KPZ scaling theory and the semidiscrete directed polymer model

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We explain how the claims of KPZ scaling theory are confirmed by a recent proof of Borodin and Corwin on the asymptotics of the semidiscrete directed polymer.

1. Introduction

The Kardar–Parisi–Zhang (KPZ) equation [1986] is a stochastic partial differential equation modeling surface growth and, more generally, the motion of an interface bordering a stable against a metastable phase. Scaling theory is an educated guess on the nonuniversal coefficients in the asymptotics for models in the KPZ universality class. Scaling theory has been developed in a landmark contribution by Krug, Meakin and Halpin-Healy [Krug et al. 1992]. The purpose of our article is to explain how to apply scaling theory to the semidiscrete directed polymer. This model has been discussed in depth at the 2010 random matrix workshop at MSRI and, so-to-speak as a spin-off, Borodin and Corwin [2014] developed the beautiful theory of Macdonald processes, which provides the tools for an asymptotic analysis of the semidiscrete directed polymer. As we will establish, scaling theory is consistent with the results of Borodin and Corwin, thereby providing a highly nonobvious control check.

To place the issue in focus, let me start with a simple example. Assume as given the stationary sequence X_j, $j \in \mathbb{Z}$, of mean zero random variables and let us consider the partial sums

$$S_n = \sum_{j=1}^{n} X_j. \tag{1-1}$$

As well studied, it is fairly common that $S_n/\sqrt{n}$ converges to a Gaussian as $n \to \infty$, i.e.,

$$\lim_{n\to\infty} \mathbb{P}\big(S_n \leq \sqrt{D}\sqrt{n}s\big) = F_{\mathrm{G}}(s), \tag{1-2}$$

where F_{G} is the distribution function of a unit Gaussian random variable. Here F_{G} is the universal object, while the coefficient $D > 0$ depends on the law $\mathbb{P}$ and

is in this sense model dependent, resp. nonuniversal. However, using stationarity, D is readily guessed as

$$D = \sum_{j=-\infty}^{\infty} \mathbb{E}(X_0 X_j). \qquad (1\text{-}3)$$

The KPZ class deals with strongly dependent random variables, for which partial sums are of size $n^{1/3}$ rather than $n^{1/2}$. F_G is to be substituted by the GUE Tracy–Widom distribution function, F_{GUE}, which first appeared in the context of the largest eigenvalue of a GUE random matrix [Forrester 1993; Tracy and Widom 1994]. F_{GUE} is defined through a Fredholm determinant as

$$F_{GUE}(s) = \det(1 - P_s K_{Ai} P_s). \qquad (1\text{-}4)$$

Here K_{Ai} is the Airy kernel,

$$K_{Ai}(x, y) = \int_0^\infty d\lambda \; Ai(x + \lambda) \, Ai(y + \lambda), \qquad (1\text{-}5)$$

with Ai the Airy function, and P_x projects onto the interval $[x, \infty)$. For each x, $P_x K_{Ai} P_x$ is a trace class operator in $L^2(\mathbb{R})$, hence (1-4) is well-defined. To determine the scale coefficient is less obvious than in the example above, but will be explained in due course. Let me stress that scaling theory is crucial for the proper statistical analysis of either physics [Takeuchi and Sano 2010; Takeuchi et al. 2011] or computer experiments [Alves et al. 2011]. Without this input, the comparison with theoretical results would be considerably less reliable.

Our paper is divided into two parts. We first explain scaling theory in the context of a specific class of growth models. In the second part the theory is applied to the semidiscrete directed polymer model. The convergence to the GUE Tracy–Widom distribution is established in [Borodin and Corwin 2014], of course including an expression for the nonuniversal scale coefficient. Our goal is to explain, how this coefficient can be determined independently, not using the import from the proof in [Borodin and Corwin 2014].

2. Scaling theory for the single-step model

In the single step growth model the moving surface is described by the graph of the height function $h(t) : \mathbb{Z} \to \mathbb{Z}$ with the constraint

$$|h(j + 1, t) - h(j, t)| = 1, \qquad (2\text{-}1)$$

hence the name. The random deposition/evaporation events are modeled by a Markov jump process constrained to satisfy (2-1). The allowed local moves are then transitions from $h(j, t)$ to $h(j, t) \pm 2$. The dynamics should be invariant

under a shift in the h-direction. Hence the rates for the deposition/evaporation events are allowed to depend only on the local slopes. It is then convenient to switch to height differences

$$\eta_j(t) = h(j+1, t) - h(j, t), \quad \eta_j(t) = \pm 1. \tag{2-2}$$

A single growth step at the bond $(j, j+1)$ is given by

$$\eta \to \eta^{j,j+1}, \tag{2-3}$$

where $\eta^{j,j+1}$ is the configuration with the slopes at j and $j+1$ interchanged. If the corresponding rates are denoted by $c_{j,j+1}(\eta)$, depending on the local neighborhood of $(j, j+1)$, the Markov generator reads

$$Lf(\eta) = \sum_{j \in \mathbb{Z}} c_{j,j+1}(\eta)(f(\eta^{j,j+1}) - f(\eta)). \tag{2-4}$$

The slope field η is locally conserved, in the sense that the sum $\sum_{j=a}^{b} \eta(j, t)$ changes only through the fluxes at the two boundaries a, b. To keep things concretely, in the following we consider only the wedge initial condition

$$h(j, 0) = |j|. \tag{2-5}$$

Scaling theory is based on the following.

Assumption. *The spatially ergodic and time stationary measures of the slope process $\eta(t)$ are precisely labeled by the average density*

$$\rho = \lim_{a \to \infty} \frac{1}{2a+1} \sum_{|j| \leq a} \eta_j, \tag{2-6}$$

with $|\rho| \leq 1$.

Our assumption was formulated more than 30 years ago. Except for special cases, it remains open even today; see [Liggett 1999] for more details. The stationary measures from the assumption are denoted by μ_ρ, as a probability measure on $\{-1, 1\}^{\mathbb{Z}}$.

Given μ_ρ one defines two natural quantities:

- the average steady state current:

$$\mathrm{j}(\rho) = \mu_\rho(c_{0,1}(\eta)(\eta_0 - \eta_1)); \tag{2-7}$$

- the integrated covariance of the conserved slope field:

$$A(\rho) = \sum_{j \in \mathbb{Z}} (\mu_\rho(\eta_0 \eta_j) - \mu_\rho(\eta_0)^2). \tag{2-8}$$

We first notice that for long times there is a law of large numbers stating that

$$h(j, t) \simeq t\phi(j/t), \qquad (2\text{-}9)$$

for large j, t with a deterministic profile function ϕ. In fact ϕ is the Legendre transform of j in the sense that

$$\phi(y) = \sup_{|\rho| \leq 1} (y\rho - \mathrm{j}(\rho)). \qquad (2\text{-}10)$$

The argument is based on the hydrodynamic limit for nonreversible lattice gases [Spohn 1991], which asserts that on the macroscopic scale the density $\rho(x, t)$ of the conserved field satisfies

$$\frac{\partial}{\partial t} \rho(x, t) + \frac{\partial}{\partial x} \mathrm{j}(\rho(x, t)) = 0, \qquad (2\text{-}11)$$

with initial condition

$$\rho(x, 0) = \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{if } x < 0. \end{cases} \qquad (2\text{-}12)$$

The entropy solution to (2-11), (2-12) is indeed given by (2-9), (2-10).

The average current can be fairly arbitrary, except for the linear behavior near $\rho = \pm 1$. On the other hand ϕ is convex up with $\phi(x) = |x|$ for $|x| \geq x_c$. At points, where ϕ is either linear or cusp-like, the fluctuations may be different from the generic case and have to be discussed separately. We set

$$\lambda(\rho) = -\mathrm{j}''(\rho). \qquad (2\text{-}13)$$

Conjecture (KPZ class). *Let y be such that ϕ is twice differentiable at y with $\phi''(y) \neq 0$ and set $\rho = \phi'(y)$, $|\rho| < 1$. If $A(\rho) < \infty$ and $\lambda(\rho) \neq 0$, then*

$$\lim_{t \to \infty} \mathbb{P}\big(h(\lfloor yt \rfloor, t) - t\phi(y) \leq -\big(-\tfrac{1}{2}\lambda A^2\big)^{1/3} t^{1/3} s\big) = F_{\mathrm{GUE}}(s). \qquad (2\text{-}14)$$

We use $\lfloor \cdot \rfloor$ to denote the integer part. Since $\phi''(y) > 0$, we have $\lambda(\rho) < 0$ by Legendre transform.

On the scale $\big(-\tfrac{1}{2}\lambda A^2 t\big)^{1/3}$ the height fluctuations are governed by the Tracy–Widom distribution. λA^2 is the model dependent coefficient which, at least in principle, can be computed once μ_ρ is available. $\lambda \equiv 0$ for reversible slope dynamics, since $\mathrm{j} = 0$. In that case the fluctuations are of scale $t^{1/4}$ and Gaussian; see [Spohn 1991, Part II, Chapter 3] for a discussion. For nonreversible slope dynamics one can still arrange for $\lambda \equiv 0$. If j is nonzero, there could be isolated points at which λ vanishes. At a cubic inflection point of j the height fluctuations are expected to be of order $(t (\log t)^{1/2})^{1/2}$ [Paczuski et al. 1992; Derrida et al. 1991].

In our set-up the conjecture has been proved by Tracy and Widom [2009] for the PASEP. In this case the exchange $+-$ to $-+$ occurs with rate p and the exchange $-+$ to $+-$ with rate $1-p$, $0 \le p < \frac{1}{2}$. Then μ_ρ is a Bernoulli measure, hence $A(\rho) = 1 - \rho^2$, $\mathrm{j}(\rho) = \frac{1}{2}(2p-1)(1-\rho^2)$, and

$$-\tfrac{1}{2}\lambda A^2 = \tfrac{1}{2}(1-2p)(1-\rho^2)^2. \tag{2-15}$$

The profile function is

$$\phi(y) = \begin{cases} \frac{1}{2}(1-2p)(1+(y/(1-2p))^2) & \text{for } |y| \le 1-2p, \\ |y| & \text{for } |y| \ge 1-2p. \end{cases}$$

For the totally asymmetric case, $p = 0$, the limit (2-14) has been established before by Johansson [2000]. The PushTASEP falls also under our scheme with a proof by Borodin and Ferrari [2008]. Scaling theory is further confirmed for growth models different from single-step, to mention the discrete time TASEP [Johansson 2000], the polynuclear growth model [Prähofer and Spohn 2000], and the KPZ equation [Amir et al. 2011; Sasamoto and Spohn 2010]. For a discussion of the scaling theory in the context of TASEP we refer to [Prähofer and Spohn 2002].

The theory of Macdonald process [Borodin and Corwin 2014] has brought a q-deformed version of the TASEP in focus. For us here it is a further example for which the nonuniversal constants can be computed. For the rate in (2-4) we set

$$c_{j,j+1}(\eta) = \tfrac{1}{4}(1-\eta_j)(1+\eta_{j+1})g((n^-_{j+1}(\eta)), \tag{2-16}$$

where n^-_j is the number of consecutive $-$ slopes to the left of site j. We have $g(0) = 0$, $g(j) > 0$ for $j > 0$, and g increases at most linearly. The q-TASEP is the special case where $g(j) = 1 - q^j$, $0 \le q < 1$, with the TASEP recovered in the limit $q \to 0$. The slope system maps onto the totally asymmetric zero range process for length of consecutive gaps between the $+$ slope, denoted by Y_j, $j \in \mathbb{Z}$, $Y_j = 0, 1, \ldots$. In the stationary measure the Y_j are i.i.d. and

$$\mathbb{P}(Y_0 = k) = \begin{cases} Z(\alpha)^{-1} & \text{if } k = 0, \\ Z(\alpha)^{-1}\big(\prod_{j=1}^k g(j)\big)^{-1}\alpha^k & \text{if } k = 1, 2, \ldots, \end{cases} \tag{2-17}$$

where

$$Z(\alpha) = 1 + \sum_{k=1}^\infty \bigg(\prod_{j=1}^k g(j)\bigg)^{-1}\alpha^k, \tag{2-18}$$

and $\alpha > 0$ such that $Z(\alpha) < \infty$. The translation invariant, time stationary measures for the $\eta(t)$-process with rates (2-16) are stationary renewal processes on $\mathbb{Z}$ with renewal distribution (2-17), (2-18).

The coefficient A can be computed from

$$\lim_{N\to\infty} \frac{1}{N} \log \left\langle \exp\left[\lambda \sum_{j=1}^{N} \eta_j\right]\right\rangle_\alpha = r(\lambda), \tag{2-19}$$

where the average is over the stationary renewal process with parameter α. Then

$$\rho = r'(\lambda) \quad \text{and} \quad A = r''(\lambda) \quad \text{at } \lambda = 0.$$

The rate function r is implicitly determined by

$$r(\lambda) = -\lambda - \log z(\lambda), \quad \frac{1}{Z(\alpha)} z(\lambda) Z(\alpha z(\lambda)) e^{2\lambda} = 1. \tag{2-20}$$

A and ρ are computed by successive differentiations. The result is best expressed through $G(\alpha) = \log Z(\alpha)$. Then

$$\tfrac{1}{2}(1 + \rho) = (1 + \alpha G')^{-1}, \tag{2-21}$$

$$A = 4(1 + \alpha G')^{-3} \alpha (\alpha G')' = -\alpha(1 + \rho)\frac{d\rho}{d\alpha}. \tag{2-22}$$

The average current is given by

$$\mathrm{j}(\rho) = -2\langle c_{0,1}\rangle_\alpha, \quad \mathrm{j} = -\alpha(1 + \rho). \tag{2-23}$$

One can use (2-23) together with (2-21) to work out λ. But no particularly illuminating formula results for the combination λA^2.

In addition to (2-14) there is a second scale, which will play no role here, but should be mentioned. Instead of the height statistics at the single point $\lfloor yt \rfloor$ one could consider, for example, the joint distribution of $h(j_1, t)$, $h(j_2, t)$, referred to as transverse correlations. The transverse scale tells us at which separation $|j_1 - j_2|$ there are nontrivial correlations in the limit $t \to \infty$. KPZ scaling theory asserts that this scale is

$$(2\lambda^2 A t^2)^{1/3}. \tag{2-24}$$

The factor 2 comes from the requirement that the limit joint distribution is the two-point distribution of the Airy process. Corresponding predictions hold for the multipoint statistics. Also when considering the two-point function of the stationary $\eta(t)$ process, $\mathbb{E}(\eta_0(0)\eta_j(t)) - \mathbb{E}(\eta_0(0))^2$, up to a shift linear in t, j has to vary on the scale of (2-24) to have a nontrivial limit as $t \to \infty$. The scale (2-24) is confirmed for the PNG [Prähofer and Spohn 2004], TASEP [Ben Arous and Corwin 2011], and PushTASEP [Borodin and Ferrari 2008] two-point function in case of step initial conditions, for the stationary TASEP [Ferrari and Spohn 2006] and stationary KPZ equation [Imamura and Sasamoto 2012], and for TASEP and PNG [Borodin et al. 2007; 2008] in case of flat initial conditions.

3. The semidiscrete directed polymer model

Our starting point is a very particular discretization of the stochastic heat equation as

$$dZ_j = Z_{j-1} \, dt + Z_j \, db_j. \tag{3-1}$$

Here $j \in \mathbb{Z}$, $t \geq 0$, and $\{b_j(t), j \in \mathbb{Z}\}$ is a collection of independent standard Brownian motions. The analogue of the wedge initial condition is

$$Z_j(0) = \delta_{j,0}. \tag{3-2}$$

Hence $Z_j(t) = 0$ for $j < 0$ and

$$dZ_j = Z_{j-1} \, dt + Z_j \, db_j, \quad j = 1, 2, \ldots,$$
$$dZ_0 = Z_0 db_0. \tag{3-3}$$

Let us introduce the totally asymmetric random walk, $w(t)$, on $\mathbb{Z}$ moving with rate 1 to the right. Denoting by $\mathbb{E}_0$ expectation for $w(t)$ with $w(0) = 0$, one can represent

$$Z_j(t) = \mathbb{E}_0 \left(\exp\left[\int_0^t db_{w(s)}(s) \right] \delta_{w(t),j} \right) e^t. \tag{3-4}$$

$Z_j(t)$ is the random partition function of the directed polymer $w(t)$, length t, endpoints 0 and j, in the random potential $db_j(s)/ds$. This model was first introduced by O'Connell and Yor [2001]; see also [Moriarty and O'Connell 2007; O'Connell 2012]. In the zero temperature limit one would maximize over the term in the exponential at fixed $\{b_j(s)\}$ and fixed end points; see [Glynn and Whitt 1991] for an early study. The statistics of the maximizer is closely related to GUE and Dyson's Brownian motion [Baryshnikov 2001].

The height corresponds to the random free energy and we set

$$h_j(t) = \log Z_j(t), \quad j \geq 0, \ t > 0. \tag{3-5}$$

Then h_j is the solution to

$$dh_j = e^{h_{j-1}-h_j} \, dt + db_j, \tag{3-6}$$

and the slope $u_j = h_{j+1} - h_j$ is governed by

$$du_j = (e^{-u_j} - e^{-u_{j-1}}) \, dt + db_{j+1} - db_j, \quad j = 1, 2, \ldots, \tag{3-7}$$
$$du_0 = e^{-u_0} + db_1 - db_0. \tag{3-8}$$

The slope $u_j(t)$ is locally conserved.

Somewhat unexpectedly, one can still find the stationary and translation invariant measures for the interacting diffusions (3-7) on the lattice $\mathbb{Z}$. They are

labeled by a parameter $r > 0$ and are of product form. The single site measure is a double exponential of the form

$$\mu_r(dx) = \Gamma(r)^{-1}e^{-e^{-x}}e^{-rx}\,dx, \quad r > 0. \tag{3-9}$$

Averages with respect to μ_r are denoted by $\langle\,\cdot\,\rangle_r$. The parameters of scaling theory are now easily computed. We find

$$\rho = \langle u_0\rangle_r = -\psi(r), \tag{3-10}$$

with $\psi = \Gamma'/\Gamma$ the digamma function on $\mathbb{R}^+$. Note that $\psi' > 0$, $\psi'' < 0$, and ρ ranges over $\mathbb{R}$. From (3-7) the random current is $-e^{-u_j}dt - db_{j+1}$ and hence the average current is

$$\mathrm{j} = -\langle e^{-u_0}\rangle_r = -r. \tag{3-11}$$

Finally

$$A(r) = \langle u_0^2\rangle_r - \langle u_0\rangle_r^2 = \psi'(r). \tag{3-12}$$

Since the initial conditions force ϕ to be convex down, the signs from Section 2 are reversed. In particular, the sup in (2-10) is replaced by the inf and

$$\phi(y) = \inf_{\rho\in\mathbb{R}}(-y\rho - \mathrm{j}(-\rho)), \quad y \geq 0. \tag{3-13}$$

Then $\phi(0) = 0$, $\phi'' < 0$, and ϕ has a single strictly positive maximum before dropping to $-\infty$ as $y \to \infty$. Thus $t\phi(y/t)$ reproduces the singular initial conditions for (3-6) as $t \to 0$. Moriarty and O'Connell [2007] prove that

$$\lim_{N\to\infty}\frac{1}{N}h_N(\kappa N) = f(\kappa), \tag{3-14}$$

with

$$f(\kappa) = \inf_{s\geq 0}(\kappa s - \psi(s)). \tag{3-15}$$

Scaling theory claims that

$$\lim_{t\to\infty}\frac{1}{t}h_{\lfloor yt\rfloor}(t) = \phi(y), \quad y > 0. \tag{3-16}$$

Hence, using that $\psi' > 0$ and $y\kappa = 1$, we have

$$\phi(y) = \frac{1}{\kappa}f(\kappa) = \inf_{s\geq 0}(s - y\psi(s)) = \inf_{\tilde{s}\in\mathbb{R}}(\psi^{-1}(\tilde{s}) - y\tilde{s}), \tag{3-17}$$

in agreement with (3-13).

We turn to the asymptotic analysis of the height fluctuations:

Theorem [Borodin and Corwin 2014, Theorem 5.12]. *There exists a κ^* such that, for $0 < \kappa^* < \kappa$,*

$$\lim_{n\to\infty}\mathbb{P}\big(h_n(\kappa n) - nf(\kappa) \leq (-2f''(\kappa))^{-1/3}n^{1/3}s\big) = F_{\mathrm{GUE}}(s). \tag{3-18}$$

According to (3-11), $\lambda = -\mathrm{j}'' > 0$. Hence in (2-14) $-\lambda$ is replaced by λ and

the sign to the right of $\leq$ is $+$. To see whether with these changes scaling theory is confirmed, we start from

$$h_{\lfloor yt \rfloor}(t) = t\phi(y) + \left(\tfrac{1}{2}\lambda A^2 t\right)^{1/3}\xi_{\mathrm{TW}}, \qquad (3\text{-}19)$$

with ξ_{TW} a GUE Tracy–Widom distributed random variable, hence

$$h_n(\kappa n) = \kappa n\phi(\kappa^{-1}) + \left(\tfrac{1}{2}\lambda A^2 \kappa n\right)^{1/3}\xi_{\mathrm{TW}}. \qquad (3\text{-}20)$$

Now $\rho = -\psi(r(\rho))$ is differentiated as

$$1 = -\psi' r', \quad 0 = \psi''(r')^2 + \psi' r''. \qquad (3\text{-}21)$$

Since $-\lambda = \mathrm{j}''(\rho) = -r''(\rho)$ and $A(r) = \psi'(r)$, one has

$$\lambda A^2 = \psi'' r'. \qquad (3\text{-}22)$$

Since $y = \mathrm{j}'(\rho) = -r'(\rho)$ and $y\kappa = 1$, we conclude

$$\lambda A^2 \kappa = -\psi'', \qquad (3\text{-}23)$$

and, since f is the Legendre transform of ψ,

$$\lambda A^2 \kappa = -\frac{1}{f''}, \qquad (3\text{-}24)$$

in agreement with (3-18).

Conclusion

KPZ scaling theory makes a prediction on the nonuniversal coefficients for models in the KPZ class and has been confirmed for PASEP, discrete TASEP, and PNG. We add to this list the semidiscrete directed polymer. The corresponding stochastic "particle" model is a system of diffusions, $u_j(t)$, with nearest neighbor interactions such that the sums $\sum_j u_j(t)$ are locally conserved. This model has a flavor rather distinct from driven lattice gases. Still the long time asymptotics in all models is the Tracy–Widom statistics.

References

[Alves et al. 2011] S. G. Alves, T. J. Oliveira, and S. Ferreira, "Universal fluctuations in radial growth models belonging to the KPZ universality class", *Europhys. Lett.* **96** (2011), 48003.

[Amir et al. 2011] G. Amir, I. Corwin, and J. Quastel, "Probability distribution of the free energy of the continuum directed random polymer in $1+1$ dimensions", *Comm. Pure Appl. Math.* **64**:4 (2011), 466–537.

[Baryshnikov 2001] Y. Baryshnikov, "GUEs and queues", *Probab. Theory Related Fields* **119**:2 (2001), 256–274.

[Ben Arous and Corwin 2011] G. Ben Arous and I. Corwin, "Current fluctuations for TASEP: a proof of the Prähofer–Spohn conjecture", *Ann. Probab.* **39**:1 (2011), 104–138.

[Borodin and Corwin 2014] A. Borodin and I. Corwin, "Macdonald processes", *Probab. Theory Related Fields* **158**:1-2 (2014), 225–400.

[Borodin and Ferrari 2008] A. Borodin and P. L. Ferrari, "Large time asymptotics of growth models on space-like paths, I: PushASEP", *Electron. J. Probab.* **13** (2008), no. 50, 1380–1418.

[Borodin et al. 2007] A. Borodin, P. L. Ferrari, M. Prähofer, and T. Sasamoto, "Fluctuation properties of the TASEP with periodic initial configuration", *J. Stat. Phys.* **129**:5-6 (2007), 1055–1080.

[Borodin et al. 2008] A. Borodin, P. L. Ferrari, and T. Sasamoto, "Large time asymptotics of growth models on space-like paths, II: PNG and parallel TASEP", *Comm. Math. Phys.* **283**:2 (2008), 417–449.

[Derrida et al. 1991] B. Derrida, J. L. Lebowitz, E. R. Speer, and H. Spohn, "Fluctuations of a stationary nonequilibrium interface", *Phys. Rev. Lett.* **67**:2 (1991), 165–168.

[Ferrari and Spohn 2006] P. L. Ferrari and H. Spohn, "Scaling limit for the space-time covariance of the stationary totally asymmetric simple exclusion process", *Comm. Math. Phys.* **265**:1 (2006), 1–44.

[Forrester 1993] P. J. Forrester, "The spectrum edge of random matrix ensembles", *Nuclear Phys. B* **402**:3 (1993), 709–728.

[Glynn and Whitt 1991] P. W. Glynn and W. Whitt, "Departures from many queues in series", *Ann. Appl. Probab.* **1**:4 (1991), 546–572.

[Imamura and Sasamoto 2012] T. Imamura and T. Sasamoto, "Exact Solution for the Stationary Kardar-Parisi-Zhang Equation", *Phys. Rev. Lett.* **108** (2012), 190603.

[Johansson 2000] K. Johansson, "Shape fluctuations and random matrices", *Comm. Math. Phys.* **209**:2 (2000), 437–476.

[Kardar et al. 1986] M. Kardar, G. Parisi, and Y.-C. Zhang, "Dynamic scaling of growing interfaces", *Phys. Rev. Lett.* **56** (1986), 889–892.

[Krug et al. 1992] J. Krug, P. Meakin, and T. Halpin-Healy, "Amplitude universality for driven interfaces and directed polymers in random media", *Phys. Rev. A* **45** (1992), 638–653.

[Liggett 1999] T. M. Liggett, *Stochastic interacting systems: contact, voter and exclusion processes*, Grundlehren der Mathematischen Wissenschaften **324**, Springer, Berlin, 1999.

[Moriarty and O'Connell 2007] J. Moriarty and N. O'Connell, "On the free energy of a directed polymer in a Brownian environment", *Markov Process. Related Fields* **13**:2 (2007), 251–266.

[O'Connell 2012] N. O'Connell, "Directed polymers and the quantum Toda lattice", *Ann. Probab.* **40**:2 (2012), 437–458.

[O'Connell and Yor 2001] N. O'Connell and M. Yor, "Brownian analogues of Burke's theorem", *Stochastic Process. Appl.* **96**:2 (2001), 285–304.

[Paczuski et al. 1992] M. Paczuski, M. Barma, S. N. Majumdar, and T. Hwa, "Fluctuations of a nonequililbrium interface", *Phys. Rev. Lett.* **69** (1992), 2735–2735.

[Prähofer and Spohn 2000] M. Prähofer and H. Spohn, "Universal distributions for growth processes in $1 + 1$ dimensions and random matrices", *Phys. Rev. Lett.* **84** (2000), 4882–4885.

[Prähofer and Spohn 2002] M. Prähofer and H. Spohn, "Current fluctuations for the totally asymmetric simple exclusion process", pp. 185–204 in *In and out of equilibrium* (Mambucaba, 2000), edited by V. Sidoravicius, Progr. Probab. **51**, Birkhäuser, Boston, 2002.

[Prähofer and Spohn 2004] M. Prähofer and H. Spohn, "Exact scaling functions for one-dimensional stationary KPZ growth", *J. Statist. Phys.* **115**:1-2 (2004), 255–279.

[Sasamoto and Spohn 2010] T. Sasamoto and H. Spohn, "Exact height distributions for the KPZ equation with narrow wedge initial condition", *Nuclear Phys. B* **834**:3 (2010), 523–542.

[Spohn 1991] H. Spohn, *Large scale dynamics of interacting particles*, Springer, Heidelberg, 1991.

[Takeuchi and Sano 2010] K. A. Takeuchi and M. Sano, "Universal fluctuations of growing interfaces: evidence in turbulent liquid crystals", *Phys. Rev. Lett.* **104** (2010), 230601.

[Takeuchi et al. 2011] K. A. Takeuchi, M. Sano, T. Sasamoto, and H. Spohn, "Growing interfaces uncover universal fluctuations behind scale invariance", *Scientific Reports* **1** (2011), 34.

[Tracy and Widom 1994] C. A. Tracy and H. Widom, "Level-spacing distributions and the Airy kernel", *Comm. Math. Phys.* **159**:1 (1994), 151–174.

[Tracy and Widom 2009] C. A. Tracy and H. Widom, "Asymptotics in ASEP with step initial condition", *Comm. Math. Phys.* **290**:1 (2009), 129–154.

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