Asymptotic expansions for β matrix models and their applications to the universality conjecture

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We consider β matrix models with real analytic potentials for both one-cut and multi-cut regimes. We discuss recent results on the asymptotic expansion of the correlators and partition functions and their applications to the studies of random matrices.

1. Introduction

We consider the probability measure on $\mathbb{R}^n$ of the form

$$p_{n,\beta}(\lambda_1, \ldots, \lambda_n) = Q_{n,\beta}^{-1}[V] \prod_{i=1}^{n} e^{-n\beta V(\lambda_i)/2} \prod_{1 \le i < j \le n} |\lambda_i - \lambda_j|^{\beta} \qquad (1\text{-}1)$$

$$=: Q_{n,\beta}^{-1}[V] e^{\beta H(\lambda_1, \ldots, \lambda_n)/2},$$

$$\mathbf{E}_{n,\beta}\{(\ldots)\} = \int (\ldots) p_{n,\beta}(\lambda_1, \ldots, \lambda_n) \, d\bar{\lambda}, \qquad (1\text{-}2)$$

where the function H, which we call the Hamiltonian to stress the analogy with statistical mechanics, and the normalizing constant $Q_{n,\beta}[V]$ (partition function) have the form

$$H(\lambda_1, \ldots, \lambda_n) = -n \sum_{i=1}^{n} V(\lambda_i) + \sum_{i \ne j} \log |\lambda_i - \lambda_j|$$

$$= \int e^{\beta H(\lambda_1, \ldots, \lambda_n)/2} d\bar{\lambda}. \qquad (1\text{-}3)$$

The function V (called the potential) is a real-valued Hölder function satisfying the condition

$$V(\lambda) \ge 2(1 + \epsilon) \log(1 + |\lambda|). \qquad (1\text{-}4)$$

We will study the asymptotic behavior (for large n) of $Q_{n,\beta}[V]$ and the marginal densities of (1-1) (correlation functions)

$$p_{l,\beta}^{(n)}(\lambda_1, \ldots, \lambda_l) = \int_{\mathbb{R}^{n-l}} p_{n,\beta}(\lambda_1, \ldots \lambda_l, \lambda_{l+1}, \ldots, \lambda_n) \, d\lambda_{l+1} \ldots d\lambda_n. \quad (1\text{-}5)$$

The distribution (1-1) can be considered for any $\beta > 0$, but the cases $\beta = 1, 2, 4$ are especially important, since they correspond to the eigenvalue distribution of real symmetric, hermitian, and symplectic matrix models respectively.

Since the papers [Boutet de Monvel et al. 1995; Johansson 1998] it is known that if V is a Hölder function, then

$$n^{-2} \log Q_{n,\beta}[V] = \frac{\beta}{2} \mathcal{E}[V] + O(\log n / n),$$

where

$$\mathcal{E}[V] = - \min_{m \in \mathcal{M}_1} \left\{ L[dm, dm] + \int V(\lambda) m(d\lambda) \right\} = \mathcal{E}_V(m^*), \quad (1\text{-}6)$$

and the minimizing measure m^* (called the equilibrium measure) has a compact support $\sigma := \operatorname{supp} m^*$. Here and below we denote

$$
\begin{aligned}
L[dm, dm] &= \int \log |\lambda - \mu|^{-1} dm(\lambda) \, dm(\mu), \\
L[f](\lambda) &= \int \log |\lambda - \mu|^{-1} f(\mu) \, d\mu, \quad L[f, g] = (L[f], g),
\end{aligned}
\quad (1\text{-}7)
$$

where $(. , .)$ is a standard inner product of $L^2[\mathbb{R}]$.

Moreover, it was proved in [Boutet de Monvel et al. 1995] that for any h whose first derivative is bounded in σ_ε (ε-neighborhood of σ) we have

$$\left| \int h(\lambda) \big(p_{1,\beta}^{(n)} d\lambda - dm(\lambda) \big) \right| \le C \|h'\|_\infty (\log n / n)^{1/2}. \quad (1\text{-}8)$$

Here and below $\|\varphi\|_\infty = \sup_{\lambda \in \sigma_\varepsilon} |\varphi(\lambda)|$.

If V' is a Hölder function, then the equilibrium measure m^* has a density ρ (equilibrium density). The support σ and the density ρ are uniquely defined by the conditions:

$$
\begin{aligned}
v(\lambda) := 2 \int \log |\mu - \lambda| \rho(\mu) \, d\mu - V(\lambda) &= \sup v(\lambda) := v^*, \quad \lambda \in \sigma, \\
v(\lambda) &\le \sup v(\lambda), \quad \lambda \notin \sigma, \quad \sigma = \operatorname{supp}\{\rho\}.
\end{aligned}
\quad (1\text{-}9)
$$

Without loss of generality we will assume below that $\sigma \subset (-1, 1)$ and $v^* = 0$.

In this paper we discuss the asymptotic expansion of the partition function $Q_{n,\beta}[V]$ and of the Stieltjes transforms of the marginal densities. Problems of this kind appear in many fields of mathematics, including the statistical

mechanics of log-gases, combinatorics (graphical enumeration), and the theory of orthogonal polynomials (see [Ercolani and McLaughlin 2003] for the detailed and interesting discussion on the motivation of the problem). Here we are going to discuss with more details the applications of the problems to studies of the eigenvalue distribution of random matrices.

The first important problem of the eigenvalue distribution is the behavior of the random variables called the linear eigenvalue statistics, which correspond to the smooth test function h

$$\mathcal{N}_n[h] = \sum_{i=1}^{n} h(\lambda_i). \tag{1-10}$$

The result of (1-8) gives us the main term of the expectation of $\mathbf{E}_{n,\beta}\{\mathcal{N}_n[h]\}$. It was also proved in [Boutet de Monvel et al. 1995] that the variance of $\mathcal{N}_n[h]$ tends to zero, as $n \to \infty$. But the behavior of the fluctuations of $\mathcal{N}_n[h]$ was studied only in the case of one-cut potentials (see [Johansson 1998]). Even the bound for $\mathbf{Var}_{n,\beta}\{\mathcal{N}_n[h]\}$ in the multi-cut regime was known only for $\beta = 2$. Thus the behavior of the characteristic functional, corresponding to the linear eigenvalue statistics (1-10) of the test function h

$$Z_{n,\beta}[h] = \mathbf{E}_{n,\beta}\big\{e^{\mathcal{N}_n[h] - \mathbf{E}_{n,\beta}\{\mathcal{N}_n[h]\}}\big\} = \frac{Q_{n,\beta}\big[V - \frac{2}{\beta}(h - \mathbf{E}_{n,\beta}\{n^{-1}\mathcal{N}_n[h]\})\big]}{Q_{n,\beta}[V]}$$
$$\tag{1-11}$$

is one of the questions of primary interest in the random matrix theory. It is evident from the right-hand side of (1-11) that since $Z_{n,\beta}[h]$ is a ratio of two partition functions, to study the behavior of $Z_{n,\beta}[h]$, it suffices to find the coefficients of the expansion of $\log Q_{n,\beta}[V]$ up to the order $o(1)$.

The other very important question of the theory of random matrices is so-called the universality conjecture for the local eigenvalue statistics. According to this conjecture, for example, for the bulk of the spectrum, the behavior of the scaled correlation functions of (1-5)

$$p_{k,\beta}^{(n)}\big(\lambda_0 + x_1/(n\rho(\lambda_0)), \ldots, \lambda_0 + x_k/(n\rho(\lambda_0))\big)$$

in the limit $n \to \infty$ is universal, that is, do not depend on V and depends only on β. For $\beta = 2$ this problem is very well studied now. It is well known (see, e.g., [Mehta 1991]) that for $\beta = 2$ all correlation functions of (1-5) can be expressed in the terms of the reproducing kernel of the system of polynomials orthogonal with a varying weight $e^{-n\beta V}$. The orthogonal polynomial machinery, in particular, the Christoffel–Darboux formula and Christoffel function simplify considerably the studies of marginal densities (1-5). This allows to study the local eigenvalue statistics in many different cases: bulk of the spectrum, edges of the spectrum,

special points, etc. (see [Pastur and Shcherbina 1997; 2008; 2011; Deift et al. 1999; Bleher and Its 2003; Claeys and Kuijlaars 2006; Levin and Lubinsky 2008; McLaughlin and Miller 2008; Shcherbina 2011]).

For $\beta = 1, 4$ the situation is more complicated. It was shown in [Tracy and Widom 1998] that all correlation functions can be expressed in terms of some 2×2-matrix kernels. But the representation is less convenient than that in the case $\beta = 2$. It makes difficult the problems, which for $\beta = 2$ are just simple exercises. For example, the bound for the variance of linear eigenvalue statistics for $\beta = 2$ is a trivial corollary of the Christoffel–Darboux formula for any σ, while for $\beta = 1, 4$, as it was mentioned above, in the multi-cut regime the bound was not known till the recent time. As for the universality conjecture, there were a number of papers with improving results, first for monomial $V = \lambda^{2m} + o(1)$, (see [Stojanovic 2000; Deift and Gioev 2007b; 2007a; Deift et al. 2007]) proving the bulk and edge universality for $\beta = 1, 4$, then for arbitrary real analytic one-cut potential (see [Shcherbina 2009b; 2009a]). But combining interesting observations of the papers [Widom 1999; Stojanovic 2000], we conclude that to prove the bulk universality for $\beta = 1, 4$, it is enough to control $\log Q_{n,\beta}[V]$ up to the $O(1)$ terms. This was done first for the one-cut case in [Kriecherbauer and Shcherbina 2010] and then in the multi-cut case in [Shcherbina 2011] (see Section 3 for a more detailed discussion of the universality proof).

Let us mention now the most important results on the expansion of $\log Q_{n,\beta}[V]$ and the correlators. The CLT for linear eigenvalue statistics in the one-cut regime for any β and polynomial V was proved in [Johansson 1998]. The expansion for the first and the second correlators for $\beta = 2$ and one-cut real analytic V and $\beta = 2$ was proved in [Albeverio et al. 2001]. The expansion of $\log Q_{n,\beta}[V]$ for a one-cut polynomial V and $\beta = 2$ was obtained in [Ercolani and McLaughlin 2003]. The formal expansion for any β and polynomial V were obtained in the physical papers [Chekhov and Eynard 2006; Eynard 2009]. The CLT for real analytic multi-cut V and special $h = V$ for $\beta = 2$ was obtained in [Pastur 2006]. The control of $\log Q_{n,\beta}[V]$ up to $O(1)$ for one-cut real analytic V and multi-cut real analytic V was performed in [Kriecherbauer and Shcherbina 2010] and [Shcherbina 2011], respectively. The expansion of the partition function and all the correlators for the one-cut real analytic V and any β was constructed in [Borot and Guionnet 2013]. And the CLT for linear eigenvalue statistics in the multi-cut regime for any β and polynomial V was proved recently in [Shcherbina 2013].

The paper is organized as follows. In Section 2 we discuss the CLT and the expansion of the partition function and correlators in the one-cut regime, obtained in [Johansson 1998; Kriecherbauer and Shcherbina 2010; Borot and Guionnet 2013]. In Section 3 we discuss the applications of the results on the

control of $\log Q_{n,\beta}[V]$ up to $O(1)$ in the multi-cut case to the proof of the bulk
universality for $\beta = 1, 4$ in the multi-cut case, following [Kriecherbauer and
Shcherbina 2010; Shcherbina 2011], and in Section 4 we discuss the results of
[Shcherbina 2013] on the CLT in the multi-cut case.

Throughout the paper we assume the following conditions on the potential V:

C1. *V is a Hölder function satisfying* (1-4)*, which is analytic in some open
domain of* $\mathbf{D} \subset \mathbb{C}$ *containing the support* σ *of the corresponding equilibrium
measure, and*

$$\sigma = \bigcup_{\alpha=1}^{q} \sigma_\alpha, \quad \sigma_\alpha = [a_\alpha, b_\alpha]; \tag{1-12}$$

C2. *The equilibrium density* ρ *can be represented in the form*

$$\rho(\lambda) = \frac{1}{2\pi} P(\lambda) \Im X^{1/2}(\lambda + i0), \quad \inf_{\lambda \in \sigma} |P(\lambda)| > 0, \tag{1-13}$$

where

$$X(z) = \prod_{\alpha=1}^{q} (z - a_\alpha)(z - b_\alpha), \tag{1-14}$$

and we choose a branch of $X^{1/2}(z)$ *such that* $X^{1/2}(z) \sim z^q$, *as* $z \to +\infty$.
Moreover, the function v *defined by* (1-9) *attains its maximum only if* λ
belongs to σ.

Remark. It is known (see, e.g., [Albeverio et al. 2001]) that for analytic V the
equilibrium density ρ always has the form (1-13)–(1-14). The function P in
(1-13) is analytic and can be represented in the form

$$P(z) = \frac{1}{2\pi i} \oint_{\mathscr{L}} \frac{V'(z) - V'(\zeta)}{(z - \zeta) X^{1/2}(\zeta)} \, d\zeta. \tag{1-15}$$

Hence condition C2 means that ρ has no zeros in the internal points of σ and
behaves like square root near the edge points. This behavior of V is usually
called generic (see [Kuijlaars and McLaughlin 2000] for the results explaining
the term).

2. Asymptotic expansion and CLT for β matrix models in the one-cut regime

The one-cut case is the simplest version of the possible spectrum of the β-models.
As it is clear from the physical papers [Chekhov and Eynard 2006; Eynard 2009],
it is the only case when it is expected that fluctuations of eigenvalue statistics are
asymptotically Gaussian and the asymptotic expansions of $\log Q_{n,\beta}[V]$ does not
contain some kind of θ-function. Hence, almost all known before results on the

expansions of $\log Q_{n,\beta}[V]$ and the correlators (see the definition in (2-4) below) were obtained for the one-cut potentials V. One of the first results in this direction is the CLT for linear eigenvalue statistics, which was proved by Johansson [1998] and improved in [Kriecherbauer and Shcherbina 2010; Shcherbina 2013].

Theorem 1. *Let V satisfy condition C1–C2 and $\sigma = \mathrm{supp}\, \rho = [a, b]$. Then for any real-valued h with $\|h^{(4)}\|_\infty$, $\|h'\|_\infty \le \log n$ the characteristic functional $Z_{n,\beta}[h]$ of (1-11) has the form*

$$Z_{n,\beta}[h] = \exp\left\{ \frac{\beta}{2}\left(\left(\frac{2}{\beta} - 1 \right)(h, \nu) + \tfrac{1}{4}(\bar{D}h, h) \right) \right\}$$
$$\cdot \left(1 + n^{-1} O\left(\|h'\|_\infty^3 + \|h^{(4)}\|_\infty^3 \right) \right), \quad (2\text{-}1)$$

where the operator $\bar{D}_\sigma$ is defined as

$$\bar{D}_\sigma = \frac{1}{2}(D_\sigma + D_\sigma^*), \quad D_\sigma h(\lambda) = \frac{X^{-1/2}(\lambda)}{\pi^2} \int_\sigma \frac{h'(\mu)X^{1/2}(\mu)d\mu}{(\lambda - \mu)}, \quad (2\text{-}2)$$

and the nonpositive measure ν has the form

$$(\nu, h) := \tfrac{1}{4}(h(b) + h(a)) - \frac{1}{2\pi} \int_\sigma \frac{h(\lambda)\, d\lambda}{X^{1/2}(\lambda)} + \tfrac{1}{2}(D_\sigma \log P, h), \quad (2\text{-}3)$$

with P defined by (1-15) and $X^{1/2}(\lambda) := \Im X^{1/2}(\lambda + i0)$ with X of (1-14).

The method of the proof proposed in [Johansson 1998] was based on the analysis of the first loop equation (see (2-10) below) combined with a priory bound (1-8), obtained in [Boutet de Monvel et al. 1995]. But it was used essentially in the proof that V is a polynomial. Then in [Kriecherbauer and Shcherbina 2010] the method of [Johansson 1998] was generalized to any one-cut analytic potential. Moreover, $\log Q_{n,\beta}[V]$ was found up to $O(1)$ term by using the idea of the interpolation between the Gaussian potential and the arbitrary one-cut potential. The last step in the construction of the asymptotic expansion in n^{-1} was done recently in [Borot and Guionnet 2013]. The authors studied the asymptotic expansion of all correlators, defined as

$$w_k(z_1, \ldots, z_k) := \frac{1}{n} \frac{\partial^k}{\partial t_1 \ldots \partial t_k} \log Q_{n,\beta}\left[V - \frac{2}{\beta n} \sum_{j=1}^k t_k \phi_{z_k} \right]\bigg|_{t_1 = \cdots = t_k = 0}, \quad (2\text{-}4)$$

where

$$\phi_z(\lambda) = \frac{1}{z - \lambda}.$$

One can easily see that then, for example, w_1 is the Stieltjes transform of the first marginal density (1-5) and

$$w_1(z) = n^{-1} \mathbf{E}_{n,\beta}\{\mathcal{N}_n[\phi_z]\} = (\phi_z, p_{1,\beta}^{(n)}),$$
$$w_2(z_1, z_2) = n^{-1} \mathbf{Cov}_{n,\beta}\{\mathcal{N}_n[\phi_{z_1}], \mathcal{N}_n[\phi_{z_2}]\}. \tag{2-5}$$

The main result of [Borot and Guionnet 2013] is the following theorem.

Theorem 2. *Under conditions* C1–C2, *any correlator* (2-4) *admits an asymptotic expansion of any order* m, *which means that*

$$w_k(z_I) = \sum_{j=k-1}^{m} n^{-j} w_k^{(j)}(z_I) + O(n^{-m-1}), \tag{2-6}$$

where the bound is uniform in $z_1, \ldots, z_k$ varying in any compact K of the upper half-plane.

Moreover, $\log Q_{n,\beta}$ *also admits the asymptotic expansion in n of any order m:*

$$\log(Q_{n,\beta}/n!) = \frac{\beta n^2}{2}\mathscr{E}[V] + F_\beta(n) + n\left(\frac{\beta}{2}-1\right)\left((\log \rho, \rho) - 1 - \log 2\pi\right)$$
$$+ \sum_{j=0}^{m} n^{-j} q^{(j)}[\rho] + O(n^{-m-1}), \tag{2-7}$$

where the coefficients of the expansion $q^{(j)}[\rho]$ are defined in terms of the integrals with the Stieltjes transform of the equilibrium density ρ, $F_\beta(n)$ collects the term which appears in the Gaussian case

$$F_\beta(n) = \log(Q_{n,\beta}^*/n!) + \frac{3\beta n^2}{8},$$

and $Q_{n,\beta}^*$ is the partition function of the Gaussian case, that is, corresponds to $V(\lambda) = \frac{1}{2}\lambda^2$.

Remark. By the Selberg formula (see, e.g., [Forrester 2010]), we have

$$Q_{n,\beta}^*/n! = \left(\frac{n\beta}{2}\right)^{-\beta n^2/4 - n(1-\beta/2)/2} (2\pi)^{n/2} \prod_{j=1}^{n} \frac{\Gamma(\beta j/2)}{\Gamma(\beta/2)}. \tag{2-8}$$

Moreover, it is known (see [Forrester 2010]) that

$$F_\beta(n) = n\left(\frac{\beta}{2}-1\right)\left(\log\frac{n\beta}{2} - \frac{1}{2}\right) + n\log\frac{\sqrt{2\pi}}{\Gamma(\beta/2)} - c_\beta \log n + c_\beta^{(1)} + o(1), \tag{2-9}$$

where

$$c_\beta = \frac{\beta}{24} - \frac{1}{4} + \frac{1}{6\beta},$$

and $c_\beta^{(1)}$ is some constant, depending only on β (for $\beta = 2$, $c_\beta^{(1)} = \zeta'(1)$).

Sketch of the proof. As was mentioned above, the proof (given in [Borot and Guionnet 2013]) is a nice combination of the methods and results of [Johansson 1998; Kriecherbauer and Shcherbina 2010] with the analysis of the loop equations given in the physical papers [Chekhov and Eynard 2006; Eynard 2009]. The first loop equation is well known and used in many papers [Pastur and Shcherbina 1997; Johansson 1998; Kriecherbauer and Shcherbina 2010]:

$$w_1^2(z) - V'(z)w_1(z) + \frac{1}{2\pi i} \oint_L \frac{V'(z)-V'(\zeta)}{z-\zeta} w_1(\zeta)\, d\zeta$$
$$= \frac{1}{n}\left(\frac{2}{\beta}-1\right)\partial_z w_1(z) - \frac{1}{n}w_2(z,z). \quad (2\text{-}10)$$

Here and below the contours L, L' (and so on) in $\mathbf{D}$ encircle the ε-neighborhood of the spectrum Σ, but do not contain z and zeros of P of (1-15). The other loop equations can be obtained from the first one by differentiating as in (2-4):

$$(2w_1(z) - V'(z))w_{k+1}(z, z_I) + \frac{1}{2\pi i} \oint \frac{V'(z) - V'(\zeta)}{z - \zeta} w_{k+1}(\zeta, z_I)\, d\zeta$$
$$= F_{k+1}\big(z; \{w_j\}_{j=2}^{k+2}\big),$$

where

$$F_{k+1}\big(z; \{w_j\}_{j=2}^{k+2}\big)$$
$$:= \frac{1}{n}\left(\frac{2}{\beta}-1\right)\partial_z w_{k+1}(z, z_I) - \sum_{\substack{J \subset I \\ |J| \neq 0, k}} w_{|J|+1}(z, z_J) w_{k+1-|J|}(z, z_{I \setminus J})$$
$$- \frac{2}{\beta}\sum_{j=1}^{k} \partial_{z_j} \frac{w_k(z, z_{I \setminus \{j\}}) - w_k(z_I)}{z - z_j} - \frac{1}{n}w_{k+2}(z, z, z_I).$$

It was proved in [Johansson 1998; Kriecherbauer and Shcherbina 2010] that

$$w_1(z) = g(z) + n^{-1}w_1^{(1)}(z) + O(n^{-2}),$$
$$g(z) = \tfrac{1}{2}\big(V'(z) - P(z)X^{1/2}(z)\big). \quad (2\text{-}11)$$

Substituting this expression in (2-10) and multiplying the result by n, we obtain an equation with respect to $w_1^{(1)}$, which (combined with equations for $\{w_k\}_{k \geq 2}$ above) gives us the system of equations:

$$\mathcal{K}w_1^{(1)}(z) = \left(1 - \frac{2}{\beta}\right)\partial_z\left(g(z) + \frac{1}{n}w_1^{(1)}(z)\right) - w_2(z, z) - \frac{1}{n}\big(w_1^{(1)}(z)\big)^2$$
$$=: F_1\big(z; w_1^{(1)}, w_2\big),$$

$$\mathcal{K}w_{k+1}(z, z_I) = F_{k+1}(z; \{w_j\}_{j=2}^{k+2}) - \frac{2}{n}w_1^{(1)}(z)w_{k+1}(z, z_I), \quad (2\text{-}12)$$

where the linear operator $\mathcal{H} : \mathrm{Hol}\,[\mathbf{D} \setminus \sigma] \to \mathrm{Hol}\,[\mathbf{D} \setminus \sigma]$ is defined as

$$\mathcal{H} f(z) = -P(z) X^{1/2}(z) f(z) + \frac{1}{2\pi i} \oint \frac{V'(z) - V'(\zeta)}{z - \zeta} f(\zeta)\, d\zeta.$$

Consider also the operator $\mathcal{H}^{(-1)} : \mathrm{Hol}\,[\mathbf{D} \setminus \sigma] \to \mathrm{Hol}\,[\mathbf{D} \setminus \sigma]$ of the form

$$\mathcal{H}^{(-1)} f(z) := \frac{1}{2\pi i\, X^{1/2}(z)} \oint_{\mathscr{L}} \frac{f(\zeta)\, d\zeta}{P(\zeta)(z - \zeta)}. \tag{2-13}$$

Till now we have not used that we have a one-cut potential. The loop equations can be written in the multi-cut case as well as in the one-cut and the operators $\mathcal{H}$ and $\mathcal{H}^{(-1)}$ can be constructed by the same formulas, if we use P and X of (1-15) and (1-14). It is straightforward to check that if we apply to the both parts of (2-12) the operator (2-13), then in the multi-cut case (when $X^{1/2}(z) \sim z^q$) we obtain

$$w_{k+1}(z, z_I) + \frac{p_{k+1}(z; z_I)}{X^{1/2}(z)} = \\ \mathcal{H}^{(-1)} F_{k+1}\big(z; \{w_j\}_{j=2}^{k+2}\big) - \frac{2}{n}\mathcal{H}^{(-1)} w_1^{(1)}(z) w_{k+1}(z, z_I),$$

where $p_{k+1}(z; z_I)$ is a polynomial with respect to z of degree $q - 2$, whose coefficients are the linear combinations of the first $q - 1$ coefficient in the asymptotic expansion of $w_{k+1}(z, z_I)$ with respect to z^{-j}. The main technical obstacle to study the multi-cut case by the method described in this section is that for $q \neq 1$ we do not know these coefficients, while in the case $q = 1$ (one-cut case) it is easy to see that $p_{k+1}(z; z_I) = 0$ for all $k \geq 0$ and we obtain the system of equations

$$w_{k+1}(z, z_I) = \mathcal{H}^{(-1)} F_{k+1}(z; \{w_j\}_{j=2}^{k+2}) - \frac{2}{n}\mathcal{H}^{(-1)} w_1^{(1)}(z) w_{k+1}(z, z_I). \tag{2-14}$$

The key technical point in the analysis of the last equations is a priory estimate

$$|w_{k+1}(z, z_I)| \leq n^{-1} C(z, z_I), \quad k \geq 1. \tag{2-15}$$

It can be derived from the bound proven in [Johansson 1998] (see also [Kriecherbauer and Shcherbina 2010]). Let $\mathscr{L}$ be any contour enclosed σ. Then there is a constant C_L such that for any real analytic function φ,

$$\mathbf{E}_{n,\beta}\big\{\exp\{\overset{\circ}{\mathcal{N}}_n[\varphi]/\big(C_L \sup_{\zeta \in \mathscr{L}} |\varphi(\zeta)|\big)\}\big\} \leq 6 \\ \Rightarrow \mathbf{E}_{n,\beta}\{|\overset{\circ}{\mathcal{N}}_n[\varphi]|^p\} \leq C_p (C_L \sup_{\zeta \in \mathscr{L}} |\varphi(\zeta)|)^p,$$

where $\overset{\circ}{\mathcal{N}}_n[\varphi] = \mathcal{N}_n[\varphi] - \mathbf{E}_{n,\beta}\{\mathcal{N}_n[\varphi]\}$. The last bound implies (2-15). With this bound in hands it is easy to see that (2-14) has "triangle" form: the right-hand side of the equation for w_{k+1} contains $w_2, \ldots, w_k$, the derivative of $n^{-1} w_{k+1}$,

$n^{-1}w_1^{(1)}w_{k+1}$, and $n^{-1}w_{k+2}$. Hence it can be solved in each order in n^{-1} starting from the first equation and going down step by step. This leads to the assertion (2-6).

To derive the assertion (2-7) from (2-6), we use the idea of [Kriecherbauer and Shcherbina 2010] of interpolation between the Gaussian (quadratic) potential with the same support $\sigma = [a, b]$ and the potential V. Consider the functions $V^{(0)}$ and V_t of the form

$$
\begin{aligned}
V^{(0)}(\lambda) &= 2(\lambda - c)^2/d, \quad c = \tfrac{1}{2}(a+b), \ d = b - a, \\
V_t(\lambda) &= tV(\lambda) + (1-t)V^{(0)}(\lambda).
\end{aligned}
\tag{2-16}
$$

Let $Q_{n,\beta}(t) := Q_{n,\beta}[V_t]$ be defined by (1-3) with V replaced by V_t. Then, evidently, $Q_{n,\beta}(1) = Q_{n,\beta}[V]$, and $Q_{n,\beta}(0) = Q_{n,\beta}[V^{(0)}]$. Hence

$$
\begin{aligned}
\frac{1}{n^2}\log Q_{n,\beta}(1) - \frac{1}{n^2}\log Q_{n,\beta}(0) &= \frac{1}{n^2}\int_0^1 dt\,\frac{d}{dt}\log Q_{n,\beta}(t) \\
&= -\frac{\beta}{2\pi i}\int_0^1 dt \oint_L dz(V(z)-V^{(0)}(z))w_1(z;t),
\end{aligned}
\tag{2-17}
$$

where $w_1(z;t)$ is defined by (2-4) for V_t. Using (1-9), one can check that for the distribution (1-1) with V replaced by V_t the equilibrium density ρ_t has the form

$$
\rho_t(\lambda) = t\rho(\lambda) + (1-t)\rho^{(0)}(\lambda), \quad \rho^{(0)}(\lambda) = \frac{2X^{1/2}(\lambda)}{\pi d^2},
\tag{2-18}
$$

with X of (1-14). Hence, substituting (2-11) for V_t into (2-17), we get

$$
\begin{aligned}
\log Q_{n,\beta}[V] = \log Q_{n,\beta}[V^{(0)}] &- n^2\frac{\beta}{2}\mathscr{E}[V^{(0)}] + n^2\frac{\beta}{2}\mathscr{E}[V] \\
&+ \frac{\beta n}{2}\frac{1}{(2\pi i)}\int_0^1 dt\oint_{\mathscr{L}}(V(z)-V^{(0)}(z))w_1^{(1)}(z;t)\,dz,
\end{aligned}
$$

Then we use the expression for $w_1^{(1)}(z;t)$ which follow from the first equations of (2-12). After some transformations we arrive at (2-7). $\qquad\square$

3. Bulk universality for orthogonal and symplectic ensembles

As it was mentioned in Introduction one of the most important applications of the asymptotic expansion of $\log Q_{n,\beta}[V]$ is the proof of the universality of the local regime in the case of $\beta = 1, 4$ (for real symmetric and symplectic matrix models). Throughout this section we will assume that V is a polynomial of degree $2m$, satisfying condition C2, and n is even, but the result can be generalized on V, satisfying conditions C1–C2. According to the results of [Tracy and Widom

1998], the matrix kernels for $\beta = 1, 4$ can be expressed in terms of the scalar kernels

$$S_{n,1}(\lambda, \mu) = -\sum_{j,k=0}^{n-1} \psi_j^{(n)}(\lambda)(M_n^{(n)})_{jk}^{-1}(\epsilon \psi_k^{(n)})(\mu), \qquad (3\text{-}1)$$

$$S_{n/2,4}(\lambda, \mu) = -\sum_{j,k=0}^{n-1} (\psi_j^{(n)})'(\lambda)(D_n^{(n)})_{jk}^{-1} \psi_k^{(n)}(\mu); \qquad (3\text{-}2)$$

here $\epsilon(\lambda) = \frac{1}{2}\,\text{sgn}(\lambda)$ (sgn denoting the standard signum function),

$$(\epsilon f)(\lambda) := \int_{\mathbb{R}} \epsilon(\lambda - \mu) f(\mu)\, d\mu,$$

and $D_n^{(n)}$ and $M_n^{(n)}$ are the top left corner $n \times n$ blocks of the semiinfinite matrices that correspond to the differentiation operator and to some integration operator, respectively:

$$D_\infty^{(n)} := \big((\psi_j^{(n)})', \psi_k^{(n)}\big)_{j,k \geq 0}, \quad D_n^{(n)} = \big\{D_{jk}^{(n)}\big\}_{j,k=0}^{n-1},$$
$$M_\infty^{(n)} := \big(\epsilon \psi_j^{(n)}, \psi_k^{(n)}\big)_{j,k \geq 0}, \quad M_n^{(n)} = \big\{M_{jk}^{(n)}\big\}_{j,k=0}^{n-1}. \qquad (3\text{-}3)$$

Both matrices $D_\infty^{(n)}$ and $M_\infty^{(n)}$ are skew-symmetric, and since $\epsilon(\psi_j^{(n)})' = \psi_j^{(n)}$, we have for any $j, l \geq 0$ that

$$\delta_{jl} = \big(\epsilon(\psi_j^{(n)})', \psi_l\big) = \sum_{k=0}^{\infty} (D_\infty^{(n)})_{jk}(M_\infty^{(n)})_{kl} \iff D_\infty^{(n)} M_\infty^{(n)} = 1 = M_\infty^{(n)} D_\infty^{(n)}.$$

It was observed in [Widom 1999] that if V is a rational function, in particular, a polynomial of degree $2m$, then the kernels $S_{n,1}$, $S_{n,4}$ can be written as

$$S_{n,1}(\lambda, \mu) = K_{n,2}(\lambda, \mu) + n \sum_{j,k=-(2m-1)}^{2m-1} F_{jk}^{(1)} \psi_{n+j}^{(n)}(\lambda) \epsilon \psi_{n+k}^{(n)}(\mu),$$

$$\qquad (3\text{-}4)$$

$$S_{n/2,4}(\lambda, \mu) = K_{n,2}(\lambda, \mu) + n \sum_{j,k=-(2m-1)}^{2m-1} F_{jk}^{(4)} \psi_{n+j}^{(n)}(\lambda) \epsilon \psi_{n+k}^{(n)}(\mu),$$

where $F_{jk}^{(1)}$, $F_{jk}^{(4)}$ can be expressed in terms of the matrix T_n^{-1}, where T_n is the $(2m-1) \times (2m-1)$ block in the bottom right corner of $D_n^{(n)} M_n^{(n)}$:

$$(T_n)_{jk} := (D_n^{(n)} M_n^{(n)})_{n-2m+j, n-2m+k}, \quad 1 \leq j, k \leq 2m-1. \qquad (3\text{-}5)$$

The main technical obstacle to study the kernels $S_{n,1}$, $S_{n,4}$ is the problem to prove that $(T_n^{-1})_{jk}$ are bounded uniformly in n. Till the recent time this technical

problem was solved only in a few cases. In [Deift and Gioev 2007b; 2007a] the case $V(\lambda) = \lambda^{2m}(1 + o(1))$ (in our notations) was studied and the problem of invertibility of T_n was solved by computing the entries of T_n explicitly. Similar method was used in [Deift et al. 2007] to prove bulk and edge universality (including the case of the hard edge) for the Laguerre type ensembles with monomial V. In [Stojanovic 2000] the problem of invertibility of T_n was solved also by computing the entries of T_n for V being an even quartic polynomial. In [Shcherbina 2009b; Shcherbina 2009a] similar problem was solved without explicit computation of the entries of T_n. It was shown that for any real analytic V with one interval support of the equilibrium density $(M_n^{(n)})^{-1}$ is uniformly bounded in the operator norm.

But there is also a possibility to prove that T_n is invertible with another technique. As a by product of the calculation in [Tracy and Widom 1998] one also obtains relations between the partition functions $Q_{n,\beta}$ and the determinants of $M_n^{(n)}$ and $D_n^{(n)}$:

$$\det M_n^{(n)} = \left(\frac{Q_{n,1}\Gamma_n}{n!\, 2^{n/2}} \right)^2, \quad \det D_n^{(n)} = \left(\frac{Q_{n/2,4}\Gamma_n}{(n/2)!\, 2^{n/2}} \right)^2,$$

where

$$\Gamma_n := \prod_{j=0}^{n-1} \gamma_j^{(n)},$$

$\gamma_j^{(n)}$ being the leading coefficient of the j-th orthogonal polynomial $p_j^{(n)}(\lambda)$. It is also known (see [Mehta 1991]) that $Q_{n,2} = n!\,/\,\Gamma_n^2$. Since $d_\infty^{(n)} M_\infty^{(n)} = 1$ and since $(D_\infty^{(n)})_{j,k} = n\,\mathrm{sign}(j-k)V'(J^{(n)})_{jk}$ implies that

$$
\begin{aligned}
(D_\infty^{(n)})_{j,k} &= 0 \quad &\text{if } |j-k| \ge 2m, \\
|(D_\infty^{(n)})_{j,k}| &\le nC \quad &\text{if } |j-n| \le nc,
\end{aligned}
\tag{3-6}
$$

we have $D_n^{(n)} M_n^{(n)} = 1 + \Delta_n$ with Δ_n being zero except for the bottom $2m-1$ rows, and we arrive at this formula, first observed in [Stojanovic 2000]:

$$\det(T_n) = \det(D_n^{(n)} M_n^{(n)}) = \left(\frac{Q_{n,1} Q_{n/2,4}}{Q_{n,2}(n/2)!\, 2^n} \right)^2. \tag{3-7}$$

Hence to control $\det(T_n)$, it suffices to control $\log Q_{n,\beta}$ for $\beta = 1, 2, 4$ up to the order $O(1)$. One can easily see that for a one-cut case the control can be done by using Theorem 1. But, as it was mentioned in the previous section, the method used there does not work for the multi-cut case (see discussion after Equation (2-13)).

In [Shcherbina 2011] the problem to control $Q_{n,\beta}[V]$ for $\beta = 1, 2, 4$ is solved in a little bit different way. It is proved that for any analytical potential V with

q-interval support σ of the equilibrium density $Q_{n,\beta}[V]$ up to the order $O(1)$ can be factorized to a product of some one-cut partition functions with appropriate "effective potentials" $V_\alpha^{(a)}$ defined in terms of σ, V and ρ.

Set

$$\mu_\alpha = \int_{\sigma_\alpha} \rho(\lambda)\,d\lambda, \quad n_\alpha^* := [n\mu_\alpha] + d_\alpha, \tag{3-8}$$

where $[x]$ means an integer part of x, and $d_\alpha = 0, \pm 1, \pm 2$ are chosen in a way which makes n_α^* even (recall that n is even) and

$$\sum n_\alpha^* = n. \tag{3-9}$$

Note, that the choice of d_α is not unique, but a different choice differs only by $O(1)$ and leads to the same expression for $Q_{n,\beta}[V]$ in (3-12) which is up to $O(1)$.

Introduce the "effective potentials"

$$V_\alpha^{(a)}(\lambda) = \mathbf{1}_{\sigma_{\alpha,\varepsilon}}(\lambda)\left(V(\lambda) - 2\int_{\sigma\setminus\sigma_\alpha} \log|\lambda - \mu|\rho(\mu)\,d\mu\right), \tag{3-10}$$

and denote by Σ^* the "cross energy"

$$\Sigma^* := \sum_{\alpha \neq \alpha'} \int_{\sigma_\alpha} d\lambda \int_{\sigma_{\alpha'}} d\mu \log|\lambda - \mu|\rho(\lambda)\rho(\mu). \tag{3-11}$$

Theorem 3. *Let V be a polynomial of degree $2m$ satisfying condition C2 and n be even. Then the matrices $F^{(1)}$ and $F^{(4)}$ in (3-4) are bounded in the operator norm uniformly in n. Moreover, the logarithm of the partition function $Q_{n,\beta}[V]$ can be obtained up to $O(1)$ term from the representation*

$$\log(Q_{n,\beta}[V]/n!) = \sum_{\alpha=1}^{q} \log(Q_{n_\alpha^*,\beta}[V_\alpha^{(a)}]/n_\alpha^*!) - \frac{\beta n^2}{2}\Sigma^* + O(1), \tag{3-12}$$

where $V_\alpha^{(a)}$ and Σ^* are defined in (3-10) and (3-11).

As it was mentioned above, Theorem 3 together with some asymptotic results for orthogonal polynomials of [Deift et al. 1999] proves the universality conjecture for local eigenvalue statistics of the matrix models (1-1).

Theorem 4. *Let V be a polynomial of degree $2m$ satisfying condition C2. Then we have for (even) $n \to \infty$, $\lambda_0 \in \mathbb{R}$ with $\rho(\lambda_0) > 0$, and for $\beta \in \{1, 4\}$ that*

$$(n\rho(\lambda_0))^{-1}S_{n,1}\left(\lambda_0 + \xi/n\rho(\lambda_0), \lambda_0 + \eta/n\rho(\lambda_0)\right) = \frac{\sin\pi(\xi - \eta)}{\pi(\xi - \eta)} + O(n^{-1/2}),$$

$$(n\rho(\lambda_0))^{-1}S_{n/2,4}\left(\lambda_0 + \xi/n\rho(\lambda_0), \lambda_0 + \eta/n\rho(\lambda_0)\right) = \frac{\sin\pi(\xi - \eta)}{\pi(\xi - \eta)} + O(n^{-1/2}).$$

the error bound is uniform for bounded ξ, η *and for* λ_0 *contained in some compact subset of* $\bigcup_{\alpha=1}^q (a_\alpha, b_\alpha)$.

It is an immediate consequence of Theorem 4 and of the formulas which express the correlation functions in terms of $S_{n,1}$ or $S_{n,4}$ (see [Tracy and Widom 1998]) that the corresponding rescaled l-point correlation functions

$$p_{l,1}^{(n)} (\lambda_0 + \xi_1/n\rho(\lambda_0), \ldots, \lambda_0 + \xi_l/n\rho(\lambda_0)),$$

$$p_{l,4}^{(n/2)} (\lambda_0 + \xi_1/n\rho(\lambda_0), \ldots, \lambda_0 + \xi_l/n\rho(\lambda_0))$$

converge for n (even) $\to \infty$ to some limit that depends on $\beta = 1, 4$ but not on the choice of V.

Sketch the proof of Theorem 3. Set

$$\sigma_\varepsilon = \bigcup_{\alpha=1}^q \sigma_{\alpha,\varepsilon}, \quad \sigma_{\alpha,\varepsilon} = [a_\alpha - \varepsilon, b_\alpha + \varepsilon], \quad \text{dist}\,\{\sigma_{\alpha,\varepsilon}, \sigma_{\alpha',\varepsilon}\} > \delta > 0, \quad \alpha \neq \alpha'. \quad (3\text{-}13)$$

First of all we replace the integration domain in the definition of $Q_{n,\beta}[V]$ and $p_{k,\beta}^{(n)}$ from $\mathbb{R}$ to σ_ε. Then, according to [Pastur and Shcherbina 2008], $Q_{n,\beta}[V]$ and $p_{k,\beta}^{(n)}$ will be changed by $(1 + O(e^{-nc}))$ factor.

To understand how the potentials $V_\alpha^{(a)}$ of (3-10) appear, let us represent $H(\bar\lambda)$ as

$$-n \sum_{i=1}^n V(\lambda_i) + \sum_{\substack{i \neq j \\ \alpha,\alpha'=1}}^q \chi_\alpha(\lambda_i)\chi_{\alpha'}(\lambda_j) \log|\lambda_i - \lambda_j|$$

$$= -n \sum_{i=1}^n V(\lambda_i) + \sum_{i \neq j} \sum_{\alpha=1}^q \chi_\alpha(\lambda_i)\chi_\alpha(\lambda_j) \log|\lambda_i - \lambda_j|$$

$$+ 2n \sum_{j=1}^n \chi_\alpha(\lambda_i) \sum_{\alpha' \neq \alpha} \int \log|\lambda_i - \mu|\chi_{\alpha'}(\mu)\rho(\mu)\,d\mu - n^2\Sigma^*$$

$$+ \sum_{\substack{i,j=1,\ldots,n \\ \alpha \neq \alpha'}} \int d\lambda\,d\mu \log|\lambda - \mu|\chi_\alpha(\lambda)\chi_{\alpha'}(\mu)(\delta_{\lambda_i}(\lambda) - \rho(\lambda))(\delta_{\lambda_j}(\mu) - \rho(\mu))$$

$$= H_a(\bar\lambda) + \Delta H(\bar\lambda), \quad (3\text{-}14)$$

where χ_α is the indicator function of the interval $\sigma_{\alpha,\varepsilon}$, $\delta_{\lambda_i}(\lambda) = \delta(\lambda - \lambda_i)$ is a delta-function, the "cross energy" Σ^* is defined in (3-11), and we introduce

$$H_a(\lambda_1 \ldots \lambda_n)$$

$$= -n \sum_{\alpha=1}^q \sum_{i=1}^n V_\alpha^{(a)}(\lambda_i) + \sum_{i \neq j} \log|\lambda_i - \lambda_j|\left(\sum_{\alpha=1}^q \chi_\alpha(\lambda_i)\chi_\alpha(\lambda_j)\right) - n^2\Sigma^*,$$

in which the "effective potential" $V_\alpha^{(a)}(\lambda)$ is defined by (3-10).

Consider

$$Q_{n,\beta}^{(a)}[V] = \int_{\sigma_\varepsilon^n} e^{\beta H_a(\lambda_1,\ldots,\lambda_n)} d\lambda_1 \ldots d\lambda_n.$$

By the Jensen inequality

$$\frac{\beta}{2}\langle \Delta H \rangle_{H_a} \le \log Q_{n,\beta}[V] - \log Q_{n,\beta}^{(a)}[V] \le \frac{\beta}{2}\langle \Delta H \rangle_H.$$

Then it was shown that both the right-hand side and the left-hand side of this inequality are $O(1)$ and

$$\log Q_{n,\beta}^{(a)}[V] = \sum_{\alpha=1}^{q} \log Q_{n_\alpha^*,\beta}[(n/n_\alpha^*)V^{(a)}] - n^2\Sigma^* + O(1),$$

where the n_α^*, $\alpha = 1,\ldots,q$, are chosen to satisfy (3-8) and (3-9). We do not give more details here, because the result follows from (4-10) in the next section. $\square$

4. CLT for β-model in the multi-cut regime

The idea of using some factorization of $Q_{n,\beta}[V]$ into a product of one-cut partition functions for the effective potentials $V_\alpha^{(a)}$ was used in [Shcherbina 2013] to prove the CLT for linear eigenvalue statistics (1-10). In order to formulate corresponding result we need some extra definitions. Consider the Hilbert space

$$\mathcal{H} = \bigoplus_{\alpha=1}^{q} L^2[\sigma_\alpha] \tag{4-1}$$

with the standard inner product $(.\,,.)$. Define the operator $\mathscr{L}$ (cf. (1-7)) by

$$\mathscr{L}f = \mathbf{1}_\sigma L[f], \quad \mathscr{L}_\alpha f := \mathbf{1}_{\sigma_\alpha} L[f\mathbf{1}_{\sigma_\alpha}], \tag{4-2}$$

the block diagonal operators

$$\bar{D} := \bigoplus_{\alpha=1}^{q} \bar{D}_\alpha, \quad \widehat{\mathscr{L}} := \bigoplus_{\alpha=1}^{q} \widehat{\mathscr{L}}_\alpha, \tag{4-3}$$

where $\bar{D}_\alpha$ is defined by (2-2) for σ_α. Moreover, denote

$$\widetilde{\mathscr{L}} := \mathscr{L} - \widehat{\mathscr{L}}, \quad \mathscr{G} := (1 + \bar{D}\widetilde{\mathscr{L}})^{-1}. \tag{4-4}$$

An important role below belongs to a positive definite matrix $\mathscr{Q} = \{\mathscr{Q}_{\alpha\alpha'}\}_{\alpha,\alpha'=1}^{q}$ of the form

$$\mathscr{Q}_{\alpha\alpha'} = (\mathscr{L}\psi^{(\alpha)}, \psi^{(\alpha')}), \tag{4-5}$$

where $\psi^{(\alpha)}(\lambda) = p_\alpha(\lambda)X^{-1}(\lambda)$ (p_α is a polynomial of degree $q-1$) is the unique solution of the system of equations

$$(\mathscr{L}\psi^{(\alpha)})_{\alpha'} = \delta_{\alpha\alpha'}, \quad \alpha' = 1, \ldots, q. \tag{4-6}$$

One can easily see that the function $\Psi_\alpha(z) = \int \log|z - \lambda| \psi^{(\alpha)}(\lambda)\, d\lambda$ is the harmonic measure of σ_α with respect to $\mathbb{C} \setminus \sigma$. Denote also

$$I[h] = (I_1[h], \ldots, I_q[h]), \quad I_\alpha[h] := \sum_{\alpha'} \mathscr{D}_{\alpha\alpha'}^{-1}(h, \psi^{(\alpha')}). \tag{4-7}$$

The main result of [Shcherbina 2013] is this:

Theorem 5. *Let the potential V satisfy conditions C1–C2, and let $\|h^{(4)}\|_\infty < \infty$. Then*

$$Z_{n,\beta}[h] = \exp\left\{ \frac{\beta}{8}(\mathscr{G}\bar{D}h, h) + \left(\frac{\beta}{2} - 1\right)(\mathscr{G}v, h) \right\} \frac{\Theta(\bar{I}[h]; \{n\bar{\mu}\})}{\Theta(0; \{n\bar{\mu}\})}\,(1 + O(n^{-\delta})), \tag{4-8}$$

where $\delta > 0$ and

$$\Theta(I[h]; \{n\bar{\mu}\}) :=$$
$$\sum_{n_1+\cdots+n_q=n_0} \exp\left\{ -\frac{\beta}{2}(\mathscr{D}^{-1}\Delta\bar{n}, \Delta\bar{n}) + \frac{\beta}{2}(\Delta\bar{n}, I[h]) + \left(\frac{\beta}{2}-1\right)(\Delta\bar{n}, I[\log\bar\rho]) \right\},$$
$$(\{n\bar\mu\})_\alpha = \{n\mu_\alpha\}, \quad (\Delta\bar{n})_\alpha = n_\alpha - \{n\mu_\alpha\},$$
$$(\log\bar\rho)_\alpha = \log\rho_\alpha, \quad n_0 = \sum_{\alpha=1}^{q}\{n\mu_\alpha\}, \tag{4-9}$$

with a positive definite matrix $\mathscr{D}$ of (4-5) and $I[h]$ defined by (4-7).
   For $h = 0$ we have

$$Q_{n,\beta}[V] = \mathscr{L}_{n,\beta}\, \frac{\exp\left\{ \frac{2}{\beta}\left(\frac{\beta}{2}-1\right)^2(\widetilde{\mathscr{L}}\mathscr{G}v, v) \right\}}{\det^{1/2}(1 - \bar{D}\widetilde{\mathscr{L}})}\,\Theta(0; \{n\bar\mu\})(1 + O(n^{-\kappa})),$$

$$Q_{n,\beta}[V] = \exp\left\{ \frac{n^2\beta}{2}\mathscr{E}[V] + F_\beta(n) + n\left(\frac{\beta}{2}-1\right)\left((\log\rho, \rho) - 1 - \log 2\pi\right) \right.$$
$$\left. -c_\beta(q-1)\log n + \sum_{\alpha=1}^{q}(q_\beta^{(0)}[\mu_\alpha^{-1}\rho_\alpha] - c_\beta\log\mu_\alpha) \right\}, \tag{4-10}$$

where μ_α, ρ_α are defined in (3-8), $q_\beta^{(0)}[\rho]$ is defined in (2-7), $F_\beta(n)$ and c_β are defined in (2-9) and $\det$ means the Fredholm determinant of $\bar{D}\widetilde{\mathscr{L}}$ on σ.

Let us remark that since the kernel of $\mathcal{L}$ is an analytic function, it is easy to prove that $\bar{D}\widetilde{\mathcal{L}}$ is a trace class operator. Moreover, it is proven in [Shcherbina 2013] that $\|\bar{D}\widetilde{\mathcal{L}}\| < 1$. Hence $\det^{1/2}(1 + \bar{D}\widetilde{\mathcal{L}})^{-1}$ is well defined.

Note also that since $\widetilde{\mathcal{L}} = 0$ in the one-cut case, the formulas of Theorem 5 for $q = 1$ coincide with that of Theorem 1.

According to Theorem 5, the fluctuations of the linear eigenvalue statistics $\mathcal{N}_n[h]$ are not Gaussian in the multi-cut regime, because the exponent of the generating functional $Z_{n,\beta}[h]$ is not quadratic with respect to h. We obtain that $Z_{n,\beta}[h]$ contains some quasiperiodic Θ-function in which the quadratic form $\mathcal{D}_{\alpha,\alpha'=1}^{q}$ is determined by the geometrical structure of σ. The fluctuations of $\mathcal{N}_n[h]$ become Gaussian if and only if all parameters $I_\alpha[h] = 0$ (see (4-7)). Similar results for $\beta = 2$ were predicted in [Pastur 2006] on the basis of the analysis of the asymptotics of orthogonal polynomials obtained in [Deift et al. 1999]. One more interesting observation is that the operator $\widetilde{\mathcal{L}}\mathcal{G}$ which appears in the place of the "variance" differ from $\mathcal{L}^{-1}$ (see (4-2) for the definition of $\mathcal{L}$) only by the final rank perturbation. This perturbation provides, in particular, that $\widetilde{\mathcal{L}}\mathcal{G}f = 0$, if $f(\lambda) = \text{const}, \ \lambda \in \sigma$.

Sketch of the proof of Theorem 5. Let $\bar{n} := (n_1, \ldots, n_q)$ and set

$$|\bar{n}| := \sum_{\alpha=1}^{q} n_\alpha, \quad \mathbf{1}_{\bar{n}}(\bar{\lambda}) := \prod_{j=1}^{n_1} \mathbf{1}_{\sigma_{1,\varepsilon}}(\lambda_j) \ldots \prod_{j=|\bar{n}|-n_q+1}^{n} \mathbf{1}_{\sigma_{q,\varepsilon}}(\lambda_j). \quad (4\text{-}11)$$

It is evident that

$$Q_{n,\beta}[V]/n! = \sum_{|\bar{n}|=n} \frac{\int \mathbf{1}_{\bar{n}}(\bar{\lambda}) e^{\beta H(\bar{\lambda})/2}}{n_1! \ldots n_q!} = \sum_{|\bar{n}|=n} \frac{\int \mathbf{1}_{\bar{n}}(\bar{\lambda}) e^{\beta(H_a(\bar{\lambda}) + \Delta H(\bar{\lambda}))/2}}{n_1! \ldots n_q!}. \quad (4\text{-}12)$$

Since $\log|\lambda - \mu|$ for $\lambda \in \sigma_{\alpha,\varepsilon}, \ \mu \in \sigma_{\alpha',\varepsilon}, \ \alpha \neq \alpha'$ is an analytic function, the expansion of it in the Fourier series with respect to some appropriate basis (e.g., Chebyshev polynomials $\{p_k^{(\alpha)}(\lambda)\}, \{p_k^{(\alpha')}(\mu)\}$) will converge exponentially fast. Hence, if we choose $M = [\log^2 n]$, then

$$\log|\lambda - \mu| = \sum_{k,m=1}^{M} L_{k,m}^{(\alpha,\alpha')} p_k^{(\alpha)}(\lambda) p_m^{(\alpha')}(\mu) + O(e^{-c\log^2 n}),$$

$$\lambda \in \sigma_{\alpha,\varepsilon}, \ \mu \in \sigma_{\alpha',\varepsilon}, \ \alpha \neq \alpha'. \quad (4\text{-}13)$$

Thus,

$$\Delta H(\bar{\lambda})\mathbf{1}_{\bar{n}}(\bar{\lambda})$$

$$= \mathbf{1}_{\bar{n}}(\bar{\lambda}) \sum_{\substack{j,j'=1 \\ \alpha \neq \alpha'}}^{n} \sum_{k,m=1}^{M} L_{k,m}^{(\alpha,\alpha')} \left(p_k^{(\alpha)}(\lambda_j) - \frac{n}{n_\alpha} c_k^{(\alpha)} \right) \left(p_m^{(\alpha')}(\lambda_{j'}) - \frac{n}{n_{\alpha'}} c_k^{(\alpha)} \right) + O(e^{-c\log^2 n})$$

with

$$c_k^{(\alpha)} := (p_k^{(\alpha)}, \rho \mathbf{1}_{\sigma_\alpha}).$$

Then we represent the matrix

$$\widetilde{\mathcal{L}}^{(M)} := \left\{ L_{k,m}^{(\alpha,\alpha')} \right\}_{k,m=1,\ldots m, \alpha,\alpha'=1,\ldots q},$$

which consists of q^2 blocks $M \times M$, as a difference of two positive block matrix of the same dimensionality

$$\widetilde{\mathcal{L}}^{(M)} = \hat{\mathcal{A}}^{(M)} - \mathcal{A}^{(M)}, \quad \mathcal{A}^{(M)}, \hat{\mathcal{A}}^{(M)} > 0$$

and apply the Hubbard–Stratonovich transformation to $e^{\beta \Delta H(\bar{\lambda})/2} \mathbf{1}_{\bar{n}}(\bar{\lambda})$:

$$e^{\beta(\hat{\mathcal{A}}^{(M)}\bar{x},\bar{x})/2} = \left(\frac{\beta}{2\pi} \right)^{(Mq)^2/2} \int_{\mathbb{R}^{(Mq)^2}} d\bar{u}^{(1)} e^{\beta((\hat{\mathcal{A}}^{(M)})^{1/2}\bar{x},\bar{u}^{(1)})/2 - \beta(\bar{u}^{(1)},\bar{u}^{(1)})/8},$$

$$e^{-\beta(\mathcal{A}^{(M)}\bar{x},\bar{x})/2} = \left(\frac{\beta}{2\pi} \right)^{(Mq)^2/2} \int_{\mathbb{R}^{(Mq)^2}} d\bar{u}^{(2)} e^{i\beta((\mathcal{A}^{(M)})^{1/2}\bar{x},\bar{u}^{(2)})/2 - \beta(\bar{u}^{(2)},\bar{u}^{(2)})/8}.$$

$$(4\text{-}14)$$

We obtain the linear with respect to $p_k^{(\alpha)}(\lambda_j)$ expression $\tilde{h}(\bar{u}_1, \bar{u}_2)$ in the exponent. Then apply Theorem 1 to

$$V = \mu_\alpha^{-1} V_\alpha^{(a)}, \quad h = \frac{\beta}{2}(\mu_\alpha^{-1} - n/n_\alpha) V_\alpha^{(a)} + \tilde{h}(u_1, \bar{u}_2).$$

This gives a quadratic form with respect to $(\bar{u}^{(1)}, \bar{u}^{(2)})$ in the exponent, and the quasi periodic quadratic form in the exponent appears due to the coefficient in front of $V_\alpha^{(a)}$. Then we integrate with respect to $(\bar{u}^{(1)}, \bar{u}^{(2)})$. After some transformations we obtain the assertion of Theorem 5. □

In principle, this way could be used to construct the asymptotic expansion of $Q_{n,\beta}[V]$ with respect to n^{-1}, because after the Hubbard–Stratonovich transformation we can apply the result Theorem 2. But there is a problem that in this case we have to apply (2-7) to a non real perturbation of V (see (4-14)). There is a way to extend the bounds obtained for th with a real t to a non real t, but for $|t| \le |\log n|^{1/2}$. It is enough for the CLT but not enough for the construction of the expansion.

Acknowledgements

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