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Asymptotics of spacing distributions 50 years later

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In 1962 Dyson used a physically based, macroscopic argument to deduce the first two terms of the large spacing asymptotic expansion of the gap probability for the bulk state of random matrix ensembles with symmetry parameter β. In the ensuing years, the question of asymptotic expansions of spacing distributions in random matrix theory has shown itself to have a rich mathematical content. As well as presenting the main known formulas, we give an account of the mathematical methods used for their proofs, and provide some new formulas. We also provide a high precision numerical computation of one of the spacing probabilities to illustrate the accuracy of the corresponding asymptotics.

1. Introduction

Random matrices were introduced in physics by Wigner in the 1950s; see [Porter 1965]. Wigner's original hypothesis was that the statistical properties of energy levels of complex nuclei could be reproduced by considering an ensemble of systems rather than a single system in which all interactions are completely described. This allowed for an entirely mathematical approach where statistical properties of the spectrum of an ensemble of random matrices were considered. But coming from physics, the aim was to use mathematics to compute experimentally measurable statistical quantities, and to compare against the data.

One viewpoint on a real spectrum from a random matrix is as a point process on the real line. As such, perhaps the most natural statistical characterization is that of the distribution of the eigenvalue spacing. This choice of statistic becomes even more compelling when one considers that in many cases of interest, eigenvalue spectra can be "unfolded". This means that unlike many statistical mechanical systems, the density is not an independent control variable, but rather fixes the length scale only. Unfolding then is scaling the eigenvalues in the bulk of the spectrum so that the mean density is unity. It is indeed the bulk spacing distribution for the Gaussian orthogonal ensemble of real symmetric matrices — albeit in an approximate form known as the Wigner surmise (see, e.g., [Mehta 1991]) — which was compared against the empirical spacing distribution for the energy level of highly excited nuclei (again, see [Mehta 1991], and references therein).

Fixing length scales at the edge of the spectrum is, as a practical exercise, a more difficult task. In addition to the bulk, we will have interest in the soft and hard spectrum edges when the eigenvalue spectrum exhibits a square root profile and inverse square root profile, respectively. To specify realizations of the bulk and edge regions of the eigenvalue spectrum, we recall (see, e.g., [Forrester 2010]) that the so-called classical random matrix ensembles have their eigenvalue probability density functions (PDFs) of the form

$$\frac{1}{C}\prod_{l=1}^{N} g(\lambda_l) \prod_{1\le j<k\le N} |\lambda_k-\lambda_j|^{\beta}, \tag{1-1}$$

with β corresponding to the underlying global symmetry ($\beta = 1, 2$ or 4 for invariance under orthogonal, unitary or symplectic unitary transformations, respectively); C denotes the normalization. This is extended to general $\beta > 0$, giving the β-ensembles [Dumitriu and Edelman 2002] as specified by the eigenvalue PDF (1-1), to be denoted $\mathrm{ME}_{N,\beta}(g(\lambda))$. In particular the choice $g(\lambda) = \mathrm{e}^{-\beta\lambda^2/2}$ defines the Gaussian β-ensemble and the choice $g(\lambda) = \lambda^{\beta a/2}\mathrm{e}^{-\beta\lambda/2}$, ($\lambda > 0$), defines the Laguerre β-ensemble.

The bulk state can be realized by scaling $\lambda_l \mapsto x_l/\sqrt{2N}$ in the Gaussian β-ensemble. The soft edge is realized by the scalings

$$\lambda_l \mapsto \sqrt{2N} + \frac{x_l}{\sqrt{2}N^{1/6}} \quad \text{and} \quad \lambda_l \mapsto 4N + 2\sqrt{2}x_l$$

in the Gaussian and Laguerre β-ensembles, respectively [Forrester 1993]. Only the Laguerre β-ensemble has a hard edge, as it requires the eigenvalue density to be strictly zero on one side; it is realized by the scaling $\lambda_l \mapsto x_l/(4N)$. In all cases the limit $N \to \infty$ needs to be taken after the scaling. At an edge, the spacing between consecutive eigenvalues is not the natural observable. Instead, it is most natural to measure the distribution of the largest, second largest, etc., eigenvalue (or smallest, second smallest etc.). It is well known, and easy to verify, that all these quantities can be expressed in terms of the (conditional) gap probabilities $E_\beta^{(\cdot)}(n; J)$ for there being exactly n eigenvalues in the interval J, for the scaled state $(\cdot) =$ bulk, soft or hard indexed by β. In the case of the hard edge, the probability depends on the exponent $\beta a/2$ in the Laguerre weight $\lambda^{\beta/2}e^{-\beta\lambda/2}$, so we write $E_\beta^{\mathrm{hard}}(n; J; a\beta/2)$.

Our interest in this review is on the asymptotic form of spacing distributions in the bulk, and of the distribution of large and small eigenvalues at the edge. This is a topic which (in the bulk case) occupied the attention of Dyson in one of the pioneering papers on random matrix theory in the early 1960s [Dyson

1962], and is still being written on as we stand today some 50 years later. We are seeking to catalog both the results and the methods which underlie them, and also to contribute some new formulas. Section 2 deals with results founded on Dyson's heuristic physical hypothesis; these are in the form of conjectures. The various mathematical techniques which can both prove, and build on these asymptotic expressions, are covered in Section 3. A numerical illustration of the accuracy of the asymptotic form is given in Section 4, as is a discussion of asymptotic results for the gap probability in the case that each eigenvalue is independently deleted with probability $(1-\xi)$.

2. Macroscopic heuristics

2.1. ***Zero eigenvalues in the gap.*** The eigenvalue PDF (1-1) can be interpreted as the Boltzmann factor of a classical log-gas system interacting at inverse temperature β. The particles repel via the logarithmic potential and are subject to a one body potential with Boltzmann factor $g(\lambda) = \mathrm{e}^{-\beta V(\lambda)}$. This interpretation led Dyson [1962] to hypothesize an ansatz for the asymptotic form of the gap probability $E_\beta(0; (-\alpha, \alpha); \mathrm{C}\beta\mathrm{E}_N)$, where $\mathrm{C}\beta\mathrm{E}_N$ denotes Dyson's circular ensembles (see, e.g., [Forrester 2010, Chapter 2]) of random unitary matrices (all eigenvalues are therefore on the unit circle; the interval $(-\alpha, \alpha)$ refers to a sector of the circumference specified by its angles):

$$E_\beta(0; (-\alpha, \alpha); \mathrm{C}\beta\mathrm{E}_N) \underset{N\to\infty}{\sim} e^{-\beta\delta F}. \tag{2-1}$$

Here and below the symbol $\sim$ is used to denote that the right-hand side gives leading terms, up to some order to be further specified, of the asymptotic expansion of the left-hand side. In (2-1) δF is the energy cost of conditioning the equilibrium particle density so that $\rho_{(1)}(\theta) = 0$ for $\theta \in (-\alpha, \alpha)$. This energy cost consists of an electrostatic energy

$$V_1 = -\frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} \big(\rho_{(1)}(\theta_1) - N/2\pi\big)\big(\rho_{(1)}(\theta_2) - N/2\pi\big) \log|\mathrm{e}^{i\theta_1} - \mathrm{e}^{i\theta_2}|\, d\theta_1\, d\theta_2 \tag{2-2}$$

and an entropy term

$$V_2 = \left(\frac{1}{\beta} - \frac{1}{2}\right)\int_0^{2\pi} \rho_{(1)}(\theta) \log\frac{\rho_{(1)}(\theta)}{N/2\pi} d\theta. \tag{2-3}$$

The density is chosen to minimize V_1 and then V_1 and V_2 evaluated, and we have

$$\delta F = (V_1 + V_2). \tag{2-4}$$

Proposition 1 [Dyson 1962]. *With the requirements that* $\rho_{(1)}(\theta) = 0$ *for* $\theta \in (-\alpha, \alpha)$ *and* $\int_0^{2\pi} \rho_{(1)}(\theta)\, d\theta = N$, V_1 *is minimized by*

$$\rho_{(1)}(\theta) = \frac{N}{2\pi} \frac{\sin\theta/2}{\sqrt{\sin^2\theta/2 - \sin^2\alpha/2}}. \tag{2-5}$$

We then have

$$\beta V_1 = -\frac{\beta}{2} N^2 \log\cos\frac{\alpha}{2}, \quad \beta V_2 = \Big(1 - \frac{\beta}{2}\Big) N \log\Big(\sec\frac{\alpha}{2} + \tan\frac{\alpha}{2}\Big). \tag{2-6}$$

We remark that explicit calculations in [Dyson 1962] showed that requiring $\rho_{(1)}(\theta)$ to minimize $V_1 + V_2$ (rather than V_1) results in a correction to βV_2 which for large N is of order $\log(N\alpha)$, indicating that the asymptotic expansion (2-1) will not correctly give terms of this order.

Substituting (2-6) in (2-4), and substituting the result in (2-1) gives a large deviation formula, telling us (as a conjecture) the probability of there being no eigenvalues in the interval $(-\alpha, \alpha)$. This probability decays as a Gaussian in N. An $O(1)$ expression should result from choosing the excluded interval as $(-\pi s/N, \pi s/N)$, as then there are $O(1)$ eigenvalues in the gap. Replacing α by $\pi\alpha/2$ in (2-6), then taking $N \to \infty$ (this is a double scaling limit) gives the prediction

$$\lim_{N\to\infty} E_\beta(0; (-\pi s/N, \pi s/N); \mathrm{C}\beta\mathrm{E}_N) \sim \mathrm{e}^{-\beta(\pi s)^2/16 + (\beta/2-1)\pi s/2}. \tag{2-7}$$

Dyson was well aware that the $\sim$ symbol should be interpreted as agreeing in the large s asymptotic expansion to the order given. But the left-hand side is the definition of $E_\beta^{\mathrm{bulk}}(0; (-s/2, s/2))$, thus providing the following conjecture.

Conjecture 2 [Dyson 1962]. *We have*

$$E_\beta^{\mathrm{bulk}}(0; (0, s)) \underset{s\to\infty}{\sim} e^{-\beta(\pi s)^2/16 + (\beta/2-1)\pi s/2}. \tag{2-8}$$

As remarked above, Dyson [1962] carried through the details of the minimization of $V_1 + V_2$, resulting in a logarithmic correction to the exponent of the right-hand side of (2-8): $((1 - \beta/2)^2/(2\beta))\log s$. However, this was later put in doubt by Mehta and des Cloizeaux [1972], who, using a method based on eigenvalues (see Section 3.3 below), obtained $-\frac{1}{8}$, $-\frac{1}{4}$ and $-\frac{1}{8}$ for the prefactor of $\log s$ for $\beta = 1$, 2 and 4, respectively. Dyson himself [1976] used inverse scattering methods applied to the Fredholm determinant form of $E_1^{\mathrm{bulk}}(0; (0, s))$ (see Section 3.1) to also give the prediction $-\frac{1}{8}$ for the prefactor in the case $\beta = 1$. In fact the correct extension of (2-8) for general β, as proved for the

Gaussian β ensemble, is [Valkó and Virág 2010]:

$$E_\beta^{\text{bulk}}(0;(0,s)) \underset{s\to\infty}{\sim} \exp\Bigl(-\beta(\pi s)^2/16+(\beta/2-1)\pi s/2 \\ +\tfrac{1}{4}(\beta/2+2/\beta-3)\log s+O(1)\Bigr). \quad (2\text{-}9)$$

Its derivation will be reviewed in Section 3.2.

The ansatz (2-1) was applied to the gap probability at the hard edge of the Laguerre ensemble by Chen and Manning [1994]. They considered the probability of there being no eigenvalues in an interval $(0,t)$.

Proposition 3 [Chen and Manning 1994]. *For the Laguerre ensemble specified by* (1-1) *with* $g(\lambda)=\lambda^a e^{-\lambda}$, *with the eigenvalues constrained to the interval* (t,b), *with* $t>0$ *given, the minimizing solution for the level density* $\rho_{(1)}(x)$ *is*

$$\rho_{(1)}(x)=\frac{1}{\pi\beta}\sqrt{\frac{b-x}{x-t}}\biggl(1-\frac{a}{x}\sqrt{\frac{t}{b}}\biggr). \quad (2\text{-}10)$$

Normalization of the density requires that b *is related to* N *by*

$$N=\frac{b-t}{2\beta}+\frac{a}{\beta}\biggl(\sqrt{\frac{t}{b}}-1\biggr). \quad (2\text{-}11)$$

Using (2-10) appropriate analogues of (2-2) and (2-3) were computed (see also [Chen and Manning 1996]), thus giving a prediction for the large N form of $E_\beta(0;(0,t);\mathrm{ME}_{N,\beta}(\lambda^\alpha e^{-\lambda}))$. This is exponentially small in N. But with $t=s/(4N)$, the number of eigenvalues in $(0,t)$ will be $O(1)$. With the resulting expression interpreted as the large s asymptotic form of $E_\beta^{\text{hard}}(0;(0,s);a)$ (s must be scaled $s\mapsto(\beta/2)^2 s$ to account for the latter being defined as the large N limit of $E_\beta(0;(0,s/(4N));\mathrm{ME}_{N,\beta}(\lambda^a e^{-\beta\lambda/2}))$), the following conjecture was obtained.

Conjecture 4 [Chen and Manning 1994]. *We have*

$$E_\beta^{\text{hard}}(0;(0,s);a) \underset{s\to\infty}{\sim} \exp\biggl(-\frac{\beta s}{8}+a\sqrt{s}-\frac{a^2}{2\beta}\log s+\Bigl(1-\frac{\beta}{2}\Bigr)\frac{a}{2\beta}\log s\biggr). \quad (2\text{-}12)$$

This asymptotic had already been proved in [Forrester 1994] for $a\in\mathbb{Z}_{\geq 0}$ and $2/\beta\in\mathbb{Z}_{>0}$ before the work of Chen and Manning [1994]. Moreover, [Forrester 1994], which was based on a-dimensional integral forms for $E_\beta^{\text{hard}}(0;(0,s);a)$, gave the explicit form of the constant term in the extension of (2-12) to next order (see Section 3.4).

The first application of the log-gas ansatz (2-1) at the soft edge was due to Dean and Majumdar [2006; 2008].

Proposition 5 [Dean and Majumdar 2006]. *Consider the Gaussian* β*-ensemble* $\mathrm{ME}_{\beta,N}(e^{-\beta N x^2})$. *Suppose the eigenvalues are confined to the interval* $(-b,t)$

where $t < 1$ and $b > 0$ is determined by charge neutrality. The corresponding density is given by

$$\rho_{(1)}(x) = \frac{2N}{\pi}\left(\frac{l-t+x}{t-x}\right)^{1/2}\left(\frac{l}{2}-x\right),$$

where $l := b + t = \frac{2}{3}(t + \sqrt{t^2+3})$.

Only the corresponding form of V_1 was computed, and this gave the large deviation formula

$$E_\beta(0;\,(t,\infty);\,\mathrm{ME}_{\beta,N}(e^{-\beta N x^2})) \underset{N\to\infty}{\sim} \exp\left(-\beta N^2\left(\frac{2t^2}{3}-\frac{t^4}{27}-\frac{5}{18}t\sqrt{3+t^2}-\frac{1}{27}t^3\sqrt{3+t^2}-\frac{1}{2}\log\frac{t+\sqrt{t^2+3}}{3}\right)\right), \quad (2\text{-}13)$$

and thus, upon the appropriate soft edge scaling $\sqrt{2N}(1-t) = -\frac{s}{\sqrt{2}N^{1/6}}$, the asymptotic formula

$$E_\beta^{\mathrm{soft}}(0;\,(s,\infty)) \underset{s\to-\infty}{\sim} e^{-\beta|s|^3/24}. \quad (2\text{-}14)$$

This latter prediction was already implied in [Forrester 1993; Tracy and Widom 1994a].

2.2. ***Loop equations.*** Borot, Eynard, Majumdar and Nadal [Borot et al. 2011] gave an alternative heuristic formalism to the Dyson log-gas ansatz, for purposes of computing the soft edge gap probability. This in based on the so-called loop equations associated with the large N form of the multiple integral definition of the latter. The approach allows for the Dyson ansatz (2-1) to be extended to include higher order terms; in practice two new terms are computed — one is termed the Polyakov anomaly, and the following result is obtained.

Conjecture 6 [Borot et al. 2011]. *We have*

$$E_\beta^{\mathrm{soft}}(0;\,(s,\infty)) \underset{s\to-\infty}{\sim} \exp\left(-\beta\frac{|s|^3}{24}+\frac{\sqrt{2}(\beta/2-1)}{3}|s|^{3/2} + \frac{\beta/2+2/\beta-3}{8}\log|s| + \log\tau_\beta^{\mathrm{soft}} + O(|s|^{-3/2})\right), \quad (2\text{-}15)$$

where

$$\log\tau_\beta^{\mathrm{soft}} = \left(\tfrac{17}{8}-\tfrac{25}{24}(\beta/2+2/\beta)\right)\log 2 - \frac{\log 2\pi}{2} - \frac{\log\beta/2}{2} + \kappa_{\beta/2}, \quad (2\text{-}16)$$

with κ_β the constant term in the large N expansion of

$$F(N+1) := \sum_{j=1}^{N}\log\Gamma(1+j\beta/2).$$

(Note that in [Borot et al. 2011] what we call $\beta/2$ is written as β.)

In [Borot et al. 2011], for β rational, κ_β was evaluated in terms of the Barnes G-function, while for general $\beta > 0$ it was shown that

$$\kappa_{\beta/2} = \frac{\log 2\pi}{4} + \frac{\beta}{2}\left(\frac{1}{12} - \zeta'(-1)\right) + \frac{\gamma}{6\beta} + \int_0^\infty \frac{1}{e^{\beta s/2}-1}\left(\frac{s}{e^s-1} - 1 + \frac{s}{2} - \frac{s^2}{12}\right) ds,$$

where γ denotes Euler's constant. In fact $\kappa_{\beta/2}$ can be expressed in terms of the so-called Stirling modular form $\rho_2(1,\tau)$, which from a computational viewpoint can be defined by the infinite product [Shintani 1980]

$$\rho_2(1,\tau) = (2\pi)^{3/4}\tau^{-1/4+(\tau+1/\tau)/12} e^{P(\tau)} \prod_{n=1}^\infty \frac{e^{Q(n\tau)}}{\Gamma(1+n\tau)},$$

where

$$P(\tau) = -\frac{\gamma}{12\tau} - \frac{\tau}{12} + \tau\zeta'(-1), \qquad Q(x) = \left(\tfrac{1}{2}+x\right)\log x - x + \log\sqrt{2\pi} + \frac{1}{12x}.$$

The quantity $\rho_2(1,\tau)$ is fundamental to the theory of the Barnes double gamma function $\Gamma_2(z;1,\tau)$ [Barnes 1904], the latter being related to the usual gamma function through the two functional equations

$$\frac{1}{\Gamma_2(z+1;1,\tau)} = \frac{\tau^{z/\tau-1/2}}{\sqrt{2\pi}} \frac{\Gamma(z/\tau)}{\Gamma_2(z;1,\tau)}, \qquad \frac{1}{\Gamma_2(z+\tau;1,\tau)} = \frac{1}{\sqrt{2\pi}} \frac{\Gamma(z)}{\Gamma_2(z;1,\tau)}, \tag{2-17}$$

and furthermore is normalized by requiring $\lim_{z\to 0} z\Gamma_2(z;1,\tau) = 1$.

Proposition 7. *Let $\tau = 2/\beta$ and specify $F(N+1)$ and $\kappa_{\beta/2}$ as in Conjecture 6. We have*

$$F(N+1) = (2\pi)^{N/2}\tau^{-(N^2-N(1-\tau))/2\tau} \frac{\Gamma(N)\Gamma(1+N/\tau)}{\Gamma_2(N;1,\tau)}, \tag{2-18}$$

$$\kappa_{1/\tau} = -\tfrac{1}{2}\log\tau + \log 2\pi - \log\rho_2(1,\tau), \tag{2-19}$$

with the latter equation substituted into (2-16) *giving*

$$\log\tau_\beta^{\text{soft}} = \left(\tfrac{17}{8} - \tfrac{25}{24}(\beta/2 + 2/\beta)\right)\log 2 + \frac{\log 2\pi}{2} - \log\rho_2(1, 2/\beta). \tag{2-20}$$

Equation (2-18) has appeared in [Brini et al. 2011]; it follows immediately by characterizing $F(N+1)$ as a first order recurrence, and using (2-17). The formula for $\kappa_{1/\tau}$ then follows by extracting the term independent of N in the corresponding asymptotic expansion. Here one uses the fact that for $\log \Gamma_2(N; 1, \tau)$ this is $\log \rho_2(1, \tau)$ [Quine et al. 1993]. A consequence of (2-20) is that

$$\log \frac{\tau^{\rm soft}_{\beta/2}}{\tau^{\rm soft}_{2/\beta}} = -\log \frac{\rho_2(1, 2/\beta)}{\rho_2(1, \beta/2)} = \frac{\beta/2+2/\beta-3}{8}\left(\log \frac{\beta}{2}\right)^{2/3} - \frac{1}{2}\log \frac{\beta}{2}, \qquad (2\text{-}21)$$

where the final equality follows from the inversion formula for the Stirling modular form [Katayama and Ohtsuki 1998, Proposition 7(iv)]. Using this in (2-15) gives that (cf. [Borot et al. 2011, Equation (6.2)])

$$E^{\rm soft}_\beta(0; (s, \infty)) \underset{s\to-\infty}{\sim} \left(\frac{2}{\beta}\right)^{1/2} \widetilde{E^{\rm soft}_{4/\beta}}\left(0; \left(\left(\frac{\beta}{2}\right)^{2/3} s, \infty\right)\right), \qquad (2\text{-}22)$$

where $\widetilde{E^{\rm soft}_\beta}$ refers to the right-hand side of (2-15) with $|s|^{3/2}$ replaced by $-|s|^{3/2}$.

2.3. ***Conditioning n eigenvalues in the gap.*** Dyson [1995], and independently Fogler and Shklovskii [1995], further developed the log-gas argument by the consideration of the setting that the gap $(-t, t)$ is required to contain exactly n eigenvalues, with $0 \ll n \ll t$. Moreover, a change of viewpoint was introduced: the log-gas was taken to be infinite in extent, with the bulk state characterized by a uniform density, normalized to unity. The n eigenvalues are modeled as a continuous conductive fluid occupying the interval $(-b, b) \subset (-t, t)$. The electrostatic potential in this region must therefore be equal to a constant $-v$ say, $v > 0$, with the potential in the other conducting region $\mathbb{R}\backslash(-t, t)$ taken to be zero. The explicit form of the density was determined, and this substituted in the appropriate modification of (2-2) and (2-3) gave after some calculation the simple results

$$V_1 = -\frac{nv}{2} + \frac{\pi^2}{4}(t^2 - b^2), \qquad V_2 = v. \qquad (2\text{-}23)$$

The end point b is determined by n via a certain elliptic integral, and similarly v in terms of an elliptic integral of modulus b/t. Expansion of these quantities for $t \to \infty$, and substitution in (2-1) provides a generalization of (2-9).

Conjecture 8 [Dyson 1995; Fogler and Shklovskii 1995]. *For* $0 \ll n \ll s$ *we have*

$$\log E^{\rm bulk}_\beta(n; (0, s)) \underset{s\to\infty}{\sim} -\beta\frac{(\pi s)^2}{16} + \left(\beta n + \frac{\beta}{2} - 1\right)\frac{\pi s}{2} + \left\{\frac{n}{2}\left(1 - \frac{\beta}{2} - \frac{\beta n}{2}\right) + \frac{1}{4}\left(\frac{\beta}{2} + \frac{2}{\beta} - 3\right)\right\}\log s. \qquad (2\text{-}24)$$

(Here we have added the $n = 0$ contribution to the term $\log s$ as implied by (2-9) — we then expect (2-24) to hold for $0 \le n \ll s$; this is not a consequence of the calculations in [Dyson 1995; Fogler and Shklovskii 1995].)

Only very recently has this infinite log-gas formalism been applied to predict the asymptotic forms of the conditioned gap probabilities at the hard and soft edges [Forrester and Witte 2012]. Since the system is (semi-) infinite, this relies on characterizing these edges in terms of the respective background densities: $\sqrt{x}/\pi$ for the soft edge, and $1/(2\pi\sqrt{x})$ for the hard edge. In both cases the coordinates are chosen so that the edge occurs at $x=0$. It was found in [Forrester and Witte 2012] that applying the ansatz (2-1) with δF given by (2-4) in this setting to the $n=0$ case gave results inconsistent with both (2-12) and its soft edge analogue in the second order term. Thus the ansatz (2-1) with δF given by (2-4) is incorrect in the infinite log-gas formalism applied to the hard and soft edges. On the other hand, it was observed that replacing V_2 by the potential drop v in going from the region containing the infinite mobile log-charges, to the region containing the n charges — which, according to (2-23), is an identity for the bulk — restores the correct value for these terms. Making this replacement for general n then gives the following predictions.

Conjecture 9 [Forrester and Witte 2012]. *We have*, *for* $0 \ll n \ll |s|$ (*or more strongly* $0 \le n \ll |s|$),

$$\log E_\beta^{\mathrm{hard}}(n;\,(0,s);\,\beta a/2) \underset{s\to\infty}{\sim} -\beta\left\{\frac{s}{8}-\sqrt{s}\left(n+\frac{a}{2}\right)+\left[\frac{n^2}{2}+\frac{na}{2}+\frac{a(a-1)}{4}+\frac{a}{2\beta}\right]\log s^{1/2}\right\} \tag{2-25}$$

and

$$\log E_\beta^{\mathrm{soft}}(n;\,(s,\infty)) \underset{s\to-\infty}{\sim} -\frac{\beta|s|^3}{24}+\frac{2\sqrt{2}}{3}|s|^{3/2}\left(\beta n+\frac{\beta}{2}-1\right) + \left[\frac{\beta}{2}n^2+\left(\frac{\beta}{2}-1\right)n+\frac{1}{6}\left(1-\frac{2}{\beta}\left(1-\frac{\beta}{2}\right)^2\right)\right]\log|s|^{-3/4}. \tag{2-26}$$

(*As for* (2-24), *the results coming from the log-gas calculation have*, *in the case of the logarithmic term*, *been supplemented by knowledge of the asymptotic expansion at that order for* $n=0$.)

We remark that a check on (2-24)–(2-26) is that they obey certain asymptotic functional equations, implied by exact functional equations for spacing distributions obtained in [Forrester 2009]. For example, at the hard edge one requires

$$E_\beta^{\mathrm{hard}}(n;\,(0,s/\tilde{s}_\beta);\,\beta a/2) \underset{\substack{s\to\infty\\ n\ll t}}{\sim} E_{4/\beta}^{\mathrm{hard}}\big(\tfrac{1}{2}\beta(n+1)-1;\,(0,s/\tilde{s}_{4/\beta});\,a-2+4/\beta\big),$$

where $\tilde{s}_\beta$ is an arbitrary length scale that satisfies $\tilde{s}_{4/\beta}(\beta/2)^2=\tilde{s}_\beta$. This is indeed a property of (2-25).

Precise asymptotic statements can also be made concerning the asymptotic form of $E_\beta^{(\,\cdot\,)}(n; J)$, for $|J| \to \infty$ and $n \approx \langle n_J \rangle$, where n_J ($\langle n_J \rangle$) denotes the number (expected number) of particles in J for the unconstrained system. Thus macroscopic heuristics applied to this linear statistic (see, e.g., [Forrester 2012, §14.5.1]) predict that $(n_J - \langle n_J \rangle)/\sqrt{\operatorname{Var} n_J}$ has a Gaussian distribution with zero mean and unit variance, and so suggesting the following result.

Conjecture 10. *For $n \approx \langle n_J \rangle$,*

$$E_\beta^{(\,\cdot\,)}(n; J) \underset{|J|\to\infty}{\sim} \frac{1}{(2\pi \operatorname{Var} n_J)^{1/2}} e^{-(n-\langle n_J \rangle)^2/2\operatorname{Var} n_J}. \tag{2-27}$$

Moreover, for $(\,\cdot\,) =$ bulk, soft and hard we have

$$\langle n_{(0,s)} \rangle \underset{s\to\infty}{\sim} s, \quad \langle n_{(s,\infty)} \rangle \underset{s\to-\infty}{\sim} \frac{2(-s)^{3/2}}{3\pi}, \quad \langle n_{(0,s)} \rangle \underset{s\to\infty}{\sim} \frac{s^{1/2}}{\pi}, \tag{2-28}$$

and

$$\operatorname{Var} n_{(0,s)} \underset{s\to\infty}{\sim} \frac{2}{\pi^2\beta} \log s, \quad \operatorname{Var} n_{(s,\infty)} \underset{s\to-\infty}{\sim} \frac{1}{\pi^2\beta} \log |s|^{3/2},$$
$$\operatorname{Var} n_{(0,s)} \underset{s\to\infty}{\sim} \frac{1}{\pi^2\beta} \log s^{1/2}. \tag{2-29}$$

The results (2-28) are immediate consequences of the corresponding asymptotic density profiles (recall the second sentence below Conjecture 8), while (2-29) can be derived heuristically from knowledge of the asymptotic form of the two-point correlation function (see [Forrester 2010, paragraph below (14.87)]). In the case of $(\,\cdot\,) =$ bulk, (2-27), with the corresponding values of $\langle n_{(0,s)} \rangle$ and $\operatorname{Var} n_J$ as implied by (2-28) and (2-29), was derived in the context of the infinite log-gas formalism by Dyson [1995] and by Fogler and Shklovskii [1995].

3. Rigorous methods

3.1. ***Toeplitz/Hankel asymptotics.*** It is a fundamental result in random matrix theory (see, e.g., [Forrester 2010, § 9.1]) that in the scaled limit $(\,\cdot\,)$ equal to bulk, hard or soft, and $\beta = 2$ the probability of there being no eigenvalues in an interval J, may be written in terms of a determinant of a Fredholm integral operator

$$E_2^{(\,\cdot\,)}(0; J) = \det(1 - K_J^{(\,\cdot\,)}),$$

where $K_J^{(\,\cdot\,)}$) is the integral operator on the interval J with well known sine, Bessel and Airy kernels (see, e.g., [Forrester 2010] for the precise definitions). This is related to the fact that for $\beta = 2$ the gap probabilities can be written in terms of either Toeplitz or Hankel determinants. For example, the Toeplitz determinant of a function $f(\theta)$, integrable over the unit circle, is defined as

$$D_n(f) := \det\left(\frac{1}{2\pi}\int_0^{2\pi} e^{-i(j-k)\theta} f(\theta)\,d\theta\right)_{j,k=0}^{n-1}, \tag{3-1}$$

and one has the well known formula

$$E_2(0;(-\alpha,\alpha);\mathrm{CUE}_N) = D_N(f_\alpha), \quad f_\alpha = \begin{cases} 1, & \theta \in (\alpha, 2\pi-\alpha), \\ 0, & \text{otherwise.} \end{cases}$$

In particular $\lim_{n\to\infty} D_n(f_{2s/n}) = \det(\mathbb{I} - K_s^{\text{bulk}})$, allowing for a strategy whereby the $s \to \infty$ behavior can be extracted from the asymptotics of the Toeplitz determinant. On the other hand the Toeplitz determinant has a representation in terms of quantities associated with orthonormal polynomials $\phi_k(z) = \chi_k z^k + \cdots$ with weight $f(\theta)$ on the unit circle; explicitly $D_n(f) = \prod_{k=0}^{n-1} \chi_k^{-2}$. Krasovsky [2004] used a Riemann–Hilbert formulation to compute the large n form of $\frac{d}{d\mu} \ln D_n(f_\mu)$, uniformly in μ, providing both a proof and refinement of (2-8) in the case $\beta = 2$.

Theorem 11 [Krasovsky 2004; Ehrhardt 2006]. *We have*

$$\log E_2^{\text{bulk}}(0;(0,s)) = -\frac{(\pi s)^2}{8} - \tfrac{1}{4}\log\frac{\pi s}{2} + \tfrac{1}{12}\log 2 + 3\zeta'(-1) + O\left(\frac{1}{s}\right), \tag{3-2}$$

where $\zeta(z)$ is the Riemann zeta function.

We remark that up to the constant term this result, deduced by Dyson [Dyson 1976] using a scaling argument from known Toeplitz determinant asymptotics, was first rigorously proved by Deift, Its and Zhou [Deift et al. 1997]; also the proof of Ehrhardt [2006] is operator theoretic, and does not make use of orthogonal polynomials.

Analogous strategies can be used to analyze the hard and soft edges for $\beta = 2$, giving the following results, proving and extending (2-12) and (2-14), respectively.

Theorem 12. *We have*

$$\log E_2^{\text{soft}}(0;(s,\infty)) = -\tfrac{1}{12}|s|^3 - \tfrac{1}{8}\log|s| + \tfrac{1}{24}\ln 2 + \zeta'(-1) + O\big(|s|^{-\frac{3}{2}}\big) \tag{3-3}$$

for $s \to -\infty$ (see [Deift et al. 2008]), *and*

$$\log E_2^{\text{hard}}(0;(0,s);a) = -\frac{s}{4} + a\sqrt{s} - \frac{a^2}{4}\log s + \log\frac{G(1+a)}{(2\pi)^{a/2}} + O\big(s^{-\frac{1}{2}}\big) \tag{3-4}$$

for $s \to \infty$ (see [Deift et al. 2011]), *where $G(x)$ denotes the Barnes G-function.*

An alternative proof of (3-3) has been given by Baik, Buckingham and DiFranco [Baik et al. 2008], using the Painlevé form of $E_2^{\text{soft}}(0;(s,\infty))$ [Tracy and Widom 1994a]. This method carries over to the cases $\beta = 1$ and 4, and in [Baik et al. 2008] the expansion (2-15) with

$$\begin{aligned} \log \tau_1^{\text{soft}} &= -\frac{11 \log 2}{48} + \frac{\zeta'(-1)}{2}, \\ \log \tau_4^{\text{soft}} &= -\frac{37 \log 2}{48} + \frac{\zeta'(-1)}{2}, \end{aligned} \tag{3-5}$$

was obtained. These confirm the values implied by (2-20).

With regards to (3-4), as noted above, for $a \in \mathbb{Z}_{\geq 0}$ it was first proved by Forrester [1994]. More recently a proof of (3-4) valid for $|a| < 1$ was given by Ehrhardt [2010]. Furthermore, let the next order (constant) term in the exponent of (2-12) be included by adding $\log \tau_{a,\beta}^{\text{hard}}$. We read off from (3-4) that $\tau_{a,2}^{\text{hard}} = G(1+a)/(2\pi)^{a/2}$. For $a \in \mathbb{Z}_{\geq 0}$ a multiple integral form for E_1^{hard} [Forrester and Witte 2002], and an identity [Forrester and Rains 2001] relating E_4^{hard} to E_2^{hard} and E_1^{hard} for general $a > -1$ tells us that

$$\begin{aligned} \tau_{a,1} &= 2^{-a(a+1/2)} \frac{G(3/2)G(2a+2)}{G(a+3/2)G(a+2)}, \\ \tau_{a+1,4} &= 2^{-a(a+1)/4-1} \frac{\tau_{a,2}}{\tau_{(a-1)/2,1}}. \end{aligned} \tag{3-6}$$

In the case of bulk scaling, include a constant term by adding $\log \tau_\beta^{\text{bulk}}$ to the exponent of (2-9) with s replaced by s/π (thus the bulk density is now $1/\pi$). It follows from (3-2) that $\log \tau_2^{\text{bulk}} = \frac{1}{3}\log 2 + 3\zeta'(-1)$. And interrelations between the bulk gap probability for $\beta = 1$ and 4 with $\beta = 2$ quantities give that [Basor et al. 1992]

$$\tau_1^{\text{bulk}} = 2^{5/12} e^{(3/2)\zeta'(-1)}, \quad \tau_4^{\text{bulk}} = 2^{-29/24} e^{(3/2)\zeta'(-1)}. \tag{3-7}$$

We observe that (3-7) is consistent with a relation analogous to (2-20).

Conjecture 13. *Let $\rho_2(1,\tau)$ denote the Stirling modular form. We have*

$$\log \tau_{\beta/2}^{\text{bulk}} = \bigl(3 - \tfrac{4}{3}(\beta/2 + 2/\beta)\bigr)\log 2 + 3\bigl(\tfrac{1}{2}\log 2\pi - \log \rho_2(1, 2/\beta)\bigr), \tag{3-8}$$

and consequently

$$\begin{aligned} \log \frac{\tau_{\beta/2}^{\text{bulk}}}{\tau_{2/\beta}^{\text{bulk}}} &= -3 \log \frac{\rho_2(1, 2/\beta)}{\rho_2(1, \beta/2)}, \\ E_\beta^{\text{bulk}}(0; (0, s/\pi)) &\underset{s\to\infty}{\sim} \Bigl(\frac{2}{\beta}\Bigr)^{3/2} \widetilde{E_{4/\beta}^{\text{bulk}}}\Bigl(0; \Bigl(0, \frac{\beta}{2} s/\pi\Bigr)\Bigr), \end{aligned} \tag{3-9}$$

where $\widetilde{E_\beta^{\text{bulk}}}$ refers to the right-hand side of (2-9) *with s replaced by $-s$ in the second term.*

3.2. ***Stochastic differential equations.*** The Gaussian and Laguerre β-ensemble, defined as eigenvalue PDFs below (1-1), admit realizations as real symmetric tridiagonal matrices [Dumitriu and Edelman 2002]. In the scaled $N \to \infty$ limit, this in turn leads to explicit characterization of gap probabilities in terms of stochastic differential equations. The first result of this type was done for the soft edge, by Ramirez, Rider and Valkó [Ramírez et al. 2011a]. With N fixed, it relies on expressing the number of eigenvalues greater than μ as the number of sign changes of the shooting vector for the tridiagonal matrix. Similarly at the hard edge [Ramírez and Rider 2009]. In the bulk, the shooting eigenvector must be parametrized in terms of the corresponding Prüfer phase [Killip and Stoiciu 2009; Valkó and Virág 2009]. The following results are obtained.

Proposition 14. *Let b_t denote standard Brownian motion. At the soft edge, define a diffusion by the Ito process by* (see [Ramírez et al. 2011a]):

$$dp(t) = \frac{2}{\sqrt{\beta}}\, db_t + (\lambda + t - p^2(t))\, dt, \qquad p(0) = \infty;$$

at the hard edge with parameter $\beta(a+1)/2 - 1$ by (see [Ramírez et al. 2011b])

$$dp(t) = db_t + \left(\tfrac{1}{4}\beta\left(a+\tfrac{1}{2}\right) - \tfrac{1}{2}\beta\sqrt{\lambda}e^{-\beta t/8}\cosh p(t)\right) dt, \qquad p(0) = \infty;$$

and in the bulk (see [Valkó and Virág 2010]) *by*

$$dp(t) = db_t + \left(\tfrac{1}{2}\tanh p(t) - \tfrac{1}{8}\beta\lambda e^{-\beta t/4}\cosh p(t)\right) dt, \qquad p(0) = \infty.$$

Let $J = (0, s/2\pi)$ for $(\,\cdot\,) = bulk$, $J = (0, s)$ for $(\,\cdot\,) = hard$, and $J = (s, \infty)$ for $(\,\cdot\,) = soft$. We have

$$E_\beta^{(\,\cdot\,)}(0; J) = \Pr\big(p(t) > -\infty \text{ for all } t \in \mathbb{R}^+ \cup \{\infty\}\big). \tag{3-10}$$

The utility of these characterizations for the purpose of asymptotics is that, via the Cameron–Martin–Girsanov formula, they allow (3-10) to be rewritten as the expectation of a functional of a transformed stochastic process. In contrast to (3-10), this functional allows for a systematic, rigorous $s \to \infty$ asymptotic analysis resulting in a proof of (2-9) — giving in the process the correct form of the general $\beta > 0$, $\log s$ term, for the first time — and a proof of (2-12) for general $\beta > 0$ and $a > -1$. At the soft edge only the leading asymptotic form (2-14) has been proved using this approach [Ramírez et al. 2011a].

For the large N limit of the circular β-ensemble, the Prüfer phase has been used to prove the analogue of the Gaussian fluctuation formula (2-27), namely

$$E_\beta(n, (-\alpha, \alpha); \mathrm{C}\beta\mathrm{E}_N) \sim (1/(2\pi \operatorname{Var} n_{(-\alpha,\alpha)})) \exp\left(-\frac{(n - N\alpha/\pi)^2}{2\operatorname{Var} n_{(-\alpha,\alpha)}}\right),$$

where $\operatorname{Var} n_{(-\alpha,\alpha)} \sim \dfrac{1}{\pi^2\beta} \log N$ [Killip 2008].

3.3. ***Fredholm determinant/eigenvalue forms for $E_\beta^{(\cdot)}(n,J)/E_\beta^{(\cdot)}(0,J)$.*** With $(\cdot)$ denoting bulk, soft or hard, let $E_\beta^{(\cdot)}(J;\xi)$ be the generating function for $\{E_\beta^{(\cdot)}(n;J)\}$, so that

$$E_\beta^{(\cdot)}(J;\xi) = \sum_{n=0}^{\infty} (1-\xi)^n E_\beta^{(\cdot)}(n;J). \tag{3-11}$$

Generalizing the Fredholm determinant expressions for $E_\beta^{(\cdot)}(0;J)$ from Section 3.1, one has that for $\beta = 2$

$$E_2^{(\cdot)}(J;\xi) = \det(1-\xi K_J^{(\cdot)}) = \prod_{l=0}^{\infty}(1-\xi\lambda_l), \tag{3-12}$$

where $1 > \lambda_0 > \lambda_1 > \lambda_2 > \cdots > 0$ are the eigenvalues of $K_J^{(\cdot)}$. Consequently

$$\frac{E_2^{(\cdot)}(n;J)}{E_2^{(\cdot)}(0;J)} = \sum_{0\le j_1<\cdots<j_n} \frac{\lambda_{j_1}\ldots\lambda_{j_n}}{(1-\lambda_{j_1})\ldots(1-\lambda_{j_n})}. \tag{3-13}$$

It has been known since the work of Gaudin [1961] that associated with $K_{(0,s)}^{\text{bulk}}$ is a commuting differential operator. Furthermore, the work of Fuchs [1964] uses this, together with a WKB asymptotic analysis, to deduce the $s\to\infty$ asymptotic form of λ_j (j fixed). It was noted by Tracy and Widom [1993] that the latter implies the term with $(j_1, j_2, \ldots, j_n) = (0, 1, \ldots, n-1)$ dominates as $t\to\infty$. These authors carried out a similar analysis in the soft and hard edge cases [Tracy and Widom 1994a; 1994b], so arriving at the following result (stated as Proposition 9.6.6 in [Forrester 2010]).

Proposition 15. *Let $G(x)$ denote the Barnes G-function. For n fixed*,

$$\begin{aligned}
\frac{E_2^{\text{bulk}}(n;(0,s))}{E_2^{\text{bulk}}(0;(0,s))} &\underset{s\to\infty}{\sim} G(n+1)\pi^{-n/2}2^{-n^2-n}(\pi s)^{-n^2/2}e^{n\pi s},\\
\frac{E_2^{\text{soft}}(n;(0,s))}{E_2^{\text{soft}}(0;(0,s))} &\underset{s\to\infty}{\sim} \frac{G(n+1)}{\pi^{n/2}2^{(5n^2+n)/2}}(-s/2)^{-3n^2/4}\exp\left(\frac{8n}{3}\left(-\frac{s}{2}\right)^{3/2}\right),\\
\frac{E_2^{\text{hard}}(n;(0,s))}{E_2^{\text{hard}}(0;(0,s))} &\underset{s\to\infty}{\sim} \frac{G(a+n+1)G(n+1)}{G(a+1)}\pi^{-n}2^{-n(2n+2a+1)}s^{-n^2/2-an/2}e^{2n\sqrt{s}}.
\end{aligned} \tag{3-14}$$

In [Forrester 2010, § 9.6.2], as $t\to\infty$, $E_1^{\text{bulk}}(n;t)$ and $E_4^{\text{bulk}}(n;t)$ are related to $E_2^{\text{hard}}(\cdot\,;\cdot)$ for particular choices of the parameters. The asymptotics of the latter are known as noted in the above proposition, allowing us to extend the first result in (3-14) to $\beta = 1$ and 4 [Forrester 2010, Equations (9.100) and (9.102)].

Proposition 16. *For n fixed and* $\beta = 1$ *and* 4 *we have*

$$\frac{E_\beta^{\mathrm{bulk}}(n;(0,s))}{E_\beta^{\mathrm{bulk}}(0;(0,s))} = c_{\beta,n}\frac{e^{\beta n\pi s/2}}{(\pi s)^{\beta n^2/4+(\beta/2-1)n/2}}\Big(1+O\Big(\frac{1}{s}\Big)\Big), \tag{3-15}$$

where

$$c_{1,n} = \frac{G(n/2+1/2)G(n/2+1)}{G(1/2)}\pi^{-n/2}2^{-n(n+1)/4},$$

$$c_{4,n} = \frac{G(n+3/2)G(n+1)}{G(3/2)}\pi^{-n}2^{-2n(n+1)}.$$

According to the first asymptotic formula in (3-14), (3-15) is, for a specific $c_{2,n}$, valid too for $\beta = 2$. Furthermore the functional form (3-15) for general $\beta > 0$ coincides with the log-gas prediction (2-26), and thus validates the latter for $\beta = 1, 2$ and 4, and furthermore extends it by the evaluation of $c_{\beta,n}$.

We would like to extend Proposition 16 to the soft and hard edge cases. For this, let

$$V_{\tilde J}^{(\,\cdot\,)} \quad \text{for } (\,\cdot\,) = \text{soft, hard}$$

and $\tilde J = (0,\infty)$, $(0,1)$, respectively, denote the integral operators on $\tilde J$, dependent on a parameter s, with kernels $\mathrm{Ai}(x+y+s)$ and $\frac{\sqrt{s}}{2}J_a(\sqrt{sxy})$. Write

$$E_\pm^{(\,\cdot\,)}(\xi;J) = \det\big(\mathbb{I} \mp \sqrt{\xi}\,V_{\tilde J}^{(\,\cdot\,)}\big),$$

and define

$$E_\pm^{(\,\cdot\,)}(n;J) := \frac{(-1)^n}{n!}\frac{\partial^n}{\partial\xi^n}E_\pm^{(\,\cdot\,)}(\xi;J)\Big|_{\xi=1}.$$

Results contained in [Forrester 2006], and further refined in [Bornemann 2010b], tell us that for $s \to \infty$

$$\begin{aligned} E_1^{\mathrm{soft}}(2k;(s,\infty)) &= E_+^{\mathrm{soft}}(k;(s,\infty)) + \cdots, \\ E_1^{\mathrm{soft}}(2k+1;(s,\infty)) &= \tfrac{1}{2}E_-^{\mathrm{soft}}(;(s,\infty)) + \cdots, \\ E_4^{\mathrm{soft}}(k;(s,\infty)) &= E_1^{\mathrm{soft}}(2k+1;(2^{2/3}s,\infty)) + \cdots. \end{aligned} \tag{3-16}$$

Here terms not written on the right-hand side are exponentially smaller (in s) than the given term. To proceed further requires a property of the eigenvalues of $V_{\tilde J}^{(\,\cdot\,)}$ which although supported by numerical computations, to our knowledge is yet to be proven.

Conjecture 17. *Let* $\upsilon_j^{(\,\cdot\,)}$ $(j = 0, 1, 2, \ldots)$ *denote the eigenvalues of* $V_{\tilde J}^{(\,\cdot\,)}$, *ordered so that*

$$|\upsilon_0^{(\,\cdot\,)}| < |\upsilon_1^{(\,\cdot\,)}| < |\upsilon_2^{(\,\cdot\,)}| < \cdots.$$

Then $\upsilon_{2j}^{(\,\cdot\,)} > 0$ *while* $\upsilon_{2j+1}^{(\,\cdot\,)} < 0$ *for each* $j = 0, 1, 2, \ldots$.

It is well known, and easy to verify (see, e.g., [Forrester 2010, § 9.6.1]) that $(\upsilon_j^{(\cdot)})^2 = \lambda_j^{(\cdot)}$, where $\{\lambda_j^{(\cdot)}\}$ are the eigenvalues of $K_J^{(\cdot)}$. This fact, together with Conjecture 17 and the analogue of (3-13) relating to $E_\pm^{\rm soft}(k;\,(s,\infty))$, tells us that for $s \to \infty$

$$\begin{aligned} \frac{E_1^{\rm soft}(2k;\,(s,\infty))}{E_1^{\rm soft}(0;\,(s,\infty))} &= \frac{1}{(1-\lambda_0^{\rm s})(1-\lambda_2^{\rm s})\dots(1-\lambda_{2k-2}^{\rm s})} + \cdots, \\ \frac{E_1^{\rm soft}(2k+1;\,(s,\infty))}{E_4^{\rm soft}(0;\,(s/2^{2/3},\infty))} &= \frac{1}{(1-\lambda_1^{\rm s})(1-\lambda_3^{\rm s})\dots(1-\lambda_{2k-1}^{\rm s})} + \cdots. \end{aligned} \tag{3-17}$$

Knowledge of the explicit asymptotic form of λ_j^s from [Tracy and Widom 1994a], together with the asymptotic form of $E_1^{\rm soft}(0;\,(s,\infty))/E_4^{\rm soft}(0;\,(s/2^{2/3},\infty))$ implied by (2-15) and (3-5) then allows us to extend the second result of (3-14) to $\beta = 1$ and 4.

Proposition 18 (under the assumption of Conjecture 17). *We have*

$$\frac{E_1^{\rm soft}(n;\,(s,\infty))}{E_1^{\rm hard}(0;\,(s,\infty)} \underset{s\to-\infty}{\sim} \frac{G(n/2+1/2)G(n/2+1)}{\pi^{n/2}G(1/2)} 2^{-\frac{5}{8}n^2+\frac{1}{8}n}(-s)^{-\frac{3}{8}n^2+\frac{3}{8}n} \times \exp\Big(\frac{4n}{3}\Big(-\frac{s}{2}\Big)^{3/2}\Big),$$

$$\frac{E_4^{\rm soft}(n;\,(s/2^{2/3},\infty))}{E_4^{\rm soft}(0;\,(s/2^{2/3},\infty))} \underset{s\to-\infty}{\sim} \sqrt{2}e^{-\frac{\sqrt{2}}{3}(-s)^{3/2}} \frac{E_1^{\rm soft}(2n+1;\,(s,\infty))}{E_1^{\rm soft}(0;\,(s,\infty))}. \tag{3-18}$$

At the hard edge, formulas structurally identical to (3-16) hold [Forrester 2006; Bornemann 2010b], with the important qualification that the additional label need to specify the hard edge gap probabilities is $(a-1)/2$ on the left-hand side of the first two equations, and $a+1$ on the left-hand side of the third equation; on the right-hand sides it is a, a and $a-1$, respectively, and in the third equation s is scaled by 4 instead of $2^{2/3}$. The analogue of (3-17) then allows the analogue of Proposition 18 to be deduced.

Proposition 19 (under the assumption of Conjecture 17). *We have*

$$\frac{E_1^{\rm hard}(n;\,(0,s);\,(a-1)/2)}{E_1^{\rm hard}(0;\,(0,s);\,(a-1)/2)} \underset{s\to\infty}{\sim} 2^{-n(n-1+a)/2}(2\pi)^{-n} \times \prod_{p=1}^{2} \frac{G((n+p)/2)G((n+p+a)/2)}{G(p/2)G((p+a)/2)} s^{-(n^2+n(a-1))/4} e^{n\sqrt{s}},$$

$$\frac{E_4^{\rm hard}(n;\,(0,s/4);\,a+1)}{E_4^{\rm hard}(0;\,(0,s/4);\,a+1)} \underset{s\to\infty}{\sim} e^{-\sqrt{s}} s^{a/4} \frac{2^{(a+1)/2}(2\pi)^{1/2}}{\Gamma((a+1)/2)} \times \frac{E_1^{\rm hard}(2n+1;\,(0,s);\,(a-1)/2)}{E_1^{\rm hard}(0;\,(0,s);\,(a-1)/2)}.$$

As a final point in this subsection, we remark that the Gaussian fluctuation formula (2-27) can be proved for $\beta = 2$, using only the determinantal structure (3-12) and the fact that $\operatorname{Var} n_J \to \infty$ [Costin and Lebowitz 1995; Soshnikov 2000].

3.4. ***Hard edge: generalized hypergeometric functions.*** In the case that $a \in \mathbb{Z}^+$ and general $\beta > 0$, the hard edge gap probability $E_\beta^{\text{hard}}(0; (0,s); a)$ permits evaluation in terms of a generalized hypergeometric function based on Jack polynomials $P_\kappa^{(\alpha)}(z_1, \dots, z_N)$. The latter are labeled by a partition

$$\kappa_1 \ge \kappa_2 \ge \cdots \ge \kappa_N \ge 0$$

of nonnegative integers, and depend on the parameter α. For $\alpha = 1$ they are the Schur polynomials, while for $\alpha = 2$ they are the zonal polynomials of mathematical statistics; their precise definition can be found in, e.g., [Forrester 2010, § 12.6]. Defining $C_\kappa^{(a)}(z_1, \dots, z_N)$ as proportional to $P_\kappa^{(\alpha)}(z_1, \dots, z_N)$ with a specific proportionality depending on α and κ [Forrester 2010, Equation (13.1)], and the generalized Pochhammer symbol $[u]_\kappa^{(a)}$ [Forrester 2010, Equation (12.46)], the generalized hypergeometric function ${}_pF_q^{(\alpha)}$ is specified by (see, e.g., [Forrester 2010, § 13.1])

$$\begin{aligned} {}_pF_q^{(\alpha)}&(a_1, \dots, a_p; b_1, \dots, b_q; x_1, \dots, x_m) \\ &:= \sum_\kappa \frac{1}{|\kappa|!} \frac{[a_1]_\kappa^{(\alpha)} \dots [a_p]_\kappa^{(\alpha)}}{[b_1]_\kappa^{(\alpha)} \dots [b_q]_\kappa^{(\alpha)}} C_\kappa^{(\alpha)}(x_1, \dots, x_m). \end{aligned} \tag{3-19}$$

Like their classical counterpart, these exhibit the confluence property

$$\begin{aligned} \lim_{a_p \to \infty} {}_pF_q^{(\alpha)}&\left(a_1, \dots, a_p; b_1, \dots, b_q; \frac{x_1}{a_p}, \dots, \frac{x_m}{a_p}\right) \\ &= {}_{p-1}F_q^{(\alpha)}(a_1, \dots, a_{p-1}; b_1, \dots, b_q; x_1, \dots, x_m). \end{aligned}$$

Using this in the case $p = q = 1$, together with an integral expression for ${}_1F_1$ [Forrester 2010, § 13.2.5] we can readily express the conditional gap probability $E_\beta^{\text{hard}}(n; (0,s); a)$ for $a, \beta \in \mathbb{Z}_{\ge 0}$ in terms of the generalized hypergeometric function ${}_0F_1^{\beta/2}$, extending the $n = 0$ result of [Forrester 1994].

Proposition 20. *Let $\beta a/2, \beta \in \mathbb{Z}_{\ge 0}$. We have*

$$\begin{aligned} E_\beta^{\text{hard}}(n; (0,s); \beta a/2) &= A_\beta(n,a) s^{n+(\beta/2)n(n+a-1)} e^{-\beta s/8} \\ &\times \int_0^1 dy_1 \dots \int_0^1 dy_n \prod_{j=1}^n (1-y_j)^{\beta a/2} \prod_{1 \le j < k \le n} |y_k - y_j|^\beta \\ &\times {}_0F_1^{(\beta/2)}\big(\_; a+2n; (s/4)^{\beta a/2}, (sy_1/4)^\beta, \dots, (sy_n/4)^\beta\big), \end{aligned} \tag{3-20}$$

where

$$A_\beta(n,a) = \frac{2^{-2n}}{n!}\left(\frac{\beta}{2}\right)^n \left(\frac{\beta}{4}\right)^{n(a+n-1)\beta} \frac{(\Gamma(1+\beta/2))^n}{\prod_{j=0}^{2n-1}\Gamma(a\beta/2+1+j\beta/2)}, \qquad (3\text{-}21)$$

and in the argument of ${}_0F_1^{\beta/2}$ *the notation* $(u)^r$ *means* u *repeated* r *times. Furthermore, in the case* $n=0$, *this remains valid for general* $\beta>0$.

An integral representation of ${}_0F_1^{(\beta/2)}$ allows for the rigorous determination of the $x\to\infty$ asymptotic expansion of ${}_0F_1^{(\beta/2)}(\_;c+2(m-1)/\beta;(x)^m)$ for $c, 2/\beta \in \mathbb{Z}^+$ [Forrester 1994], implying the corresponding asymptotic form of $E_\beta^{\text{hard}}(0;(0,s))$.

Proposition 21 [Forrester 1994]. *Let* $2/\beta\in\mathbb{Z}^+$, *and* $a\beta/2=m\in Z_{\geq 0}$. *For* $s\to\infty$ *we have*

$$E_\beta^{\text{hard}}(0;(0,s);m) = \tau_{m,\beta}\left(\frac{1}{s}\right)^{m(m+1)/2\beta-m/4} e^{-\beta s/8+ms^{\frac{1}{2}}}\left(1+O\left(\frac{1}{s^{1/2}}\right)\right), \quad (3\text{-}22)$$

where

$$\tau_{m,\beta}^{\text{hard}} = 2^{(2/\beta-1)m}\left(\frac{1}{2\pi}\right)^{m/2}\prod_{j=1}^{m}\Gamma(2j/\beta). \qquad (3\text{-}23)$$

We see that (3-22) is in agreement with the log-gas prediction (2-12) for general $a>-1,\ \beta>0$, and furthermore gives the explicit form of the constant in the asymptotic expansion (to use (3-23) for $m\notin\mathbb{Z}_{\geq 0}$ and check for example (3-6) requires an appropriate rewrite of the product using (2-17)).

In the case $\beta=4$ of ${}_0F_1^{\beta/2}$, an integral representation not available for general β shows that, for $s\to\infty$ and $y_1,\dots,y_n\approx 1$ (see [Muirhead 1978]),

$$\begin{aligned}&{}_0F_1^{(\beta/2)}\big(\_;c;(s/4)^{\beta a/2},(sy_1/4)^\beta,\dots,(sy_n/4)^\beta\big)\\ &\quad = {}_0F_1^{(\beta/2)}(\_;c;(s/4)^{\beta(a+2n)/2})e^{\beta\sqrt{s}\sum_{j=1}^n(1-y_j)/2}\left(1+O\left(\frac{1}{s^{1/2}}\right)\right). \qquad (3\text{-}24)\end{aligned}$$

This, substituted in (3-20), implies, as a conjecture, the extension of the asymptotic formula (2-26) to include the constant term.

Conjecture 22. *For* $\beta n\in\mathbb{Z}_{\geq 0}$, *let*

$$\tau_{\beta a/2,\beta}^{\text{hard}}(n) = \frac{2^{-(a+n)\beta n}}{n!}\left(\frac{\beta}{2}\right)^{n(a+n-1)\beta/2}\prod_{j=1}^{\beta n}\frac{\Gamma(a+2j/\beta)}{(2\pi)^{1/2}}\frac{\prod_{j=0}^{n-1}\Gamma(1+(j+1)\beta/2)}{\prod_{j=n}^{2n-1}\Gamma(1+(j+a)\beta/2)}.$$

For $s \to \infty$ we have

$$\frac{E_\beta^{\rm hard}(n;(0,s);\beta a/2)}{E_\beta^{\rm hard}(0;(0,s);\beta a/2)} = \tau^{\rm hard}_{\beta a/2,\beta}(n)\exp\left(-\beta\left\{-\sqrt{s}n+\left(\frac{n^2}{2}+\frac{na}{2}\right)\log s^{1/2}\right\}\right)\left(1+O\left(\frac{1}{s^{1/2}}\right)\right). \tag{3-25}$$

We can check that (3-25) is consistent with the results of Proposition 19.

3.5. ***Approach to unity of $E_\beta^{(\cdot)}(0;J)$ for $|J| \to 0$.*** Generally the gap probability is given in terms of the k-point correlation functions $\{\rho_{(k)}^{(\cdot)}\}$ according to

$$E_\beta^{(\cdot)}(0;J) = 1 - \int_J \rho_{(1)}^{(\cdot)}(x)\,dx + \frac{1}{2!}\int_J\int_J \rho_{(2)}^{(\cdot)}(x,y)\,dx\,dy - \cdots.$$

Thus the leading $|J| \to 0$ asymptotic form of $E_\beta^{(\cdot)}(0;J)$ is determined by the asymptotic form of $\rho_{(1)}^{\rm hard}(x)$ for $x \to 0$, $\rho_{(1)}^{\rm soft}(x)$ for $x \to \infty$ and $\rho_{(2)}^{\rm bulk}(x,y)$ for $x, y \to 0$ (for $(\cdot)$ = bulk, $\rho_{(1)}(x) = 1$ and so gives no distinguishing information). The calculation of the first and third is elementary [Forrester 1992; 1994], while direct calculation of $\rho_{(1)}^{\rm soft}(x)$ is only known for $\beta = 1$, and β even [Desrosiers and Forrester 2006]. Collecting these together, we have the following result.

Proposition 23. *Let*

$$A_{a,\beta} = 4^{-(a+1)}(\beta/2)^{2a+1}\frac{\Gamma(1+\beta/2)}{\Gamma(1+a)\Gamma(1+a+\beta/2)},$$

$$B_\beta = (\pi\beta)^\beta \frac{(\Gamma(\beta/2+1))^3}{\Gamma(\beta+1)\Gamma(3\beta/2+1)}.$$

For $t \to 0$,

$$\begin{aligned} E_\beta^{\rm hard}(0;(0,t);a) &= 1 - A_{a,\beta}\int_0^t s^a\,ds + O(t^{a+2}), \\ E_\beta^{\rm bulk}(0;(0,t)) &= 1 - t + \tfrac{1}{2}B_\beta \int_0^t\int_0^t (s_1-s_2)^\beta\,ds_1\,ds_2 + O(t^{\beta+3}), \end{aligned} \tag{3-26}$$

while for $t \to \infty$,

$$E_\beta^{\rm soft}(0;(t,\infty)) = 1 - \frac{\Gamma(1+\beta/2)}{\pi(4\beta)^{\beta/2}}\int_t^\infty \frac{e^{-2\beta X^{3/2}/3}}{X^{3\beta/4-1/2}}\,dX + O\left(\int_t^\infty \frac{e^{-2\beta X^{3/2}/3}}{X^{3\beta/4+1}}\,dX\right). \tag{3-27}$$

Two distinct derivations of (3-27) for general $\beta > 0$ are known, both involving use of a nonrigorous double scaling limit [Forrester 2012; Borot and Nadal 2011]. In [Dumaz and Virág 2011], the stochastic differential equation characterization (recall Section 3.2) is used to give a rigorous proof for general $\beta > 0$, but without determining the prefactor of the integral.

4. Other aspects

4.1. *Numerical results.* Bornemann [2010a; 2010b] has given a detailed study of the numerical analysis relating to the precise numerical evaluation of spacing distributions for $\beta = 1, 2$ and 4, working from the Fredholm determinant forms. As an end product he has provided a suite of Matlab programs implementing the theoretical procedures. The implementation in Matlab, with the arithmetic done in the hardware, means that the tails of the spacing distributions cannot be computed: their numerical values written as decimals are typically smaller than 10^{-15}, and so double precision arithmetic typically truncates significant nonzero digits, leading to unreliable results. But with there being numerous exact and conjectured results relating to the asymptotics of spacing distributions, there is much interest in implementing the theory of [Bornemann 2010a; 2010b] using an arbitrary precision package. As a start, we have done this for the Fredholm determinant form for $E_2^{\text{bulk}}(0; (0, s))$ (in fact we have modified the procedure of [Bornemann 2010a; 2010b] by using instead of Gauss–Legendre or Clenshaw–Curtis quadrature rules, the tanh-sinh quadrature rule (see, e.g., [Ye 2006])). As a result we are able to tabulate

$$r(s) = \frac{E_2^{\text{b},as}(0; (0, s))}{E_2^{\text{bulk}}(0; (0, s))}, \tag{4-1}$$

where $E_2^{\text{b},as}(0; (0, s))$ is the asymptotic form of $E_2^{\text{bulk}}(0; (0, s))$ as given by (3-2), extended to the next two terms: $1/(8(\pi s)^2) + 5/(8(\pi s)^4)$ (these follow from the Painlevé transcendent characterization of $E_2^{\text{bulk}}(0; (0, s))$ (see, e.g., [Forrester 2010, § 9.6.7])). The values in Table 1 clearly illustrate the accuracy of the asymptotic expansion, even for relatively small values of s.

4.2. *Diluted spectra.* For a general one-dimensional point process, the generating function (3-11) can also be interpreted as the probability that there are no eigenvalues in the interval J, given that each eigenvalue has independently been deleted with probability $(1 - \xi)$. In this setting the $|J| \to \infty$ asymptotics can readily be deduced, by making use of a heuristic analysis based on (2-27) [Bohigas and Pato 2004].

Conjecture 24. *For* $0 < \xi < 1$ *we have*

$$E_\beta^{(\,\cdot\,)}(J; \xi) \underset{|J|\to\infty}{\sim} e^{\langle n_J \rangle \log(1-\xi)}, \tag{4-2}$$

where $\langle n_J \rangle$ *is given by* (2-28) *for* $(\,\cdot\,) =$ *bulk*, *soft and hard.*

s	$r(s)$
1	1.0046735914726577
2	0.9998383226940526
3	0.9999753765440204
4	0.9999961026171116
5	0.9999991096965057
6	0.9999997235559452
7	0.9999998946139279
8	0.9999999537746553
9	0.9999999775313906
10	0.9999999881794448

Table 1. Tabulation of the ratio of the asymptotic to exact bulk gap probability for $\beta = 2$.

We see from (2-28) and (2-8), (2-12), (2-14) that as a function of s the decay exhibited by (4-2) is proportional to the square root of the leading decay of $E_\beta^{(\cdot)}$. A method to prove (4-2) for $\beta = 2$, making use of (3-12), has been given in [Pastur and Shcherbina 2011]. Alternatively, for this β, (4-2) can be verified by using known asymptotics of the Painlevé transcendent evaluations, as done for $(\cdot) =$ soft in [Bohigas et al. 2009].

An interesting feature of the asymptotic expansion of the relevant Painlevé transcendents with $0 < \xi < 1$ is that they contain oscillatory terms, in contrast to their asymptotic expansion with $\xi = 1$. It is indeed the case that oscillations can clearly be seen in plots of $(d/ds)E_2^{\text{soft}}((s,\infty);\xi)$ with $0 < \xi < 1$ [Bohigas et al. 2009]. Dyson [1995] has combined Coulomb gas and Painlevé theory to deduce the asymptotic form $E_2^{\text{bulk}}((0,s);\xi)$ when $\xi \to 1$ and simultaneously $s \to \infty$, which is shown to involve an elliptic theta function; for fixed ξ the asymptotic expansion of the relevant Painlevé transcendent [McCoy and Tang 1986] involves only trigonometric functions.

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