On the asymptotics of a Toeplitz determinant with singularities

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We provide an alternative proof of the classical single-term asymptotics for Toeplitz determinants whose symbols possess Fisher–Hartwig singularities. We also relax the smoothness conditions on the regular part of the symbols and obtain an estimate for the error term in the asymptotics. Our proof is based on the Riemann–Hilbert analysis of the related systems of orthogonal polynomials and on differential identities for Toeplitz determinants. The result discussed in this paper is crucial for the proof of the asymptotics in the general case of Fisher–Hartwig's singularities and extensions to Hankel and Toeplitz+Hankel determinants.

1. Introduction

Let $f(z)$ be a complex-valued function integrable over the unit circle. Denote its Fourier coefficients

$$f_j = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-ij\theta} d\theta, \quad j = 0, \pm 1, \pm 2, \ldots$$

We are interested in the n-dimensional Toeplitz determinant with symbol $f(z)$,

$$D_n(f(z)) = \det(f_{j-k})_{j,k=0}^{n-1}, \quad n \geq 1, \tag{1-1}$$

where $f(e^{i\theta})$ has a fixed number of Fisher–Hartwig singularities [Fisher and Hartwig 1968; Lenard 1964; 1972], i.e., f has the following form on the unit circle:

$$f(z) = e^{V(z)} z^{\sum_{j=0}^m \beta_j} \prod_{j=0}^m |z - z_j|^{2\alpha_j} g_{z_j,\beta_j}(z) z_j^{-\beta_j},$$
$$z = e^{i\theta}, \quad \theta \in [0, 2\pi), \tag{1-2}$$

for some $m = 0, 1, \ldots$, where

$$z_j = e^{i\theta_j}, \quad j = 0, \ldots, m, \quad 0 = \theta_0 < \theta_1 < \cdots < \theta_m < 2\pi, \tag{1-3}$$

$$g_{z_j,\beta_j}(z) := g_{\beta_j}(z) = \begin{cases} e^{i\pi\beta_j} & \text{if } 0 \leq \arg z < \theta_j, \\ e^{-i\pi\beta_j} & \text{if } \theta_j \leq \arg z < 2\pi, \end{cases} \tag{1-4}$$

$$\Re \alpha_j > -\tfrac{1}{2}, \quad \beta_j \in \mathbb{C}, \quad j = 0, \dots, m, \qquad (1\text{-}5)$$

and $V(e^{i\theta})$ is a sufficiently smooth function on the unit circle (see below). The condition on the α_j insures integrability. Note that a single Fisher–Hartwig singularity at z_j consists of a root-type singularity

$$|z - z_j|^{2\alpha_j} = \left| 2 \sin \frac{\theta - \theta_j}{2} \right|^{2\alpha_j} \qquad (1\text{-}6)$$

and a jump $e^{i\pi\beta} \to e^{-i\pi\beta}$. We assume that z_j, $j = 1, \dots, m$, are genuine singular points, i.e., either $\alpha_j \neq 0$ or $\beta_j \neq 0$. However, we always include $z_0 = 1$ explicitly in (1-2), even when $\alpha_0 = \beta_0 = 0$: this convention was adopted in [Deift et al. 2011] in order to facilitate the application of our Toeplitz methods to Hankel determinants. Note that $g_{\beta_0}(z) = e^{-i\pi\beta_0}$. Observe that for each $j \neq 0$, $z^{\beta_j} g_{\beta_j}(z)$ is continuous at $z = 1$, and so for each j each "beta" singularity produces a jump only at the point z_j. The factors $z_j^{-\beta_j}$ are singled out to simplify comparisons with the existing literature. Indeed, (1-2) with the notation $b(\theta) = e^{V(e^{i\theta})}$ is exactly the symbol considered in [Fisher and Hartwig 1968; Basor 1978; 1979; Böttcher and Silbermann 1981; 1985; 1986; Ehrhardt and Silbermann 1997; Ehrhardt 2001; Widom 1973]. However, we write the symbol in a form with $z^{\sum_{j=0}^{m} \beta_j}$ factored out. The representation (1-2) is more natural for our analysis.

On the unit circle, $V(z)$ is represented by its Fourier expansion:

$$V(z) = \sum_{k=-\infty}^{\infty} V_k z^k, \quad V_k = \frac{1}{2\pi} \int_0^{2\pi} V(e^{i\theta}) e^{-ki\theta} d\theta. \qquad (1\text{-}7)$$

The canonical Wiener–Hopf factorization of $e^{V(z)}$ is given by

$$e^{V(z)} = b_+(z) e^{V_0} b_-(z), \quad b_+(z) = e^{\sum_{k=1}^{\infty} V_k z^k}, \quad b_-(z) = e^{\sum_{k=-\infty}^{-1} V_k z^k}. \quad (1\text{-}8)$$

Define a seminorm

$$\|\!|\beta|\!\| = \max_{j,k} |\Re\beta_j - \Re\beta_k|, \qquad (1\text{-}9)$$

where the indices $j, k = 0$ are omitted if $z = 1$ is not a singular point, i.e., if $\alpha_0 = \beta_0 = 0$. If $m = 0$, set $\|\!|\beta|\!\| = 0$.

In this paper we consider the asymptotics of $D_n(f)$, $n \to \infty$, in the case $\|\!|\beta|\!\| < 1$. The asymptotic behavior of $D_n(f)$ has been studied by many authors (see [Deift et al. 2011; Ehrhardt 2001] for a review). An expansion of $D_n(f)$ for the case $V \in C^\infty$, $\|\!|\beta|\!\| < 1$, was obtained by Ehrhardt in [2001]. The aim of this paper is to provide an alternative proof of this result based on differential identities for $D_n(f)$ and a Riemann–Hilbert-problem analysis of the corresponding system of orthogonal polynomials. We also obtain estimates for the error term and

extend the validity of the result to less smooth V. The behavior of $D_n(f)$ as $n \to \infty$ for $\|\!|\beta|\!\| \geq 1$ was conjectured by Basor and Tracy [1991], and their conjecture was eventually proved in [Deift et al. 2011]. It turns out that the case $\|\!|\beta|\!\| < 1$ plays a crucial role in our proof of the conjecture for $\|\!|\beta|\!\| \geq 1$.

Theorem 1.1. *Let $f(e^{i\theta})$ be defined in* (1-2), $\|\!|\beta|\!\| < 1$, $\Re\alpha_j > -\frac{1}{2}$, $\alpha_j \pm \beta_j \neq -1, -2, \ldots$ *for* $j, k = 0, 1, \ldots, m$, *and let $V(z)$ satisfy the condition* (1-11), (1-12) *below. Then, as $n \to \infty$,*

$$D_n(f) = \exp\left[nV_0 + \sum_{k=1}^{\infty} kV_kV_{-k}\right]\prod_{j=0}^{m} b_+(z_j)^{-\alpha_j+\beta_j}b_-(z_j)^{-\alpha_j-\beta_j}$$

$$\times n^{\sum_{j=0}^{m}(\alpha_j^2-\beta_j^2)}\prod_{0\leq j<k\leq m}|z_j-z_k|^{2(\beta_j\beta_k-\alpha_j\alpha_k)}\left(\frac{z_k}{z_je^{i\pi}}\right)^{\alpha_j\beta_k-\alpha_k\beta_j}$$

$$\times\prod_{j=0}^{m}\frac{G(1+\alpha_j+\beta_j)G(1+\alpha_j-\beta_j)}{G(1+2\alpha_j)}(1+o(1)), \quad (1\text{-}10)$$

where $G(x)$ is Barnes' G-function. The double product over $j < k$ is set to 1 if $m = 0$.

Remark 1.2. As indicated above, this result was first obtained by Ehrhardt in the case $V \in C^{\infty}$. We prove the theorem for $V(z)$ satisfying the smoothness condition

$$\sum_{k=-\infty}^{\infty}|k|^s|V_k| < \infty, \quad (1\text{-}11)$$

where

$$s > \frac{1+\sum_{j=0}^{m}\left[(\Im\alpha_j)^2+(\Re\beta_j)^2\right]}{1-\|\!|\beta|\!\|}. \quad (1\text{-}12)$$

Remark 1.3. In the case of a single singularity, i.e., when $m = 1$ and $\alpha_0 = \beta_0 = 0$, or when $m = 0$, the seminorm $\|\!|\beta|\!\| = 0$, and the theorem implies that the asymptotic form (1-10) holds for all

$$\Re\alpha_m > -\tfrac{1}{2}, \quad \beta_m \in \mathbb{C}, \quad \alpha_m \pm \beta_m \neq -1, -2, \ldots. \quad (1\text{-}13)$$

In fact, if there is only one singularity, say $m = 0$, and $V \equiv 0$, an explicit formula is known for $D_n(f)$ for any n in terms of the G-functions:

$$D_n(f) = \frac{G(1+\alpha_0+\beta_0)G(1+\alpha_0-\beta_0)}{G(1+2\alpha_0)}\frac{G(n+1)G(n+1+2\alpha_0)}{G(n+1+\alpha_0+\beta_0)G(n+1+\alpha_0-\beta_0)},$$

$$\Re\alpha_0 > -\tfrac{1}{2}, \quad \alpha_0 \pm \beta_0 \neq -1, -2, \ldots, \quad n \geq 1, \quad (1\text{-}14)$$

and (1-10) can then be read off from the known asymptotics of the G-function (see, e.g., [Barnes 1900]). The formula (1-14) was found by Böttcher and Silbermann

[1985]. However, it also follows from a more general one for a Selberg-type integral which was discovered by Selberg himself (see [Forrester and Warnaar 2008]) a few years earlier.

Remark 1.4. Assume that the function $V(z)$ is sufficiently smooth, i.e., such that s in (1-11) is, in addition to satisfying (1-12), sufficiently large in comparison with $\||\beta\||$. Then we show that the error term $o(1) = O(n^{\||\beta\||-1})$ in (1-10). In particular, the error term $o(1) = O(n^{\||\beta\||-1})$ if $V(z)$ is analytic in a neighborhood of the unit circle. Moreover, for analytic $V(z)$, our methods would allow us to calculate, in principle, the full asymptotic expansion rather than just the leading term presented in (1-10). Various regularity properties of the expansion (uniformity, differentiability) in compact sets of parameters satisfying $\Re \alpha_j > -\frac{1}{2}$, $\||\beta\|| < 1, \alpha_j \pm \beta_j \neq -1, -2, \ldots, \theta_j \neq \theta_k$, are easy to deduce from our analysis.

Remark 1.5. Since $G(-k) = 0$, $k = 0, 1, \ldots$, the formula (1-10) no longer represents the leading asymptotics if $\alpha_j + \beta_j$ or $\alpha_j - \beta_j$ is a negative integer for some j. Although our method applies, we do not address these cases in this paper. It is simply a matter of going deeper in the asymptotic expansion for $D_n(f)$. It can happen that $D_n(f)$ vanishes to all orders (cf. discussion of the Ising model at temperatures above the critical temperature in [Deift et al. 2012]), and $e^{-nV_0} D_n(f)$ is exponentially decreasing.

We prove Theorem 1.1 in the following way. We begin by deriving differential identities for the logarithm of $D_n(f)$ in Section 3, in the spirit of [Deift 1999; Krasovsky 2004; 2007; Its and Krasovsky 2008; Deift et al. 2007; 2008], utilizing the polynomials orthogonal with respect to the weight $f(z)$ on the unit circle. Then, assuming that $V(z)$ is analytic in a neighborhood of the unit circle, we analyze in Section 4 the asymptotics of these polynomials using Riemann–Hilbert/steepest-descent methods as in [Deift et al. 2011]. This gives in turn the asymptotics of the differential identities from which the formula (1-10) follows in the $V \equiv 0$ case by integration with respect to α_j, β_j in Section 5A. However, the error term that results is of order $n^{2\||\beta\||-1} \ln n$ (see (5-35)), which is asymptotically small only for $\||\beta\|| < \frac{1}{2}$, rather than in the full range $\||\beta\|| < 1$. To prove (1-10) for all $\||\beta\|| < 1$, we need a finer analysis of cancellations in the Riemann–Hilbert problem as $n \to \infty$. We carry this out in Section 5B and reduce the leading order terms in $D_n(f)$ to a telescopic form (see (5-65)), which leads to a uniform bound on $D_n(f) n^{\sum_{j=0}^{m} (\beta_j^2 - \alpha_j^2)}$ for large n which is valid for all $\||\beta\|| < 1$ and away from the points $\alpha_j \pm \beta_j = -1, -2, \ldots$. We then apply Vitali's theorem together with the previous result for $\||\beta\|| < \frac{1}{2}$. This proves Theorem 1.1 in the $V \equiv 0$ case as desired, with the error term of order $n^{\||\beta\||-1}$. In Sections 5C and 5D, we then extend the result to the case of analytic $V \not\equiv 0$ by applying another differential identity from Section 3.

Ehrhardt proves (1-10) using a "localization" or "separation" technique, introduced by Basor [1979], where the effect of adding in Fisher–Hartwig singularities one at a time is controlled. One may also view our approach as a "separation" technique, but in contrast to [Basor 1979; Ehrhardt 2001], we add in the Fisher–Hartwig singularities, as well as the regular term $e^{V(z)}$, in a continuous fashion.

Remark 1.6. An alternative approach to proving Theorem 1.1 in the V-analytic case is to apply, ab initio, the finer analysis of Section 5B to the orthogonal polynomials which appear in the differential identities. This approach is more direct, but is considerably more involved technically. The analysis is resolved, as above, by reduction of the problem to an appropriate telescopic form.

Finally in Section 5E, we extend our result to the case when $V(z)$ is not analytic and only satisfies the smoothness condition (1-11), (1-12). We approximate such $V(z)$ by trigonometric polynomials $V^{(n)}(z) = \sum_{k=-p}^{p} V_k z^k$ with an appropriate $p = p(n)$ and modify the Riemann–Hilbert analysis accordingly. This produces asymptotics of a Toeplitz determinant in whose symbol $f^{(n)}(z)$ the function $V(z)$ is replaced by $V^{(n)}(z)$. We then use the Heine representation of Toeplitz determinants by multiple integrals to show that $D_n(f^{(n)})$ approximates $D_n(f)$ as $n \to \infty$, sufficiently strongly to conclude Theorem 1.1 in the general case.

2. Riemann–Hilbert problem

In this section we formulate a Riemann–Hilbert problem (RHP) for the polynomials orthogonal on the unit circle (which, oriented in the positive direction, we denote C) with weight $f(z)$ given by (1-2). We use this RHP in Section 4 to find the asymptotics of the polynomials in the case of analytic $V(z)$. Suppose that all $D_k(f) \neq 0$, $k = k_0, k_0 + 1 \ldots$, for some sufficiently large k_0 (see discussion below). Then the polynomials $\phi_k(z) = \chi_k z^k + \cdots$, $\hat{\phi}_k(z) = \chi_k z^k + \cdots$ of degree k, $k = k_0, k_0 + 1, \ldots$, with nonzero leading coefficients χ_k, satisfying the orthogonality conditions

$$\frac{1}{2\pi} \int_0^{2\pi} \phi_k(z) z^{-j} f(z)\, d\theta = \chi_k^{-1} \delta_{jk}, \quad \frac{1}{2\pi} \int_0^{2\pi} \hat{\phi}_k(z^{-1}) z^j f(z)\, d\theta = \chi_k^{-1} \delta_{jk},$$
$$z = e^{i\theta}, \quad j = 0, 1, \ldots, k, \quad (2\text{-}1)$$

exist and are given by the following expressions:

$$\phi_k(z) = \frac{1}{\sqrt{D_k D_{k+1}}} \begin{vmatrix} f_{00} & f_{01} & \cdots & f_{0k} \\ f_{10} & f_{11} & \cdots & f_{1k} \\ \vdots & \vdots & & \vdots \\ f_{k-1\,0} & f_{k-1\,1} & \cdots & f_{k-1\,k} \\ 1 & z & \cdots & z^k \end{vmatrix}, \quad (2\text{-}2)$$

$$\hat{\phi}_k(z^{-1}) = \frac{1}{\sqrt{D_k D_{k+1}}} \begin{vmatrix} f_{00} & f_{01} & \cdots & f_{0\,k-1} & 1 \\ f_{10} & f_{11} & \cdots & f_{1\,k-1} & z^{-1} \\ \vdots & \vdots & & \vdots & \vdots \\ f_{k0} & f_{k1} & \cdots & f_{k\,k-1} & z^{-k} \end{vmatrix}, \tag{2-3}$$

where

$$f_{st} = \frac{1}{2\pi} \int_0^{2\pi} f(z) z^{-(s-t)} d\theta, \quad s, t = 0, 1, \ldots, k.$$

We obviously have

$$\chi_k = \sqrt{\frac{D_k}{D_{k+1}}}. \tag{2-4}$$

Consider the matrix-valued function $Y^{(k)}(z) = Y(z)$, $k \geq k_0$, given by

$$Y^{(k)}(z) := \begin{pmatrix} \chi_k^{-1} \phi_k(z) & \chi_k^{-1} \int_C \frac{\phi_k(\xi)}{\xi - z} \frac{f(\xi) d\xi}{2\pi i \xi^k} \\ -\chi_{k-1} z^{k-1} \hat{\phi}_{k-1}(z^{-1}) & -\chi_{k-1} \int_C \frac{\hat{\phi}_{k-1}(\xi^{-1})}{\xi - z} \frac{f(\xi) d\xi}{2\pi i \xi} \end{pmatrix}. \tag{2-5}$$

It is easy to verify that $Y(z)$ solves the following Riemann–Hilbert problem:

(a) $Y(z)$ is analytic for $z \in \mathbb{C} \setminus C$.

(b) Let $z \in C \setminus \bigcup_{j=0}^m z_j$. Y has continuous boundary values $Y_+(z)$ as z approaches the unit circle from the inside, and $Y_-(z)$, from the outside, related by the jump condition

$$Y_+(z) = Y_-(z) \begin{pmatrix} 1 & z^{-k} f(z) \\ 0 & 1 \end{pmatrix}, \quad z \in C \setminus \bigcup_{j=0}^m z_j. \tag{2-6}$$

(c) $Y(z)$ has the following asymptotic behavior at infinity:

$$Y(z) = \left(I + O\left(\frac{1}{z}\right) \right) \begin{pmatrix} z^k & 0 \\ 0 & z^{-k} \end{pmatrix} \quad \text{as } z \to \infty. \tag{2-7}$$

(d) As $z \to z_j$, $j = 0, 1, \ldots, m$, $z \in \mathbb{C} \setminus C$,

$$Y(z) = \begin{pmatrix} O(1) & O(1) + O(|z - z_j|^{2\alpha_j}) \\ O(1) & O(1) + O(|z - z_j|^{2\alpha_j}) \end{pmatrix} \quad \text{if } \alpha_j \neq 0, \tag{2-8}$$

and

$$Y(z) = \begin{pmatrix} O(1) & O(\ln|z - z_j|) \\ O(1) & O(\ln|z - z_j|) \end{pmatrix} \quad \text{if } \alpha_j = 0, \beta_j \neq 0. \tag{2-9}$$

(Here and below $O(a)$ stands for $O(|a|)$.) If $\alpha_0 = \beta_0 = 0$, $Y(z)$ is bounded at $z = 1$.

A general fact that orthogonal polynomials can be so represented as a solution of a Riemann–Hilbert problem was noticed in [Fokas et al. 1992] for polynomials on the line and extended to polynomials on the circle in [Baik et al. 1999]. This fact is important because it turns out that the RHP can be efficiently analyzed for large k by a steepest-descent type method found in [Deift and Zhou 1993] and developed further in many subsequent works. Thus, we first find the solution to the problem (a)–(d) for large k (applying this method) and then interpret it as the asymptotics of the orthogonal polynomials by (2-5).

Recall the Heine representation for a Toeplitz determinant:

$$D_n(f) = \frac{1}{(2\pi)^n n!} \int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{1 \le j < k \le n} |e^{i\phi_j} - e^{i\phi_k}|^2 \prod_{j=1}^n f(e^{i\phi_j}) d\phi_j. \quad (2\text{-}10)$$

If $f(z)$ is positive on the unit circle, it follows from (2-10) that $D_k(f) > 0$, $k = 1, 2, \ldots$, i.e., $k_0 = 1$. In the general case, let Λ be a compact subset in the subset $\|\beta\| < 1$, $\alpha_j \pm \beta_j \neq -1, -2, \ldots$ of the parameter space $\mathcal{P} = \{(\alpha_0, \beta_0, \ldots, \alpha_m, \beta_m) : \alpha_j, \beta_j \in \mathbb{C}, \Re\alpha_j > -\frac{1}{2}\}$. We will show in Section 4 that the RHP (a)–(d) is solvable in Λ, in particular χ_k are finite and nonzero, for all sufficiently large k ($k \ge k_0(\Lambda)$). Let Ω_{k_0} be the set of parameters in $\mathcal{P}$ such that $D_k(f) = 0$ for some $k = 1, 2, \ldots, k_0 - 1$. We will then have $D_k(f) \neq 0$, $k = 1, 2, \ldots$ for all points in $\Lambda \setminus \Omega_{k_0}$. Note that $D_n(f)$ depends analytically on α_j, β_j in $\mathcal{P}$: this is true, in particular, on (the interior of) $\Lambda \setminus \Omega_{k_0}$.

The solution to the RHP (a)–(d) is unique. Note first that $\det Y(z) = 1$. Indeed, from the conditions on $Y(z)$, $\det Y(z)$ is analytic across the unit circle, has all singularities removable, and tends to 1 as $z \to \infty$. It is then identically 1 by Liouville's theorem. Now if there is another solution $\widetilde{Y}(z)$, we easily obtain by Liouville's theorem that $\widetilde{Y}(z)Y(z)^{-1} \equiv 1$.

3. Differential identities

In this section we derive expressions for the derivative $(\partial/\partial\gamma) \ln D_n(f(z))$, where either $\gamma = \alpha_j$ or $\gamma = \beta_j$, $j = 0, 1, \ldots, m$, in terms of the matrix elements of (2-5). These will be exact differential identities valid for all $n = 1, 2, \ldots$ (see Proposition 3.1 below), provided all the $D_n(f) \neq 0$. We will use these expressions in Section 5A to obtain the asymptotics (1-10) in the case $V \equiv 0$, $\|\beta\| < \frac{1}{2}$ (improved to $\|\beta\| < 1$ in Section 5B). Furthermore, in this section we will derive a differential identity (see Proposition 3.3 below) which will enable us in Sections 5C and 5D to extend the results to analytic $V \not\equiv 0$ (improved to sufficiently smooth V in Section 5E).

Set $D_0 \equiv 1$, $\phi_0(z) \equiv \hat{\phi}_0(z) \equiv 1$, and suppose that $D_n(f) \neq 0$ for all $n = 1, 2, \dots$. Then the orthogonal polynomials (2-2), (2-3) exist and are analytic in the α_j and β_j for all $k = 1, 2, \dots$. Moreover, (2-4) implies that

$$D_n(f(z)) = \prod_{j=0}^{n-1} \chi_j^{-2}. \tag{3-1}$$

Note that by orthogonality,

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \phi_j(z)}{\partial \gamma} \hat{\phi}_j(z^{-1}) f(z) \, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\partial \chi_j}{\partial \gamma} z^j + \text{polynomial of degree } j-1 \right) \hat{\phi}_j(z^{-1}) f(z) \, d\theta$$

$$= \frac{1}{\chi_j} \frac{\partial \chi_j}{\partial \gamma}. \tag{3-2}$$

Similarly,

$$\frac{1}{2\pi} \int_0^{2\pi} \phi_j(z) \frac{\partial \hat{\phi}_j(z^{-1})}{\partial \gamma} f(z) \, d\theta = \frac{1}{\chi_j} \frac{\partial \chi_j}{\partial \gamma}. \tag{3-3}$$

Therefore, using (3-1), we obtain

$$\frac{\partial}{\partial \gamma} \ln D_n(f(z)) = \frac{\partial}{\partial \gamma} \ln \prod_{j=0}^{n-1} \chi_j^{-2} = -2 \sum_{j=0}^{n-1} \frac{\frac{\partial \chi_j}{\partial \gamma}}{\chi_j}$$

$$= -\frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial \gamma} \left(\sum_{j=0}^{n-1} \phi_j(z) \hat{\phi}_j(z^{-1}) \right) f(z) \, d\theta. \tag{3-4}$$

Using here the Christoffel–Darboux identity (see Lemma 2.3 of [Deift et al. 2011], for example),

$$\sum_{k=0}^{n-1} \hat{\phi}_k(z^{-1}) \phi_k(z) = -n\phi_n(z)\hat{\phi}_n(z^{-1}) + z\left(\hat{\phi}_n(z^{-1}) \frac{d}{dz}\phi_n(z) - \phi_n(z) \frac{d}{dz}\hat{\phi}_n(z^{-1}) \right),$$

and then orthogonality, we can write

$$\frac{\partial}{\partial \gamma} \ln D_n(f(z))$$

$$= 2n \frac{\frac{\partial \chi_n}{\partial \gamma}}{\chi_n} + \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial \gamma} \left(\phi_n(z) \frac{d\hat{\phi}_n(z^{-1})}{dz} - \hat{\phi}_n(z^{-1}) \frac{d\phi_n(z)}{dz} \right) z f(z) \, d\theta. \tag{3-5}$$

Writing out the derivative with respect to γ in the integral and using orthogonality, we obtain

$$\frac{\partial}{\partial\gamma}\ln D_n(f(z)) = I_1 - I_2, \tag{3-6}$$

where

$$I_1 = \frac{1}{2\pi i}\int_0^{2\pi}\frac{\partial\phi_n(z)}{\partial\gamma}\frac{\partial\hat{\phi}_n(z^{-1})}{\partial\theta}f(z)\,d\theta,$$
$$I_2 = \frac{1}{2\pi i}\int_0^{2\pi}\frac{\partial\phi_n(z)}{\partial\theta}\frac{\partial\hat{\phi}_n(z^{-1})}{\partial\gamma}f(z)\,d\theta. \tag{3-7}$$

It turns out that the particular structure of Fisher–Hartwig singularities allows us to reduce (3-6) to a local formula, i.e., to replace the integrals by the polynomials (and their Cauchy transforms) evaluated only at several points (compare [Its and Krasovsky 2008; Krasovsky 2007]).

Let us encircle each of the points z_j by a sufficiently small disc,

$$U_{z_j} = \left\{z : |z - z_j| < \varepsilon\right\}. \tag{3-8}$$

Denote

$$C_\varepsilon = \bigcup_{j=0}^{m}\left(U_{z_j}\cap C\right). \tag{3-9}$$

We now integrate I_1 by parts. First assume that $V(z)\equiv 0$. Then, using the expression

$$\frac{\partial f(z)}{\partial\theta} = \sum_{j=0}^{m}\left(\alpha_j\cot\frac{\theta-\theta_j}{2} + i\beta_j\right)f(z) = \left(\sum_{j=0}^{m}\alpha_j\frac{z+z_j}{z-z_j} + \beta_j\right)if(z),$$

we obtain

$$I_1 = -\chi_n^{-1}\frac{\partial\chi_n}{\partial\gamma}\left(n + \sum_{j=0}^{m}\beta_j\right)$$
$$- \lim_{\varepsilon\to 0}\left[\frac{1}{2\pi}\int_{C\backslash C_\varepsilon}\frac{\partial\phi_n(z)}{\partial\gamma}\hat{\phi}_n(z^{-1})\left(\sum_{j=0}^{m}\alpha_j\frac{z+z_j}{z-z_j}\right)f(z)\,d\theta\right.$$
$$\left. - \frac{1}{2\pi i}\sum_{j=0}^{m}\frac{\partial\phi_n(z_j)}{\partial\gamma}\hat{\phi}_n(z_j^{-1})(f(z_je^{-i\varepsilon}) - f(z_je^{i\varepsilon}))\right], \tag{3-10}$$

where the integration is over $C\setminus C_\varepsilon$ in the positive direction around the unit circle.

Note that by adding and subtracting $\dfrac{\partial \phi_n(z_j)}{\partial \gamma}$, and by using orthogonality we can write

$$
\int_{C \backslash C_\varepsilon} \frac{\partial \phi_n(z)}{\partial \gamma} \hat{\phi}_n(z^{-1}) \frac{z + z_j}{z - z_j} f(z)\, d\theta =
$$

$$
\int_{C \backslash C_\varepsilon} \hat{\phi}_n(z^{-1}) \frac{\partial \phi_n(z)/\partial \gamma - \partial \phi_n(z_j)/\partial \gamma}{z - z_j} (z + z_j) f(z)\, d\theta
$$

$$
+ \frac{\partial \phi_n(z_j)}{\partial \gamma} \int_{C \backslash C_\varepsilon} \hat{\phi}_n(z^{-1}) \frac{2z_j}{z - z_j} f(z)\, d\theta + O(\varepsilon^{2\Re \alpha_j + 1}). \quad (3\text{-}11)
$$

Obviously, the fraction in the first integral on the right-hand side is a polynomial in z of degree $n - 1$ with leading coefficient $\partial \chi_n / \partial \gamma$. Therefore, the integral equals $2\pi (\partial \chi_n / \partial \gamma)/\chi_n$ up to $O(\varepsilon^{2\Re \alpha_j + 1})$. The second integral can be written in terms of the element Y_{22} of (2-5) for $z \to z_j$. Let us estimate therefore the following expression for $\alpha_j \neq 0$:

$$
\int_C \hat{\phi}_n(s^{-1}) \frac{2z_j f(s)}{s - z} \frac{ds}{is}
$$

$$
- \lim_{\varepsilon \to 0} \left[\int_{C \backslash C_\varepsilon} \hat{\phi}_n(s^{-1}) \frac{2z_j f(s)}{s - z_j} \frac{ds}{is} - \frac{1}{i\alpha_j} \hat{\phi}_n(z_j^{-1})(f(z_j e^{-i\varepsilon}) - f(z_j e^{i\varepsilon})) \right],
$$

$$
z \to z_j, \quad |z| > 1. \quad (3\text{-}12)
$$

This difference tends to zero as $z \to z_j$, for $\Re \alpha_j > 0$. When $\Re \alpha_j < 0$, it is a growing function as $z \to z_j$, and when $\Re \alpha_j = 0$, $\Im \alpha_j \neq 0$, an oscillating one. The analysis is similar to that of Section 3 in [Krasovsky 2007]. For future use, we now fix an analytical continuation of the absolute value, namely, write for z on the unit circle,

$$
|z - z_j|^{\alpha_j} = (z - z_j)^{\alpha_j/2}(z^{-1} - z_j^{-1})^{\alpha_j/2} = \frac{(z - z_j)^{\alpha_j}}{(z z_j e^{i \ell_j})^{\alpha_j/2}}, \quad z = e^{i\theta}, \quad (3\text{-}13)
$$

where ℓ_j is found from the condition that the argument of the above function is zero on the unit circle. Let us fix the cut of $(z - z_j)^{\alpha_j}$ going along the line $\theta = \theta_j$ from z_j to infinity. Fix the branch by the condition that on the line going from z_j to the right parallel to the real axis, $\arg(z - z_j) = 2\pi$. For $z^{\alpha_j/2}$ in the denominator, $0 < \arg z < 2\pi$. If $z_0 = 1$ let $0 < \arg(z - 1) < 2\pi$. (This choice will enable us to use the standard asymptotics for a confluent hypergeometric function in the RH analysis in Section 4 below.) Then, a simple consideration of triangles shows that

$$
\ell_j = \begin{cases} 3\pi & \text{if } 0 < \theta < \theta_j, \\ \pi & \text{if } \theta_j < \theta < 2\pi. \end{cases} \quad (3\text{-}14)
$$

Thus (3-13) is continued analytically to neighborhoods of the arcs $0 < \theta < \theta_j$ and $\theta_j < \theta < 2\pi$. We now analyze (3-12) in the same way that equation (26) was analyzed in [Krasovsky 2007]. For this analysis, however, we will need two other choices of the function $(z - z_j)^{2\alpha_j}$: one choice with the cut going a short distance clockwise along the unit circle C from z_j, and another, with the cut going a short distance anticlockwise along C from z_j. Let c_j and d_j be some points on C between z_j and the neighboring singularity in the clockwise and anticlockwise directions, respectively. In a neighborhood of z_j, let $g(s)$ be defined by

$$\frac{\hat{\phi}_n(s^{-1}) f(s)}{is} = |s - z_j|^{2\alpha_j} g(s).$$

We then obtain as in [Krasovsky 2007] for the part of (3-12) on the arc (c_j, d_j):

$$\int_{c_j}^{d_j} \frac{|s - z_j|^{2\alpha_j}}{s - z} g(s)\, ds$$

$$- \lim_{\varepsilon \to 0}\left[\left(\int_{c_j}^{z_j e^{-i\varepsilon}} + \int_{z_j e^{i\varepsilon}}^{d_j}\right) \frac{|s - z_j|^{2\alpha_j}}{s - z_j} g(s)\, ds - \frac{\varepsilon^{2\alpha_j}}{2\alpha_j}(g(z_j e^{-i\varepsilon}) - g(z_j e^{i\varepsilon}))\right]$$

$$= \lim_{\varepsilon \to 0} \frac{\pi(z - z_j)^{2\alpha_j}}{(zz_j)^{\alpha_j} \sin(2\pi\alpha_j)}(e^{2\pi i\alpha_j - i\alpha\ell_R} g(z_j e^{-i\varepsilon}) - e^{-2\pi i\alpha_j - i\alpha\ell_L} g(z_j e^{i\varepsilon}))$$

$$+ \alpha_j^{-1} O(z - z_j), \quad z \to z_j, \quad |z| > 1, \quad (3\text{-}15)$$

for $\alpha_j \neq 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$ (when $\alpha_j = \frac{1}{2}, 1, \frac{3}{2}, \dots$ one obtains terms involving $(z - z_j)^k \ln(z - z_j)$ vanishing as $z \to z_j$). Here the constants ℓ_R, ℓ_L depend on the choice of a branch for $(z - z_j)^{2\alpha}$ (whose cut is, recall, along the circle) and their values will not be important below.

Introduce a "regularized" version of the integral in a neighborhood of z_j:

$$\int_{c_j}^{d_j{}^{(r)}} \frac{|s - z_j|^{2\alpha_j}}{s - z} g(s)\, ds \equiv \int_{c_j}^{d_j} \frac{|s - z_j|^{2\alpha_j}}{s - z} g(s)\, ds$$

$$- \lim_{\varepsilon \to 0} \frac{\pi(z - z_j)^{2\alpha_j}}{(zz_j)^{\alpha_j} \sin(2\pi\alpha_j)}(e^{2\pi i\alpha_j - i\alpha\ell_R} g(z_j e^{-i\varepsilon}) - e^{-2\pi i\alpha_j - i\alpha\ell_L} g(z_j e^{i\varepsilon})),$$

for z in a complex neighborhood of z_j and $-\frac{1}{2} < \Re\alpha_j \leq 0$, $\alpha_j \neq 0$. If $\Re\alpha_j > 0$, we set the "regularized" integral equal to the integral itself.

Denote by $\widetilde{Y}$ the matrix (2-5), in which the integrals of the second column are replaced by their "regularized" values in a neighborhood of each z_j.

Then, collecting our observations together, we can write (3-10) in the form

$$I_1 = -\chi_n^{-1} \frac{\partial \chi_n}{\partial \gamma}\left(n + \sum_{j=0}^{m}(\alpha_j + \beta_j)\right) + \sum_{j=0}^{m} G_j^+(z_j), \quad (3\text{-}16)$$

where

$$G_j^+(z_j) = \begin{cases} \dfrac{2\alpha_j z_j}{\chi_n} \dfrac{\partial}{\partial\gamma}\big(\chi_n Y_{11}^{(n)}(z_j)\big)\widetilde{Y}_{22}^{(n+1)}(z_j) & \text{if } \alpha_j \neq 0, \\[3mm] \dfrac{1}{2\pi i}\dfrac{\partial\phi_n(z_j)}{\partial\gamma}\hat{\phi}_n(z_j^{-1})\Delta f(z_j) & \text{if } \alpha_j = 0, \end{cases} \qquad (3\text{-}17)$$

with

$$\Delta f(z_j) = \lim_{\varepsilon\to 0}(f(z_j e^{-i\varepsilon}) - f(z_j e^{i\varepsilon})). \qquad (3\text{-}18)$$

A similar analysis yields

$$I_2 = \chi_n^{-1}\frac{\partial\chi_n}{\partial\gamma}\left(n + \sum_{j=0}^m(\alpha_j - \beta_j)\right) + \sum_{j=0}^m G_j^-(z_j),$$

where

$$G_j^-(z_j) = \begin{cases} 2\chi_n\alpha_j\dfrac{\partial}{\partial\gamma}\big(\chi_n^{-1}Y_{21}^{(n+1)}(z_j)\big)\widetilde{Y}_{12}^{(n)}(z_j) & \text{if } \alpha_j \neq 0, \\[3mm] \dfrac{1}{2\pi i}\dfrac{\partial\hat{\phi}_n(z_j^{-1})}{\partial\gamma}\phi_n(z_j)\Delta f(z_j) & \text{if } \alpha_j = 0. \end{cases} \qquad (3\text{-}19)$$

Substituting these results into (3-6) we obtain:

Proposition 3.1. *Let $V(z) \equiv 0$. Let $\gamma = \alpha_k$ or $\gamma = \beta_k$, $k = 0, 1, \ldots, m$, and $D_n(f(z)) \neq 0$ for all n. Then for any $n = 1, 2, \ldots$,*

$$\frac{\partial}{\partial\gamma}\ln D_n(f(z)) = -2\chi_n^{-1}\frac{\partial\chi_n}{\partial\gamma}\left(n + \sum_{j=0}^m\alpha_j\right) + \sum_{j=0}^m(G_j^+(z_j) - G_j^-(z_j)), \quad (3\text{-}20)$$

where $G_j^+(z_j)$ and $G_j^-(z_j)$ are defined in (3-17) *and* (3-19), *with reference to* (3-18).

In Section 5A we substitute the asymptotics for Y (found in Section 4) in (3-20) and, by integrating, obtain part of Theorem 1.1 for $f(z)$ with $V(z) \equiv 0$, $\|\beta\| < \frac{1}{2}$. Further analysis of Section 5B extends the result to $\|\beta\| < 1$.

Remark 3.2. The differential identities (3-20) admit an interesting interpretation in the context of the monodromy theory of the Fuchsian system of linear ODEs canonically related to the Riemann–Hilbert problem (2-6)–(2-9). We explain this connection in some detail in the Appendix. The results presented there, however, are not used in the main body of the paper.

To extend the theorem to nonzero $V(z)$ we will use another differential identity. Let us introduce a parametric family of weights and the corresponding orthogonal

polynomials indexed by $t \in [0, 1]$. Namely, let

$$f(z, t) = (1 - t + t e^{V(z)}) e^{-V(z)} f(z). \tag{3-21}$$

Thus $f(z, 0)$ corresponds to $f(z)$ with $V = 0$, whereas $f(z, 1)$ gives the function (1-2) we are interested in.

Note that

$$\frac{\partial f(z, t)}{\partial t} = \frac{f(z, t) - f(z, 0)}{t}. \tag{3-22}$$

Set now $\gamma = t$ and replace the function $f(z)$ and the orthogonal polynomials in (3-5) by $f(z, t)$ and the polynomials orthogonal with respect to $f(z, t)$. Then the integral in the right-hand side of (3-5) can be written as follows (we assume $D_n \neq 0$ for all n):

$$\frac{\partial}{\partial t} \left[\frac{1}{2\pi} \int_0^{2\pi} \left(\phi_n(z, t) \frac{d\hat{\phi}_n(z^{-1}, t)}{dz} - \hat{\phi}_n(z^{-1}, t) \frac{d\phi_n(z, t)}{dz} \right) z f(z, t) \, d\theta \right]$$

$$- \frac{1}{2\pi} \int_0^{2\pi} \left(\phi_n(z, t) \frac{d\hat{\phi}_n(z^{-1}, t)}{dz} - \hat{\phi}_n(z^{-1}, t) \frac{d\phi_n(z, t)}{dz} \right) z \frac{f(z, t) - f(z, 0)}{t} \, d\theta$$

$$= \frac{2n}{t} + \frac{1}{2\pi t} \int_C \left(\phi_n(z, t) \frac{d\hat{\phi}_n(z^{-1}, t)}{dz} - \hat{\phi}_n(z^{-1}, t) \frac{d\phi_n(z, t)}{dz} \right) z f(z, 0) \, d\theta.$$

Therefore, we obtain

$$\frac{\partial}{\partial t} \ln D_n(f(z, t)) = 2n \left(\frac{1}{t} + \chi_n(t)^{-1} \frac{\partial \chi_n(t)}{\partial t} \right)$$

$$+ \frac{1}{2\pi t} \int_C \left(\phi_n(z, t) \frac{d\hat{\phi}_n(z^{-1}, t)}{dz} - \hat{\phi}_n(z^{-1}, t) \frac{d\phi_n(z, t)}{dz} \right) z f(z, 0) \, d\theta. \tag{3-23}$$

To write this identity in terms of the solution to the RHP (2-6)–(2-7), note first that using the recurrence relation (see, e.g., [Deift et al. 2011, Lemma 2.2])

$$\chi_n z^{-1} \hat{\phi}_n(z^{-1}) = \chi_{n+1} \hat{\phi}_{n+1}(z^{-1}) - \hat{\phi}_{n+1}(0) z^{-n-1} \phi_{n+1}(z), \tag{3-24}$$

we have

$$Y_{21}^{(n)}(z, t) = -\chi_{n-1}(t) z^{n-1} \hat{\phi}_{n-1}(z^{-1}, t)$$

$$= -\chi_n(t) z^n \hat{\phi}_n(z^{-1}, t) + \hat{\phi}_n(0, t) \phi_n(z, t). \tag{3-25}$$

Now using the orthogonality relations (2-1) and the formulae (3-2) and (3-3), we obtain from (3-23):

Proposition 3.3. *Let $f(z, t)$ be given by* (3-21) *and $D_n(f(z, t)) \neq 0$ for all n. Let $\phi_k(z, t)$, $\hat{\phi}_k(z, t)$, $k = 0, 1, \ldots,$ be the corresponding orthogonal polynomials.*

Then, for any $n = 1, 2, \ldots,$

$$\frac{\partial}{\partial t} \ln D_n(f(z, t))$$

$$= \frac{1}{2\pi i} \int_C z^{-n} \left(Y_{11}(z, t) \frac{\partial Y_{21}(z, t)}{\partial z} - Y_{21}(z, t) \frac{\partial Y_{11}(z, t)}{\partial z} \right) \frac{\partial f(z, t)}{\partial t} \, dz, \quad (3\text{-}26)$$

where the integration is over the unit circle and $Y(z, t) := Y^{(n)}(z, t)$.

4. Asymptotics for the Riemann–Hilbert problem

The RHP of Section 2 was solved in [Deift et al. 2011]. In this section we list the results from that paper needed for the proof of Theorem 1.1. We always assume (for the rest of the paper) that $f(z)$ is given by (1-2) and that $\alpha_j \pm \beta_j \neq -1, -2, \ldots$ for all $j = 0, 1, \ldots, m$. In this section we also assume for simplicity that $z_0 = 1$ is a singularity. However, the results trivially extend to the case $\alpha_0 = \beta_0 = 0$. In this section, we further assume that $V(z)$ is analytic in a neighborhood of the unit circle.

Let $Y(z) := Y^{(n)}(z)$. First, set

$$T(z) = Y(z) \begin{cases} z^{-n\sigma_3} & \text{if } |z| > 1 \\ I & \text{if } |z| < 1. \end{cases} \quad (4\text{-}1)$$

Now split the contour as shown in Figure 1. We call lenses the $m + 1$ regions of the complex plane containing the arcs Σ'_j of the unit circle and bounded by the curves Σ_j, Σ''_j.
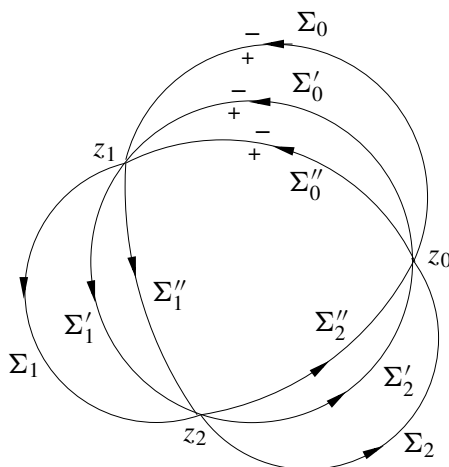

Figure 1. Contour for the S-Riemann–Hilbert problem ($m = 2$).

Set

$$S(z) = \begin{cases} T(z) & \text{for } z \text{ outside the lenses,} \\[2ex] T(z) \begin{pmatrix} 1 & 0 \\ f(z)^{-1}z^{-n} & 1 \end{pmatrix} & \text{for } |z| > 1 \text{ and inside the lenses,} \\[2ex] T(z) \begin{pmatrix} 1 & 0 \\ -f(z)^{-1}z^{n} & 1 \end{pmatrix} & \text{for } |z| < 1 \text{ and inside the lenses.} \end{cases} \tag{4-2}$$

Here $f(z)$ is the analytic continuation of $f(z)$ off the unit circle into the inside of the lenses as discussed following (3-13).

The function $S(z)$ satisfies the following Riemann–Hilbert problem:

(a) $S(z)$ is analytic for $z \in \mathbb{C} \setminus \Sigma$, where $\Sigma = \bigcup_{j=0}^{m}(\Sigma_j \cup \Sigma'_j \cup \Sigma''_j)$.

(b) The boundary values of $S(z)$ are related by the jump condition

$$S_+(z) = S_-(z) \begin{pmatrix} 1 & 0 \\ f(z)^{-1}z^{\mp n} & 1 \end{pmatrix}, \quad z \in \bigcup_{j=0}^{m}(\Sigma_j \cup \Sigma''_j), \tag{4-3}$$

where the minus sign in the exponent is on Σ_j, and plus on Σ''_j,

$$S_+(z) = S_-(z) \begin{pmatrix} 0 & f(z) \\ -f(z)^{-1} & 0 \end{pmatrix}, \quad z \in \bigcup_{j=0}^{m} \Sigma'_j. \tag{4-4}$$

(c) $S(z) = I + O(1/z)$ as $z \to \infty$,

(d) As $z \to z_j$, $j = 0, \ldots, m$, $z \in \mathbb{C} \setminus C$ outside the lenses,

$$S(z) = \begin{pmatrix} O(1) & O(1) + O(|z - z_j|^{2\alpha_j}) \\ O(1) & O(1) + O(|z - z_j|^{2\alpha_j}) \end{pmatrix} \tag{4-5}$$

if $\alpha_j \neq 0$, and

$$S(z) = \begin{pmatrix} O(1) & O(\ln|z - z_j|) \\ O(1) & O(\ln|z - z_j|) \end{pmatrix} \tag{4-6}$$

if $\alpha_j = 0$, $\beta_j \neq 0$. The behavior of $S(z)$ for $z \to z_j$ in other sectors is obtained from these expressions by application of the appropriate jump conditions.

We now present formulae for the parametrices which solve the model Riemann–Hilbert problems outside the neighborhoods U_{z_j} of the points z_j, and inside those neighborhoods, respectively. These parametrices match to the leading order in n on the boundaries of the neighborhoods U_{z_j}, and this matching allows us to construct the asymptotic solution to the RHP for Y.

The parametrix outside the U_{z_j} is

$$N(z) = \begin{cases} \mathscr{D}(z)^{\sigma_3} & \text{if } |z| > 1, \\ \mathscr{D}(z)^{\sigma_3} \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) & \text{if } |z| < 1, \end{cases} \qquad z \in \mathbb{C} \setminus \bigcup_{j=0}^{m} U_{z_j}, \qquad (4\text{-}7)$$

where the Szegő function

$$\mathscr{D}(z) = \exp \frac{1}{2\pi i} \int_C \frac{\ln f(s)}{s - z} \, ds \qquad (4\text{-}8)$$

is analytic away from the unit circle, and we have

$$\mathscr{D}(z) = e^{V_0} b_+(z) \prod_{k=0}^{m} \left(\frac{z - z_k}{z_k e^{i\pi}} \right)^{\alpha_k + \beta_k}, \quad |z| < 1. \qquad (4\text{-}9)$$

and

$$\mathscr{D}(z) = b_-(z)^{-1} \prod_{k=0}^{m} \left(\frac{z - z_k}{z} \right)^{-\alpha_k + \beta_k}, \quad |z| > 1, \qquad (4\text{-}10)$$

where V_0, $b_\pm(z)$ are defined in (1-8). Note that the branch of $(z - z_k)^{\pm \alpha_k + \beta_k}$ in (4-9) and (4-10) is taken as discussed following Equation (3-13) above. In (4-10) for any k, the cut of the root $z^{-\alpha_k + \beta_k}$ is the line $\theta = \theta_k$ from $z = 0$ to infinity, and $\theta_k < \arg z < 2\pi + \theta_k$.

Inside each neighborhood U_{z_j} the parametrix is given in terms of a confluent hypergeometric function. First, set

$$\zeta = n \ln \frac{z}{z_j}, \qquad (4\text{-}11)$$

where $\ln x > 0$ for $x > 1$, and has a cut on the negative half of the real axis. Under this transformation the neighborhood U_{z_j} is mapped onto a neighborhood of zero in the ζ-plane. Note that the transformation $\zeta(z)$ is analytic, one-to-one, and it takes an arc of the unit circle to an interval of the imaginary axis. Let us now choose the exact form of the cuts Σ in U_{z_j} so that their images under the mapping $\zeta(z)$ are straight lines (Figure 2).

We add one more jump contour to Σ in U_{z_j} which is the preimage of the real line Γ_3 and Γ_7 in the ζ-plane. This is needed below because of the nonanalyticity of the function $|z - z_j|^{\alpha_j}$. Note that we can construct two different analytic continuations of this function off the unit circle to the preimages of the upper and lower half ζ-plane, respectively. Namely, let

$$h_{\alpha_j}(z) = |z - z_j|^{\alpha_j}, \quad z = e^{i\theta} \qquad (4\text{-}12)$$

with the branches chosen as in (3-13). As remarked above, (3-13) is continued analytically to neighborhoods of the arcs $0 < \theta < \theta_j$ and $\theta_j < \theta < 2\pi$. In U_{z_j}, we
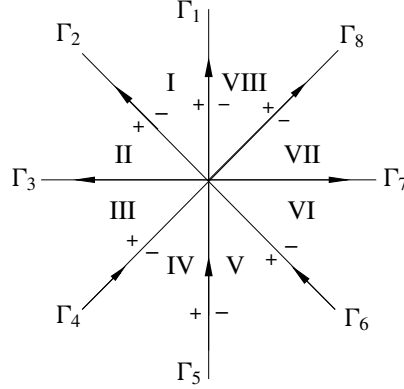
Figure 2. The auxiliary contour for the parametrix at z_j.

extend these neighborhoods to the preimages of the lower and upper half ζ-plane (intersected with $\zeta(U_{z_j})$), respectively. The cut of h_{α_j} is along the contours Γ_3 and Γ_7 in the ζ-plane.

For $z \to z_j$, $\zeta = n(z - z_j)/z_j + O((z - z_j)^2)$. We have $0 < \arg \zeta < 2\pi$, which follows from the choice of $\arg(z - z_j)$ in (3-13).

From now on we will provide the formulae for the parametrix only in the region I (see [Deift et al. 2011] for complete results). Set

$$F_j(z) = e^{V(z)/2} \prod_{k=0}^{m} \left(\frac{z}{z_k}\right)^{\beta_k/2} \prod_{k \neq j} h_{\alpha_k}(z) g_{\beta_k}(z)^{1/2} h_{\alpha_j}(z) e^{-i\pi\alpha_j},$$
$$\zeta \in I, \quad z \in U_{z_j}, \quad j \neq 0. \quad \text{(4-13)}$$

Note that this function is related to $f(z)$ as follows:

$$F_j(z)^2 = f(z) e^{-2\pi i \alpha_j} g_{\beta_j}^{-1}(z) \quad \zeta \in I. \quad \text{(4-14)}$$

The functions $g_{\beta_k}(z)$ are defined in (1-4). The formulae for $F_0(z)$ are the same, but with $g_{\beta_0}(z)$ replaced by

$$\widehat{g}_{\beta_0}(z) = \begin{cases} e^{-i\pi\beta_0} & \text{if } \arg z > 0, \\ e^{i\pi\beta_0} & \text{if } \arg z < 2, \pi, \end{cases} \quad z \in U_{z_0}. \quad \text{(4-15)}$$

We then have the following expression for the parametrix $P_j(z)$ in the region $z(I)$ of U_{z_j}:

$$P_{z_j}(z) = E(z) \Psi_j(\zeta) F_j(z)^{-\sigma_3} z^{n\sigma_3/2}, \quad \zeta \in I. \quad \text{(4-16)}$$

Here

$$E(z) = N(z) \zeta^{\beta_j \sigma_3} F_j^{\sigma_3}(z) z_j^{-n\sigma_3/2} \begin{pmatrix} e^{-i\pi(2\beta_j + \alpha_j)} & 0 \\ 0 & e^{i\pi(\beta_j + 2\alpha_j)} \end{pmatrix} \quad \text{(4-17)}$$

and

$$\Psi_j(\zeta)$$

$$= \begin{pmatrix} \zeta^{\alpha_j}\psi(\alpha_j+\beta_j, 1+2\alpha_j, \zeta)e^{i\pi(2\beta_j+\alpha_j)}e^{-\zeta/2} \\ -\zeta^{-\alpha_j}\psi(1-\alpha_j+\beta_j, 1-2\alpha_j, \zeta)e^{i\pi(\beta_j-3\alpha_j)}e^{-\zeta/2}\frac{\Gamma(1+\alpha_j+\beta_j)}{\Gamma(\alpha_j-\beta_j)} \\ \qquad -\zeta^{\alpha_j}\psi(1+\alpha_j-\beta_j, 1+2\alpha_j, e^{-i\pi}\zeta)e^{i\pi(\beta_j+\alpha_j)}e^{\zeta/2}\frac{\Gamma(1+\alpha_j-\beta_j)}{\Gamma(\alpha_j+\beta_j)} \\ \zeta^{-\alpha_j}\psi(-\alpha_j-\beta_j, 1-2\alpha_j, e^{-i\pi}\zeta)e^{-i\pi\alpha_j}e^{\zeta/2} \end{pmatrix}, \quad (4\text{-}18)$$

where $\psi(a, b, x)$ is the confluent hypergeometric function of the second kind, and $\Gamma(x)$ is Euler's Γ-function. Recall our assumption that $\alpha_j \pm \beta_j \neq -1, -2, \ldots$.

The matching condition for the parametrices P_{z_j} and N is the following for any $k = 1, 2, \ldots$:

$$P_{z_j}(z)N^{-1}(z) = I + \Delta_1(z) + \Delta_2(z) + \cdots + \Delta_k(z) + \Delta_{k+1}^{(r)}, \quad z \in \partial U_{z_j}. \quad (4\text{-}19)$$

Every $\Delta_p(z)$, $\Delta_p^{(r)}(z)$, $p = 1, 2, \ldots$, with $z \in \partial U_{z_j}$ is of the form

$$a_j^{-\sigma_3}O(n^{-p})a_j^{\sigma_3}, \quad a_j := n^{\beta_j}z_j^{-n/2}. \quad (4\text{-}20)$$

In particular, explicitly, on the part of ∂U_{z_j} whose ζ-image is in I,

$$\Delta_1(z) = \frac{1}{\zeta}\begin{pmatrix} -(\alpha_j^2-\beta_j^2) \\ -\frac{\Gamma(1+\alpha_j-\beta_j)}{\Gamma(\alpha_j+\beta_j)}\left(\frac{\mathcal{D}(z)}{\zeta^{\beta_j}F_j(z)}\right)^{-2}z_j^{-n}e^{-i\pi(2\beta_j-\alpha_j)} \\ \qquad \frac{\Gamma(1+\alpha_j+\beta_j)}{\Gamma(\alpha_j-\beta_j)}\left(\frac{\mathcal{D}(z)}{\zeta^{\beta_j}F_j(z)}\right)^{2}z_j^{n}e^{i\pi(2\beta_j-\alpha_j)} \\ \alpha_j^2-\beta_j^2 \end{pmatrix}, \quad (4\text{-}21)$$

which extends to a meromorphic function in a neighborhood of U_{z_j} with a simple pole at $z = z_j$.

The error term $\Delta_{k+1}^{(r)}$ in (4-19) is uniform in z on ∂U_{z_j}.

At the point z_j we have

$$F_j(z) = \eta_j e^{-3i\pi\alpha_j/2}z_j^{-\alpha_j}u^{\alpha_j}(1 + O(u)), \quad u = z - z_j, \quad \zeta \in I, \quad (4\text{-}22)$$

where

$$\eta_j = e^{V(z_j)/2}\exp\left\{-\frac{i\pi}{2}\left(\sum_{k=0}^{j-1}\beta_k - \sum_{k=j+1}^{m}\beta_k\right)\right\}\prod_{k\neq j}\left(\frac{z_j}{z_k}\right)^{\beta_k/2}|z_j - z_k|^{\alpha_k}, \quad (4\text{-}23)$$
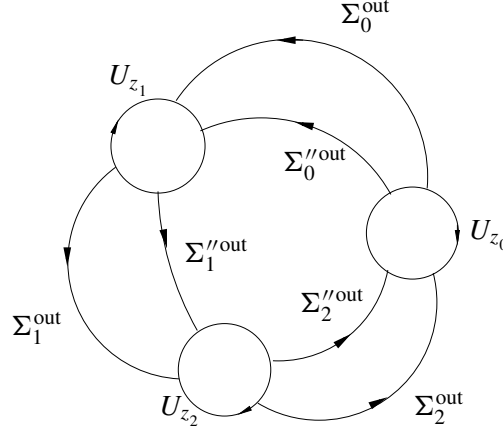
Figure 3. Contour Γ for the R and $\widetilde{R}$ Riemann–Hilbert problems ($m = 2$).

and

$$\left(\frac{\mathcal{D}(z)}{\zeta^{\beta_j} F_j(z)}\right)^2 = \mu_j^2 e^{i\pi(\alpha_j - 2\beta_j)} n^{-2\beta_j}(1 + O(u)), \quad u = z - z_j, \quad \zeta \in I, \quad (4\text{-}24)$$

$$\mu_j = \left(e^{V_0}\frac{b_+(z_j)}{b_-(z_j)}\right)^{1/2} \exp\left\{-\frac{i\pi}{2}\left(\sum_{k=0}^{j-1}\alpha_k - \sum_{k=j+1}^{m}\alpha_k\right)\right\}\prod_{k\neq j}\left(\frac{z_j}{z_k}\right)^{\alpha_k/2}|z_j - z_k|^{\beta_k}.$$

$$(4\text{-}25)$$

The sums from 0 to -1 for $j = 0$ and from $m + 1$ to m for $j = m$ are set to zero.

4A. ***R-RHP.*** Let

$$R(z) = \begin{cases} S(z)N^{-1}(z) & \text{if } z \in U_\infty \setminus \Gamma, \text{ with } U_\infty = \mathbb{C} \setminus \bigcup_{j=0}^{m} U_{z_j}, \\ S(z)P_{z_j}^{-1}(z) & \text{if } z \in U_{z_j} \setminus \Gamma \text{ for } j = 0, \ldots, m. \end{cases} \quad (4\text{-}26)$$

It is easy to verify that this function has jumps only on ∂U_{z_j}, and the parts of Σ_j, Σ_j'' lying outside the neighborhoods U_{z_j} (we denote these parts without the end-points Σ^{out}, Σ''^{out}). The full contour Γ is shown in Figure 3. Away from Γ, as a standard argument shows, $R(z)$ is analytic. Moreover, we have: $R(z) = I + O(1/z)$ as $z \to \infty$.

The jumps of $R(z)$ are as follows, with $j = 0, \ldots, m$:

$$R_+(z) = R_-(z)N(z)\begin{pmatrix} 1 & 0 \\ f(z)^{-1}z^{-n} & 1 \end{pmatrix}N(z)^{-1}, \qquad z \in \Sigma_j^{\text{out}}, \qquad (4\text{-}27)$$

$$R_+(z) = R_-(z)N(z)\begin{pmatrix} 1 & 0 \\ f(z)^{-1}z^{n} & 1 \end{pmatrix}N(z)^{-1}, \qquad z \in \Sigma_j''^{\text{out}}, \qquad (4\text{-}28)$$

$$R_+(z) = R_-(z)P_{z_j}(z)N(z)^{-1}, \quad z \in \partial U_{z_j} \setminus \{\text{intersection points}\}, \qquad (4\text{-}29)$$

The jump matrix on Σ^{out}, Σ''^{out} can be estimated uniformly in α_j, β_j as $I + O(\exp(-\varepsilon n))$, where ε is a positive constant. The jump matrices on ∂U_{z_j} admit a uniform expansion (4-19) in inverse powers of n conjugated by $n^{\beta_j \sigma_3} z_j^{-n\sigma_3/2}$, and (4-20) is of order $n^{2\max_j |\Re\beta_j|-p}$. To obtain the standard solution of the R-RHP in terms of a Neuman series (see, e.g., [Deift et al. 1999]) we must have $n^{2\max_j |\Re\beta_j|-1} = o(1)$, that is $\Re\beta_j \in (-\frac{1}{2}, \frac{1}{2})$ for all $j = 0, 1, \ldots, m$. However, it is possible to obtain the solution in any half-closed or open interval of length 1, i.e., for $\||\beta\|| < 1$, as follows.

Let $\||\beta\|| < 1$ and consider the transformation

$$\widetilde{R}(z) = n^{\omega\sigma_3} R(z) n^{-\omega\sigma_3} \quad z \in \mathbb{C} \setminus \Gamma, \tag{4-30}$$

where

$$\omega = \tfrac{1}{2}(\min_j \Re\beta_j + \max_j \Re\beta_j) \tag{4-31}$$

which "shifts" all $\Re\beta_j$ (in the conjugation n^{β_j} terms of (4-20) for the jump matrix in (4-29)) into the interval $(-\frac{1}{2}, \frac{1}{2})$. Note that $\omega = \Re\beta_{j_0}$ if only one $\Re\beta_{j_0} \neq 0$, and $\omega = 0$ if all $\Re\beta_j = 0$.

Now in the RHP for $\widetilde{R}(z)$, the condition at infinity and the uniform exponential estimate $I + O(\exp(-\varepsilon n))$ (with different ε) of the jump matrices on Σ^{out}, Σ''^{out} is preserved, while the jump matrices on ∂U_{z_j} have the form

$$I + n^{\omega\sigma_3}\Delta_1(z)n^{-\omega\sigma_3} + \cdots + n^{\omega\sigma_3}\Delta_k(z)n^{-\omega\sigma_3} + n^{\omega\sigma_3}\Delta_{k+1}^{(r)}(z)n^{-\omega\sigma_3},$$
$$z \in \partial U_{z_j}, \quad (4\text{-}32)$$

where the order of each $n^{\omega\sigma_3}\Delta_p(z)n^{-\omega\sigma_3}$, $n^{\omega\sigma_3}\Delta_p^{(r)}(z)n^{-\omega\sigma_3}$, $p = 1, 2, \ldots$, with $z \in \bigcup_{j=0}^m \partial U_{z_j}$ is

$$O(n^{2\max_j |\Re\beta_j-\omega|-p}) = O(n^{\||\beta\||-p}).$$

This implies that the standard analysis can be applied to the $\widetilde{R}$-RHP problem in the range $\Re\beta_j \in (q - \frac{1}{2}, q + \frac{1}{2})$, $j = 0, 1, \ldots, m$, for any $q \in \mathbb{R}$, and we obtain the asymptotic expansion

$$\widetilde{R}(z) = I + \sum_{p=1}^k \widetilde{R}_p(z) + \widetilde{R}_{k+1}^{(r)}(z), \quad p = 1, 2 \ldots \tag{4-33}$$

uniformly for all z and for β_j in bounded sets of the strip $q - \frac{1}{2} < \Re\beta_j < q + \frac{1}{2}$, $j = 0, 1, \ldots m$, i.e., $\||\beta\|| < 1$, provided $\alpha_j \pm \beta_j$ are outside neighborhoods of the points $-1, -2, \ldots$ (compare (4-21)).

The functions $\widetilde{R}_j(z)$ are computed recursively. We will need explicit expressions only for the first two. The first one is found from the conditions that $\widetilde{R}_1(z)$

is analytic outside $\partial U = \bigcup_{j=0}^{m} \partial U_{z_j}$, $\widetilde{R}_1(z) \to 0$ as $z \to \infty$, and

$$\widetilde{R}_{1,+}(z) = \widetilde{R}_{1,-}(z) + n^{\omega\sigma_3}\Delta_1(z)n^{-\omega\sigma_3}, \quad z \in \partial U. \qquad (4\text{-}34)$$

The solution is easily found. First set

$$R_k(z) := n^{-\omega\sigma_3}\widetilde{R}_k(z)n^{\omega\sigma_3}, \quad R_k^{(r)}(z) := n^{-\omega\sigma_3}\widetilde{R}_k^{(r)}(z)n^{\omega\sigma_3}, \quad k = 1, 2, \ldots,$$

and write for R:

$$R_1(z) = \frac{1}{2\pi i}\int_{\partial U}\frac{\Delta_1(x)\,dx}{x-z}$$

$$= \begin{cases} \displaystyle\sum_{k=0}^{m}\frac{A_k}{z-z_k} & \text{if } z \in \mathbb{C}\setminus\bigcup_{j=0}^{m}U_{z_j}, \\[3mm] \displaystyle\sum_{k=0}^{m}\frac{A_k}{z-z_k} - \Delta_1(z) & \text{if } z \in U_{z_j}, \quad j = 0, 1, \ldots, m, \end{cases} \qquad (4\text{-}35)$$

where

$$\partial U = \bigcup_{j=0}^{m}\partial U_{z_j}, \qquad (4\text{-}36)$$

the contours in the integral are traversed in the negative direction, and the A_k are the coefficients in the Laurent expansion of $\Delta_1(z)$:

$$\Delta_1(z) = \frac{A_k}{z-z_k} + B_k + O(z-z_k), \quad z \to z_k, \quad k = 0, 1, \ldots, m. \qquad (4\text{-}37)$$

The coefficients are easy to compute using (4-19) and (4-25):

$$A_k = A_k^{(n)} = \frac{z_k}{n}\begin{pmatrix} -(\alpha_k^2 - \beta_k^2) & \dfrac{\Gamma(1+\alpha_k+\beta_k)}{\Gamma(\alpha_k-\beta_k)}z_k^n\mu_k^2 n^{-2\beta_k} \\[4mm] -\dfrac{\Gamma(1+\alpha_k-\beta_k)}{\Gamma(\alpha_k+\beta_k)}z_k^{-n}\mu_k^{-2}n^{2\beta_k} & \alpha_k^2 - \beta_k^2 \end{pmatrix}. \qquad (4\text{-}38)$$

The function $\widetilde{R}_2$ is now found from the conditions that $\widetilde{R}_2(z) \to 0$ as $z \to \infty$, is analytic outside ∂U, and

$$\widetilde{R}_{2,+}(z) = \widetilde{R}_{2,-}(z) + \widetilde{R}_{1,-}(z)n^{\omega\sigma_3}\Delta_1(z)n^{-\omega\sigma_3} + n^{\omega\sigma_3}\Delta_2(z)n^{-\omega\sigma_3}, \quad z \in \partial U.$$

The solution to this RHP is

$$\widetilde{R}_2(z) = \frac{1}{2\pi i}\int_{\partial U}\left(\widetilde{R}_{1,-}(x)n^{\omega\sigma_3}\Delta_1(x)n^{-\omega\sigma_3} + n^{\omega\sigma_3}\Delta_2(x)n^{-\omega\sigma_3}\right)\frac{dx}{x-z}. \qquad (4\text{-}39)$$

At the k-th step we have the RHP for $\widetilde{R}_k(z)$ with the same analyticity condition and the condition at infinity, and the following jump, where $\widetilde{R}_0(z) \equiv I$:

$$\widetilde{R}_{k,+}(z) = \widetilde{R}_{k,-}(z) + \sum_{p=1}^{k}\widetilde{R}_{k-p,-}(z)n^{\omega\sigma_3}\Delta_p(z)n^{-\omega\sigma_3}, \quad z \in \partial U. \qquad (4\text{-}40)$$

We will now discuss the way in which the general $\widetilde{R}_k(z)$ depends on n. In particular, we will discuss its order in n. First note that

$$\widetilde{R}_1(z) \sim \frac{1}{n} \begin{pmatrix} 1 & \sum_j b_j^{-2} \\ \sum_j b_j^2 & 1 \end{pmatrix}, \tag{4-41}$$

$$\widetilde{R}_2(z) \sim \frac{1}{n^2} \begin{pmatrix} 1 + \delta'n^2 & \sum_j b_j^{-2} \\ \sum_j b_j^2 & 1 + \delta'n^2 \end{pmatrix}, \tag{4-42}$$

where

$$b_j := n^{\beta_j - \omega} z_j^{-n/2}, \quad \delta' \sim \sum_{j,k} n^{2((\beta_j - \beta_k) - 1)} \left(\frac{z_k}{z_j} \right)^n. \tag{4-43}$$

Here the notation $A \sim B$ means $A' = B'$, where X' is X in which each matrix element and each term in the sums is multiplied by a suitable constant *independent* of n. Starting with these expressions, and noting from (4-40) that

$$\widetilde{R}_k(z) \sim \sum_{p=1}^{k} \widetilde{R}_{k-p,-}(z) n^{\omega\sigma_3} \Delta_p(z) n^{-\omega\sigma_3}$$

$$\sim \widetilde{R}_{k-1,-}(z) n^{\omega\sigma_3} \Delta_1(z) n^{-\omega\sigma_3} + \frac{1}{n} \widetilde{R}_{k-1,-}(z), \tag{4-44}$$

we obtain by induction, for $p = 0, 1, \ldots$:

$$\widetilde{R}_{2p+1}(z) \sim \frac{1}{n^{2p+1}} \sum_{k=0}^{p} (\delta'n^2)^k \begin{pmatrix} 1 & \sum_j b_j^{-2} \\ \sum_j b_j^2 & 1 \end{pmatrix}, \tag{4-45}$$

$$\widetilde{R}_{2p+2}(z) \sim \frac{1}{n^{2p+2}} \sum_{k=0}^{p} (\delta'n^2)^k \begin{pmatrix} 1 + \delta'n^2 & \sum_j b_j^{-2} \\ \sum_j b_j^2 & 1 + \delta'n^2 \end{pmatrix}. \tag{4-46}$$

In particular, as $n \to \infty$, and again for $p = 0, 1, \ldots$ we have,

$$\widetilde{R}_{2p+1}(z) = \frac{\delta'^p}{n} O \begin{pmatrix} 1 & \sum_j b_j^{-2} \\ \sum_j b_j^2 & 1 \end{pmatrix}, \tag{4-47}$$

$$\widetilde{R}_{2p+2}(z) = \frac{\delta'^p}{n^2} O \begin{pmatrix} 1 + \delta'n^2 & \sum_j b_j^{-2} \\ \sum_j b_j^2 & 1 + \delta'n^2 \end{pmatrix}, \tag{4-48}$$

$$O(\delta') = O(\delta), \quad \delta = \max_{j,k} n^{2(\Re(\beta_j - \beta_k) - 1)} = n^{2(|||\beta||| - 1)}, \tag{4-49}$$

Here $O(A)$ represent 2×2 matrices with elements of the corresponding order.

Finally, note that the error term in (4-33) is

$$\widetilde{R}_k^{(r)}(z) = O(|\widetilde{R}_k(z)| + |\widetilde{R}_{k+1}(z)|).$$

In particular, as is clear from the above, if there is only one nonzero β_{j_0}, we obtain the expansion purely in inverse integer powers of n valid in fact for all $\beta_{j_0} \in \mathbb{C}$ uniformly in bounded sets of the complex plane.

It is clear from the construction and the properties of the asymptotic series of the confluent hypergeometric function that the error terms $\widetilde{R}_k^{(r)}(z)$ are uniform for β_j in bounded subsets of the strip $q - \frac{1}{2} < \Re\beta_j < q + \frac{1}{2}$, $j = 0, 1, \ldots m$, for α_j in bounded sets of the half-plane $\Re\alpha_j > -\frac{1}{2}$, and for $\alpha_j \pm \beta_j$ away from neighborhoods of the negative integers. Moreover, the series (4-33) is differentiable in α_j, β_j.

5. Asymptotics for differential identities and integration: proof of Theorem 1.1

5A. *Pure Fisher–Hartwig singularities: the case $|\!|\!|\beta|\!|\!| < \frac{1}{2}$.* First, we will prove the theorem for $V(z) \equiv 0$ and $|\!|\!|\beta|\!|\!| = \max_{j,k} |\Re\beta_j - \Re\beta_k| < \frac{1}{2}$. The proof is based on the differential identity (3-20). First, we show that (3-20) has the following asymptotic form.

Proposition 5.1. *Let $(\alpha_0, \beta_0, \ldots, \alpha_m, \beta_m)$ be in a compact subset, denote it Λ, belonging to the subset $|\!|\!|\beta|\!|\!| < 1$, $\alpha_j \pm \beta_j \neq -1, -2, \ldots$ of the parameter space $\mathscr{P} = \{(\alpha_0, \beta_0, \ldots, \alpha_m, \beta_m) : \alpha_j, \beta_j \in \mathbb{C}, \Re\alpha_j > -\frac{1}{2}\}$ and including the point $\alpha_j = \beta_j = 0$, $j = 0, 1, \ldots, m$. Let $\beta_j = 0$ if $\alpha_j = 0$, $j = 0, 1, \ldots, m$, $\delta = n^{2(|\!|\!|\beta|\!|\!|-1)}$. Then for $n \to \infty$, and $\nu = 0, 1, \ldots, m$,*

$$\frac{\partial}{\partial\alpha_\nu} \ln D_n(f(z)) = 2\alpha_\nu + (\alpha_\nu + \beta_\nu)\left[\frac{\partial}{\partial\alpha_\nu} \ln \frac{\Gamma(1 + \alpha_\nu + \beta_\nu)}{\Gamma(1 + 2\alpha_\nu)} + \ln n\right]$$

$$+ (\alpha_\nu - \beta_\nu)\left[\frac{\partial}{\partial\alpha_\nu} \ln \frac{\Gamma(1 + \alpha_\nu - \beta_\nu)}{\Gamma(1 + 2\alpha_\nu)} + \ln n\right]$$

$$- \sum_{j \neq \nu}\left[(\alpha_j + \beta_j) \ln \frac{z_j - z_\nu}{z_j} + (\alpha_j - \beta_j) \ln \frac{z_j - z_\nu}{z_\nu e^{i\pi}}\right]$$

$$+ 2\pi i \sum_{j=0}^{\nu-1}(\alpha_j + \beta_j) + O(n^{-1} \ln n) + O(\delta n \ln n), \quad (5\text{-}1)$$

$$\frac{\partial}{\partial\beta_\nu} \ln D_n(f(z)) = -2\beta_\nu + (\alpha_\nu + \beta_\nu)\left[\frac{\partial}{\partial\beta_\nu} \ln \Gamma(1 + \alpha_\nu + \beta_\nu) - \ln n\right]$$

$$+ (\alpha_\nu - \beta_\nu)\left[\frac{\partial}{\partial\beta_\nu} \ln \Gamma(1 + \alpha_\nu - \beta_\nu) + \ln n\right]$$

$$+ \sum_{j \neq \nu}\left[(\alpha_j + \beta_j) \ln \frac{z_j - z_\nu}{z_j} - (\alpha_j - \beta_j) \ln \frac{z_j - z_\nu}{z_\nu e^{i\pi}}\right]$$

$$- 2\pi i \sum_{j=0}^{\nu-1}(\alpha_j + \beta_j) + O(n^{-1} \ln n) + O(\delta n \ln n). \quad (5\text{-}2)$$

Remark 5.2. The error term $O(\delta n \ln n)$ is $o(1)$ uniformly in Λ if $\||\beta|\| \leq \frac{1}{2} - \varepsilon$, $\varepsilon > 0$. In fact, the estimate for the error term can be considerably improved: see next section.

Remark 5.3. The case $\beta_j = 0$ if $\alpha_j = 0$ is all we need below. After the proof of Proposition 5.1, we will integrate the identity (5-1) to obtain the asymptotics of D_n for all $\alpha_j \neq 0$, $\beta_j = 0$. We then integrate the identity (5-2) and obtain the asymptotics of D_n for all $\alpha_j \neq 0$, $\beta_j \neq 0$. This gives the general result for $V \equiv 0$, $\||\beta|\| < \frac{1}{2}$, since we can set any $\alpha_j = 0$ using the uniformity of the asymptotic expansion in the α_j.

Proof. Assume that for all j, $\beta_j = 0$ if $\alpha_j = 0$, and $D_k(f) \neq 0$, $k = 1, 2, \ldots$. Then we can rewrite (3-20) in the form

$$
\frac{\partial}{\partial \gamma} \ln D_n(f(z))
$$
$$
= -2 \frac{\frac{\partial \chi_n}{\partial \gamma}}{\chi_n} \left(n + \sum_{j=0}^{m} \alpha_j \left\{ 1 - \widetilde{Y}_{12}^{(n)}(z_j) Y_{21}^{(n+1)}(z_j) - z_j Y_{11}^{(n)}(z_j) \widetilde{Y}_{22}^{(n+1)}(z_j) \right\} \right)
$$
$$
- 2 \sum_{j=0}^{m} \alpha_j \left(\widetilde{Y}_{12}^{(n)}(z_j) \frac{\partial}{\partial \gamma} Y_{21}^{(n+1)}(z_j) - z_j \frac{\partial}{\partial \gamma} Y_{11}^{(n)}(z_j) \widetilde{Y}_{22}^{(n+1)}(z_j) \right), \quad (5\text{-}3)
$$

where $\gamma = \alpha_j$ or $\gamma = \beta_j$. We now estimate the right-hand side of this identity as $n \to \infty$. The asymptotics of χ_n were found in [Deift et al. 2011, Theorem 1.8]. We need these asymptotics here in the case $V \equiv 0$:

$$
\chi_{n-1}^2 = 1 - \frac{1}{n} \sum_{k=0}^{m} (\alpha_k^2 - \beta_k^2)
$$
$$
+ \sum_{j=0}^{m} \sum_{k \neq j} \frac{z_k}{z_j - z_k} \left(\frac{z_j}{z_k} \right)^n n^{2(\beta_k - \beta_j - 1)} \frac{v_j}{v_k} \frac{\Gamma(1 + \alpha_j + \beta_j) \Gamma(1 + \alpha_k - \beta_k)}{\Gamma(\alpha_j - \beta_j) \Gamma(\alpha_k + \beta_k)}
$$
$$
+ O(\delta^2) + O(\delta/n), \quad \delta = n^{2(\||\beta|\| - 1)}, \quad n \to \infty, \quad (5\text{-}4)
$$

where

$$
v_j = \exp\left\{ -i\pi \left(\sum_{p=0}^{j-1} \alpha_p - \sum_{p=j+1}^{m} \alpha_p \right) \right\} \prod_{p \neq j} \left(\frac{z_j}{z_p} \right)^{\alpha_p} |z_j - z_p|^{2\beta_p}. \quad (5\text{-}5)
$$

The asymptotics of $\widetilde{Y}^{(n)}(z_j)$ were also found in [Deift et al. 2011, (7.11)–(7.21)]. Namely,

$$
\widetilde{Y}^{(n)}(z_j) = (I + r_j^{(n)}) L_j^{(n)}, \quad (5\text{-}6)
$$

where

$$L_j^{(n)} = \begin{pmatrix} M_{21}\mu_j\eta_j^{-1}n^{\alpha_j-\beta_j}z_j^n & M_{22}\mu_j\eta_jn^{-\alpha_j-\beta_j} \\ -M_{11}\mu_j^{-1}\eta_j^{-1}n^{\alpha_j+\beta_j} & -M_{12}\mu_j^{-1}\eta_jn^{-\alpha_j+\beta_j}z_j^{-n} \end{pmatrix},$$

and $r_j = R_1^{(r)}(z_j)$, the parameters η_j, μ_j are given by (4-23), (4-25), and

$$M = \begin{pmatrix} \dfrac{\Gamma(1+\alpha_j-\beta_j)}{\Gamma(1+2\alpha_j)} & -\dfrac{\Gamma(2\alpha_j)}{\Gamma(\alpha_j+\beta_j)} \\ \dfrac{\Gamma(1+\alpha_j+\beta_j)}{\Gamma(1+2\alpha_j)} & \dfrac{\Gamma(2\alpha_j)}{\Gamma(\alpha_j-\beta_j)} \end{pmatrix}.$$

Note that the matrix $L_j^{(n)}$ has the structure

$$L_j^{(n)} = n^{-\beta_j\sigma_3}\hat{L}_j^{(n)}n^{\alpha_j\sigma_3}, \tag{5-7}$$

where $\hat{L}$ depends on n only via the oscillatory terms z_j^n.

Let us now obtain the asymptotics for the following combination appearing in (5-3):

$$\frac{\partial}{\partial\gamma}Y_{11}^{(n)}(z_j)\widetilde{Y}_{22}^{(n+1)}(z_j)$$

$$= \frac{\partial}{\partial\gamma}\big((1+r_{11}^{(n)})L_{11}^{(n)} + r_{12}^{(n)}L_{21}^{(n)}\big)\big(r_{21}^{(n+1)}L_{12}^{(n+1)} + (1+r_{22}^{(n+1)})L_{22}^{(n+1)}\big)$$

$$= \big(L_{11}^{(n)}L_{22}^{(n+1)}(1+r_{22}^{(n+1)}) + L_{11}^{(n)}L_{12}^{(n+1)}r_{21}^{(n+1)}\big)\Big[(1+r_{11}^{(n)})\frac{\partial}{\partial\gamma}\ln L_{11}^{(n)} + \frac{\partial}{\partial\gamma}r_{11}^{(n)}\Big]$$

$$+ \big(L_{21}^{(n)}L_{22}^{(n+1)}(1+r_{22}^{(n+1)}) + L_{21}^{(n)}L_{12}^{(n+1)}r_{21}^{(n+1)}\big)\Big[r_{12}^{(n)}\frac{\partial}{\partial\gamma}\ln L_{21}^{(n)} + \frac{\partial}{\partial\gamma}r_{12}^{(n)}\Big].$$

We omit the lower index j of r and L for simplicity of notation.

Now the explicit formula for L and the estimates for $\widetilde{R}(z)$ imply that

$$L_{11}^{(n)}L_{22}^{(n+1)} = O(1), \quad L_{21}^{(n)}L_{12}^{(n+1)} = O(1), \tag{5-8}$$

$$L_{11}^{(n)}L_{12}^{(n+1)}r_{21}^{(n+1)} = O(n\delta), \quad L_{21}^{(n)}L_{22}^{(n+1)}r_{12}^{(n)} = O(n\delta), \tag{5-9}$$

$$r_{21}^{(n+1)}r_{12}^{(n)} = O(\delta), \tag{5-10}$$

$$\frac{\partial}{\partial\gamma}\ln L^{(n)} = O(\ln n), \quad \frac{\partial}{\partial\gamma}r = O(r)\ln n, \tag{5-11}$$

where as before

$$\delta = n^{2(\|\|\beta\|\|-1)}.$$

Therefore,

$$\frac{\partial}{\partial\gamma}Y_{11}^{(n)}(z_j)\widetilde{Y}_{22}^{(n+1)}(z_j) = L_{11}^{(n)}L_{22}^{(n+1)}\frac{\partial}{\partial\gamma}\ln L_{11}^{(n)} + O\left(\frac{\ln n}{n}\right) + O(\delta n \ln n)$$

$$= \frac{\alpha_j+\beta_j}{2\alpha_j z_j}\frac{\partial}{\partial\gamma}\ln L_{11}^{(n)} + O\left(\frac{\ln n}{n}\right) + O(\delta n \ln n). \quad (5\text{-}12)$$

Similarly, we obtain

$$\widetilde{Y}_{12}^{(n)}(z_j)\frac{\partial}{\partial\gamma}Y_{21}^{(n+1)}(z_j) = -\frac{\alpha_j-\beta_j}{2\alpha_j}\frac{\partial}{\partial\gamma}\ln L_{21}^{(n+1)} + O\left(\frac{\ln n}{n}\right) + O(\delta n \ln n),$$
$$(5\text{-}13)$$

and furthermore,

$$\widetilde{Y}_{12}^{(n)}(z_j)Y_{21}^{(n+1)}(z_j) = O(\delta n) + O(1),$$
$$Y_{11}^{(n)}(z_j)\widetilde{Y}_{22}^{(n+1)}(z_j) = O(\delta n) + O(1). \qquad (5\text{-}14)$$

Note that because of the special structure of (5-7), the quantity n^{α_j} does not appear in any of the products (5-12)–(5-14). Substituting (5-12)–(5-14) into (5-3) and using the asymptotics (5-4), we obtain

$$\frac{\partial}{\partial\gamma}\ln D_n(f(z)) =$$

$$\frac{\partial}{\partial\gamma}\left[\sum_{j=0}^{m}(\alpha_j^2-\beta_j^2)\right] + \sum_{j=0}^{m}\left[(\alpha_j+\beta_j)\frac{\partial}{\partial\gamma}\ln L_{j,11}^{(n)} + (\alpha_j-\beta_j)\frac{\partial}{\partial\gamma}\ln L_{j,21}^{(n+1)}\right]$$

$$+ O\left(\frac{\ln n}{n}\right) + O(\delta n \ln n). \quad (5\text{-}15)$$

Let us calculate the logarithmic derivatives appearing in (5-15). From (5-6), (4-25), and (4-23) it is easy to obtain for the derivatives with respect to α_ν, $\nu = 0, 1, \ldots, m$:

$$\frac{\partial}{\partial\alpha_\nu}\ln L_{\nu,11}^{(n)} = \frac{\partial}{\partial\alpha_\nu}\ln\frac{\Gamma(1+\alpha_\nu+\beta_\nu)}{\Gamma(1+2\alpha_j)} + \ln n, \qquad (5\text{-}16)$$

$$\frac{\partial}{\partial\alpha_\nu}\ln L_{j,11}^{(n)} = \begin{cases} -\ln\dfrac{z_j-z_\nu}{z_j} + 2\pi i & \text{if } j < \nu, \\[2mm] -\ln\dfrac{z_j-z_\nu}{z_j} & \text{if } j > \nu, \end{cases} \qquad (5\text{-}17)$$

$$\frac{\partial}{\partial\alpha_\nu}\ln L_{\nu,21}^{(n+1)} = \frac{\partial}{\partial\alpha_\nu}\ln\frac{\Gamma(1+\alpha_\nu-\beta_\nu)}{\Gamma(1+2\alpha_j)} + \ln n, \qquad (5\text{-}18)$$

$$\frac{\partial}{\partial\alpha_\nu}\ln L_{j,21}^{(n+1)} = -\ln\frac{z_j-z_\nu}{z_\nu e^{i\pi}}, \quad j \neq \nu. \qquad (5\text{-}19)$$

Similarly, we obtain for the derivatives with respect to β_ν:

$$\frac{\partial}{\partial \beta_\nu} \ln L_{\nu,11}^{(n)} = \frac{\partial}{\partial \beta_\nu} \ln \Gamma(1 + \alpha_\nu + \beta_\nu) - \ln n, \tag{5-20}$$

$$\frac{\partial}{\partial \beta_\nu} \ln L_{j,11}^{(n)} = \begin{cases} \ln \dfrac{z_j - z_\nu}{z_j} - 2\pi i & \text{if } j < \nu, \\[2mm] \ln \dfrac{z_j - z_\nu}{z_j} & \text{if } j > \nu, \end{cases} \tag{5-21}$$

$$\frac{\partial}{\partial \beta_\nu} \ln L_{\nu,21}^{(n+1)} = \frac{\partial}{\partial \beta_\nu} \ln \Gamma(1 + \alpha_\nu - \beta_\nu) + \ln n, \tag{5-22}$$

$$\frac{\partial}{\partial \beta_\nu} \ln L_{j,21}^{(n+1)} = -\ln \frac{z_j - z_\nu}{z_\nu e^{i\pi}}, \quad j \neq \nu. \tag{5-23}$$

Combining these results with (5-15) we obtain (5-1), (5-2) on condition that $D_k(f) \neq 0$, $k = 1, 2, \ldots$, or equivalently (see Section 2), $(\alpha_0, \beta_0, \ldots, \alpha_m, \beta_m) \in \Lambda \setminus \Omega_{k_0}$. This condition can be replaced simply by $(\alpha_0, \beta_0, \ldots, \alpha_m, \beta_m) \in \Lambda$ in the following way. Let

$$\beta_0 = 0, \quad \alpha_j = \beta_j = 0, \quad j = 1, \ldots, m. \tag{5-24}$$

Let $\Omega_{k_0}(\alpha_0)$ be the subset of Ω_{k_0} with α_j, $j = 1, \ldots, m$, and β_j, $j = 0, \ldots, m$, fixed by (5-24). Since $D_k(f) \equiv D_k(f(\alpha_0; z))$ is an analytic function of α_0 and $D_k(1) \neq 0$, the set $\Omega_{k_0}(\alpha_0)$ is finite. Let us rewrite the identity (5-1) with $\nu = 0$ and assuming (5-24) in the form $H'(\alpha_0) = 0$, where $H(\alpha_0) = D_n(f(\alpha_0; z)) \exp(-\int_0^{\alpha_0} r(n, s)\, ds)$ and where $r(n, \alpha_0)$ is the right-hand side of (3-20) with $\gamma = \alpha_0$ and assuming (5-24). Since the expression (5-1) for $r(n, \alpha_0)$ holds uniformly and is continuous for $\alpha_0 \in \Lambda$ provided n is larger than some $k_0(\Lambda)$, and $D_n(f(\alpha_0; z))$ and its derivative are continuous, the function $H(\alpha_0)$ is continuously differentiable for all $n > k_0(\Lambda)$. Hence, $H'(\alpha_0) = 0$ for *all* $\alpha_0 \in \Lambda$ and $n > k_0(\Lambda)$. Taking into account that $H(0) = D_n(1) \neq 0$, we conclude that $D_n(f(\alpha_0; z))$ is nonzero, and that the identity (5-1) under (5-24) is, in fact, true for all $\alpha_0 \in \Lambda$ if n is sufficiently large (larger than $k_0(\Lambda)$). Now fix $\alpha_0 \in \Lambda$ and assume the condition $\alpha_1 = \cdots = \alpha_m = \beta_m = 0$. A similar argument as above then gives that $D_n(f(\alpha_0, \beta_0; z))$ is nonzero and the identity (5-2) with $\nu = 0$ is true for all $\beta_0 \in \Lambda$ if n is large. Continuing this way, we complete the proof of Proposition 5.1 by induction.

Remark 5.4. A similar argument applies to the asymptotic form of the differential identity (3-26) we need in Section 5C below. We omit the discussion. $\square$

We will now complete the proof of Theorem 1.1 in the case $\|\|\beta\|\| < \frac{1}{2}$, $V(z) \equiv 0$ by integrating the identities of Proposition 5.1. In this case we denote

$$D_n(f(z)) \equiv D_n(\alpha_0, \ldots, \alpha_m; \beta_0, \ldots, \beta_m). \tag{5-25}$$

First, set $m = 0$ and $\beta_0 = 0$. Then (5-1) becomes

$$\frac{\partial}{\partial \alpha_0} \ln D_n(\alpha_0) = 2\alpha_0 \left(1 + \ln n + \frac{d}{d\alpha_0} \frac{\Gamma(1+\alpha_0)}{\Gamma(1+2\alpha_0)} \right) + O(n^{-1} \ln n) + O(\delta n \ln n).$$

Integrating both sides over α_0 from 0 to some α_0 and using the fact that $D_n(0) = 1$, we obtain

$$D_n(\alpha_0) = n^{\alpha_0^2} \frac{G(1+\alpha_0)^2}{G(1+2\alpha_0)}, \tag{5-26}$$

where $G(x)$ is Barnes G-function. To perform the integration we used the identity

$$\int_0^z \left(1 + \frac{d}{dx} \frac{\Gamma(1+x)}{\Gamma(1+2x)} \right) 2x\,dx = \ln \frac{G(1+z)^2}{G(1+2z)}, \tag{5-27}$$

which easily follows from the standard formula (see, e.g., [Whittaker and Watson 1927]):

$$\int_0^z \ln \Gamma(x+1)\,dx = \frac{z}{2} \ln 2\pi - \frac{z(z+1)}{2} + z \ln \Gamma(z+1) - \ln G(z+1). \tag{5-28}$$

Now set $m = 1$, α_0 fixed. Set $\beta_0 = \beta_1 = 0$. Relation (5-1) for $\nu = 1$ is then

$$\frac{\partial}{\partial \alpha_1} \ln D_n(\alpha_0, \alpha_1) = 2\alpha_1 \left(1 + \ln n + \frac{d}{d\alpha_1} \frac{\Gamma(1+\alpha_1)}{\Gamma(1+2\alpha_1)} \right)$$
$$+ \alpha_0 \ln z_0 + \alpha_0 (\ln z_1 + 3\pi i) - 2\alpha_0 \ln(z_0 - z_1) + O(n^{-1} \ln n) + O(\delta n \ln n). \tag{5-29}$$

Integrating this over α_1 from 0 to some fixed α_1 along a path lying in Λ (see Proposition 5.1) and using (5-27), we obtain

$$\ln \frac{D_n(\alpha_0, \alpha_1)}{D_n(\alpha_0, 0)} = \alpha_1^2 \ln n + 2 \ln G(1+\alpha_1) - \ln G(1+2\alpha_1) + \alpha_0 \alpha_1 \ln(z_0 z_1 e^{3\pi i})$$
$$- 2\alpha_0 \alpha_1 \ln(z_0 - z_1) + O(n^{-1} \ln n) + O(\delta n \ln n). \tag{5-30}$$

Substituting (5-26) here, we obtain

$$D_n(\alpha_0, \alpha_1)$$
$$= n^{\alpha_0^2 + \alpha_1^2} \prod_{j=0}^1 \frac{G(1+\alpha_j)^2}{G(1+2\alpha_j)} \left[\frac{(z_0 - z_1)^2}{z_0 z_1 e^{3\pi i}} \right]^{-\alpha_0 \alpha_1} (1 + O(n^{-1} \ln n) + O(\delta n \ln n))$$
$$= n^{\alpha_0^2 + \alpha_1^2} \prod_{j=0}^1 \frac{G(1+\alpha_j)^2}{G(1+2\alpha_j)} |z_0 - z_1|^{-\alpha_0 \alpha_1} (1 + O(n^{-1} \ln n) + O(\delta n \ln n)) \tag{5-31}$$

(to write the last equation we recall (3-13), the way the branch of $(z - z_j)^{\alpha_j}$ was fixed there, and the fact that $\arg z_0 < \arg z_1$).

Continuing this way, we finally obtain by induction for any fixed m, $\beta_j = 0$, $j = 0, 1, \ldots, m$, the asymptotic expression

$$D_n(\alpha_0, \ldots, \alpha_m) = n^{\sum_{j=0}^{m} \alpha_j^2}$$

$$\times \prod_{j=0}^{m} \frac{G(1+\alpha_j)^2}{G(1+2\alpha_j)} \prod_{0 \le j < k \le m} |z_j - z_k|^{-\alpha_j \alpha_k} (1 + O(n^{-1} \ln n) + O(\delta n \ln n)). \quad (5\text{-}32)$$

We now add in the β-singularities. We will make use of one more identity, which follows from (5-28):

$$\int_0^{\beta} \left((\alpha + x) \frac{d}{dx} \ln \Gamma(1 + \alpha + x) + (\alpha - x) \frac{d}{dx} \ln \Gamma(1 + \alpha - x) - 2x \right) dx$$

$$= \ln \frac{G(1 + \alpha + \beta) G(1 + \alpha - \beta)}{G(1 + \alpha)^2}. \quad (5\text{-}33)$$

First, we obtain the result for the case when $-\frac{1}{4} < \Re\beta_j < \frac{1}{4}$. This implies that the order of the error term $O(\delta n \ln n)$ remains $o(1)$ as we integrate over the β_j starting at zero. As before, we always assume integration along a path in Λ. Setting $\nu = 0$, $\beta_j = 0$, $j = 1, \ldots, m$ in (5-2), and integrating this identity over β_0 from zero to a fixed β_0, we obtain using (5-33):

$$\ln \frac{D_n(\alpha_0, \ldots, \alpha_m; \beta_0)}{D_n(\alpha_0, \ldots, \alpha_m; 0)} = -\beta_0^2 \ln n + \ln \frac{G(1 + \alpha_0 + \beta_0) G(1 + \alpha_0 - \beta_0)}{G(1 + \alpha_0)^2}$$

$$+ \beta_0 \sum_{j \ne 0} \alpha_j \ln \frac{z_0 e^{i\pi}}{z_j} + O(n^{-1} \ln n) + O(\delta n \ln n). \quad (5\text{-}34)$$

Substituting (5-32) here, we obtain

$$D_n(\alpha_0, \ldots, \alpha_m; \beta_0)$$

$$= n^{\sum_{j=0}^{m} \alpha_j^2 - \beta_0^2} \frac{G(1 + \alpha_0 + \beta_0) G(1 + \alpha_0 - \beta_0)}{G(1 + \alpha_0)^2} \prod_{j=1}^{m} \frac{G(1 + \alpha_j)^2}{G(1 + 2\alpha_j)}$$

$$\times \prod_{0 \le j < k \le m} |z_j - z_k|^{-\alpha_j \alpha_k} \prod_{j=1}^{m} \left(\frac{z_0 e^{i\pi}}{z_j} \right)^{\alpha_j \beta_0} (1 + O(n^{-1} \ln n) + O(\delta n \ln n)). \quad (5\text{-}35)$$

Next, set $\nu = 1$, $\beta_j = 0$, $j = 2, \ldots n$ in (5-2) and integrate over β_1. We obtain then the determinant $D_n(\alpha_0, \ldots, \alpha_m; \beta_0, \beta_1)$. Continuing this procedure, we finally obtain by induction at step m the asymptotics (1-10) for the case $-\frac{1}{4} < \Re\beta_j < \frac{1}{4}$ with $V \equiv 0$ and the error term $O(n^{-1} \ln n) + O(\delta n \ln n) = o(1)$.

Consider now the general case $\|\beta\| < \frac{1}{2}$. We can choose q such that all $\Re\beta_j \in (q - \frac{1}{4}, q + \frac{1}{4})$. Divide $(0, q)$ into subintervals of length less than $\frac{1}{2}$. Apply the above integration procedure to move all β_j from zero to the line where the

real part of all β_j is the right end of the first subinterval. Since the length of subintervals is less than $\frac{1}{2}$, the error term $O(\delta n \ln n)$ remains $o(1)$. Recall that during the integration we avoid any points where $\alpha_j + \beta_j$ or $\alpha_j - \beta_j$ is a negative integer. Next, move the β_j to the right end of the second subinterval, and so on, until the point $\Re \beta_j = q$. From that point move the β_j as needed. We thus obtain Theorem 1.1 with $V \equiv 0$ and $\||\beta\|| < \frac{1}{2}$.

5B. *Pure Fisher–Hartwig singularities: extension to $\||\beta\|| < 1$.* We now show that in fact the error term in (1-10) remains $o(1)$ for the full range $\||\beta\|| < 1$. First, recall the definition of $\widetilde{R}$ and ω in (4-30) and (4-31). We have $\||\beta\|| = 2 \max_j (\Re \beta_j - \omega)$ and

$$-\tfrac{1}{2} < \Re \beta_j - \omega < \tfrac{1}{2}$$

for all singular points z_j. We denote $p_j = z_j$ if $\Re \beta_j - \omega > 0$, and m_+ the number of such points. Furthermore, denote $q_j = z_j$ if $\Re \beta_j - \omega < 0$, and m_- the number of such points. Finally, let $r_j = z_j$ if $\Re \beta_j - \omega = 0$.

Separating the main contributions in n (see (4-21)), we write the jump matrix for $\widetilde{R}$ on ∂U_{z_j} (cf. (4-32)) in the form

$$I + n^{\omega \sigma_3} \Delta_1(z) n^{-\omega \sigma_3} + \cdots = I + \hat{\Delta}_1(z) + \hat{D}(z) + O(n^{-1-\rho}), \quad z \in \partial U_{z_j}, \quad (5\text{-}36)$$

where $\hat{\Delta}_1(z)$ is about to be defined and

$$\rho = 1 - \||\beta\||, \tag{5-37}$$

$$\hat{D}(z) = n^{\omega \sigma_3} \Delta_1(z) n^{-\omega \sigma_3} - \hat{\Delta}_1(z), \quad z \in \partial U_{z_j}. \tag{5-38}$$

For $z \in \partial U_{p_j}$ we set

$$\hat{\Delta}_1(z) = (n^{\omega \sigma_3} \Delta_1(z) n^{-\omega \sigma_3})_{21} \sigma_- = \frac{b_j(z)}{z - p_j} \sigma_-, \quad \text{with } \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and}$$

$$b_j(z) = -\frac{\Gamma(1+\alpha_j-\beta_j)}{\Gamma(\alpha_j+\beta_j)} \left(\frac{\mathscr{D}(z)}{\zeta^{\beta_j} F_j(z)} \right)^{-2} p_j^{-n} e^{-i\pi(2\beta_j-\alpha_j)} \frac{z - p_j}{n \ln(z/p_j)} n^{-2\omega}. \tag{5-39}$$

For $z \in \partial U_{q_j}$ we set

$$\hat{\Delta}_1(z) = (n^{\omega \sigma_3} \Delta_1(z) n^{-\omega \sigma_3})_{12} \sigma_+ = \frac{a_j(z)}{z - q_j} \sigma_+, \quad \text{with } \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and}$$

$$a_j(z) = \frac{\Gamma(1+\alpha_j+\beta_j)}{\Gamma(\alpha_j-\beta_j)} \left(\frac{\mathscr{D}(z)}{\zeta^{\beta_j} F_j(z)} \right)^{2} q_j^{n} e^{i\pi(2\beta_j-\alpha_j)} \frac{z - q_j}{n \ln(z/q_j)} n^{2\omega}. \tag{5-40}$$

Finally,

$$\hat{\Delta}_1(z) = 0 \quad \text{when } z \in \partial U_{r_j}. \tag{5-41}$$

Note that $\hat{D}(z)$ and $\hat{\Delta}_1(z)$ are meromorphic functions in a neighborhood of U_{z_j} with a simple pole at $z = z_j$. We have (see (4-24), (4-25), and recall that $V \equiv 0$)

$$b_j = b_j^{(n)} \equiv \lim_{z \to p_j} b_j(z) = -n^{2(\beta_j - \omega) - 1} p_j^{-n+1} \mu_j^{-2} \frac{\Gamma(1 + \alpha_j - \beta_j)}{\Gamma(\alpha_j + \beta_j)} = O(n^{-\rho}),$$

$$\text{(5-42)}$$

$$a_j = a_j^{(n)} \equiv \lim_{z \to q_j} a_j(z) = n^{-2(\beta_j - \omega) - 1} q_j^{n+1} \mu_j^2 \frac{\Gamma(1 + \alpha_j + \beta_j)}{\Gamma(\alpha_j - \beta_j)} = O(n^{-\rho}). \quad \text{(5-43)}$$

Also note that

$$\hat{D}(z) = O(n^{-1}), \quad z \in \partial U_{z_j}. \tag{5-44}$$

The main idea which will allow us to give the required estimate for the error term in (1-10) is the following. Write $\widetilde{R}$ in the form

$$\widetilde{R}(z) = Q(z)\hat{R}(z),$$

where $\hat{R}(z)$ is the solution to the following RHP:

$$\hat{R}(z) \text{ is analytic for } z \in \mathbb{C} \setminus \bigcup_j \partial U_{z_j}. \tag{5-45}$$

$$\hat{R}(z)_+ = \hat{R}(z)_-(I + \hat{\Delta}_1) \quad \text{if } z \in \bigcup_j \partial U_{z_j}, \tag{5-46}$$

$$\hat{R}(z) = I + O(1/z) \quad \text{as } z \to \infty. \tag{5-47}$$

We solve this RHP below explicitly, however, note first that the solution exists and is unique and

$$\hat{R}(z) = I + O(n^{-\rho}) \tag{5-48}$$

uniformly in z by standard arguments. Using (5-48), the jump condition (5-46), and the nilpotency of $\hat{\Delta}_1$, we then have on ∂U_{z_j}

$$Q_+ = \widetilde{R}_+ \hat{R}_+^{-1} = \widetilde{R}_-(I + \hat{\Delta}_1(z) + \hat{D}(z) + O(n^{-1-\rho}))\hat{R}_+^{-1}$$
$$= Q_-(I + \hat{D}(z) + O(n^{-1-\rho})).$$

The jump matrix for Q on Σ^{out}, Σ''^{out} remains exponentially close to the identity. Therefore,

$$Q(z) = I + Q_1(z) + O(n^{-1-\rho}), \quad Q_1(z) = \frac{1}{2\pi i} \int_{\bigcup_j \partial U_{z_j}} \frac{\hat{D}(s)}{s - z} \, ds \tag{5-49}$$

In what follows, we will be interested in the matrix element $Y_{21}(z)$ of our original RHP in a neighborhood of $z = 0$. In this neighborhood, we have using

(5-49),

$$Y(z) = R(z)N(z) = n^{-\omega\sigma_3}Q(z)\hat{R}(z)n^{\omega\sigma_3}N(z)$$

$$= n^{-\omega\sigma_3}(I + Q_1(z) + O(n^{-1-\rho}))\hat{R}(z)n^{\omega\sigma_3}\mathcal{D}(z)^{\sigma_3}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \qquad (5\text{-}50)$$

From here, noting that $\mathcal{D}(0) = 1$ as $V(z) \equiv 0$, and using the estimates $\hat{R}(z) = I + O(n^{-\rho})$, $Q_1(0) = O(n^{-1})$, we obtain

$$\chi^2_{n-1} = -Y_{21}(0) = (1 + Q_{1,22}(0) + O(n^{-1-\rho}))\hat{R}_{22}(0).$$

Using the expression for Q_1 from (5-49) and (5-38), we can write this equation in the form (recall that the contours ∂U_{z_j} are oriented in the negative direction):

$$\chi^2_{n-1} = \left(1 - \frac{1}{n}\sum_{j=0}^{m}(\alpha_j^2 - \beta_j^2) + O(n^{-1-\rho})\right)\hat{R}_{22}(0). \qquad (5\text{-}51)$$

Eventually, we will use the product of these quantities over n to represent the determinant. Before doing that, we now solve the RHP for $\hat{R}$ and find $\hat{R}_{22}(0)$.

Set

$$\Phi(z) = \begin{cases} \hat{R}(z) & \text{if } z \in \mathbb{C} \setminus \bigcup_j U_{z_j}, \\ \hat{R}(z)(I + \hat{\Delta}_1(z)) & \text{if } z \in \bigcup_j U_{z_j}. \end{cases} \qquad (5\text{-}52)$$

So defined, $\Phi(z)$ is obviously a meromorphic function with simple poles at the points z_j, which tends to I at infinity. Therefore, it can be written in the form

$$\Phi(z) = I + \sum_{j=1}^{m_+}\frac{\Phi_j^+}{z - p_j} + \sum_{j=1}^{m_-}\frac{\Phi_j^-}{z - q_j} \qquad (5\text{-}53)$$

for some constant matrices $\Phi_j^\pm$.

Moreover, the function $\hat{R}(z)$, which has the expression

$$\hat{R}(z) = \Phi(z)(I + \hat{\Delta}_1)^{-1}$$

$$= \left[I + \sum_{j=1}^{m_+}\frac{\Phi_j^+}{z - p_j} + \sum_{j=1}^{m_-}\frac{\Phi_j^-}{z - q_j}\right]\left(I - \frac{b_k(z)}{z - p_k}\sigma_-\right) \qquad (5\text{-}54)$$

in U_{p_k}, $k = 1, \ldots, m_+$ (we use here the definition of $\hat{\Delta}_1$), is analytic there. Hence, the coefficients at negative powers of $z - p_k$ vanish. Equating the coefficient at $(z - p_k)^{-2}$ to zero, we obtain

$$\Phi_k^+\sigma_- = 0, \qquad (5\text{-}55)$$

and therefore the matrix Φ_k^+ has the form

$$\Phi_k^+ = \begin{pmatrix} g_k & 0 \\ f_k & 0 \end{pmatrix} \tag{5-56}$$

for some constants g_k, f_k. The vanishing of the coefficient at $(z - p_k)^{-1}$ gives the following condition on $\Phi_j^\pm$, where we used the equation $\Phi_j^+ \sigma_- = 0$,

$$\Phi_k^+ - b_k \sum_{j=1}^{m_-} \frac{\Phi_j^- \sigma_-}{p_k - q_j} = b_k \sigma_-, \quad k = 1, \ldots, m_+, \tag{5-57}$$

with b_k given by (5-42). Similarly, using the analyticity of $\Phi(z)(I + \hat{\Delta}_1(z))^{-1}$ in U_{q_k}, we obtain for all $k = 1, \ldots, m_-$

$$\Phi_k^- = \begin{pmatrix} 0 & e_k \\ 0 & h_k \end{pmatrix}, \tag{5-58}$$

$$\Phi_k^- - a_k \sum_{j=1}^{m_+} \frac{\Phi_j^+ \sigma_+}{q_k - p_j} = a_k \sigma_+ \tag{5-59}$$

for some constants e_k, h_k. Conditions (5-59), (5-57) are equations for the constants g_k, f_k, e_k, h_k. In view of (5-51), we are interested in

$$\hat{R}_{22}(0) = \Phi_{22}(0) = 1 - \sum_{j=1}^{m_-} \frac{h_j}{q_j}, \tag{5-60}$$

where we used (5-53) to write the last equation. To calculate this quantity we first substitute Φ_j^+ from (5-57) into (5-59). Then the 2,2-element of the resulting equations for $k = 1, \ldots, m_-$ can be written as follows:

$$(I - A)h = B, \tag{5-61}$$

where h and B are m_--dimensional vectors with components h_k and

$$B_k = \sum_{j=1}^{m_+} \frac{a_k b_j}{q_k - p_j}, \quad k = 1, \ldots, m_-,$$

$A = A^{(n)}$ is an $m_- \times m_-$ matrix with matrix elements

$$A_{k,\ell} = \sum_{j=1}^{m_+} \frac{a_k b_j}{(q_k - p_j)(p_j - q_\ell)},$$

and I is the $m_- \times m_-$ identity matrix.

Define the $m_- \times m_-$ diagonal matrix Δ as follows

$$\Delta = \text{diag}\{-q_1, -q_2, \ldots, -q_{m_-}\}.$$

Then (5-61) can be written in the form

$$Tx = y, \quad T = I - \Delta^{-1} A \Delta, \quad x = \Delta^{-1} h, \quad y = \Delta^{-1} B$$

By Cramer's rule,

$$x_k = \frac{\det(T_1 \cdots y \cdots T_{m_-})}{\det T},$$

where T_j are the columns of T and y is in the place of the k-th column. We are interested in

$$1 - \sum_{j=1}^{m_-} \frac{h_j}{q_j} = 1 + \sum_{j=1}^{m_-} x_j = \frac{\det(T_1 + y, T_2 + y, \cdots, T_{m_-} + y)}{\det T}. \qquad (5\text{-}62)$$

First note that

$$\det T = \det(1 - \Delta^{-1} A \Delta) = \det(I - A) = \det(I - A^{(n)}). \qquad (5\text{-}63)$$

Second, a direct calculation shows that

$$(T_1 + y, T_2 + y, \cdots, T_{m_-} + y) = I - A', \quad A'_{jk} = \sum_{\ell=1}^{m_+} \frac{a'_j b'_\ell}{(q_j - p_\ell)(p_\ell - q_k)},$$

where

$$a'_j = a_j q_j^{-1}, \quad b'_\ell = b_\ell p_\ell.$$

Using the definitions (5-43), (5-42) of a_j, b_j, we note that

$$a'_j = a_j^{(n-1)} + O(n^{-1-\rho}),$$
$$b'_j = b_j^{(n-1)} + O(n^{-1-\rho}),$$

and therefore $A' = A^{(n-1)} + O(n^{-1-\rho})$. Thus we can rewrite (5-62) as

$$\hat{R}_{22}(0) = \Phi_{22}(0) = 1 - \sum_{j=1}^{m_-} \frac{h_j}{q_j} = \frac{\det(I - A^{(n-1)})}{\det(I - A^{(n)})} \left[1 + O\left(\frac{1}{n^{1+\rho}} \right) \right]. \quad (5\text{-}64)$$

Note that

$$\det(I - A^{(n)}) = 1 + O(n^{-2\rho}).$$

Recalling now (5-51) and the representation of the Toeplitz determinant $D_n(f)$ as a product of χ_k^{-2} and using (5-64), we can write, for some sufficiently large $n_0 > 0$,

$D_n(f)$

$$= D_{n_0}(f) \prod_{k=n_0+1}^{n} \chi_{k-1}^{-2}$$

$$= D_{n_0}(f) \prod_{k=n_0+1}^{n} \left[1 + \frac{1}{k} \sum_{j=0}^{m} (\alpha_j^2 - \beta_j^2) + O\left(\frac{1}{k^{1+\rho}}\right) \right] \frac{\det(I - A^{(k)})}{\det(I - A^{(k-1)})}$$

$$\times \left[1 + O\left(\frac{1}{k^{1+\rho}}\right) \right]$$

$$= C(n_0, \alpha_0, \ldots, \alpha_m, \beta_0, \ldots, \beta_m) n^{\sum_{j=0}^{m}(\alpha_j^2 - \beta_j^2)} \left[1 + O\left(\frac{1}{n^\rho}\right) \right], \qquad (5\text{-}65)$$

where all the error terms are uniform for α_j, β_j in compact sets, and C is a constant depending analytically on α_j, β_j, and n_0 only. Recall that in the derivation of this expression we assumed that $\||\beta\|| < 1$ (and as usual $\Re \alpha_j > -\frac{1}{2}$ for all j). Under this condition $\rho > 0$ (see (5-37)), and the error term tends to zero as $n \to \infty$. In the previous section, we obtained C explicitly for β_j satisfying $\||\beta\|| < \frac{1}{2}$. Obviously, if all α_j, β_j belong to fixed compact sets and the value of n_0 is fixed, the constant C in (5-65) is bounded above in absolute value by a constant independent of any α_j, β_j. By Vitali's theorem this fact implies that C can be analytically continued in β_j to the full domain $\||\beta\|| < 1$ off its values on the domain $\||\beta\|| < \frac{1}{2}$. Thus C is given by the same expression also for $\||\beta\|| < 1$, and this concludes the proof of Theorem 1.1 for $\||\beta\|| < 1$, $V(z) \equiv 0$, with the error term $o(1) = O(n^{\||\beta\||-1})$ in (1-10).

5C. *Adding special analytic $V(z)$.* In this section and in the next one, we will add the multiplicative factor $e^{V(z)}$, where V is analytic in a neighborhood of the unit circle C, to a symbol with pure Fisher–Hartwig singularities and obtain the asymptotics of the corresponding determinant. Consider the deformation of the symbol $f(z, t)$ given by (3-21) for $t \in [0, 1]$. The analysis is based on integration of the differential identity (3-26) over t. In the present section we assume that V is such that

$$1 - t + t e^{V(z)} \neq 0, \quad t \in [0, 1], \quad z \in C, \qquad (5\text{-}66)$$

and $1 - t + t e^{V(z)}$ has no winding around C for all $t \in [0, 1]$. Then the Riemann–Hilbert problem (for the polynomials orthogonal) with $f(z, t)$ has the same singularities as the problem with $f(z)$ and is solved in the same way. In the following section we remove the condition (5-66).

Let us rewrite the identity (3-26) in terms of the function $S(z) = S(z, t)$ that is the solution of the Riemann–Hilbert problem posed on the deformed contour depicted in Figure 1. From the transformation $Y \to S$ defined in (4-1), (4-2) we

obtain

$$Y_{11} = z^n f^{-1} S_{12,+} + S_{11,+}, \quad Y_{21} = z^n f^{-1} S_{22,+} + S_{21,+}, \quad z \in C \equiv \Sigma'. \quad (5\text{-}67)$$

Substituting these expressions into (3-26) and taking into account that $\det S = 1$ we arrive at the formula

$$\frac{\partial}{\partial t} \ln D_n(f(z,t))$$
$$= n \int_C f^{-1} \dot{f} \frac{dz}{2\pi i z} + \int_C \left[-f^{-2} f' + 2 f^{-1} \left(S'_{22,+} S_{11,+} - S'_{12,+} S_{21,+} \right) \right] \dot{f} \frac{dz}{2\pi i}$$
$$+ \int_C \left[z^{-n} \left(S'_{21,+} S_{11,+} - S'_{11,+} S_{21,+} \right) + z^n \left(S'_{22,+} S_{12,+} - S'_{12,+} S_{22,+} \right) f^{-2} \right] \dot{f} \frac{dz}{2\pi i}, \quad (5\text{-}68)$$

where we introduced the notation $\dot{f} := \partial f / \partial t$ and $f' := \partial f / \partial z$. Using the jump relation (4-4) satisfied by the function $S(z,t)$ across the unit circle, we can rewrite (5-68) more symmetrically as

$$\frac{\partial}{\partial t} \ln D_n(f(z,t)) = n \int_C f^{-1} \dot{f} \frac{dz}{2\pi i z} + X(t), \quad (5\text{-}69)$$

with

$$X(t) := \int_C \left[S'_{22,+} S_{11,+} - S'_{12,+} S_{21,+} + S'_{11,-} S_{22,-} - S'_{21,-} S_{12,-} \right] f^{-1} \dot{f} \frac{dz}{2\pi i}$$
$$+ \int_C \left[z^{-n} \left(S'_{22,-} S_{12,-} - S'_{12,-} S_{22,-} \right) + z^n \left(S'_{22,+} S_{12,+} - S'_{12,+} S_{22,+} \right) \right] f^{-2} \dot{f} \frac{dz}{2\pi i}.$$

Analytically continuing the boundary values of $S(z,t)$ from the $+$ side of C to the $-$ side of Σ'', and from the $-$ side of C to the $+$ side of Σ, we can write the term $X(t)$ in (5-69) in the form

$$X(t) = \int_{\Sigma''} (J_- + z^n I_-) f^{-1} \dot{f} \frac{dz}{2\pi i} + \int_{\Sigma} (-J_+ + z^{-n} I_+) f^{-1} \dot{f} \frac{dz}{2\pi i}, \quad (5\text{-}70)$$

where

$$I = (S'_{22} S_{12} - S'_{12} S_{22}) f^{-1}, \quad J = S'_{22} S_{11} - S'_{12} S_{21}. \quad (5\text{-}71)$$

Denote by Σ_ε (resp., Σ''_ε) the part of Σ (resp., Σ'') that lies inside $\bigcup_{j=0}^m U_{z_j}$. Consider first

$$\int_{\Sigma''_\varepsilon} (J_- + z^n I_-) f^{-1} \dot{f} \frac{dz}{2\pi i}. \quad (5\text{-}72)$$

Using (4-3) on Σ'' and (5-71), we easily obtain that $J_- = J_+ - z^n I_+$ and $I_- = I_+$, and therefore,

$$\int_{\Sigma''_\varepsilon} (J_- + z^n I_-) f^{-1} \dot{f} \frac{dz}{2\pi i} = \int_{\Sigma''_\varepsilon} J_+ f^{-1} \dot{f} \frac{dz}{2\pi i} = \int_{C''_\varepsilon} J f^{-1} \dot{f} \frac{dz}{2\pi i}, \quad (5\text{-}73)$$

where C_ε'' is the part of $\bigcup_{j=0}^m \partial U_{z_j}$ lying inside the unit circle from the intersection of ∂U_{z_j} with the incoming Σ'' to the intersection with the outgoing Σ'' for each j. Note that $J(z, t)$ has no nonintegrable singularity at z_j. Indeed, for z inside the smaller sector formed by Σ'' at z_j, we can write

$$S(z, t) = Y(z, t) = \hat{Y}(z, t) \begin{pmatrix} 1 & \kappa(z, t) \\ 0 & 1 \end{pmatrix}, \tag{5-74}$$

where $\kappa(z, t) = c_1(z, t)(z - z_j)^{2\alpha_j}$, if $\alpha_j \neq 0$, and $\kappa(z, t) = c_2(z, t) \ln(z - z_j)$, if $\alpha_j = 0$, $\beta_j \neq 0$, for some $c_k(z, t)$ analytic near z_j, chosen so that $\hat{Y}(z)$ is analytic in a neighborhood of z_j. Writing J in terms of the matrix elements of $\hat{Y}$, we see that the contributions of the (singular) derivative $\kappa'(z)$ cancel, and we obtain

$$J = (\hat{Y}_{21}' \hat{Y}_{11} - \hat{Y}_{21} \hat{Y}_{11}')\kappa + \hat{Y}_{22}' \hat{Y}_{11} - \hat{Y}_{12}' \hat{Y}_{21}. \tag{5-75}$$

We now analyze (5-73) asymptotically. The asymptotic expression for $S(z, t)$ inside the unit circle and outside $\bigcup_{j=0}^m U_{z_j}$ is given by (see (4-26), (4-7))

$$S(z, t) = R(z, t)e^{g(z,t)\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{5-76}$$

where $g(z)$ is defined by the formula

$$\mathscr{D}(z, t) = e^{g(z,t)}. \tag{5-77}$$

We have

$$J = g' + R_{11}' R_{22} - R_{12} R_{21}' = g' + O_\varepsilon(1/n), \quad n \to \infty,$$

and we finally obtain

$$\int_{\Sigma_\varepsilon''} (J_- + z^n I_-) f^{-1} \dot{f} \frac{dz}{2\pi i} = \int_{C_\varepsilon''} J f^{-1} \dot{f} \frac{dz}{2\pi i}$$
$$= \int_{C_\varepsilon''} g'(z) f^{-1} \dot{f} \frac{dz}{2\pi i} + O_\varepsilon(1/n), \quad n \to \infty. \tag{5-78}$$

Similarly, using the jump condition for S and then the asymptotics for S outside the unit circle, we obtain

$$\int_{\Sigma_\varepsilon} (-J_+ + z^{-n} I_+) f^{-1} \dot{f} \frac{dz}{2\pi i} = -\int_{C_\varepsilon} J f^{-1} \dot{f} \frac{dz}{2\pi i}$$
$$= \int_{C_\varepsilon} g'(z) f^{-1} \dot{f} \frac{dz}{2\pi i} + O_\varepsilon(1/n), \quad n \to \infty, \tag{5-79}$$

where C_ε is the part of $\bigcup_{j=0}^m \partial U_{z_j}$ lying outside the unit circle from the intersection of ∂U_{z_j} with the incoming Σ to the intersection with the outgoing Σ for each j.

Returning to the integrals (5-70), we now consider the part arising from the integration over $\Sigma'' \setminus \Sigma''_\varepsilon$ and $\Sigma \setminus \Sigma_\varepsilon$. In this part the terms containing $z^{\pm n} I_\mp$ give a contribution that is exponentially small in n, while the integration of the terms with J can be replaced by the integration over $\Sigma' \setminus \Sigma'_\varepsilon$, where $\Sigma'_\varepsilon = \Sigma' \cap \left(\bigcup_{j=0}^m U_{z_j} \right)$, and over parts of the boundaries ∂U_{z_j}. Thus, recalling also (5-78), (5-79), we have

$$
X(t) = \int_{\Sigma' \setminus \Sigma'_\varepsilon} (J_+ - J_-) f^{-1} \dot{f} \frac{dz}{2\pi i}
$$

$$
+ \sum_{j=0}^m \left(\int_{\partial U_{z_j}^+} + \int_{\partial U_{z_j}^-} \right) g'(z) f^{-1} \dot{f} \frac{dz}{2\pi i} + O_\varepsilon(1/n), \quad n \to \infty, \quad (5\text{-}80)
$$

where $\partial U_{z_j}^+$ (resp., $\partial U_{z_j}^-$) is the part of the boundary of U_{z_j} inside (resp., outside) the unit circle oriented from the intersection with the incoming Σ' to the intersection with the outgoing Σ'.

Note that using the same considerations as before, we can write in (5-80) $J_+ - J_- = g'_+ + g'_- + O_\varepsilon(1/n)$, and therefore

$$
X(t) = \int_{\Sigma_+} g'_+ f^{-1} \dot{f} \frac{dz}{2\pi i} + \int_{\Sigma_-} g'_- f^{-1} \dot{f} \frac{dz}{2\pi i} + O_\varepsilon(1/n), \quad n \to \infty, \quad (5\text{-}81)
$$

where the closed anticlockwise oriented contours

$$
\Sigma_+ = (\Sigma' \setminus \Sigma'_\varepsilon) \bigcup_{j=0}^m \partial U_{z_j}^+, \quad \Sigma_- = (\Sigma' \setminus \Sigma'_\varepsilon) \bigcup_{j=0}^m \partial U_{z_j}^-. \quad (5\text{-}82)
$$

(Note that one can deform Σ_+ (resp., Σ_-) to a circle around zero of radius $1 - \varepsilon$ (resp., $1 + \varepsilon$).)

By (3-21),

$$
f^{-1} \dot{f} = \frac{-1 + e^{V(z)}}{1 - t + t e^{V(z)}} = \frac{\partial}{\partial t} \ln \left(1 - t + t e^{V(z)} \right). \quad (5\text{-}83)
$$

Furthermore, writing g in the form

$$
g(z, t) = g^{Sz}(z, t) + g^{FH}(z), \quad (5\text{-}84)
$$

we have, by (5-77), (4-8), (4-9), (4-10),

$$
g^{Sz}(z, t) = \int_C \frac{\ln(1 - t + t e^{V(s)})}{s - z} \frac{ds}{2\pi i},
$$

$$
g^{FH}(z) = \begin{cases} \displaystyle\sum_{k=1}^m (\alpha_k + \beta_k) \ln \frac{z - z_k}{z_k e^{i\pi}} & \text{if } |z| < 1, \\[4mm] \displaystyle\sum_{k=1}^m (-\alpha_k + \beta_k) \ln \frac{z - z_k}{z} & \text{if } |z| > 1. \end{cases} \quad (5\text{-}85)
$$

The solution of the Riemann–Hilbert problem is uniform in $t \in [0, 1]$. From this fact and the explicit formulas (5-83), (5-84) and (5-85), we conclude, by an argument similar to the argument following (5-23) that the identity (5-69) holds for all $t \in [0, 1]$. Now integrating (5-69) from $t = 0$ to $t = 1$, we connect the Toeplitz determinant $D_n(f(z, 1))$ with the Toeplitz determinant $D_n(f(z, 0))$ that represents the "pure" Fisher–Hartwig case and whose asymptotics we evaluated in the previous section. First, using (5-83) and changing the order of integration in the first term of (5-69), we obtain

$$\int_0^1 dt \int_C f^{-1} \dot{f} \frac{dz}{2\pi i z} = \frac{1}{2\pi} \int_0^{2\pi} V(e^{i\theta}) \, d\theta = V_0. \qquad (5\text{-}86)$$

Furthermore, by (5-81), (5-84) and (5-85),

$$\int_0^1 dt \, X(t) = I^{Sz} + I^{FH} + O_\varepsilon(1/n), \quad n \to \infty, \qquad (5\text{-}87)$$

where (compare [Deift 1999, (86), (87)])

$$I^{Sz} = \int_0^1 dt \int_C \left((g^{Sz})'_+ + (g^{Sz})'_- \right) f^{-1} \dot{f} \frac{dz}{2\pi i} = \sum_{k=1}^{\infty} k V_k V_{-k}, \qquad (5\text{-}88)$$

and

$$I^{FH} = \int_{\Sigma_+} \left(g^{FH}(z) \right)'_+ V(z) \frac{dz}{2\pi i} + \int_{\Sigma_-} \left(g^{FH}(z) \right)'_- V(z) \frac{dz}{2\pi i} \qquad (5\text{-}89)$$

$$= \sum_{k=0}^{m} \left[(\alpha_k + \beta_k) \int_{\Sigma_+} \frac{V(z)}{z - z_k} \frac{dz}{2\pi i} + (-\alpha_k + \beta_k) \int_{\Sigma_-} \left(\frac{V(z)}{z - z_k} - \frac{V(z)}{z} \right) \frac{dz}{2\pi i} \right].$$

Since

$$g^{Sz}(z, 1) = \int_C \frac{V(s)}{s - z} \frac{ds}{2\pi i} = \begin{cases} \ln b_+(z) + V_0 & \text{if } |z| < 1, \\ -\ln b_-(z) & \text{if } |z| > 1, \end{cases} \qquad (5\text{-}90)$$

we obtain

$$\ln b_+(z_k) = \int_{\Sigma_-} \frac{V(z)}{z - z_k} \frac{dz}{2\pi i} - V_0, \quad \ln b_-(z_k) = -\int_{\Sigma_+} \frac{V(z)}{z - z_k} \frac{dz}{2\pi i}, \qquad (5\text{-}91)$$

which finally gives

$$I^{FH} = \sum_{k=0}^{m} \left[-(\alpha_k + \beta_k) \ln b_-(z_k) + (-\alpha_k + \beta_k) \ln b_+(z_k) \right]. \qquad (5\text{-}92)$$

Collecting (5-86), (5-87), (5-88), and (5-92), we obtain from (5-69)

$$\ln D_n(f(z, 1)) - \ln D_n(f(z, 0))$$

$$= nV_0 + \sum_{k=1}^{\infty} kV_kV_{-k} + \sum_{k=0}^{m} \left[-(\alpha_k + \beta_k)\ln b_-(z_k) + (-\alpha_k + \beta_k)\ln b_+(z_k) \right]$$

$$+ O_\varepsilon(1/n), \quad n \to \infty, \quad (5\text{-}93)$$

which, in view of the result of the previous section, concludes the proof of Theorem 1.1 for analytic $V(z)$ satisfying the condition (5-66).

5D. *Extension to general analytic $V(z)$.* Now let $V(z)$ be any function analytic in a neighborhood of the unite circle. Since zeros of the expression $1 - t + te^{V(z)}$, $t \in [0, 1]$, $z \in C$, can only occur if $\Im V(z) = \pi(2k + 1)$, $k \in \mathbb{Z}$, there exists a positive integer q such that $\frac{1}{q}V(z)$ satisfies the condition (5-66) of the previous section, i.e.,

$$1 - t + te^{\frac{1}{q}V(z)} \neq 0 \quad \text{for all } t \in [0, 1] \text{ and } z \in C,$$

and this function has no winding around C for all $t \in [0, 1]$. Let

$$f_0(z) = f^{FH}(z), \quad f_\ell(z) = e^{\frac{1}{q}V(z)}f_{\ell-1}(z), \quad \ell = 1, \ldots, q, \quad (5\text{-}94)$$

where $f^{FH}(z)$ is the symbol for the "pure" Fisher–Hartwig case. Note that

$$f(z) \equiv e^{V(z)}f^{FH}(z) = f_q(z). \quad (5\text{-}95)$$

Consider $f_\ell(z)$, $\ell = 1, \ldots, q$, and introduce the deformation,

$$f_\ell(z, t) = \left(1 - t + te^{\frac{1}{q}V(z)}\right)f_{\ell-1}(z) = \left(1 - t + te^{\frac{1}{q}V(z)}\right)e^{\frac{\ell-1}{q}V(z)}f^{FH}(z). \quad (5\text{-}96)$$

All the considerations of the part of the previous section between equations (5-67) and (5-81) go through with $f(z, t)$ replaced by $f_\ell(z, t)$ and we arrive at the formula

$$\frac{\partial}{\partial t}\ln D_n(f_\ell(z, t)) = n\int_C f_\ell^{-1}\dot{f}_\ell\frac{dz}{2\pi iz} + X(t), \quad (5\text{-}97)$$

with

$$X(t) = \int_{\Sigma_+} g_+'f_\ell^{-1}\dot{f}_\ell\frac{dz}{2\pi i} + \int_{\Sigma_-} g_-'f_\ell^{-1}\dot{f}_\ell\frac{dz}{2\pi i} + O_\varepsilon(1/n), n \to \infty,$$

where we have, as before, the closed anticlockwise oriented contours

$$\Sigma_+ = (\Sigma' \setminus \Sigma_\varepsilon') \bigcup_{j=0}^{m} \partial U_{z_j}^+, \quad \Sigma_- = (\Sigma' \setminus \Sigma_\varepsilon') \bigcup_{j=0}^{m} \partial U_{z_j}^-, \quad (5\text{-}98)$$

and $g(z) := g_\ell(z)$ now corresponds to f_ℓ.

The first term in (5-97) yields (cf. (5-86))

$$\int_0^1 dt \int_C f_\ell^{-1} \dot{f}_\ell \frac{dz}{2\pi i z} = \frac{1}{2\pi q} \int_0^{2\pi} V(e^{i\theta}) \, d\theta = \frac{1}{q} V_0. \qquad (5\text{-}99)$$

In order to evaluate $X(t)$, we write g in the form

$$g(z,t) = g^{Sz}(z,t) + \tilde{g}^{Sz}(z) + g^{FH}(z), \qquad (5\text{-}100)$$

where $g^{FH}(z)$ is the same as in (5-85), and

$$g^{Sz}(z,t) = \int_C \frac{\ln(1-t+te^{\frac{1}{q}V(s)})}{s-z} \frac{ds}{2\pi i}, \quad \tilde{g}^{Sz}(z) = \frac{\ell-1}{q} \int_C \frac{V(s)}{s-z} \frac{ds}{2\pi i}. \quad (5\text{-}101)$$

Then we obtain

$$\int_0^1 dt\, X(t) = I^{Sz} + \tilde{I}^{Sz} + I^{FH} + O_\varepsilon(1/n), \quad n \to \infty, \qquad (5\text{-}102)$$

where, up to the replacement $V \to V/q$, the integrals I^{Sz} and I^{FH} are the respective integrals from the previous section:

$$I^{Sz} = \frac{1}{q^2} \sum_{k=1}^\infty k V_k V_{-k},$$

$$I^{FH} = \frac{1}{q} \sum_{k=0}^m \left[-(\alpha_k + \beta_k) \ln b_-(z_k) + (-\alpha_k + \beta_k) \ln b_+(z_k) \right]. \qquad (5\text{-}103)$$

The term $\tilde{I}^{Sz}$ in (5-102) is given by the equation

$$\tilde{I}^{Sz} = \frac{1}{q} \int_C \left((\tilde{g}^{Sz}(z))'_+ + (\tilde{g}^{Sz}(z))'_- \right) V(z) \frac{dz}{2\pi i}. \qquad (5\text{-}104)$$

Note that

$$\left(\tilde{g}^{Sz}(z) \right)'_+ = \frac{\ell-1}{q} \sum_{k=1}^\infty k z^{k-1} V_k, \quad \left(\tilde{g}^{Sz}(z) \right)'_- = \frac{\ell-1}{q} \sum_{k=1}^\infty k z^{-k-1} V_{-k}.$$

Therefore, after a simple calculation we obtain

$$\tilde{I}^{Sz} = \frac{2\ell-2}{q^2} \sum_{k=1}^\infty k V_k V_{-k}. \qquad (5\text{-}105)$$

Integrating (5-97) from $t = 0$ to $t = 1$ and taking into account (5-99), (5-102), (5-103), and (5-105), we obtain the following equation for the determinant $D_n(f_\ell(z))$:

$$\ln D_n(f_\ell(z)) - \ln D_n(f_{\ell-1}(z))$$

$$= \frac{1}{q} n V_0 + \frac{2\ell - 1}{q^2} \sum_{k=1}^{\infty} k V_k V_{-k}$$

$$+ \frac{1}{q} \sum_{k=0}^{m} \left[-(\alpha_k + \beta_k) \ln b_-(z_k) + (-\alpha_k + \beta_k) \ln b_+(z_k) \right]$$

$$+ O_\varepsilon(1/n), \quad n \to \infty. \quad (5\text{-}106)$$

This equation holds for any $\ell = 1, \ldots, q$. Summing up from $\ell = 1$ to $\ell = q$ we again arrive at the formula

$$\ln D_n(f(z)) - \ln D_n(f^{FH}(z))$$

$$= n V_0 + \sum_{k=1}^{\infty} k V_k V_{-k} + \sum_{k=0}^{m} \left[-(\alpha_k + \beta_k) \ln b_-(z_k) + (-\alpha_k + \beta_k) \ln b_+(z_k) \right]$$

$$+ O_\varepsilon(1/n), \quad n \to \infty, \quad (5\text{-}107)$$

which conludes the proof of Theorem 1.1 in the case of $V(z)$ analytic in a neighborhood of the unit circle.

5E. *Extension to smooth $V(z)$.* If $V(z)$ is just sufficiently smooth, in particular C^∞, on the unit circle C so that (1-11) holds for s from zero up to and including some $s \geq 0$, we can approximate $V(z)$ by trigonometric polynomials $V^{(n)}(z) = \sum_{k=-p(n)}^{p(n)} V_k z^k$, $z \in C$. First, consider the case when

$$\|\!|\beta|\!\| = \max_{j,k} |\Re\beta_j - \Re\beta_k| = 2 \max_j |\Re\beta_j - \omega| < 1,$$

where ω is defined by (4-31). (The indices $j, k = 0$ are omitted if $\alpha_0 = \beta_0 = 0$.) We set

$$p = [n^{1-\nu}], \quad \nu = \|\!|\beta|\!\| + \varepsilon_1, \quad (5\text{-}108)$$

where $\varepsilon_1 > 0$ is chosen sufficiently small so that $\nu < 1$ (square brackets denote the integer part).

First, we need to extend the RH analysis of the previous sections to symbols which depend on n, namely to the case when V in f is replaced by $V^{(n)}$. (We will denote such f by $f(z, V^{(n)})$, and the original f by $f(z, V)$.) We need to have a suitable estimate for the behavior of the error term in the asymptotics with n. For a fixed f, our analysis depended, in particular, on the fact that $f(z)^{-1} z^{-n}$ is of order $e^{-\varepsilon' n}$, $\varepsilon' > 0$, for $z \in \Sigma^{\text{out}}$ (see Section 4A), and similarly, $f(z)^{-1} z^n = O(e^{-\varepsilon' n})$ for $z \in \Sigma''^{\text{out}}$. Here the contours Σ^{out}, Σ''^{out} are outside a *fixed* neighborhood of the unit circle (outside and inside C, respectively). If V is replaced by $V^{(n)}$, let us define the curve Σ outside $\bigcup_{j=0}^{m} U_{z_j}$ by

$$z = \left(1 + \gamma \frac{\ln p}{p}\right) e^{i\theta}, \quad \gamma > 0, \tag{5-109}$$

and Σ'' outside $\bigcup_{j=0}^{m} U_{z_j}$ by

$$z = \left(1 - \gamma \frac{\ln p}{p}\right) e^{i\theta}. \tag{5-110}$$

Inside all the sets U_{z_j}, the curves still go to z_j as discussed in Section 4. Let the radius of all U_{z_j} be $2\gamma \ln p/p$. We now fix the value of γ as follows. Using the condition (1-11) we can write (here and below c stands for various positive constants independent of n)

$$\begin{aligned}
|V^{(n)}(z)| - |V_0| &\leq \sum_{\substack{k=-p \\ k \neq 0}}^{p} |k^s V_k| \frac{|z|^k}{|k|^s} \\
&< c \left(\sum_{\substack{k=-p \\ k \neq 0}}^{p} |k^s V_k|^2\right)^{1/2} \left(\sum_{k=1}^{p} \frac{(1 \pm 3\gamma \ln p/p)^{\pm 2k}}{k^{2s}}\right)^{1/2} \\
&< c \left(\sum_{k=1}^{p} \frac{(1 \pm 3\gamma \ln k/k)^{\pm 2k}}{k^{2s}}\right)^{1/2} \\
&< c \left(\sum_{k=1}^{p} \frac{1}{k^{2(s-3\gamma)}} \left[1 + O\left(\frac{\ln^2 k}{k}\right)\right]\right)^{1/2}, \tag{5-111}
\end{aligned}$$

where $z \in \Sigma^{\text{out}}$, $z \in \partial U_{z_j} \cap \{|z| > 1\}$ (with the "+" sign in "$\pm$"), and $z \in \Sigma''^{\text{out}}$, $z \in \partial U_{z_j} \cap \{|z| < 1\}$ (with the "$-$" sign). We now set

$$3\gamma = s - (1 + \varepsilon_2)/2, \quad \varepsilon_2 > 0, \tag{5-112}$$

and then

$$|V^{(n)}(z)| < c, \quad |b_+(z, V^{(n)})| < c, \quad |b_-(z, V^{(n)})| < c, \quad \text{for all } n \tag{5-113}$$

uniformly on Σ^{out}, Σ''^{out} and the ∂U_{z_j}, and in fact in the whole annulus

$$1 - 3\gamma \frac{\ln p}{p} < |z| < 1 + 3\gamma \frac{\ln p}{p}.$$

It is easy to adapt the considerations of the previous sections to the present case, and we again obtain the expansion (4-19) for the jump matrix of R on ∂U_{z_j}. Note that now $|\zeta(z)| = O(n^\nu \ln n)$ and $|z - z_j| = \ln n/n^{1-\nu}$ as $n \to \infty$ for $z \in \partial U_{z_j}$, and therefore using (4-19), (4-13), (3-13), (4-9) and the definition of ν in (5-108), we obtain, in particular,

$$n^{\omega\sigma_3}\Delta_1(z)n^{-\omega\sigma_3} = O\left(\frac{1}{n^{\varepsilon_1}\ln n}\right), \quad z \in \bigcup_{j=0}^{m}\partial U_{z_j}. \tag{5-114}$$

Furthermore, as follows from (5-109), (5-110), (5-113), and (4-27), (4-28), the jump matrix on Σ^{out} and Σ''^{out} is now the identity plus a function uniformly bounded in absolute value by

$$c\left(\frac{n^{1-\nu}}{\ln n}\right)^{2\max_j|\Re\beta_j|}\left(1 \pm \gamma(1-\nu)\frac{\ln n}{n^{1-\nu}}\right)^{\mp n}$$

$$< c\exp\left\{-\frac{\gamma}{2}(1-\nu)n^\nu\ln n\right\}n^{2(1-\nu)\max_j|\Re\beta_j|}, \tag{5-115}$$

where the upper sign corresponds to Σ^{out}, and the lower to Σ''^{out}.

The RH problem for $R(z)$ (see Section 4A) is therefore solvable, and we obtain $R(z)$ as a series where the first term R_1 is the same as before, and for the error term the same estimate holds for z outside a fixed neighborhood of the unit circle, e.g., for z large.

This implies that Theorem 1.1 holds for $f(z, V^{(n)})$. Note that it also holds for $|f(z, V^{(n)})|$.

We will now show that replacing $V^{(n)}$ with V in the symbol of the determinant $D_n(f(z, V^{(n)}))$ results, under a condition on s, in a small error only, so that Theorem 1.1 holds for $D_n(f(z, V))$ as well.

Using the Heine representation (2-10) for a Toeplitz determinant with (any) symbol $f(z)$, the straightforward estimate

$$b_\pm(z, V^{(n)}) = b_\pm(z, V)\left[1 + O\left(\frac{1}{n^{(1-\nu)s}}\right)\right] \quad \text{uniformly for } |z| = 1, \tag{5-116}$$

which follows from (1-11), and Theorem 1.1 applied to $D_n(|f(z, V^{(n)})|)$ and $D_n(f(z, V^{(n)}))$, we have, if $s(1-\nu) > 1$,

$$\left|D_n(f(z, V)) - D_n(f(z, V^{(n)}))\right|$$

$$< \frac{1}{(2\pi)^n n!}\int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{1\le j<k\le n}|e^{i\phi_j} - e^{i\phi_k}|^2\prod_{j=0}^{n}|f(e^{i\phi_j}, V^{(n)})|\,d\phi_j$$
$$\times\left(\left|1 + c/n^{(1-\nu)s}\right|^n - 1\right)$$

$$< ce^{\Re V_0 n}n^{\sum_{j=0}^{m}((\Re\alpha_j)^2+(\Im\beta_j)^2)}(e^{c/n^{(1-\nu)s-1}} - 1)$$

$$< c\left|e^{V_0 n}n^{\sum_{j=0}^{m}(\alpha_j^2-\beta_j^2)}\right|n^{\sum_{j=0}^{m}((\Im\alpha_j)^2+(\Re\beta_j)^2)}\frac{1}{n^{(1-\nu)s-1}}$$

$$< c\left|D_n(f(z, V^{(n)}))\right|n^{-((1-\nu)s-1-\sum_{j=0}^{m}((\Im\alpha_j)^2+(\Re\beta_j)^2))}. \tag{5-117}$$

Therefore,

$$D_n(f(z, V)) = D_n(f(z, V^{(n)})) \left(1 + \frac{D_n(f(z, V)) - D_n(f(z, V^{(n)}))}{D_n(f(z, V^{(n)}))} \right)$$

$$= D_n(f(z, V^{(n)}))(1 + o(1)), \tag{5-118}$$

if

$$s > \frac{1 + \sum_{j=0}^{m} ((\Im \alpha_j)^2 + (\Re \beta_j)^2)}{1 - \nu}. \tag{5-119}$$

Note that this condition is consistent with (5-112) and the requirement that $\gamma > 0$. Using the expression for ν in (5-108) and noting that ε_1 can be arbitrary close to zero, we replace (5-119) with (1-12). Under the condition (1-12) we then obtain the statement of the theorem for $D_n(f(z, V))$.

Appendix: the Toeplitz determinant D_n as a tau-function

Here we construct a Fuchsian system of ODE's corresponding to the Riemann–Hilbert problem of Section 2 for $V \equiv 0$. We show that the differential identities (3-20) for the Toeplitz determinant can be viewed as monodromy deformations of the tau-function associated with this Fuchsian system.

Assume the pure Fisher–Hartwig case, $V(z) \equiv 0$. Set

$$\Phi(z) = \Lambda Y^{(n)}(z)\Lambda^{-1} \prod_{k=0}^{m} (z - z_k)^{\alpha_k \sigma_3} z^{\lambda \sigma_3}, \tag{A-1}$$

where

$$\Lambda = \prod_{k=0}^{m} z_k^{\frac{1}{2}(\beta_k + \alpha_k)\sigma_3},$$

$$\lambda = \sum_{k=0}^{m} \frac{\beta_k - \alpha_k}{2} - \frac{n}{2},$$

and the branches of all multivalued functions are chosen as in Section 3. In terms of the function $\Phi(z)$, the Riemann–Hilbert problem (2-6)- (2-7) reads as follows:

(a) $\Phi(z)$ is analytic for $z \in \mathbb{C} \setminus (C \cup [0, 1] \cup \{\bigcup_{j=0}^{m} \Gamma_j\})$, where Γ_j is the ray $\theta = \theta_j$ from z_j to infinity. The unit circle C is oriented as before, counterclockwise, the segment $[0, 1]$ is oriented from 0 to 1, and the rays Γ_j are oriented towards infinity.

(b) The boundary values of $\Phi(z)$ are related by the jump conditions, namely

$$\Phi_+(z) = \Phi_-(z) \begin{pmatrix} 1 & s_j \\ 0 & 1 \end{pmatrix} \tag{A-2}$$

if $z \in C$, $\theta_j < \arg z < \theta_{j+1}$, $0 \le j \le m$, $\theta_{m+1} = 2\pi$, and otherwise

$$\Phi_+(z) = \Phi_-(z) \begin{cases} e^{-2\pi i \alpha_j \sigma_3} & \text{if } z \in \Gamma_j, \ 1 \le j \le m, \\ e^{-2\pi i (\alpha_0 + \lambda)\sigma_3} & \text{if } z \in \Gamma_0, \\ e^{-2\pi i \lambda \sigma_3} & \text{if } z \in [0, 1], \end{cases}$$

where

$$s_j = \exp\left\{-i\pi \sum_{k=0}^{j} \beta_k + i\pi \sum_{k=j+1}^{m} \beta_k - i\pi \sum_{k=0}^{j} \alpha_k - 3i\pi \sum_{k=j+1}^{m} \alpha_k\right\} \qquad \text{(A-3)}$$

(for $j = m$, the second and fourth sums are absent).

(c) $\Phi(z)$ has the following asymptotic behavior at infinity:

$$\Phi(z) = \left(I + O\left(\frac{1}{z}\right)\right) z^{\left(\sum_{k=0}^{m} \frac{1}{2}(\beta_k + \alpha_k) + \frac{1}{2} n\right)\sigma_3} e^{2\pi i \sum_{k=j+1}^{m} \alpha_k \sigma_3}, \qquad \text{(A-4)}$$

as $z \to \infty$ and $\theta_j < \arg z < \theta_{j+1}$, $0 \le j \le m$, $\theta_{m+1} = 2\pi$ (for $j = m$ the last factor is omitted).

(d) In the neighborhoods U_{z_j} of the points z_j, $j = 0, 1, \ldots, m$, the function $\Phi(z)$ admits the following representations, which constitute a refinement of the estimates (2-8) and (2-9).

• If $\alpha_j \ne 0$, then

$$\Phi(z) = \widetilde{\Phi}_j(z)(z - z_j)^{\alpha_j \sigma_3} C_j, \qquad \text{(A-5)}$$

where $\widetilde{\Phi}_j(z)$ is holomorphic at $z = z_j$ (it is essentially the function $\widetilde{Y}(z)$ from Section 3) and the matrix C_j is given by the formula

$$C_j = \begin{pmatrix} 1 & c_j \\ 0 & 1 \end{pmatrix}, \qquad \text{(A-6)}$$

with

$$c_j = s_j \begin{cases} \dfrac{1 - e^{2\pi i(\beta_j + \alpha_j)}}{1 - e^{4\pi i \alpha_j}} & \text{if } z \in U_{z_j}, \ |z| < 1, \\[2ex] \dfrac{1 - e^{2\pi i(\beta_j - \alpha_j)}}{1 - e^{4\pi i \alpha_j}} & \text{if } z \in U_{z_j}, \ |z| > 1, \ \arg z < \theta_j, \\[2ex] e^{4\pi i \alpha_j} \dfrac{1 - e^{2\pi i(\beta_j - \alpha_j)}}{1 - e^{4\pi i \alpha_j}} & \text{if } z \in U_{z_j}, \ |z| > 1, \ \arg z > \theta_j \end{cases} \qquad \text{(A-7)}$$

in the case $j \ne 0$, and

$$C_0 = \begin{pmatrix} 1 & c_0 \\ 0 & 1 \end{pmatrix} \times \begin{cases} I & \text{if } \Im z > 0, \\ e^{2\pi i \kappa \sigma_3} & \text{if } \Im z < 0, \end{cases} \qquad \text{(A-8)}$$

with

$$c_0 = s_0 \begin{cases} \dfrac{1-e^{2\pi i(\beta_0+\alpha_0)}}{1-e^{4\pi i\alpha_0}} & \text{if } z \in U_{z_0}, \ |z| < 1 \\[2ex] \dfrac{1-e^{2\pi i(\beta_0-\alpha_0)}}{1-e^{4\pi i\alpha_0}} & \text{if } z \in U_{z_0}, \ |z| > 1, \ \Im z < 0, \\[2ex] e^{4\pi i\alpha_0}\dfrac{1-e^{2\pi i(\beta_0-\alpha_0)}}{1-e^{4\pi i\alpha_0}} & \text{if } z \in U_{z_0}, \ |z| > 1, \ \Im z > 0, \end{cases} \qquad \text{(A-9)}$$

in the case $j = 0$.

• If $\alpha_j = 0$ and $\beta_j \neq 0$, then

$$\Phi(z) = \widetilde{\Phi}_j(z) \begin{pmatrix} 1 & \frac{d_j}{2\pi i}\ln(z-z_j) \\ 0 & 1 \end{pmatrix} C_j, \quad d_j = -s_j(1-e^{2\pi i\beta_j}), \qquad \text{(A-10)}$$

where $\widetilde{\Phi}_j(z)$ is again holomorphic at $z = z_j$ and the matrix C_j this time is given by the formula

$$C_j = \begin{pmatrix} 1 & c_j \\ 0 & 1 \end{pmatrix}, \qquad \text{(A-11)}$$

with

$$c_j = \begin{cases} s_{j-1} & \text{if } z \in U_{z_j}, \ |z| < 1, \\ 0 & \text{if } z \in U_{z_j}, \ |z| > 1, \ \arg z < \theta_j, \\ d_j & \text{if } z \in U_{z_j}, \ |z| > 1, \ \arg z > \theta_j, \end{cases} \qquad \text{(A-12)}$$

in the case $j \neq 0$, and

$$C_0 = \begin{pmatrix} 1 & c_0 \\ 0 & 1 \end{pmatrix} \times \begin{cases} I & \text{if } \Im z > 0, \\ e^{2\pi i\lambda\sigma_3} & \text{if } \Im z < 0, \end{cases} \qquad \text{(A-13)}$$

with

$$c_0 = \begin{cases} s_m e^{4\pi i\lambda} & \text{if } z \in U_{z_0}, \quad |z| < 1, \\ 0 & \text{if } z \in U_{z_0}, \quad |z| > 1, \quad \Im z < 0, \\ d_0 & \text{if } z \in U_{z_0}, \quad |z| > 1, \quad \Im z > 0, \end{cases} \qquad \text{(A-14)}$$

in the case $j = 0$.

(e) In a small neighborhood of $z = 0$, the function $\Phi(z)$ admits a similar representation:

$$\Phi(z) = \widetilde{\Phi}^{(0)}(z) z^{\lambda\sigma_3}, \qquad \text{(A-15)}$$

where $\widetilde{\Phi}^{(0)}(z)$ is holomorphic at $z = 0$.

We also note that all the matrices above have determinants equal to 1.

A key feature of the Φ-RH problem is that all its jump matrices and the connection matrices C_j are piecewise constant in z. By standard arguments (see, e.g., [Jimbo et al. 1981; Its et al. 2001]), based on the Liouville theorem, this fact

implies that the function $\Phi(z)$ satisfies a linear matrix ODE of Fuchsian type:

$$\frac{d\Phi(z)}{dz} = A(z)\Phi(z), \quad A(z) = \sum_{k=0}^{m} \frac{A_k}{z - z_k} + \frac{B}{z} \tag{A-16}$$

with

$$B = \lambda \widetilde{\Phi}^{(0)}(0)\sigma_3\big(\widetilde{\Phi}^{(0)}(0)\big)^{-1} \tag{A-17}$$

and

$$A_j = \begin{cases} \alpha_j \widetilde{\Phi}_j(z_j)\sigma_3 \widetilde{\Phi}_j^{-1}(z_j) & \text{if } \alpha_j \neq 0, \\ \dfrac{d_j}{2\pi i} \widetilde{\Phi}_j(z_j)\big(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\big)\widetilde{\Phi}_j^{-1}(z_j) & \text{if } \alpha_j = 0, \ \beta_j \neq 0. \end{cases} \tag{A-18}$$

Moreover, since the jump matrices and the connection matrices C_j are all constant with respect to z_j, the function $\Phi(z)$ satisfies, in addition to (A-16), the equations

$$\frac{\partial \Phi(z)}{\partial z_j} = -\frac{A_j}{z - z_j}\Phi(z), \quad j = 1, \ldots, m. \tag{A-19}$$

The compatibility condition of (A-16) and (A-19) yields the following nonlinear systems of ODEs on the matrix coefficients B and A_j:

$$\frac{\partial B}{\partial z_j} = \frac{[A_j, B]}{z_j}, \quad \frac{\partial A_k}{\partial z_j} = \frac{[A_j, A_k]}{z_j - z_k}, \quad k \neq j, \tag{A-20}$$

$$\frac{\partial A_j}{\partial z_j} = -\sum_{\substack{k=0 \\ k \neq j}}^{m} \frac{[A_j, A_k]}{z_j - z_k} + \frac{[B, A_j]}{z_j}. \tag{A-21}$$

In the context of the Fuchsian system (A-16), the jump matrices and the connection matrices C_j of the Φ-Riemann–Hilbert problem form the monodromy data of the system. The fact that these data do not depend on the parameters z_j means that the functions $B = B(z_1, \ldots, z_m)$ and $A_j = A_j(z_1, \ldots, z_m)$, $j = 0, \ldots, m$ describe *isomonodromy deformations* of the system (A-16). Equations (A-20)–(A-21) are the classical Schlesinger equations.

An important role in the modern theory of isomonodromy deformations is played by the notion of a τ-function which was introduced by M. Jimbo, T. Miwa and K. Ueno in [Jimbo et al. 1981]. In the Fucshian case, the τ-function is defined as follows. Let

$$\widetilde{\omega} = \sum_{k=1}^{m} \text{Res}_{z=z_k} \text{trace} \, A(z) \frac{\partial \widetilde{\Phi}_k(z)}{\partial z} \widetilde{\Phi}_k^{-1}(z) \, dz_k$$

$$= \sum_{k=1}^{m} \text{trace} \, A_k \frac{\partial \widetilde{\Phi}_k(z_k)}{\partial z} \widetilde{\Phi}_k^{-1}(z_k) \, dz_k. \tag{A-22}$$

As shown in [Jimbo et al. 1981], the differential form

$$\widetilde{\omega} = \widetilde{\omega}\,(A_0, \ldots, A_m, B; z_1, \ldots, z_m)$$

is closed on the solutions of the Schlesinger system (A-20)–(A-21). The τ-function is then defined as the exponential of the antiderivative of $\widetilde{\omega}$, i.e.,

$$\frac{\partial \ln \tau}{\partial z_k} = \text{trace } A_k \frac{\partial \widetilde{\Phi}_k(z_k)}{\partial z} \widetilde{\Phi}_k^{-1}(z_k). \tag{A-23}$$

It has been observed (see, e.g., [Its et al. 2001; Bertola 2009]) that the τ-functions evaluated on solutions of the Schlesinger equations generated by the Riemann–Hilbert problems associated with Toeplitz and Hankel determinants coincide, up to trivial factors, with the determinants themselves. In particular, for the Toeplitz determinant D_n with the pure Fisher–Hartwig symbol, $V(z) \equiv 0$, one can follow the calculations of [Its et al. 2001] and obtain that

$$\frac{\partial \ln D_n}{\partial z_k} = \text{trace } A_k \frac{\partial \widetilde{\Phi}_k(z_k)}{\partial z} \widetilde{\Phi}_k^{-1}(z_k) + 2 \sum_{\substack{j=0 \\ j \neq k}}^{m} \frac{\alpha_k \alpha_j}{z_j - z_k} - 2\lambda \frac{\alpha_k}{z_k}. \tag{A-24}$$

Therefore, in the case of the Riemann–Hilbert problem (A-2)–(A-15), the relation between the associated Toeplitz determinant and the τ-function is given by

$$D_n(f(z)) \equiv D_n(z_1, \ldots, z_m | \alpha, \beta)$$

$$= \tau(z_1, \ldots, z_m | \alpha, \beta) \prod_{j < k}(z_k - z_j)^{-2\alpha_j \alpha_k} \prod_{j=0}^{m} z_j^{-2\alpha_j \lambda}. \tag{A-25}$$

Direct substitution of (A-5), (A-10), and (A-15) into (A-16) and (A-19) yields

$$\text{trace } A_k \frac{\partial \widetilde{\Phi}_k(z_k)}{\partial z} \widetilde{\Phi}_k^{-1}(z_k)$$

$$= \sum_{\substack{j=0 \\ j \neq k}}^{m} \frac{\text{trace } A_j A_k}{z_k - z_j} + \frac{1}{z_k} \text{trace } B A_k$$

$$= \sum_{j=0}^{m} \text{trace } A_j \frac{\partial \widetilde{\Phi}_j(z_j)}{\partial z_k} \widetilde{\Phi}_j^{-1}(z_j) + \text{trace } B \frac{\partial \widetilde{\Phi}^{(0)}(0)}{\partial z_k}\big(\widetilde{\Phi}^{(0)}(0)\big)^{-1}. \tag{A-26}$$

Hence (A-24) can be written as follows:

Lemma A.5. *Let the Riemann–Hilbert problem for Φ be solvable. Then for any $k = 0, 1, \ldots, m$,*

$$\frac{\partial \ln D_n}{\partial z_k} = \sum_{j=0}^{m} \operatorname{trace} A_j \frac{\partial \widetilde{\Phi}_j(z_j)}{\partial z_k} \widetilde{\Phi}_j^{-1}(z_j) + \operatorname{trace} B \frac{\partial \widetilde{\Phi}^{(0)}(0)}{\partial z_k} \left(\widetilde{\Phi}^{(0)}(0)\right)^{-1}$$

$$+ 2 \sum_{\substack{j=0 \\ j \neq k}}^{m} \frac{\alpha_k \alpha_j}{z_j - z_k} - 2\lambda \frac{\alpha_k}{z_k}. \quad \text{(A-27)}$$

What is of interest here is that the differential identities (3-20), which play a very important role in the main text, can be written in a matrix form closely related to (A-27). For simplicity, we present only the case $\alpha_j \neq 0$.

Lemma A.6. *Let the Riemann–Hilbert problem for* Φ *be solvable. Let* $\alpha_j \neq 0$, $j = 0, \ldots, m$. *Then, for any* $k = 0, 1, \ldots, m$,

$$\frac{\partial \ln D_n}{\partial \alpha_k} = \sum_{j=0}^{m} \operatorname{trace} A_j \frac{\partial \widetilde{\Phi}_j(z_j)}{\partial \alpha_k} \widetilde{\Phi}_j^{-1}(z_j) + \operatorname{trace} B \frac{\partial \widetilde{\Phi}^{(0)}(0)}{\partial \alpha_k} \left(\widetilde{\Phi}^{(0)}(0)\right)^{-1}$$

$$- 2 \sum_{\substack{j=0 \\ j \neq k}}^{m} \alpha_j \ln(z_j - z_k) - 2\lambda \ln(-z_k) + \sum_{j=0}^{m} \alpha_j \ln z_j - n \ln z_k, \quad \text{(A-28)}$$

$$\frac{\partial \ln D_n}{\partial \beta_k} = \sum_{j=0}^{m} \operatorname{trace} A_j \frac{\partial \widetilde{\Phi}_j(z_j)}{\partial \beta_k} \widetilde{\Phi}_j^{-1}(z_j) + \operatorname{trace} B \frac{\partial \widetilde{\Phi}^{(0)}(0)}{\partial \beta_k} \left(\widetilde{\Phi}^{(0)}(0)\right)^{-1}$$

$$- \sum_{j=0}^{m} \alpha_j \ln z_j - n \ln z_k. \quad \text{(A-29)}$$

Remark A.7. The significance of these equations is that they complement the isomonodromy deformation formula (A-27) by formulae which describe the *monodromy* deformations of the τ-function (represented by the Toeplitz determinants D_n). (Equations (A-28) and (A-29) should be compared with the general constructions of the recent paper [Bertola 2010].)

Proof. Although straightforward, it is rather tedious to derive (A-28) and (A-29) from the differential identities (3-20). There is, however, an alternative way to obtain (A-28) and (A-29) based on the direct analysis of the Riemann–Hilbert problem (2-6)–(2-7). First, we observe that the basic deformation formula for the Toeplitz determinant D_n, i.e., Equations (3-4) and (3-5), can be written, using [Deift et al. 2011, (2.4)], in the form

$$\frac{\partial}{\partial \gamma} \ln D_n = \frac{1}{2\pi i} \int_C z^{-n} \left(Y_{11}(z) \frac{dY_{21}(z)}{dz} - \frac{dY_{11}(z)}{dz} Y_{21}(z) \right) \frac{\partial f(z)}{\partial \gamma} \, dz. \quad \text{(A-30)}$$

Here, as in Section 3, γ is either α_k or β_k. Second, by γ–differentiating the Riemann–Hilbert problem (2-6)–(2-7) we easily obtain the following representation for the logarithmic derivative $(\partial Y(z)/\partial \gamma) Y^{-1}(z)$ of its solution (compare

[Bertola 2010, Lemma 2.1]):

$$X(z) \equiv \frac{\partial Y(z)}{\partial \gamma} Y^{-1}(z)$$

$$= \frac{1}{2\pi i} \int_C Y_-(z') \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} Y_+^{-1}(z') \frac{\partial f(z')}{\partial \gamma} (z')^{-n} \frac{dz'}{z' - z}$$

$$= \frac{1}{2\pi i} \int_C \begin{pmatrix} -Y_{11}(z')Y_{21}(z') & Y_{11}^2(z') \\ -Y_{21}^2(z') & Y_{11}(z')Y_{21}(z') \end{pmatrix} \frac{\partial f(z')}{\partial \gamma} (z')^{-n} \frac{dz'}{z' - z}. \quad \text{(A-31)}$$

Now, from (A-1) and (A-16) we see that

$$\frac{dY_{11}(z)}{dz} = A_{11}(z)Y_{11}(z) + \Lambda_{11}^{-2} A_{12}(z)Y_{21}(z) - c(z)Y_{11}(z),$$

$$\frac{dY_{21}(z)}{dz} = \Lambda_{11}^2 A_{21}(z)Y_{11}(z) + A_{22}(z)Y_{21}(z) - c(z)Y_{21}(z),$$

where $c(z) = \sum_{k=0}^{m} \frac{\alpha_k}{z - z_k} + \frac{\lambda}{z}$. Further, this allows us to rewrite (A-30) in the form

$$\frac{\partial}{\partial \gamma} \ln D_n$$

$$= \frac{1}{2\pi i} \int_C z^{-n} \big(Y_{11}^2(z) \Lambda_{11}^2 A_{21}(z) - \Lambda_{11}^{-2} A_{12}(z) Y_{21}^2(z)$$

$$+ Y_{11}(z)Y_{21}(z) \left(A_{22}(z) - A_{11}(z) \right) \big) \frac{\partial f(z)}{\partial \gamma} \, dz$$

$$= \Lambda_{11}^2 \sum_{k=0}^{m} A_{k,21} \int_C Y_{11}^2(z) \frac{\partial f(z)}{\partial \gamma} \frac{z^{-n} \, dz}{2\pi i (z - z_k)}$$

$$- \Lambda_{11}^{-2} \sum_{k=0}^{m} A_{k,12} \int_C Y_{21}^2(z) \frac{\partial f(z)}{\partial \gamma} \frac{z^{-n} \, dz}{2\pi i (z - z_k)}$$

$$+ \sum_{k=0}^{m} (A_{k,22} - A_{k,11}) \int_C Y_{11}(z)Y_{21}(z) \frac{\partial f(z)}{\partial \gamma} \frac{z^{-n} \, dz}{2\pi i (z - z_k)}$$

$$+ \Lambda_{11}^2 B_{21} \int_C Y_{11}^2(z) \frac{\partial f(z)}{\partial \gamma} \frac{z^{-n} \, dz}{2\pi i z} - \Lambda_{11}^{-2} B_{12} \int_C Y_{21}^2(z) \frac{\partial f(z)}{\partial \gamma} \frac{z^{-n} \, dz}{2\pi i z}$$

$$+ (B_{22} - B_{11}) \int_C Y_{11}(z)Y_{21}(z) \frac{\partial f(z)}{\partial \gamma} \frac{z^{-n} \, dz}{2\pi i z}.$$

A comparison with (A-31) yields the following *local* representation for the γ-derivative of $\ln D_n$:

$$\frac{\partial}{\partial \gamma} \ln D_n = \Lambda_{11}^2 \sum_{k=0}^{m} A_{k,21} X_{12}(z_k) + \Lambda_{11}^{-2} \sum_{k=0}^{m} A_{k,12} X_{21}(z_k) +$$

$$+ \sum_{k=0}^{m} A_{k,11} X_{11}(z_k) + \sum_{k=0}^{m} A_{k,22} X_{22}(z_k) + \Lambda_{11}^2 B_{21} X_{12}(0)$$
$$+ \Lambda_{11}^{-2} B_{12} X_{21}(0) + B_{11} X_{11}(0) + B_{22} X_{22}(0). \quad \text{(A-32)}$$

The last formula can be also written in the compact matrix form

$$\frac{\partial}{\partial \gamma} \ln D_n = \sum_{k=0}^{m} \text{trace} \, \Lambda^{-1} A_k \Lambda X(z_k) + \text{trace} \, \Lambda^{-1} B \Lambda X(0). \qquad \text{(A-33)}$$

By evaluating (A-33), with the help of the representation (A-5), one arrives at the formulae (A-28) and (A-29). □

References

[Baik et al. 1999] J. Baik, P. Deift, and K. Johansson, "On the distribution of the length of the longest increasing subsequence of random permutations", *J. Amer. Math. Soc.* **12**:4 (1999), 1119–1178.

[Barnes 1900] E. W. Barnes, "The theory of the *G*-function", *Quart. J. Pure and Appl. Math.* **31** (1900), 264–313.

[Basor 1978] E. L. Basor, "Asymptotic formulas for Toeplitz determinants", *Trans. Amer. Math. Soc.* **239** (1978), 33–65.

[Basor 1979] E. L. Basor, "A localization theorem for Toeplitz determinants", *Indiana Univ. Math. J.* **28**:6 (1979), 975–983.

[Basor and Tracy 1991] E. L. Basor and C. A. Tracy, "The Fisher–Hartwig conjecture and generalizations", *Phys. A* **177**:1-3 (1991), 167–173.

[Bertola 2009] M. Bertola, "Moment determinants as isomonodromic tau functions", *Nonlinearity* **22**:1 (2009), 29–50.

[Bertola 2010] M. Bertola, "The dependence on the monodromy data of the isomonodromic tau function", *Comm. Math. Phys.* **294**:2 (2010), 539–579.

[Böttcher and Silbermann 1981] A. Böttcher and B. Silbermann, "The asymptotic behavior of Toeplitz determinants for generating functions with zeros of integral orders", *Math. Nachr.* **102** (1981), 79–105.

[Böttcher and Silbermann 1985] A. Böttcher and B. Silbermann, "Toeplitz matrices and determinants with Fisher–Hartwig symbols", *J. Funct. Anal.* **63**:2 (1985), 178–214.

[Böttcher and Silbermann 1986] A. Böttcher and B. Silbermann, "Toeplitz operators and determinants generated by symbols with one Fisher–Hartwig singularity", *Math. Nachr.* **127** (1986), 95–123.

[Deift 1999] P. Deift, "Integrable operators", pp. 69–84 in *Differential operators and spectral theory*, edited by V. Buslaev et al., Amer. Math. Soc. Transl. Ser. 2 **189**, Amer. Math. Soc., Providence, RI, 1999.

[Deift and Zhou 1993] P. Deift and X. Zhou, "A steepest descent method for oscillatory Riemann–Hilbert problems: asymptotics for the MKdV equation", *Ann. of Math.* (2) **137**:2 (1993), 295–368.

[Deift et al. 1999] P. Deift, T. Kriecherbauer, K. T.-R. McLaughlin, S. Venakides, and X. Zhou, "Strong asymptotics of orthogonal polynomials with respect to exponential weights", *Comm. Pure Appl. Math.* **52**:12 (1999), 1491–1552.

[Deift et al. 2007] P. Deift, A. Its, I. Krasovsky, and X. Zhou, "The Widom–Dyson constant for the gap probability in random matrix theory", *J. Comput. Appl. Math.* **202**:1 (2007), 26–47.

[Deift et al. 2008] P. Deift, A. Its, and I. Krasovsky, "Asymptotics of the Airy-kernel determinant", *Comm. Math. Phys.* **278**:3 (2008), 643–678.

[Deift et al. 2011] P. Deift, A. Its, and I. Krasovsky, "Asymptotics of Toeplitz, Hankel, and Toeplitz+Hankel determinants with Fisher–Hartwig singularities", *Ann. of Math.* (2) **174**:2 (2011), 1243–1299.

[Deift et al. 2012] P. Deift, A. Its, and I. Krasovsky, "Toeplitz matrices and Toeplitz determinants under the impetus of the Ising model: some history and some recent results", preprint 1207.4990, 2012.

[Ehrhardt 2001] T. Ehrhardt, "A status report on the asymptotic behavior of Toeplitz determinants with Fisher–Hartwig singularities", pp. 217–241 in *Recent advances in operator theory* (Groningen, 1998), edited by A. Dijksma et al., Oper. Theory Adv. Appl. **124**, Birkhäuser, Basel, 2001.

[Ehrhardt and Silbermann 1997] T. Ehrhardt and B. Silbermann, "Toeplitz determinants with one Fisher–Hartwig singularity", *J. Funct. Anal.* **148**:1 (1997), 229–256.

[Fisher and Hartwig 1968] M. E. Fisher and R. E. Hartwig, "Toeplitz determinants: some applications, theorems, and conjectures", pp. 333–353 in *Stochastic processes in chemical physics*, edited by K. E. Shuler, Advances in Chemical Physics **15**, Wiley, 1968.

[Fokas et al. 1992] A. S. Fokas, A. R. Its, and A. V. Kitaev, "The isomonodromy approach to matrix models in 2D quantum gravity", *Comm. Math. Phys.* **147**:2 (1992), 395–430.

[Forrester and Warnaar 2008] P. J. Forrester and S. O. Warnaar, "The importance of the Selberg integral", *Bull. Amer. Math. Soc.* (*N.S.*) **45**:4 (2008), 489–534.

[Its and Krasovsky 2008] A. Its and I. Krasovsky, "Hankel determinant and orthogonal polynomials for the Gaussian weight with a jump", pp. 215–247 in *Integrable systems and random matrices*, edited by J. Baik et al., Contemp. Math. **458**, Amer. Math. Soc., Providence, RI, 2008.

[Its et al. 2001] A. R. Its, C. A. Tracy, and H. Widom, "Random words, Toeplitz determinants and integrable systems, II: advances in nonlinear mathematics and science", *Phys. D* **152/153** (2001), 199–224.

[Jimbo et al. 1981] M. Jimbo, T. Miwa, and K. Ueno, "Monodromy preserving deformation of linear ordinary differential equations with rational coefficients, I: general theory and τ-function", *Phys. D* **2**:2 (1981), 306–352.

[Krasovsky 2004] I. V. Krasovsky, "Gap probability in the spectrum of random matrices and asymptotics of polynomials orthogonal on an arc of the unit circle", *Int. Math. Res. Not.* **2004**:25 (2004), 1249–1272.

[Krasovsky 2007] I. V. Krasovsky, "Correlations of the characteristic polynomials in the Gaussian unitary ensemble or a singular Hankel determinant", *Duke Math. J.* **139**:3 (2007), 581–619.

[Lenard 1964] A. Lenard, "Momentum distribution in the ground state of the one-dimensional system of impenetrable Bosons", *J. Mathematical Phys.* **5** (1964), 930–943.

[Lenard 1972] A. Lenard, "Some remarks on large Toeplitz determinants", *Pacific J. Math.* **42** (1972), 137–145.

[Whittaker and Watson 1927] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis, an Introduction to the General Theory of Infinite Processes and of Analytic Functions*, 4th ed., Cambridge University Press, 1927. Reprinted 1969.

[Widom 1973] H. Widom, "Toeplitz determinants with singular generating functions", *Amer. J. Math.* **95** (1973), 333–383.

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