)} \mathbb{P}_x(T = \infty)$.*

Proof. This is well known; see for example [Biane et al. 2005]. The function $u(x) = \mathbb{P}_x(T = \infty)$, $x \in \Omega$, satisfies $\frac{1}{2}\Delta u + \lambda \cdot \nabla u = 0$, vanishes on the boundary of Ω and $\lim_{x\to\infty} u(x) = 1$. Here we write $x \to \infty$ to mean $x_i - x_{i+1} \to \infty$ for $i = 1, \dots, n-1$. Hence $v(x) = e^{\lambda(x)}u(x)$ satisfies $\Delta v = \sum_i \lambda_i^2 v$, vanishes on

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the boundary of Ω and $\lim_{x\to\infty} e^{-\lambda(x)} v(x) = 1$. The function $\det(e^{\lambda_i x_j})$ also has these properties, so by uniqueness, $v(x) = \det(e^{\lambda_i x_j})$, as required. $\square$

The Harish-Chandra formula has the following interpretation. Pick U at random according to the normalised Haar measure on $U(n)$ and let $\mu^x(dy)$ denote the law of the diagonal of the random matrix UXU^*. Then the integral becomes

$$\int_{U(n)} e^{\operatorname{tr}\,\Lambda U X U^*} dU = \int_{\mathbb{R}^n} e^{\lambda(y)} \mu^x(dy).$$

Setting $m^x(dy) = J_0(x)\mu^x(dy)$, we obtain

$$\int_{\mathbb{R}^n} e^{\lambda(y)} m^x(dy) = J_\lambda(x).$$

The measure m^x is known as the Duistermaat–Heckman measure associated with the point $x \in \Omega$. It has the following properties, which are well-known. The symmetric group S_n acts naturally on $\mathbb{R}^n$ by permuting coordinates. The support of the measure m^x is the convex hull of the set of images of x under the action of S_n. It has a piecewise polynomial density. This comes from the fact, which we will now explain, that the Duistermaat–Heckman measure is the push-forward via an affine map of the Lebesgue measure on a higher dimensional polytope known as the Gelfand–Tsetlin polytope.

Let $x \in \Omega$ and denote by $GT(x)$ the polytope of Gelfand–Tsetlin patterns with bottom row equal to x:

$$GT(x) = \{P_{k,j},\ 1 \le j \le k \le n : P_{k,j+1} \le P_{k-1,j} \le P_{k,j},\ 1 \le j < k \le n,\ P_{n,\cdot} = x\}.$$

Define the *type* of a pattern P to be the vector

$$\text{type } P = \left(P_{1,1},\, P_{2,1} + P_{2,2} - P_{1,1}, \ldots, \sum_{j=1}^{n} P_{n,j} - \sum_{j=1}^{n-1} P_{n-1,j} \right). \tag{2}$$

Consider the map from $U(n)$ to $GT(x)$ defined by $U \mapsto P$ where: for each $1 \le k \le n$, $P_{k,\cdot}$ is the vector of eigenvalues of the k-th principal minor of UXU^*. It is well-known (see, for example, [Baryshnikov 2001] or [Alexeev and Brion 2004, Section 5.6] for a more general statement) that the push-forward of Haar measure under this map is the standard Euclidean measure on the polytope $GT(x)$. Moreover, the diagonal of the matrix UXU^* is equal to the type of the pattern P. From this we obtain another integral representation for the function J_λ as

$$J_\lambda(x) = \int_{GT(x)} e^{\lambda \cdot \text{type } P}\, dP. \tag{3}$$

2. Whittaker functions

Set $H = \Delta - 2\sum_{i=1}^{n-1} e^{-\alpha_i(x)}$, where $\alpha_i = e_i - e_{i+1}$, $i = 1, \ldots, n-1$. Write $H = H^{(n)}$ for the moment; we will drop the superscript again later, whenever it is unnecessary. For convenience we define $H^{(1)} = d^2/dx^2$ and $\psi_\lambda^{(1)}(x) = e^{\lambda x}$. Following [Gerasimov et al. 2006], for $n \geq 2$ and $\theta \in \mathbb{C}$, define a kernel on $\mathbb{R}^n \times \mathbb{R}^{n-1}$ by

$$Q_\theta^{(n)}(x, y) = \exp\biggl(\theta\biggl(\sum_{i=1}^n x_i - \sum_{i=1}^{n-1} y_i\biggr) - \sum_{i=1}^{n-1}\bigl(e^{y_i - x_i} + e^{x_{i+1} - y_i}\bigr)\biggr).$$

Denote the corresponding integral operator by $\mathcal{Q}_\theta^{(n)}$, defined on a suitable class of functions by

$$\mathcal{Q}_\theta^{(n)} f(x) = \int_{\mathbb{R}^{n-1}} Q_\theta^{(n)}(x, y) f(y)\, dy.$$

The Whittaker functions $\psi_\lambda^{(n)}$, $\lambda \in \mathbb{C}^n$ are defined recursively by

$$\psi_{\lambda_1, \ldots, \lambda_n}^{(n)} = \mathcal{Q}_{\lambda_n}^{(n)} \psi_{\lambda_1, \ldots, \lambda_{n-1}}^{(n-1)}. \tag{4}$$

As observed in [Gerasimov et al. 2006], the following intertwining relation holds:

$$(H^{(n)} - \theta^2) \circ \mathcal{Q}_\theta^{(n)} = \mathcal{Q}_\theta^{(n)} \circ H^{(n-1)}. \tag{5}$$

This follows from the identity $(H_x^{(n)} - \theta^2) Q_\theta^{(n)}(x, y) = H_y^{(n-1)} Q_\theta^{(n)}(x, y)$, which is readily verified. Combining (4) with the intertwining relation (5) yields the eigenvalue equation:

$$H^{(n)} \psi_\lambda^{(n)} = \biggl(\sum_{i=1}^n \lambda_i^2\biggr) \psi_\lambda^{(n)}. \tag{6}$$

Let us now drop the superscripts and write $H = H^{(n)}$, $\psi_\lambda = \psi_\lambda^{(n)}$. Iterating (4) gives the following integral formula, due to Givental [1997] (see also [Joe and Kim 2003; Gerasimov et al. 2006]):

$$\psi_\lambda(x) = \int_{\Gamma(x)} e^{\mathcal{F}_\lambda(T)} \prod_{k=1}^{n-1} \prod_{i=1}^{k} dT_{k,i}, \tag{7}$$

where $\Gamma(x)$ denotes the set of real triangular arrays $(T_{k,i},\ 1 \leq i \leq k \leq n)$ with $T_{n,i} = x_i$, $1 \leq i \leq n$, and

$$\mathcal{F}_\lambda(T) = \sum_{k=1}^n \lambda_k \biggl(\sum_{i=1}^k T_{k,i} - \sum_{i=1}^{k-1} T_{k-1,i}\biggr) - \sum_{k=1}^{n-1} \sum_{i=1}^k \bigl(e^{T_{k,i} - T_{k+1,i}} + e^{T_{k+1,i+1} - T_{k,i}}\bigr).$$

Now, it is shown in [Baudoin and O'Connell 2011] that, for each $\lambda \in \Omega$, the equation $Hf = \sum_i \lambda_i^2 f$ has a unique solution $f = f_\lambda$ such that $e^{-\lambda(x)} f_\lambda(x)$ is bounded and $\lim_{x\to+\infty} e^{-\lambda(x)} f_\lambda(x) = 1$, where we write $x \to +\infty$ to mean $\alpha_i(x) = x_i - x_{i+1} \to +\infty$ for each i. Moreover, by Feynman–Kac,

$$f_\lambda(x) = e^{\lambda(x)} \mathbb{E}_x \exp\biggl(-\sum_{i=1}^{n-1} \int_0^\infty e^{-\alpha_i(\beta_s)}\, ds\biggr), \tag{8}$$

where β_s is a Brownian motion in $\mathbb{R}^n$ with drift λ as in the previous section. The relation between the functions f_λ and the Whittaker functions ψ_λ is thus determined by the following proposition.

Proposition 2. *For* $\lambda \in \Omega$,

$$\lim_{x\to+\infty} e^{-\lambda(x)} \psi_\lambda(x) = \prod_{i<j} \Gamma(\lambda_i - \lambda_j). \tag{9}$$

Proof. We prove this by induction on n using the recursion (4). Write $\psi_\lambda = \psi_\lambda^{(n)}$ as before, setting $\psi_\lambda^{(1)}(x) = e^{\lambda x}$. Then $e^{-\lambda(x)} \psi_\lambda^{(1)}(x) = 1$ and, for $n \geq 2$,

$$\begin{aligned}
e^{-\lambda(x)} \psi_\lambda^{(n)}(x) &= \int_{\mathbb{R}^{n-1}} \exp\biggl(-\sum_{i=1}^{n} \lambda_i x_i + \lambda_n \biggl(\sum_{i=1}^{n} x_i - \sum_{i=1}^{n-1} y_i\biggr) - \sum_{i=1}^{n-1} \Bigl(e^{y_i - x_i} + e^{x_{i+1} - y_i}\Bigr)\biggr) \\
&\qquad \times \psi^{(n-1)}_{\lambda_1,\dots,\lambda_{n-1}}(y_1, \dots, y_{n-1})\, dy_1 \dots dy_{n-1} \\
&= \int_{\mathbb{R}^{n-1}} e^{\sum_{i=1}^{n-1} (\lambda_i - \lambda_n) y_i} \exp\biggl(-\sum_{i=1}^{n-1} e^{y_i} - \sum_{i=1}^{n-1} e^{x_{i+1} - x_i - y_i}\biggr) \\
&\qquad \times e^{-\sum_{i=1}^{n-1} \lambda_i (x_i + y_i)} \psi^{(n-1)}_{\lambda_1,\dots,\lambda_{n-1}}(x_1 + y_1, \dots, x_{n-1} + y_{n-1})\, dy_1 \dots dy_{n-1}.
\end{aligned}$$

By induction, we immediately conclude that, for each n, if $x, \lambda \in \Omega$ then $e^{-\lambda(x)} \psi_\lambda^{(n)}(x) \leq \prod_{i<j} \Gamma(\lambda_i - \lambda_j)$. Here we are using

$$\begin{aligned}
&\int_{\mathbb{R}^{n-1}} e^{\sum_{i=1}^{n-1} (\lambda_i - \lambda_n) y_i} \exp\biggl(-\sum_{i=1}^{n-1} e^{y_i} - \sum_{i=1}^{n-1} e^{x_{i+1} - x_i - y_i}\biggr) dy_1 \dots dy_{n-1} \\
&\qquad \leq \int_{\mathbb{R}^{n-1}} e^{\sum_{i=1}^{n-1} (\lambda_i - \lambda_n) y_i} \exp\biggl(-\sum_{i=1}^{n-1} e^{y_i}\biggr) dy_1 \dots dy_{n-1} = \prod_{i=1}^{n-1} \Gamma(\lambda_i - \lambda_n).
\end{aligned}$$

It follows, again by induction and using the dominated convergence theorem, that (9) holds for $\lambda \in \Omega$. ☐

Corollary 3. *For* $\lambda \in \Omega$,

$$\psi_\lambda(x) = \prod_{i<j} \Gamma(\lambda_i - \lambda_j) e^{\lambda(x)} \mathbb{E}_x \exp\biggl(-\sum_{i=1}^{n-1} \int_0^\infty e^{-\alpha_i(\beta_s)}\, ds\biggr). \tag{10}$$

Corollary 4. *For $x, \lambda \in \Omega$,*

$$J_\lambda(x) = \lim_{\beta\to\infty} \beta^{-n(n-1)/2}\psi_{\lambda/\beta}(\beta x).$$

Proof. By Proposition 1, the statement is equivalent to

$$\lim_{\beta\to\infty} \beta^{-n(n-1)/2}\psi_{\lambda/\beta}(\beta x) = h(\lambda)^{-1}e^{\lambda(x)}\mathbb{P}_x(T=\infty).$$

This follows directly from (10) by Brownian rescaling. □

As shown in [Baudoin and O'Connell 2011], the function ψ_λ, which can be defined by (10), is a class-one Whittaker function, as defined by Jacquet [2004] and Hashizume [1982]. In the notation of [Baudoin and O'Connell 2011] we are taking $\Pi = \{\alpha_i/2,\ i = 1, \dots, n-1\}$, $m(2\alpha) = 0$, $|\eta_\alpha|^2 = 1$ and $\psi_\nu(x) = 2^q k_\nu(x)$ where $q = n(n-1)/2$. In [Gerasimov et al. 2008], the relationship between Givental integral formula and a recursive integral formula due to Stade [1990] based on Jacquet's definition (see also [Ishii and Stade 2007]) is described.

Givental's integral formula (7) has a very similar structure to the formula (3). Indeed, if we define the type of an array $(T_{k,i},\ 1 \le i \le k \le n)$ to be the vector

$$\text{type}\, T = \left(T_{1,1},\, T_{2,1}+T_{2,2}-T_{1,1},\, \dots,\, \sum_{j=1}^{n} T_{n,j} - \sum_{j=1}^{n-1} T_{n-1,j}\right),$$

and a measure

$$g(dT) = \prod_{k=1}^{n-1}\prod_{i=1}^{k} e^{-e^{T_{k,i}-T_{k+1,i}}}\, e^{-e^{T_{k+1,i+1}-T_{k,i}}}\, dT_{k,i} = e^{\mathscr{F}_0(T)} \prod_{k=1}^{n-1}\prod_{i=1}^{k} dT_{k,i},$$

then

$$\psi_\lambda(x) = \int_{\Gamma(x)} e^{\lambda\cdot \text{type}\, T} g(dT).$$

On the other hand, if we replace the functions $e^{-e^{x-y}}$ in the reference measure g by the indicator functions $1_{x<y}$ to get a new reference measure

$$g_0(dT) = \prod_{k=1}^{n-1}\prod_{i=1}^{k} 1_{T_{k,i}<T_{k+1,i}}\, 1_{T_{k+1,i+1}<T_{k,i}},$$

then (3) can be written as

$$J_\lambda(x) = \int_{\Gamma(x)} e^{\lambda\cdot \text{type}\, T} g_0(dT).$$

We note the following. If $\lambda \in \iota\mathbb{R}^n$ then $\overline{\psi_\lambda(x)} = \psi_{-\lambda}(x)$; if $\lambda \in \iota\mathbb{R}^n$ and $\nu \in \mathbb{R}^n$, then $|\psi_{\lambda+\nu}(x)| \le \psi_\nu(x)$. For each $x \in \mathbb{R}^n$, $\psi_\lambda(x)$ is an entire, symmetric function of $\lambda \in \mathbb{C}^n$ [Gerasimov et al. 2008; Hashizume 1982; Kharchev and Lebedev

1999]. There is a Plancherel theorem [Wallach 1992; Arnold and Novikov 1994; Gerasimov et al. 2008; Kharchev and Lebedev 1999] which states that the integral transform

$$\hat{f}(\lambda) = \int_{\mathbb{R}^n} f(x)\psi_\lambda(x)\,dx \tag{11}$$

is an isometry from $L_2(\mathbb{R}^n, dx)$ onto $L_2^{\mathrm{sym}}(\iota\mathbb{R}^n, s_n(\lambda)\,d\lambda)$, where L_2^{sym} is the space of L_2 functions which are symmetric in their variables, $\iota = \sqrt{-1}$ and $s_n(\lambda)\,d\lambda$ is the *Sklyanin measure* defined by

$$s_n(\lambda) = \frac{1}{(2\pi\iota)^n n!} \prod_{j\neq k} \Gamma(\lambda_j - \lambda_k)^{-1}. \tag{12}$$

For $x, \mu \in \mathbb{R}^n$, denote by σ_μ^x the probability measure on the set of real triangular arrays $(T_{k,i})_{1\le i\le k\le n}$ defined by

$$\int f\,d\sigma_\mu^x = \psi_\mu(x)^{-1} \int_{\Gamma(x)} f(T) e^{\mathcal{F}_\mu(T)} \prod_{k=1}^{n-1}\prod_{i=1}^{k} dT_{k,i}.$$

Define a probability measure γ_μ^x by

$$\int_{\mathbb{R}^n} e^{\lambda\cdot y}\gamma_\mu^x(dy) = \frac{\psi_{\mu+\lambda}(x)}{\psi_\mu(x)}, \quad \lambda \in \mathbb{C}^n.$$

The probability measure $\gamma^x = \gamma_0^x$ is the analogue of the (normalised) Duistermaat–Heckman measure in this setting. The integral operator K with kernel

$$K(x, dy) = \psi_0(x)\gamma^x(dy)$$

satisfies the intertwining relation $HK = K\Delta$. We can write

$$K(x, dy) = k(x, y)\rho_x(dy),$$

where k is a smooth kernel from $\mathbb{R}^n$ to $\mathbb{R}^n_x = \left\{y \in \mathbb{R}^n : \sum_i y_i = \sum_i x_i\right\}$ and ρ_x denotes the Euclidean measure on $\mathbb{R}^n_x$. For $n = 2$,

$$k(x, y) = \exp(-e^{x_2-y_1} - e^{y_1-x_1})$$

and, for $n = 3$,

$$k(x, y) = \psi_0^{(2)}(a, b) = 2K_0(2e^{(b-a)/2})$$

where

$$e^{-a} = e^{x_3-y_1-y_2} + e^{-x_1}, \quad e^b = e^{y_1} + e^{y_2} + e^{y_1+y_2-x_2} + e^{x_2},$$

and K_0 denotes the Macdonald function with index 0.

3. Interpretation of γ^x in terms of Brownian motion

A reduced decomposition of an element $w \in S_n$ is a minimal expression of w as a product of adjacent transpositions, that is, $w = s_{i_1} \dots s_{i_r}$, where s_i denotes the transposition $(i, i+1)$. We will also refer to the word $\boldsymbol{i} = i_1 i_2 \dots i_r$ as a reduced decomposition. By definition, any reduced decomposition has the same length $l(w)$, defined to be the length of w. There is a unique longest element in S_n, namely the permutation

$$w_0 = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n{-}1 & \cdots & 1 \end{pmatrix}.$$

Its length is $n(n-1)/2$, as can be seen by taking the reduced decomposition

$$\boldsymbol{i} = 1\ 21\ 321\ \dots\ nn{-}1 \dots 21.$$

The symmetric group acts on $\mathbb{R}^n$ by permutation of coordinates, and as such is an example of a finite reflection group. It is generated by the hyperplane reflections $s_i = s_{\alpha_i}$, $i = 1, \dots, n-1$, defined for $x \in \mathbb{R}^n$ by

$$s_i x = x - \alpha_i(x)\alpha_i,$$

where $\alpha_i = e_i - e_{i+1}$. Note that s_i corresponds to the adjacent transposition $(i, i+1)$.

For a continuous path $\eta : (0, \infty) \to \mathbb{R}^n$, define $T_i = T_{\alpha_i}$ by

$$T_i \eta(t) = \eta(t) + \left(\log \int_0^t e^{-\alpha_i(\eta(s))}\, ds \right) \alpha_i, \quad t > 0.$$

Let $w = s_{i_1} \cdots s_{i_r}$ be a reduced decomposition. Then

$$T_w := T_{i_r} \cdots T_{i_1}$$

depends only on w, not on the chosen decomposition [Biane et al. 2005].

We now introduce a probability measure $\mathbb{P}$ under which η is a Brownian motion in $\mathbb{R}^n$ with a drift μ and $\eta(0) = 0$. In this setting, a very special role is played by the transform $T^{(n)} = T_{w_0}$. In the following we use the fact that this is well-defined for each n. Write $\eta = (\eta^1, \dots, \eta^n)$. For each $k \le n$, set

$$(T_{k,1}, \dots, T_{k,k}) = T^{(k)}(\eta^1, \dots, \eta^k).$$

The evolution of the triangular array $T_{k,j}$, $1 \le j \le k \le n$, is given recursively as follows: $dT_{1,1} = d\eta^1$ and, for $k \geq 2$,

$$\begin{aligned}
dT_{k,1} &= dT_{k-1,1} + e^{T_{k,2}-T_{k-1,1}}\,dt \\
dT_{k,2} &= dT_{k-1,2} + \left(e^{T_{k,3}-T_{k-1,2}} - e^{T_{k,2}-T_{k-1,1}}\right)dt \\
&\;\vdots \\
dT_{k,k-1} &= dT_{k-1,k-1} + \left(e^{T_{k,k}-T_{k-1,k-1}} - e^{T_{k,k-1}-T_{k-1,k-2}}\right)dt \\
dT_{k,k} &= d\eta^k - e^{T_{k,k}-T_{k-1,k-1}}\,dt.
\end{aligned} \tag{13}$$

The process, which is clearly Markov, contains a number of projections which are also Markov. For example, setting $\xi_k = T_{k,k}$, we have, for $k \leq n$,

$$d\xi_k = d\eta^k - e^{\xi_k - \xi_{k-1}}\,dt.$$

This defines a simple interacting particle system on the real line, which has very nice properties. For example, in the coordinates $\sum_i \xi_i$ and $\xi_i - \xi_{i+1}$ $1 \leq i \leq n-1$, it has a product form invariant measure, that is, a product measure which is invariant.

A remarkable fact is that each row in the pattern $T_{k,j}$ is a Markov process with respect to its own filtration. This gives an interpretation of the measures γ_μ^x and σ_μ^x defined in the previous section.

Theorem 5 [O'Connell 2012]. *$T_{w_0}\eta(t)$, $t > 0$, is a diffusion process in $\mathbb{R}^n$ with infinitesimal generator*

$$\mathscr{L}_\mu = \frac{1}{2}\psi_\mu^{-1}\left(H - \sum_{i=1}^n \mu_i^2\right)\psi_\mu = \tfrac{1}{2}\Delta + \nabla \log \psi_\mu \cdot \nabla.$$

For $t > 0$, the conditional law of $\{T_{k,j}(t),\ 1 \leq j \leq k \leq n\}$, given

$$\{T_{w_0}\eta(s),\ s \leq t;\ T_{w_0}\eta(t) = x\},$$

is σ_μ^x, and the conditional law of $\eta(t)$, given the same, is γ_μ^x. The law of $T_{w_0}\eta(t)$ is given by

$$\nu_t^\mu(dx) = e^{-\sum_i \mu_i^2 t/2}\psi_\mu(x)\theta_t(x)\,dx,$$

where

$$\theta_t(x) = \int_{\iota\mathbb{R}^n} \psi_{-\lambda}(x) e^{\sum_i \lambda_i^2 t/2} s_n(\lambda)\,d\lambda. \tag{14}$$

In the case $n = 2$, this is equivalent to a theorem of Matsumoto and Yor [1999].

Write $\mathscr{L} = \mathscr{L}_0$ and $\nu_t = \nu_t^0$. The diffusion with generator $\mathscr{L}$ is the analogue of Dyson's Brownian motion in this setting and the measures ν_t and θ_t (the latter requires normalisation) are the analogues of the Gaussian unitary and Gaussian orthogonal ensembles, respectively. The diffusion with generator $\mathscr{L}_\mu$ was introduced in [Baudoin and O'Connell 2011]. When $\mu \in \bar{\Omega}$, it can be interpreted as a Brownian motion in $\mathbb{R}^n$ killed according to the potential $\sum_i e^{x_{i+1}-x_i}$ and then conditioned to survive forever [Katori 2011; 2012]. The path-transformation T_{w_0} is closely related to the geometric (lifting of the) RSK correspondence introduced by A. N. Kirillov [2001] and studied further by Noumi and Yamada [2004]. A discrete-time version of the above theorem, which works directly in the setting of the geometric RSK correspondence, is given in [Corwin et al. 2014]. In the discrete-time setting the Whittaker functions continue to play a central role. See also [Borodin and Corwin 2014; Borodin et al. 2013; Chhaibi 2012; Gorsky et al. 2012; O'Connell et al. 2014; O'Connell and Warren 2011] for further related developments.

4. Application to random polymers

The following model was introduced in [O'Connell and Yor 2001]. The environment is given by a sequence $B_1, B_2, \ldots$ independent standard 1-dimensional Brownian motions. For up/right paths $\phi \equiv \{0 < t_1 < \cdots < t_{N-1} < t\}$ (as shown in Figure 1), define

$$E(\phi) = B_1(t_1) + B_2(t_2) - B_2(t_1) + \cdots + B_N(t) - B_N(t_{N-1}),$$
$$P(d\phi) = Z_t^n(\beta)^{-1} e^{\beta E(\phi)} d\phi, \quad Z_t^n(\beta) = \int e^{\beta E(\phi)} d\phi.$$

Set $X_1^n(t) = \log Z_t^n$ and, for $k = 2, \ldots, n$,

$$X_1^n(t) + \cdots + X_k^n(t) = \log \int e^{E(\phi_1)+\cdots+E(\phi_k)} d\phi_1 \ldots d\phi_k,$$

where the integral is over nonintersecting paths $\phi_1, \ldots, \phi_k$ from $(0, 1), \ldots, (0, k)$ to $(t, n-k+1), \ldots, (t, n)$.

Let $\eta = (B_n, \ldots, B_1)$. Then $X = T_{w_0}\eta$ and the following holds.

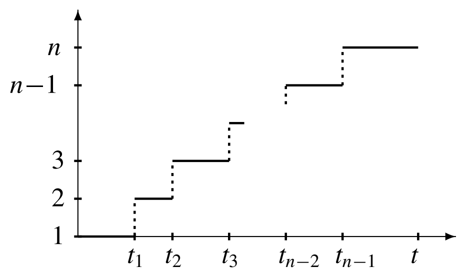

Figure 1. An up/right path $\phi \equiv \{0 < t_1 < \ldots < t_{n-1} < t\}$.

Theorem 6 [O'Connell 2012]. *The process $X(t)$, $t > 0$ is a diffusion in $\mathbb{R}^n$ with infinitesimal generator $\mathcal{L}$. The distribution of $X(t)$ is given by ν_t. For $s > 0$,*

$$Ee^{-sZ_t^n} = \int s^{-\sum \lambda_i} \prod_i \Gamma(\lambda_i)^n e^{\frac{1}{2}\sum_i \lambda_i^2 t} s_n(\lambda)\, d\lambda,$$

where the integral is along (upwards) vertical lines with $\Re\lambda_i > 0$ for all i.

The free energy for this model is given by [O'Connell and Yor 2001; Moriarty and O'Connell 2007]

$$\lim_{n\to\infty} \frac{1}{n} \log Z_n^n = \inf_{t>0}[t - \Psi(t)],$$

almost surely, where $\Psi(z) = \Gamma'(z)/\Gamma(z)$. The conjectured KPZ scaling behaviour for the fluctuations of $\log Z_n^n$ was (essentially) established by Seppäläinen and Valkó [2010]; more recently, Borodin, Corwin and Ferrari have proved the full KPZ universality conjecture for this model, namely that $\log Z_n^n$, suitably centred and rescaled, converges in law to the Tracy–Widom F_2 distribution of random matrix theory [Borodin and Corwin 2014; Borodin et al. 2014]. See also [Spohn 2014].

5. Reduced double Bruhat cells and their parametrisations

The Weyl group associated with $GL(n)$ is the symmetric group S_n. Each element $w \in S_n$ has a representative $\bar{w} \in GL(n)$ defined as follows. Denote the standard generators for $\mathfrak{gl}_n$ by h_i, e_i and f_i. For example, for $n = 3$,

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

For adjacent transpositions $s_i = (i, i+1)$, define

$$\bar{s}_i = \exp(-e_i)\exp(f_i)\exp(-e_i) = (I - e_i)(I + f_i)(I - e_i).$$

In other words, $\bar{s}_i = \varphi_i\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, where φ_i is the natural embedding of SL(2) into GL(n) given by h_i, e_i and f_i. For example, when $n = 3$,

$$\bar{s}_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \bar{s}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Now let $w = s_{i_1} \dots s_{i_r}$ be a reduced decomposition and define $\bar{w} = \bar{s}_{i_1} \dots \bar{s}_{i_r}$. Note that $\overline{uv} = \bar{u}\bar{v}$ whenever $l(uv) = l(u) + l(v)$. For $n = 2$, $w_0 = s_1$ and

$$\bar{w}_0 = \bar{s}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

For $n = 3$, $w_0 = s_1 s_2 s_1 = s_2 s_1 s_2$ is represented by

$$\bar{w}_0 = \bar{s}_1 \bar{s}_2 \bar{s}_1 = \bar{s}_2 \bar{s}_1 \bar{s}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Denote the upper (respectively lower) triangular matrices in $GL(n)$ by B and B_-, and the upper (respectively lower) uni-triangular matrices in $GL(n)$ by N and N_-. The group $GL(n)$ has two Bruhat decompositions

$$GL(n) = \bigcup_{u \in S_n} B\bar{u}B = \bigcup_{v \in S_n} B_- \bar{v} B_-.$$

The double Bruhat cells $G^{u,v}$ are defined, for $u, v \in S_n$, by

$$G^{u,v} = B\bar{u}B \cap B_- \bar{v} B_-.$$

The reduced double Bruhat cells $L^{u,v}$ are defined by

$$L^{u,v} = N\bar{u}N \cap B_- \bar{v} B_-.$$

We also define the opposite reduced double Bruhat cells $M^{u,v}$ by

$$M^{u,v} = B\bar{u}B \cap N_- \bar{v} N_-.$$

The reduced double Bruhat cell $L^{w,e}$ (where e denotes the identity in S_n) admits the following parametrisations, one for each reduced decomposition of w. See [Lusztig 1994; Berenstein and Zelevinsky 2001; Berenstein et al. 1996; Fomin and Zelevinsky 1999]. Set

$$Y_i(u) = \varphi_i \begin{pmatrix} u & 0 \\ 1 & u^{-1} \end{pmatrix}, \quad i = 1, \dots, n-1.$$

Then, for any reduced decomposition $\boldsymbol{i} = i_1 \dots i_r$ of w, the map

$$(u_1, \dots, u_r) \mapsto Y_{i_1}(u_1) \cdots Y_{i_r}(u_r)$$

defines a bijection between $\mathbb{C}^r_{\neq 0}$ and $L^{w,e}$. This bijection has the property that the totally positive part $L^{u,v}_{>0}$ of $L^{u,v}$ corresponds precisely to the subset $\mathbb{R}^r_{>0}$ of $\mathbb{C}^r_{\neq 0}$. There are explicit transition maps which relate the parameters $(u_1, \dots, u_r)$ corresponding to different reduced decompositions of w.

In the case $n=3$, the two representations of an element in $L^{w_0,e}$ corresponding to the words 121 and 212, denoting the corresponding parameters by (u_1, u_2, u_3) and (u'_1, u'_2, u'_3), respectively, are given by

$$\begin{pmatrix} u_1u_3 & 0 & 0 \\ u_3+u_2/u_1 & u_2/u_1u_3 & 0 \\ 1 & 1/u_3 & 1/u_2 \end{pmatrix} = \begin{pmatrix} u'_2 & 0 & 0 \\ u'_1 & u'_1u'_3/u'_2 & 0 \\ 1 & u'_3/u'_2+1/u'_1 & 1/u'_1u'_3 \end{pmatrix}.$$

The transition maps are given by

$$u'_1 = u_3 + u_2/u_1, \quad u'_2 = u_1u_3, \quad u'_3 = u_1u_2/(u_2 + u_1u_3). \tag{15}$$

Their is a similar parametrisation for $M^{w,e}$, due to Lusztig [1994]. For $i=1, \dots, n-1$, set $X_i(v) = I + vf_i$. Take any reduced decomposition $\boldsymbol{i} = i_1 \dots i_r$ for w. Then the map

$$(v_1, \dots, v_r) \mapsto X_{i_1}(v_1) \cdots X_{i_r}(v_r)$$

defines a bijection between $\mathbb{C}^r_{\neq 0}$ and $M^{w,e}$. This bijection also has the property that the totally positive part $M^{u,v}_{>0}$ of $M^{u,v}$ corresponds precisely to the subset $\mathbb{R}^r_{>0}$ of $\mathbb{C}^r_{\neq 0}$.

In the case $n=3$, the two representations of an element in $M^{w_0,e}$ corresponding to the words 121 and 212, denoting the corresponding parameters by (v_1, v_2, v_3) and (v'_1, v'_2, v'_3), respectively, are given by

$$\begin{pmatrix} 1 & 0 & 0 \\ v_1+v_3 & 1 & 0 \\ v_2v_3 & v_2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ v'_2 & 1 & 0 \\ v'_1v'_2 & v'_1+v'_3 & 1 \end{pmatrix},$$

with transition maps

$$v'_1 = \frac{v_2v_3}{v_1+v_3}, \quad v'_2 = v_1 + v_3, \quad v'_3 = \frac{v_1v_2}{v_1+v_3}. \tag{16}$$

We conclude this section with a simple lemma. Let $b \in G^{e,w}$ and write $b = an$ where $a = \mathrm{diag}(a_1, \dots, a_n)$, say, and $n \in N$. Then, for any $w \in S_n$, $b\bar{w}$ has a Gauss (or LDU) decomposition $b\bar{w} = [b\bar{w}]_-[b\bar{w}]_0[b\bar{w}]_+$ and $n\bar{w}$ has a Gauss decomposition $n\bar{w} = [n\bar{w}]_-[n\bar{w}]_0[n\bar{w}]_+$ [Fomin and Zelevinsky 1999]. Moreover, $[n\bar{w}]_{-0} = [n\bar{w}]_-[n\bar{w}]_0 \in L^{w,e}$ and $[b\bar{w}]_- \in M^{w,e}$. Let $\boldsymbol{i} = i_1 \dots i_r$ be a reduced decomposition for w. Then we can write

$$[n\bar{w}]_{-0} = Y_{i_1}(u_1) \dots Y_{i_r}(u_r), \quad [b\bar{w}]_- = X_{i_1}(v_1) \dots X_{i_r}(v_r).$$

Define $Z_i(u) = \varphi_i\left(\begin{smallmatrix} u & 0 \\ 0 & u^{-1} \end{smallmatrix}\right)$. Set $a^0 = a$ and, for $1 \le k \le r$, $a^k = a^{k-1}Z_{i_k}(u_k)$. Write $a^k = \mathrm{diag}(a^k_1, \dots, a^k_n)$.

Lemma 7. *The following relations holds*: $[b\bar{w}]_0 = a^r$ *and*, *for* $k = 1, \dots, r$, $v_k = u_k^{-1} a^{k-1}_{i_k+1}/a^{k-1}_{i_k}$.

Proof. Note that $a[n\bar{w}]_{-0} = [b\bar{w}]_{-0} = [b\bar{w}]_{-}[b\bar{w}]_0$, hence

$$aY_{i_1}(u_1)\dots Y_{i_r}(u_r) = X_{i_1}(v_1)\dots X_{i_r}(v_r)[b\bar{w}]_0.$$

The result follows by repeated application of the identity $aY_i(u) = X_i(v)a'$, where $a' = aZ_i(u)$ and $v = u^{-1}a_{i+1}/a_i$. □

6. An evolution on upper triangular matrices

As shown in [Biane et al. 2005], the path-transformations $T_w\eta$ can also be represented in terms of an evolution on the upper triangular matrices in $GL(n, \mathbb{R})$. Let $w = s_{i_1}\cdots s_{i_r}$ be a reduced decomposition and $\eta : (0, \infty) \to \mathbb{R}^n$ a continuous path. Set $\eta_0 = \eta$ and, for $k \le r$,

$$\eta_k = T_{i_k}\dots T_{i_1}\eta \quad x_k(t) = \log \int_0^t e^{-\alpha_{i_k}(\eta_{k-1}(s))}ds. \tag{17}$$

Then $\eta_r = T_w\eta$ and, for each $k \le r$, $\eta_k = \eta + \sum_{j=1}^k x_j\alpha_{i_j}$.

Write $\eta(t) = (\eta^1_t, \dots, \eta^n_t)$. Define a path $b(t)$ taking values in B by

$$b_{ij}(t) = e^{\eta^i(t)} \int_{0<s_{j-1}<s_{j-2}<\dots<s_i<t} \exp\left(-\sum_{k=i}^{j-1} \alpha_k(\eta(s_k))\right) ds_i\cdots ds_{j-1}.$$

If η is smooth, the b satisfies the ordinary differential equation

$$db = \left(\sum_{i=1}^n h_i\, d\eta^i + \sum_{i=1}^{n-1} e_i\, dt\right)b, \quad b(0) = I.$$

If η is a Brownian path (as in the next section) then b satisfies the equation interpreted as a Stratonovich SDE.

When $n = 2$,

$$db = \begin{pmatrix} d\eta^1 & dt \\ 0 & d\eta^2 \end{pmatrix} b, \quad b(t) = \begin{pmatrix} e^{\eta^1_t} & \int_0^t e^{\eta^2_s - \eta^1_s + \eta^1_t}\, ds \\ 0 & e^{\eta^2_t} \end{pmatrix}.$$

When $n = 3$,

$$db = \begin{pmatrix} d\eta^1 & dt & 0 \\ 0 & d\eta^2 & dt \\ 0 & 0 & d\eta^3 \end{pmatrix} b,$$

and the solution is given by

$$b(t)=\begin{pmatrix} e^{\eta^1_t} & \int_0^t e^{\eta^2_s-\eta^1_s+\eta^1_t}\,ds & \iint_{0<r<s<t} e^{\eta^3_r-\eta^2_r+\eta^2_s-\eta^1_s+\eta^1_t}\,dr\,ds \\ 0 & e^{\eta^2_t} & \int_0^t e^{\eta^3_s-\eta^2_s+\eta^2_t}\,ds \\ 0 & 0 & e^{\eta^3_t} \end{pmatrix}.$$

Write $b=an$, where $a=\mathrm{diag}(e^{\eta^1},\dots,e^{\eta^n})$ and $n\in N$. Set $u_k=e^{x_k}$ and $v_k=e^{-x_k-\alpha_k(\eta_{k-1})}$.

Theorem 8 [Biane et al. 2005; 2009]. *For each $t>0$, $b(t)\bar{w}$ has a Gauss decomposition $b\bar{w}=[b\bar{w}]_-[b\bar{w}]_0[b\bar{w}]_+$, with $[b\bar{w}]_0=\exp(T_w\eta(t))$. Moreover, $[n\bar{w}]_{-0}=Y_{i_1}(u_1)\cdots Y_{i_r}(u_r)\in L^{w,e}_{>0}$.*

By Lemma 7, we also have $[b\bar{w}]_-=X_{i_1}(v_1)\cdots X_{i_r}(v_r)\in M^{w,e}_{>0}$.

6.1. ***The case n = 2.*** From the definitions: $\alpha_1=e_1-e_2$, $w_0=s_1=s_{e_1-e_2}$,

$$u:=u_1=e^{x_1}=\int_0^t e^{-\eta^1_s+\eta^2_s}\,ds,\qquad v:=v_1=e^{-\eta^1+\eta^2}u^{-1},$$

$$e^{T_{w_0}\eta}=(e^{\eta^1}u,e^{\eta^2}u^{-1})=\left(\int_0^t e^{\eta^2_s+\eta^1_t-\eta^1_s}\,ds,\ \int_0^t e^{-(\eta^1_s+\eta^2_t-\eta^2_s)}\right),$$

$$b=\begin{pmatrix} e^{\eta^1} & \int_0^t e^{\eta^2_s-\eta^1_s+\eta^1_t}\,ds \\ 0 & e^{\eta^2}\end{pmatrix}=\begin{pmatrix} e^{\eta^1} & e^{\eta^1}u \\ 0 & e^{\eta^2}\end{pmatrix}=\begin{pmatrix} e^{\eta^1} & 0 \\ 0 & e^{\eta^2}\end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1\end{pmatrix}=an.$$

Taking $\bar{w}_0=\left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$, we see that

$$b\bar{w}_0=\begin{pmatrix} e^{\eta^1}u & -e^{\eta^1} \\ e^{\eta^2} & 0\end{pmatrix}=\begin{pmatrix} 1 & 0 \\ v & 1\end{pmatrix}\begin{pmatrix} e^{\eta^1}u & 0 \\ 0 & e^{\eta^2}u^{-1}\end{pmatrix}\begin{pmatrix} 1 & -u^{-1} \\ 0 & 1\end{pmatrix}.$$

and

$$n\bar{w}_0=\begin{pmatrix} 1 & u \\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}=\begin{pmatrix} u & -1 \\ 1 & 0\end{pmatrix}=\begin{pmatrix} u & 0 \\ 1 & u^{-1}\end{pmatrix}\begin{pmatrix} 1 & -u^{-1} \\ 0 & 1\end{pmatrix}$$

Hence

$$[b\bar{w}_0]_0=e^{T_{w_0}\eta},\quad [b\bar{w}_0]_-=\begin{pmatrix} 1 & 0 \\ v & 1\end{pmatrix}=X_1(v),\quad [n\bar{w}_0]_{-0}=\begin{pmatrix} u & 0 \\ 1 & u^{-1}\end{pmatrix}=Y_1(u),$$

as claimed.

6.2. ***The case n = 3.*** From the definitions:

$$\alpha_1=e_1-e_2,\quad \alpha_2=e_2-e_3,\quad w_0=s_1s_2s_1=s_2s_1s_2.$$

For the reduced decomposition $w_0=s_1s_2s_1$, we have

$$u_1=e^{x_1}=\int_0^t e^{-\eta^1(s)+\eta^2(s)}\,ds,\quad e^{\eta_1}=(e^{\eta^1}u_1,e^{\eta^2}/u_1,e^{\eta^3}),$$

$$u_2 = e^{x_2} = \int_0^t e^{-\eta_1^2(s)+\eta_1^3(s)}\,ds, \quad e^{\eta_2} = (e^{\eta^1}u_1, e^{\eta^2}u_2/u_1, e^{\eta^3}/u_2),$$

$$u_3 = e^{x_3} = \int_0^t e^{-\eta_2^1(s)+\eta_2^2(s)}\,ds, \quad e^{\eta_3} = e^{T_{w_0}\eta} = (e^{\eta^1}u_1u_3, e^{\eta^2}u_2/u_1u_3, e^{\eta^3}/u_2),$$

$$v_1 = e^{-\eta^1+\eta^2}/u_1, \quad v_2 = e^{-\eta^2+\eta^3}\frac{u_1}{u_2}, \quad v_3 = e^{-\eta^1+\eta^2}\frac{u_2}{u_1^2u_3},$$

$$b = \begin{pmatrix} e^{\eta^1} & \int_0^t e^{\eta_s^2-\eta_s^1+\eta_t^1}\,ds & \iint_{0<r<s<t} e^{\eta_r^3-\eta_r^2+\eta_s^2-\eta_s^1+\eta_t^1}\,dr\,ds \\ 0 & e^{\eta^2} & \int_0^t e^{\eta_s^3-\eta_s^2+\eta_t^2}\,ds \\ 0 & 0 & e^{\eta^3} \end{pmatrix}$$

$$= \begin{pmatrix} e^{\eta^1} & 0 & 0 \\ 0 & e^{\eta^2} & 0 \\ 0 & 0 & e^{\eta^3} \end{pmatrix}\begin{pmatrix} 1 & u_1 & u_1u_3 \\ 0 & 1 & u_3+u_2/u_1 \\ 0 & 0 & 1 \end{pmatrix} = an.$$

The identity

$$\int_0^t e^{\eta_s^3-\eta_s^2+\eta_t^2}\,ds = u_3+u_2/u_1$$

follows from (15). Now,

$$\bar{w}_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

so we have

$$b\bar{w}_0 = \begin{pmatrix} e^{\eta^1}u_1u_3 & -e^{\eta^1}u_1 & e^{\eta^1} \\ e^{\eta^2}(u_3+u_2/u_1) & -e^{\eta^2} & 0 \\ e^{\eta^3} & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ v_1+v_3 & 1 & 0 \\ v_2v_3 & v_2 & 1 \end{pmatrix}\begin{pmatrix} e^{\eta^1}u_1u_3 & 0 & 0 \\ 0 & e^{\eta^2}u_2/u_1u_3 & 0 \\ 0 & 0 & e^{\eta^3}/u_2 \end{pmatrix}\begin{pmatrix} 1 & -1/u_3 & 1/u_1u_3 \\ 0 & 1 & -u_3/u_2-1/u_1 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$n\bar{w}_0 = \begin{pmatrix} u_1u_3 & -u_1 & 1 \\ u_3+u_2/u_1 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} u_1u_3 & 0 & 0 \\ u_3+u_2/u_1 & u_2/u_1u_3 & 0 \\ 1 & 1/u_3 & 1/u_2 \end{pmatrix}\begin{pmatrix} 1 & -1/u_3 & 1/u_1u_3 \\ 0 & 1 & -u_3/u_2-1/u_1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus $[b\bar{w}_0]_0 = e^{T_{w_0}\eta}$, and, as claimed,

$$[b\bar{w}_0]_- = \begin{pmatrix} 1 & 0 & 0 \\ v_1+v_3 & 1 & 0 \\ v_2v_3 & v_2 & 1 \end{pmatrix} = X_1(v_1)X_2(v_2)X_3(v_3),$$

$$[n\bar{w}_0]_{-0} = \begin{pmatrix} u_1u_3 & 0 & 0 \\ u_3+u_2/u_1 & u_2/u_1u_3 & 0 \\ 1 & 1/u_3 & 1/u_2 \end{pmatrix} = Y_1(u_1)Y_2(u_2)Y_3(u_3).$$

6.3. ***Evolution of the Lusztig parameters.*** As before, we introduce a probability measure $\mathbb{P}$ under which η is a Brownian motion in $\mathbb{R}^n$ with a drift μ and $\eta(0)=0$. For each $k \le n$, set

$$(T_{k,1}, \ldots, T_{k,k}) = T^{(k)}(\eta^1, \ldots, \eta^k).$$

Note that this is given in terms of the principal minors $b^{(k)}$, $k \le n$, of b by $T^{(k)}(\eta^1, \ldots, \eta^k) = \log\bigl[b^{(k)}\bar{w}_0^{(k)}\bigr]_0$, where $w_0^{(k)}$ denotes the longest element in S_k. The evolution of the triangular array $T_{k,j}$, $1 \le j \le k \le n$, is given by (13). As remarked earlier, this process contains a number of projections which are also Markov. In particular, setting $\xi_k = T_{k,k}$, we have, for $k \le n$,

$$d\xi_k = d\eta^k - e^{\xi_k - \xi_{k-1}}\, dt.$$

This defines a simple interacting particle system on the real line which, in the coordinates $\sum_i \xi_i$ and $\xi_i - \xi_{i+1}$, $1 \le i \le n-1$, has a product form invariant measure. There is an extension of this process, involving the Lusztig parameters, which is also Markov and, moreover, also has a product form invariant measure. Let $v_1, \ldots, v_q$ be the Lusztig parameters corresponding to a reduced decomposition $w_0 = s_{i_1} \ldots s_{i_q}$, that is,

$$[b\bar{w}_0]_- = X_{i_1}(v_1) \cdots X_{i_q}(v_q).$$

Set $y_k = -\log v_k$. The evolution of y_k, $1 \le k \le q$, is given by

$$dy_k = d\alpha_{i_k}(\eta_{k-1}) + e^{-y_k}\, dt,$$

where $\eta_k = T_{i_k} \ldots T_{i_1}\eta$. Setting $x_k = y_k - \alpha_{i_k}(\eta_{k-1})$, note that $dx_k = e^{-y_k}\, dt$ and $\eta_k = \eta + \sum_{j=1}^k x_j\alpha_j$. Hence,

$$dy_k = d\alpha_{i_k}(\eta) + \sum_{j=1}^{k-1} \alpha_{i_k}(\alpha_{i_j})e^{-y_j}\, dt + e^{-y_k}\, dt. \tag{18}$$

Let $\beta_1 = \alpha_{i_1}$ and, for $2 \le k \le q$, $\beta_k = s_{i_1} \ldots s_{i_{k-1}}\alpha_{i_k}$. Set $\theta_k = -\beta_k(\mu)$. If $\mu \in w_0\Omega = -\Omega$, then $\theta_k > 0$ for all k and the diffusion has stationary distribution given by the product measure

$$\pi = \bigotimes_{k=1}^{q} \Gamma(\theta_k)^{-1} g_{\theta_k},$$

where $g_\theta(dx) = \exp(-\theta x - e^{-x})\,dx$. This can be seen as a consequence of the following fact, which is the analogue in this setting of the output theorem for the $M/M/1$ queue [O'Connell and Yor 2001]. Let x_t be a standard one-dimensional Brownian motion with negative drift $-\theta$, and consider the one-dimensional diffusion

$$dy = \sqrt{2}dx + e^{-y}\,dt.$$

This has a unique invariant distribution $\Gamma(\theta)^{-1} g_\theta$. If we start this diffusion in equilibrium and define $\tilde{x}_t = x_t + 2(y_0 - y_t)$, then $\tilde{x}$ has the same law as x and, moreover, $\tilde{x}_s$, $s \le t$, is independent of y_u, $u \ge t$, for all t. It follows that the measure π is invariant. For an analytic proof of this fact, see [O'Connell and Ortmann 2012]. See also [Biane et al. 2009, Proposition 5.9], where the equivalent property is proved in the "zero-temperature" setting.

If we choose the reduced decomposition $\boldsymbol{i} = 1\ 21\ 321\ n{-}1n{-}2\dots21$, and define, for $m \le n-1$ and $1 \le i \le n-m$, $q_{m,i} = T_{i+m-1,i} - T_{i+m,i+1}$, then

$$(y_1, y_2, \dots, y_q) = (q_{1,1}, q_{1,2}, \dots, q_{1,n}, q_{2,1}, \dots, q_{2,n-1}, \dots, q_{n-1,1}).$$

Note that $q_{1,i} = \xi_i - \xi_{i+1}$, for $1 \le i \le n-1$. In these coordinates, the evolution is given by

$$dq_{m,i} = d\alpha_i(\eta) + e^{-q_{m,i}}\,dt + \sum_{l=1}^{m-1} (2e^{-q_{l,i}} - e^{-q_{l,i+1}} - e^{-q_{l,i-1}})\,dt,$$

with the conventions that the empty sum is zero and $q_{l,0} = +\infty$. Setting $\theta_{m,i} = \mu_{m+i} - \mu_m$, an invariant measure for this diffusion is given by the product measure $\bigotimes_{m,i} g_{\theta_{m,i}}$. The dynamics of this process can be viewed as a network, as follows. Consider the dynamics

$$dQ = d(A-S) + e^{-Q}\,dt, \quad dD = dA - dQ, \quad dT = dS + dQ.$$

We think of A, S as the input and D, T as the output, and represent this system graphically as follows:

$$\begin{tikzcd} & {} \arrow[d, "A"] & \\ {} \arrow[r, "S"] & \boxed{Q} \arrow[r, "D"] \arrow[d, "T"] & {} \\ & {} & \end{tikzcd}$$

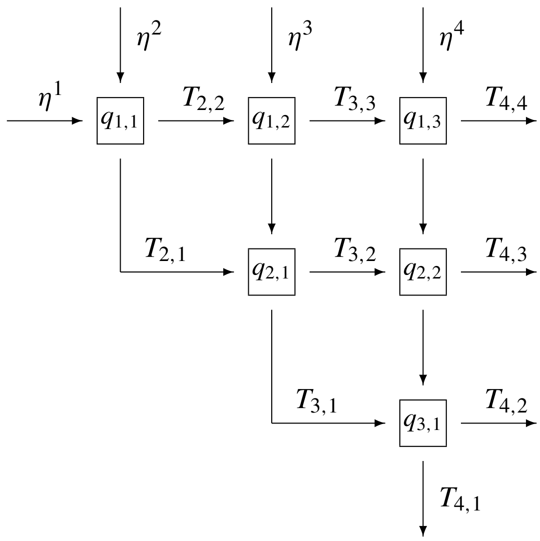

Figure 2. Graphical representation of the evolution of Lusztig parameters.

Then the evolution of the $q_{m,i}$ can be represented as in Figure 2. To see directly from this picture the product-form invariant measure, note that, if A and S are independent standard one-dimensional Brownian motions with respective drifts λ and σ, with $\lambda < \sigma$, then the diffusion Q has invariant distribution $\Gamma(\theta)^{-1} g_\theta$, where $\theta = \sigma - \lambda$. Moreover, if we start this diffusion in equilibrium, then $D_t = A_t + Q_0 - Q_t$ and $T_t = S_t - Q_0 + Q_t$ are independent standard one-dimensional Brownian motions with respective drifts λ and σ, and for each $t > 0$, (D_s, T_s), $s \le t$, is independent of Q_u, $u \ge t$. The analogue of this fact in the setting of Poisson queueing networks is the cornerstone of classical queueing theory. It is called the output, or Burke, theorem. Finally, we remark that the dynamics indicated by Figure 2 is the analogue, in this setting, of the dynamical interpretation given in [O'Connell 2003] of the RSK correspondence as a kind of "queueing network".

7. From the Feynman–Kac formula to Givental's integral formula

The fact that the evolution equation (18) for the Lusztig parameters has a product form invariant measure sheds some light on the relation between the Feynman–Kac formula (10) and the integral formula of Givental. It follows from this that, for any given reduced decomposition of w_0, the random variables

$$\int_0^\infty e^{-\alpha_i(\beta_s)}\, ds, \quad i = 1, \ldots, n-1$$

can be expressed, via the transition maps, as rational functions of a collection of

$q=n(n-1)/2$ independent Gamma-distributed random variables with respective parameters θ_k, $k\le q$, defined as above with $\beta=-\eta$. Note that β is a Brownian motion with drift $\lambda=-\mu\in\Omega$. Since the sets $\{\theta_k,\, k\le q\}$ and $\{\lambda_i-\lambda_j,\, i<j\}$ are the same, this allows (10) to be written as a q-dimensional integral

$$\begin{aligned}\psi_\lambda(x)&=\prod_{i<j}\Gamma(\lambda_i-\lambda_j)e^{\lambda(x)}\mathbb{E}_x\exp\biggl(-\sum_{i=1}^{n-1}\int_0^\infty e^{-\alpha_i(\beta_s)}\,ds\biggr)\\&=e^{\lambda(x)}\int_{\mathbb{R}_+^q}e^{-\sum_{i=1}^{n-1}e^{-\alpha_i(x)}r_i(v_1,\dots,v_q)}\prod_{i=1}^q v_i^{\theta_i-1}e^{-v_i}\,dv_i.\end{aligned}\tag{19}$$

For example, when $n=3$ and $\boldsymbol{i}=121$, we have

$$\theta_1=\lambda_1-\lambda_2,\quad \theta_2=\lambda_1-\lambda_3,\quad \theta_3=\lambda_2-\lambda_3,$$

and, using (16),

$$r_1(v_1,v_2,v_3)=\frac{1}{v_1},\quad r_2(v_1,v_2,v_3)=\frac{1}{v_1'}=\frac{v_1+v_3}{v_2v_3}.$$

In this case, the integral formula (19) becomes

$$\begin{aligned}\psi_\lambda(x)=e^{\lambda_1x_1+\lambda_2x_2+\lambda_3x_3}&\int_{\mathbb{R}_+^3}v_1^{\lambda_1-\lambda_2-1}v_2^{\lambda_1-\lambda_3-1}v_3^{\lambda_2-\lambda_3-1}\\&\times\exp\biggl(-v_1-v_2-v_3-e^{-x_1+x_2}\frac{1}{v_1}-e^{-x_2+x_3}\frac{v_1+v_3}{v_2v_3}\biggr)\,dv_1\,dv_2\,dv_3.\end{aligned}$$

Under the change of variables

$$v_1=e^{T_{32}-T_{21}},\quad v_2=e^{T_{33}-T_{22}},\quad v_3=e^{T_{22}-T_{11}},$$

where $T=(T_{ki},\,1\le i\le k\le 3)$ is an array with $(T_{31},T_{32},T_{33})=(x_1,x_2,x_3)$, this integral becomes

$$\begin{aligned}\psi_\lambda(x)=&\int_{\mathbb{R}^3}e^{\lambda_1(T_{31}+T_{32}+T_{33}-T_{21}-T_{11})+\lambda_2(T_{21}+T_{22}-T_{11})+\lambda_3T_{11}}\\&\times\exp\bigl(-e^{T_{32}-T_{21}}-e^{T_{33}-T_{22}}-e^{T_{22}-T_{11}}-e^{T_{21}-T_{31}}-e^{T_{11}-T_{22}}-e^{T_{22}-T_{32}}\bigr)\,dT_{11}\,dT_{21}\,dT_{22}.\end{aligned}$$

Since $\Psi_\lambda(x)$ is a symmetric function of λ we see that this agrees with Givental's integral formula (7). We note that this is reminiscent of the derivation of Givental's formula given in [Gerasimov et al. 2008] (see also [Gerasimov et al. 2006; 1997]).

8. Fundamental Whittaker functions

The eigenvalue equation (6) also has series solutions known as *fundamental* Whittaker functions. Define a collection of analytic functions $a_{n,m}(\nu)$, $n \geq 2$, $m \in (\mathbb{Z}_+)^{n-1}$, $\nu \in \mathbb{C}^n$ recursively by

$$a_{2,m}(\nu) = \frac{1}{m!\,\Gamma(\nu_1 - \nu_2 + m + 1)},$$

and for $n > 2$,

$$a_{n,m}(\nu) = \sum_k a_{n-1,k}(\mu) \prod_{i=1}^{n-1} \frac{1}{(m_i - k_i)!} \frac{1}{\Gamma(\nu_i - \nu_n + m_i - k_{i-1})},$$

where $\mu_i = \nu_i + \nu_n/(n-1)$, $i \leq n-1$, and the sum is over $k \in (\mathbb{Z}_+)^{n-2}$ satisfying $k_i \leq m_i$, $1 \leq i \leq n-2$, with the convention that $k_0 = k_{n-1} = 0$. Then for each n, $a_{n,m}(\nu)$ satisfies the recursion

$$\left[\sum_{i=1}^{n-1} m_i^2 - \sum_{i=1}^{n-2} m_i m_{i+1} + \sum_{i=1}^{n-1} (\nu_i - \nu_{i+1}) m_i\right] a_{n,m}(\nu) = \sum_{i=1}^{n-1} a_{n,m-e_i}(\nu)$$

[Ishii and Stade 2007, Theorem 15], with the convention that $a_{n,m} = 0$ for $m \notin (\mathbb{Z}_+)^{n-1}$, and $a_{n,0}(\nu) = \prod_{i<j} \Gamma(\nu_i - \nu_j + 1)^{-1}$. Writing $m' = \sum_{i=1}^{n-1} m_i(e_i - e_{i+1})$, the series

$$m_\nu(x) = \sum_m a_{n,m}(\nu) e^{-(m'+\nu,x)} \tag{20}$$

is a fundamental Whittaker function as defined by Hashizume [1982], and satisfies the eigenvalue equation (6). We adopt a slightly different normalisation than the ones used in [Hashizume 1982] or [Ishii and Stade 2007]. Note that, for each $x \in \mathbb{R}^n$, $m_\nu(x)$ is an analytic function of ν. Moreover:

Proposition 9. $$\psi_\nu(x) = \prod_{i<j} \frac{\pi}{\sin \pi(\nu_i - \nu_j)} \sum_{w \in S_n} (-1)^w m_{-w\nu}(x).$$

Proof. This comes from [Baudoin and O'Connell 2011]. In the notation of that paper we are taking $\Pi = \{\alpha_i/2 : i = 1, \dots, n-1\}$, $m(2\alpha) = 0$, $|\eta_\alpha|^2 = 1$ and $\psi_\nu(x) = 2^q k_\nu(x)$, where $q = n(n-1)/2$. □

Now consider the function $\theta_t(x)$ defined by (14). Note that we can write

$$s_n(\lambda) = \frac{1}{(2\pi\iota)^n n!} h(\lambda) \prod_{i>j} \frac{\sin \pi(\lambda_i - \lambda_j)}{\pi}.$$

Corollary 10. $$\theta_t(x) = \frac{1}{(2\pi\iota)^n} \int_{\iota\mathbb{R}^n} m_\lambda(x) h(\lambda) e^{\sum_i \lambda_i^2 t/2} d\lambda.$$

9. Relativistic Toda and q-deformed Whittaker functions

The algebraic structure underlying Theorem 5 is an intertwining relation between certain differential operators associated with the open quantum Toda chain with n particles. This structure should carry over to the setting of Ruijsenaars' relativistic Toda difference operators and q-deformed Whittaker functions [Ruijsenaars 1990; 1999; Etingof 1999; Gerasimov et al. 2010]. A recent (related, but different) development along these lines is given in [Borodin and Corwin 2014]. We will describe here the q-analogue of Theorem 5 in the rank one case, which corresponds to $n = 2$.

In the case $n = 2$, the Whittaker function is given by

$$\psi_\lambda(x) = 2\exp\big(\tfrac{1}{2}(\lambda_1+\lambda_2)(x_1+x_2)\big)K_{\lambda_1-\lambda_2}(2e^{(x_2-x_1)/2}),$$

where $K_\nu(z)$ is the Macdonald function. In this case, Theorem 5 is equivalent to the following theorem of Matsumoto and Yor [1999].

Theorem 11. (1) *Let* $(B_t^{(\mu)}, t \geq 0)$ *be a Brownian motion with drift* μ, *and define*

$$Z_t^{(\mu)} = \int_0^t e^{2B_s^{(\mu)}-B_t^{(\mu)}}\, ds.$$

Then $\log Z^{(\mu)}$ *is a diffusion process with infinitesimal generator*

$$\frac{1}{2}\frac{d^2}{dx^2} + \left(\frac{d}{dx}\log K_\mu(e^{-x})\right)\frac{d}{dx}.$$

(2) *The conditional law of* $B_t^{(\mu)}$, *given* $\{Z_s^{(\mu)},\ s \leq t : Z_t^{(\mu)} = z\}$, *is given by the generalised inverse Gaussian distribution*

$$\tfrac{1}{2}K_\mu(1/z)^{-1}e^{\mu x}\exp(-\cosh(x)/z)\, dx.$$

Let $0 \leq q < 1$. Denote the q-Pochhammer symbol by

$$(q)_n = (q;q)_n = (1-q)\cdots(1-q^n),$$

with the conventions that $(q)_0 = 1$ and $(0)_n = 1$. In what follows we also adopt the convention that $0^0 = 1$.

For $\lambda \in \mathbb{C}$ and $z \geq 0$, define

$$\psi_\lambda(z) = \sum_{y=0}^{z} \frac{q^{\lambda(2y-z)}}{(q)_y(q)_{z-y}}.$$

This is a q-deformed Whittaker function associated with $\mathfrak{sl}_2$ [Gerasimov et al. 2010]. It satisfies the difference equation

$$(1-q^{z+1})\psi_\lambda(z+1) + \psi_\lambda(z-1) = (q^\lambda + q^{-\lambda})\psi_\lambda(z)$$

where we set $\psi_\lambda(-1)=0$, and is related to the q-Hermite polynomials by

$$(q)_z\psi_\lambda(z) = H_z\left(\frac{q^\lambda+q^{-\lambda}}{2}\,\middle|\, q\right).$$

Fix $0\le q<1,\ 0\le p\le 1$, and let $(Y_n, Z_n)_{n\ge 0}$ be a Markov chain with state space $\{(y,z)\in\mathbb{Z}^2 : z\ge y\ge 0\}$ and transition probabilities given by

$$\Pi((y,z),(y+1,z+1)) = p, \quad \Pi((y,z),(y,z+1)) = (1-p)q^y,$$
$$\Pi((y,z),(y-1,z-1)) = (1-p)(1-q^y).$$

Note that Y is itself a Markov chain with transition probabilities

$$P(y,y+1)=p, \quad P(y,y)=(1-p)q^y, \quad P(y,y-1)=(1-p)(1-q^y),$$

and $X=2Y-Z$ is a simple random walk on the integers which increases by one with probability p and decreases by one with probability $1-p$. Choose $\nu\in\mathbb{R}$ such that $p=q^\nu/(q^\nu+q^{-\nu})$.

Theorem 12. *Let $Y_0=Z_0=0$. The process $(Z_n, n\ge 0)$ is a Markov chain with transition probabilities*

$$Q(z,z+1)=\frac{1-q^{z+1}}{q^\nu+q^{-\nu}}\frac{\psi_\nu(z+1)}{\psi_\nu(z)}, \quad Q(z,z-1)=\frac{1}{q^\nu+q^{-\nu}}\frac{\psi_\nu(z-1)}{\psi_\nu(z)}.$$

Moreover, for each $n\ge 0$, the conditional distribution of Y_n, given $\sigma\{Z_m, m\le n\}$ and $Z_n=z$, is given by

$$\pi_z(y)=\psi_\nu(z)^{-1}\frac{q^{\nu(2y-z)}}{(q)_y(q)_{z-y}}, \quad y=0,1,\ldots,z.$$

The proof is straightforward using the theory of Markov functions, by which it suffices to check that the transition operators Π and Q satisfy the intertwining relation $QK=K\Pi$ where

$$K(z,(y,z'))=\frac{\delta_{z,z'}q^{\nu(2y-z)}}{\psi_\nu(z)(q)_y(q)_{z-y}}.$$

This intertwining relation is readily verified. When $q=0$ and $\nu=0$, $\psi_\nu(z)=z$ and the above theorem can be interpreted as the discrete version of Pitman's $2M-X$ theorem, which states that if X_n is a simple symmetric random walk and $M_n=\max_{m\le n}X_m$, then $2M-X$ is a Markov chain with transition probabilities $Q(z,z+1)=(z+1)/2z,\ Q(z,z-1)=(z-1)/2z$. When $q\to 1$, it should rescale to Theorem 11.

The analogue of the output/Burke theorem in the setting of Theorem 12 is the following. If $p<1/2$, then the Markov chain Y has a stationary distribution. If Y_0 is chosen according to this distribution and $Z_0=0$, the process $(Z_n, n\ge 0)$

is a simple random walk on the integers which increases by one with probability p and decreases by one with probability $1 - p$.

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