Syzygies, finite length modules, and random curves

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We apply the theory of Gröbner bases to the computation of free resolutions over a polynomial ring, the defining equations of a canonically embedded curve, and the unirationality of the moduli space of curves of a fixed small genus.

Introduction

While a great deal of modern commutative algebra and algebraic geometry has taken a nonconstructive form, the theory of Gröbner bases provides an algorithmic approach. Algorithms currently implemented in computer algebra systems, such as Macaulay2 [Grayson and Stillman] and Singular [Decker et al. 2011], already exhibit the wide range of computational possibilities that arise from Gröbner bases.

In these lectures, we focus on certain applications of Gröbner bases to syzygies and curves. In Section 1, we use Gröbner bases to give an algorithmic proof of Hilbert's syzygy theorem, which bounds the length of a free resolution over a polynomial ring. In Section 2, we prove Petri's theorem about the defining equations for canonical embeddings of curves. We turn in Section 3 to the Hartshorne–Rao module of a curve, showing by example how a module M of finite length can be used to explicitly construct a curve whose Hartshorne–Rao module is M. Section 4 then applies this construction to the study of the unirationality of the moduli space $\mathfrak{M}_g$ of curves of genus g.

1. Hilbert's syzygy theorem

Let $R := \mathbb{K}[x_1, \dots, x_n]$ be a polynomial ring in n variables over a field $\mathbb{K}$. A *free resolution* of a finitely generated R-module M is a complex of free modules

$\cdots \to R^{\beta_2} \to R^{\beta_1} \to R^{\beta_0}$ such that the following is exact:

$$\cdots \to R^{\beta_2} \to R^{\beta_1} \to R^{\beta_0} \to M \to 0.$$

Hilbert's syzygy theorem (Theorem 1.1). *Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring in n variables over a field $\mathbb{K}$. Every finitely generated R-module M has a finite free resolution of length at most n.*

In this section, we give an algorithmic Gröbner basis proof of Hilbert's theorem, whose strategy is used in modern computer algebra systems like Macaulay2 and Singular for syzygy computations. Gröbner bases were introduced by Gordan [1899] to provide a new proof of Hilbert's basis theorem. We believe that Gordan could have given the proof of Hilbert's syzygy theorem presented here.

Definition 1.2. A (*global*) *monomial order* on R is a total order $>$ on the set of monomials in R such that:

(1) if $x^\alpha > x^\beta$, then $x^\gamma x^\alpha > x^\gamma x^\beta$ for all $\gamma \in \mathbb{N}^n$; and

(2) $x_i > 1$ for all i.

Given a global monomial order, the *leading term* of a nonzero polynomial $f = \sum_\alpha f_\alpha x^\alpha \in R$ is defined to be

$$\mathbf{L}(f) := f_\beta x^\beta, \quad \text{where } x^\beta := \max_\alpha \{x^\alpha \mid f_\alpha \neq 0\}.$$

For convenience, set $\mathbf{L}(0) := 0$.

Theorem 1.3 (division with remainder). *Let $>$ be a global monomial order on R, and let $f_1, \ldots, f_r \in R$ be nonzero polynomials. For every $g \in R$, there exist uniquely determined $g_1, \ldots, g_r \in R$ and a remainder $h \in R$ such that*:

(1) $g = g_1 f_1 + \cdots + g_r f_r + h$.

(2a) *No term of $g_i \mathbf{L}(f_i)$ is divisible by any $\mathbf{L}(f_j)$ with $j < i$.*

(2b) *No term of h is a multiple of $\mathbf{L}(f_i)$ for any i.*

Proof. The result is obvious if $f_1, \ldots, f_r$ are monomials, or more generally, if each f_i has only a single nonzero term. Thus there is always a unique expression

$$g = \sum_{i=1}^r g_i^{(1)} \mathbf{L}(f_i) + h^{(1)},$$

if we require that $g_1^{(1)}, \ldots, g_r^{(1)}$ and $h^{(1)}$ satisfy (2a) and (2b). By construction, the leading terms of the summands of the expression

$$\sum_{i=1}^r g_i^{(1)} f_i + h^{(1)}$$

are distinct, and the leading term in the difference $g^{(1)} = g - (\sum_{i=1}^r g_i^{(1)} f_i + h^{(1)})$ cancels. Thus $\mathbf{L}(g^{(1)}) < \mathbf{L}(g)$, and recursion applies. $\hfill\square$

The remainder h of the division of g by $f_1, \ldots, f_r$ depends on the order of $f_1, \ldots, f_r$, since the partition of the monomials in R given by (2a) and (2b) depends on this order. Even worse, it might not be the case that if $g \in \langle f_1, \ldots, f_r \rangle$, then $h = 0$. A Gröbner basis is a system of generators for which this desirable property holds.

Definition 1.4. Let $I \subset R$ be an ideal. The *leading ideal* of I (with respect to a given global monomial order) is

$$\mathbf{L}(I) := \langle \mathbf{L}(f) \mid f \in I \rangle.$$

A finite set $f_1, \ldots, f_r$ of polynomials is a *Gröbner basis* when

$$\langle \mathbf{L}(\langle f_1, \ldots, f_r \rangle) \rangle = \langle \mathbf{L}(f_1), \ldots, \mathbf{L}(f_r) \rangle.$$

Gordan's proof of Hilbert's basis theorem now follows from the easier statement that monomial ideals are finitely generated. In combinatorics, this result is called Dickson's lemma [1913].

If $f_1, \ldots, f_r$ is a Gröbner basis, then by definition, a polynomial g lies in $\langle f_1, \ldots, f_r \rangle$ if and only if the remainder h under division of g by $f_1, \ldots, f_r$ is zero. In particular, in this case, the remainder does not depend on the order of $f_1, \ldots, f_r$, and the monomials $x^\alpha \notin \langle \mathbf{L}(f_1), \ldots, \mathbf{L}(f_r) \rangle$ represent a $\mathbb{K}$-vector space basis of the quotient ring $R / \langle f_1, \ldots, f_r \rangle$, a fact known as Macaulay's theorem [1916]. For these reasons, it is desirable to have a Gröbner basis on hand.

The algorithm that computes a Gröbner basis for an ideal is due to Buchberger [1965; 1970]. Usually, Buchberger's criterion is formulated in terms of so-called S-pairs. In the treatment below, we do not use S-pairs; instead, we focus on the partition of the monomials of R induced by $\mathbf{L}(f_1), \ldots, \mathbf{L}(f_r)$ via (2a) and (2b) of Theorem 1.9.

Given polynomials $f_1, \ldots, f_r$, consider the monomial ideals

$$M_i := \langle \mathbf{L}(f_1), \ldots, \mathbf{L}(f_{i-1}) \rangle : \mathbf{L}(f_i) \quad \text{for } i = 1, \ldots, r. \tag{1-1}$$

For each minimal generator x^α of an M_i, let $h^{(i,\alpha)}$ denote the remainder of $x^\alpha f_i$ divided by $f_1, \ldots, f_r$ (in this order).

Buchberger's criterion (Theorem 1.5) [Buchberger 1970]. *Let $f_1, \ldots, f_r \in R$ be a collection of nonzero polynomials. Then $f_1, \ldots, f_r$ form a Gröbner basis if and only if all of the remainders $h^{(i,\alpha)}$ are zero.*

We will prove this result after a few more preliminaries.

Example 1.6. Consider the ideal generated by the 3×3 minors of the matrix

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ z_1 & z_2 & z_3 & z_4 & z_5 \end{pmatrix}.$$

Using the lexicographic order on $\mathbb{K}[x_1, \ldots, z_5]$, the leading terms of the maximal minors of this matrix and the minimal generators of the corresponding monomial ideals M_i are listed in the following table.

$x_1 y_2 z_3$	$M_1 = 0$
$x_1 y_2 z_4$	$M_2 = \langle z_3 \rangle$
$x_1 y_3 z_4$	$M_3 = \langle y_2 \rangle$
$x_2 y_3 z_4$	$M_4 = \langle x_1 \rangle$
$x_1 y_2 z_5$	$M_5 = \langle z_3, z_4 \rangle$
$x_1 y_3 z_5$	$M_6 = \langle y_2, z_4 \rangle$
$x_2 y_3 z_5$	$M_7 = \langle x_1, z_4 \rangle$
$x_1 y_4 z_5$	$M_8 = \langle y_2, y_3 \rangle$
$x_2 y_4 z_5$	$M_9 = \langle x_1, y_3 \rangle$
$x_3 y_4 z_5$	$M_{10} = \langle x_1, x_2 \rangle$

Note that only 15 of the possible $\binom{10}{2} = 45$ S-pairs are needed to test Buchberger's criterion.

Exercise 1.7. Show that the maximal minors of the matrix in Example 1.6 form a Gröbner basis by using the Laplace expansions of suitable 4×4 matrices.

In order to prove Hilbert's syzygy theorem and Buchberger's criterion, we now extend the notion of a monomial order to vectors of polynomials.

Definition 1.8. A monomial of a free module R^r with basis $e_1, \ldots, e_r$ is an expression $x^\alpha e_i$. A (*global*) *monomial order* on R^r is a total order of the monomials of R^r such that:

(1) if $x^\alpha e_i > x^\beta e_j$, then $x^\gamma x^\alpha e_i > x^\gamma x^\beta e_j$ for all i, j and $\gamma \in \mathbb{N}^n$;

(2) $x^\alpha e_i > e_i$ for all i and $\alpha \neq 0$.

Usually, it is also the case that $x^\alpha e_i > x^\beta e_i$ if and only if $x^\alpha e_j > x^\beta e_j$, i.e., the order on the monomials in the components induce a single monomial order on R.

Thanks to Definition 1.8, we may now speak of the leading term of a vector of polynomials. In this situation, the division theorem still holds.

Theorem 1.9 (division with remainder for vectors of polynomials). *Let $>$ be a global monomial order on R^{r_0}, and let $F_1, \ldots, F_r \in R^{r_0}$ be nonzero polynomial*

vectors. For every $G \in R^{r_0}$, there exist uniquely determined $g_1, \ldots, g_r \in R$ and a remainder $H \in R^{r_0}$ such that:

(1) $G = g_1 F_1 + \cdots + g_r F_r + H$.

(2a)   *No term of $g_i \, \mathbf{L}(F_i)$ is a multiple of an $\mathbf{L}(F_j)$ with $j < i$.*

(2b)   *No term of H is a multiple of $\mathbf{L}(F_i)$ for any i.*   □

Definition 1.10. Generalizing the earlier definition, given a global monomial order on R^r, the *leading term* of a nonzero vector of polynomials $F = (f_1, \ldots, f_r)$ is defined to be the monomial

$$\mathbf{L}(F) := f_{\beta_i} x^{\beta_i} e_i, \quad \text{where } x^{\beta_i} = \max_{\alpha_i}\{x^{\alpha_i} \mid f_{\alpha_i} x^{\alpha_i} \text{ is a nonzero term of } f_i\}.$$

A finite set $F_1, \ldots, F_s$ of vectors of polynomials in R^r is a *Gröbner basis* when

$$\langle \mathbf{L}(\langle F_1, \ldots, F_s \rangle) \rangle = \langle \mathbf{L}(F_1), \ldots, \mathbf{L}(F_s) \rangle.$$

Proof of Buchberger's criterion. The forward direction follows by definition. For the converse, assume that all remainders $h^{(i,\alpha)}$ vanish. Then for each minimal generator x^α in an M_i, there is an expression

$$x^\alpha f_i = g_1^{(i,\alpha)} f_1 + \cdots + g_r^{(i,\alpha)} f_r \tag{1-2}$$

such that no term of $g_j^{(i,\alpha)} \mathbf{L}(f_j)$ is divisible by an $\mathbf{L}(f_k)$ for every $k < j$, by condition (2a) of Theorem 1.3. (Of course, for a suitable $j < i$, one of the terms of $g_j^{(i,\alpha)} \mathbf{L}(f_j)$ coincides with $x^\alpha \mathbf{L}(f_i)$. This is the second term in the usual S-pair description of Buchberger's criterion.) Now let $e_1, \ldots, e_r \in R^r$ denote the basis of the free module, and let $\varphi : R^r \to R$ be defined by $e_i \mapsto f_i$. Then by (1-2), elements of the form

$$G^{(i,\alpha)} := -g_1^{(i,\alpha)} e_1 + \cdots + (x^\alpha - g_i^{(i,\alpha)}) e_i + \cdots + (-g_r^{(i,\alpha)}) e_r \tag{1-3}$$

are in the kernel of ϕ. In other words, the $G^{(i,\alpha)}$'s are syzygies between $f_1, \ldots, f_r$.

We now proceed with a division with remainder in the free module R^r, using the induced monomial order $>_1$ on R^r defined by

$$x^\alpha e_i >_1 x^\beta e_j \iff \begin{aligned} &x^\alpha \mathbf{L}(f_i) > x^\beta \mathbf{L}(f_j) \text{ or} \\ &x^\alpha \mathbf{L}(f_i) = x^\beta \mathbf{L}(f_j) \text{ (up to a scalar) with } i > j. \end{aligned} \tag{1-4}$$

With respect to this order,

$$\mathbf{L}(G^{(i,\alpha)}) = x^\alpha e_i$$

because the term $c x^\beta \mathbf{L}(f_j)$ that cancels against $x^\alpha \mathbf{L}(f_i)$ in (1-2) satisfies $j < i$, and all other terms of any $g_k^{(i,\alpha)} \mathbf{L}(f_k)$ are smaller.

Now consider an arbitrary element

$$g = a_1 f_1 + \cdots + a_r f_r \in \langle f_1, \ldots, f_r \rangle.$$

We must show that $\mathbf{L}(g) \in \langle \mathbf{L}(f_1), \ldots, \mathbf{L}(f_r) \rangle$. Let $g_1 e_1 + \cdots + g_r e_r$ be the remainder of $a_1 e_1 + \cdots + a_r e_r$ divided by the collection of $G^{(i,\alpha)}$ vectors. Then

$$g = a_1 f_1 + \cdots + a_r f_r = g_1 f_1 + \cdots + g_r f_r$$

because the $G^{(i,\alpha)}$ are syzygies, and $g_1, \ldots, g_r$ satisfy (2a) of Theorem 1.3 when $a_1, \ldots, a_n$ are divided by $f_1, \ldots, f_r$, by the definition of the M_i in (1-1). Therefore, the nonzero initial terms

$$\mathbf{L}(g_j f_j) = \mathbf{L}(g_j)\mathbf{L}(f_j)$$

are distinct and no cancellation can occur among them. The proof is now complete because

$$\mathbf{L}(g) := \max_j \{\mathbf{L}(g_j)\mathbf{L}(f_j)\} \in \langle \mathbf{L}(f_1), \ldots, \mathbf{L}(f_r) \rangle. \qquad \square$$

Corollary 1.11 [Schreyer 1980]. *If $F_1, \ldots, F_{r_1} \in R^{r_0}$ are a Gröbner basis, then the $G^{(i,\alpha)}$ of (1-3) form a Gröbner basis of $\ker(\varphi_1 \colon R^{r_1} \to R^{r_0})$ with respect to the induced monomial order $>_1$ defined in (1-4). In particular, $F_1, \ldots, F_{r_1}$ generate the kernel of φ_1.*

Proof. As mentioned in the proof of Buchberger's criterion, the coefficients $g_1, \ldots, g_r$ of a remainder $g_1 e_1 + \cdots + g_r e_r$ resulting from division by the $G^{(i,\alpha)}$ satisfy condition (2a) of Theorem 1.3 when divided by $f_1, \ldots, f_r$. Hence, no cancellation can occur in the sum $g_1 \mathbf{L}(f_1) + \cdots + g_r \mathbf{L}(f_r)$, and $g_1 f_1 + \cdots g_r f_r = 0$ only if $g_1 = \ldots = g_r = 0$. Therefore, the collection of $\mathbf{L}(G^{(i,\alpha)})$ generate the leading term ideal $\mathbf{L}(\ker \varphi_1)$. $\qquad \square$

We have reached the goal of this section, an algorithmic proof of Hilbert's syzygy theorem.

Proof of Hilbert's syzygy theorem. Let M be a finitely generated R-module with presentation

$$R^r \xrightarrow{\varphi} R^{r_0} \to M \to 0.$$

Regard φ as a matrix and, thus, its columns as a set of generators for $\mathrm{im}\,\varphi$. Starting from these generators, compute a minimal Gröbner basis $F_1, \ldots, F_{r_1}$ for $\mathrm{im}\,\varphi$ with respect to some global monomial $>_0$ order on R^{r_0}. Now consider the induced monomial order $>_1$ on R^{r_1}, and let $G^{(i,\alpha)} \in R^{r_1}$ denote the syzygies obtained by applying Buchberger's criterion to $F_1, \ldots, F_{r_1}$. By Corollary 1.11, the $G^{(i,\alpha)}$ form a Gröbner basis for the kernel of the map $\varphi_1 \colon R^{r_1} \to R^{r_0}$, so we may now repeat this process.

Let ℓ be the maximal k such that the variable x_k occurs in some leading term $\mathbf{L}(F_j)$. Sort $F_1, \ldots, F_{r_1}$ so that whenever $j < i$, the exponent of x_ℓ in $\mathbf{L}(F_j)$ is less than or equal to the exponent of x_ℓ in $\mathbf{L}(F_i)$. In this way, none of the variables $x_\ell, \ldots, x_n$ will occur in a leading term $\mathbf{L}(G^{(i,\alpha)})$. Thus the process will terminate after at most n steps. $\qquad\square$

Note that there are a number of choices allowed in the algorithm in the proof of Hilbert's syzygy theorem. In particular, we may order each set of Gröbner basis elements as we see fit.

Example 1.12. Consider the ideal $I = \langle f_1, \ldots, f_5 \rangle \subset R = \mathbb{K}[w, x, y, z]$ generated by the polynomials

$$f_1 = w^2 - xz, \quad f_2 = wx - yz, \quad f_3 = x^2 - wy, \quad f_4 = xy - z^2, \quad f_5 = y^2 - wz.$$

To compute a finite free resolution of $M = R/I$ using the method of the proof of Hilbert's syzygy theorem, we use the degree reverse lexicographic order on R. The algorithm successively produces three syzygy matrices φ_1, φ_2, and φ_3, which we present in a compact way as follows.

$\boldsymbol{w^2} - xz$	$-x$	y	0	z	0	$-y^2 + wz$
$\boldsymbol{wx} - yz$	$\boldsymbol{w}$	$-x$	$-y$	0	z	z^2
$\boldsymbol{x^2} - wy$	$-z$	$\boldsymbol{w}$	0	$-y$	0	0
$\boldsymbol{xy} - z^2$	0	0	$\boldsymbol{w}$	$\boldsymbol{x}$	$-y$	yz
$\boldsymbol{y^2} - wz$	0	0	$-z$	$-w$	$\boldsymbol{x}$	$\boldsymbol{w^2}$
	0	y	$-x$	$\boldsymbol{w}$	$-z$	1
	$-y^2 + wz$	z^2	$-wy$	yz	$-w^2$	$\boldsymbol{x}$

All initial terms are printed in bold. The first column of this table is the transpose of the matrix φ_1. It contains the original generators for I which, as Buchberger's criterion criterion shows, already form a Gröbner basis for I. The syzygy matrix φ_2 resulting from the algorithm is the 5×6 matrix in the middle of our table. Note that, for instance, $M_4 = \langle w, x \rangle$ can be read from the 4th row of φ_2.

By Corollary 1.11, we know that the columns of φ_2 form a Gröbner basis for $\ker(\varphi_1)$ with respect to the induced monomial order on R^5. Buchberger's criterion criterion applied to these Gröbner basis elements yields a 6×2 syzygy matrix φ_3, whose transpose is printed in the two bottom rows of the table above. Note that there are no syzygies on the two columns of φ_3 because the initial terms of these vectors lie with different basis vectors.

To summarize, we obtain a free resolution of the form

$$0 \longrightarrow R^2 \xrightarrow{\varphi_3} R^6 \xrightarrow{\varphi_2} R^5 \xrightarrow{\varphi_1} R \longrightarrow R/I \longrightarrow 0.$$

Observe that, in general, once we have the initial terms of a Gröbner basis for I, we can easily compute the initial terms of the Gröbner bases for all syzygy modules, that is, all bold face entries of our table. This gives us an idea on the amount of computation that will be needed to obtain the full free resolution.

If the polynomial ring is graded, say $R = S = \mathbb{K}[x_0, \ldots, x_n]$ is the homogeneous coordinate ring of $\mathbb{P}^n$, and M is a finitely generated graded S-module, then the resolution computed through the proof of Hilbert's syzygy theorem is homogeneous as well. However, this resolution is typically not minimal. In Example 1.12, the last column of φ_2 is in the span of the previous columns, as can be seen from the first row of φ_3^t.

Example 1.13. Recall that in Example 1.6, we considered the ideal I of 3×3 minors of a generic 3×5 matrix over $S = \mathbb{K}[x_1, \ldots, z_5]$ with the standard grading. The algorithm in the proof of Definition 1.2 produces a resolution of S/I of the form

$$0 \longrightarrow S(-5)^6 \longrightarrow S(-4)^{15} \longrightarrow S(-3)^{10} \longrightarrow S \longrightarrow S/I \longrightarrow 0$$

because I is generated by 10 Gröbner basis elements, there are altogether 15 minimal generators of the M_i ideals, and 6 of the monomial ideal M_i have 2 generators. In this case, the resolution is minimal for degree reasons.

Exercise 1.14. Let I be a Borel-fixed monomial ideal. Prove that in this case, the algorithm in the proof of Hilbert's syzygy theorem produces a minimal free resolution of I. Compute the differentials explicitly and compare your result with the complex of S. Eliahou and Kervaire [1990] (see also [Peeva and Stillman 2008]).

2. Petri's Theorem

One of the first theoretical applications of Gröbner bases is Petri's analysis of the generators of the homogeneous ideal of a canonically embedded curve. Petri was the last student of Max Noether, and he acknowledges help from Emmy Noether in his thesis. As Emmy Noether was a student of Gordan, it is quite possible that Petri became aware of the concept of Gröbner bases through his communication with her, but we do not know if this was the case.

Let C be a smooth projective curve of genus g over $\mathbb{C}$. Let

$$\omega_1, \ldots, \omega_g \in H^0(C, \omega_C)$$

be a basis of the space of holomorphic differential forms on C and consider the canonical map

$$\iota : C \to \mathbb{P}^{g-1} \quad \text{given by} \quad p \mapsto [\omega_1(p) : \cdots : \omega_g(p)].$$

The map ι is an embedding unless C is hyperelliptic. We will assume that C is not hyperelliptic. Let $S := \mathbb{C}[x_1, \ldots, x_g]$ be the homogeneous coordinate ring of $\mathbb{P}^{g-1}$, and let $I_C \subset S$ be the homogeneous ideal of C.

Petri's theorem (Theorem 2.1) [Petri 1923]. *The homogeneous ideal of a canonically embedded curve is generated by quadrics unless*

- *C is trigonal (i.e., there is a 3:1 holomorphic map $C \to \mathbb{P}^1$) or*

- *C is isomorphic to a smooth plane quintic. In this case, $g = 6$.*

Petri's theorem received much attention through the work of Mark Green [1984], who formulated a conjectural generalization to higher syzygies of canonical curves in terms of the Clifford index. We will not report here on the impressive progress made on this conjecture in the last two decades, but refer instead to [Aprodu and Farkas 2011; Aprodu and Nagel 2010; Aprodu and Voisin 2003; Green and Lazarsfeld 1986; Hirschowitz and Ramanan 1998; Mukai 1992; Schreyer 1986; 1991; 2003; Voisin 1988; 2002; 2005] for further reading.

In the cases of the exceptions in Petri's theorem, also Babbage [1939] observed that the ideal cannot be generated by quadrics alone. If $D := p_1 + \cdots + p_d$ is a divisor of degree d on C, then the linear system $|\omega_C(-D)|$ is cut out by hyperplanes through the span $\overline{D}$ of the points $p_i \in C \subset \mathbb{P}^{g-1}$. Thus Riemann–Roch implies that

$$h^0(C, \mathcal{O}_C(D)) = d + 1 - g + \operatorname{codim} \overline{D} = d - \dim \overline{D}.$$

Hence the three points of a trigonal divisor span only a line, and by Bézout's theorem, we need cubic generators in the generating set of its vanishing ideal.

Similarly, in the second exceptional case, the 5 points of a g_5^2 are contained in a unique conic in the plane they span, and quadrics alone do not cut out the curve.

The first step of Petri's analysis builds upon a proof by Max Noether.

Theorem 2.1 [Noether 1880]. *A nonhyperelliptic canonical curve $C \subset \mathbb{P}^{g-1}$ is projectively normal, i.e., the maps*

$$H^0(\mathbb{P}^{g-1}, \mathcal{O}(n)) \to H^0(C, \omega_C^{\otimes n})$$

are surjective for every n.

Proof. Noether's proof is a clever application of the basepoint-free pencil trick. This is a method which, according to Mumford, Zariski taught to all of his students. Let $|D|$ be a basepoint-free pencil on a curve, and let $\mathcal{L}$ be a further line bundle on C. Then the Koszul complex

$$0 \to \Lambda^2 H^0(\mathcal{O}_C(D)) \otimes \mathcal{L}(-D) \to H^0(\mathcal{O}_C(D)) \otimes \mathcal{L} \to \mathcal{L}(D) \to 0$$

is an exact sequence. To see this, note that locally, at least one section of the line bundle $\mathcal{O}_C(D)$ does not vanish. Thus the kernel of the multiplication map

$$H^0(\mathcal{O}_C(D)) \otimes H^0(\mathcal{L}) \to H^0(\mathcal{L}(D))$$

is isomorphic with $H^0(\mathcal{L}(-D))$. (Note $\Lambda^2 H^0(\mathcal{O}_C(D)) \cong \mathbb{C}$, as $h^0(\mathcal{O}_C(D)) = 2$.)

Consider $p_1, \ldots, p_g$ general points on C and the divisor $D = p_1 + \cdots + p_{g-2}$ built from the first $g - 2$ points. Then the images of these points span $\mathbb{P}^{g-1}$ and the span of any subset of less than $g - 1$ points intersects the curve in no further points. Choose a basis $\omega_1, \ldots, \omega_g \in H^0(\omega_C)$ that is, up to scalars, dual to these points, i.e., $\omega_i(p_j) = 0$ for $i \neq j$ and $\omega_i(p_i) \neq 0$. Then $|\omega_C(-D)|$ is a basepoint-free pencil spanned by ω_{g-1}, ω_g. If we apply the basepoint-free pencil trick to this pencil and $\mathcal{L} = \omega_C$, then we obtain the sequence

$$0 \to \Lambda^2 H^0 \omega_C(-D) \otimes H^0 \mathcal{O}_C(D) \to H^0 \omega_C(-D) \otimes H^0 \omega_C \xrightarrow{\mu} H^0 \omega_C^{\otimes 2}(-D),$$

and the image of

$$\mu : H^0(\omega_C(-D)) \otimes H^0(\omega_C) \to H^0(\omega_C^{\otimes 2}(-D)) \qquad (2\text{-}1)$$

is $2g - 1$ dimensional because $h^0(\omega_C(-D)) = 2$ and $h^0(\mathcal{O}_C(D)) = 1$. Thus μ in (2-1) is surjective, since $h^0(\omega_C^{\otimes 2}(-D)) = 2g - 1$ holds by Riemann–Roch. On the other hand,

$$\omega_1^{\otimes 2}, \ldots, \omega_{g-2}^{\otimes 2} \in H^0(\omega_C^{\otimes 2})$$

represent linearly independent elements of $H^0(\omega_C^{\otimes 2})/H^0(\omega_C^{\otimes 2}(-D))$, hence represent a basis, and the map $H^0(\omega_C) \otimes H^0(\omega_C) \to H^0(\omega_C^{\otimes 2})$ is surjective as well. This proves quadratic normality.

The surjectivity of the multiplication maps

$$H^0(\omega_C^{\otimes n-1}) \otimes H^0(\omega_C) \to H^0(\omega_C^{\otimes n})$$

for $n \geq 3$ is similar, but easier: $\omega_1^{\otimes n}, \ldots, \omega_{g-2}^{\otimes n} \in H^0(\omega_C^{\otimes n})$ are linearly independent modulo the codimension $g - 2$ subspace $H^0(\omega_C^{\otimes n}(-D))$, and the map

$$H^0(\omega_C^{\otimes n-1}) \otimes H^0(\omega_C(-D)) \to H^0(\omega_C^{\otimes n}(-D))$$

is surjective simply because $H^1(\omega_C^{\otimes n-2}(D)) = 0$ for $n \geq 3$. □

Corollary 2.2. *The Hilbert function of the coordinate ring of a canonical curve takes the values*

$$\dim(S/I_C)_n = \begin{cases} 1 & \text{if } n = 0 \\ g & \text{if } n = 1 \\ (2n - 1)(g - 1) & \text{if } n \geq 2. \end{cases}$$

□

Proof of Petri's theorem. Petri's analysis begins with the map μ in (2-1) above. Choose homogeneous coordinates $x_1, \ldots, x_g$ such that $x_i \mapsto \omega_i$. Since $\omega_i \otimes \omega_j \in H^0(\omega_C^{\otimes 2}(-D))$ for $1 \leq i < j \leq g-2$, we find the polynomials

$$f_{ij} := x_i x_j - \sum_{r=1}^{g-2} a_{ij}^r x_r - b_{ij} \in I_C, \tag{2-2}$$

where the a_{ij}^r and b_{ij} are linear and quadratic, respectively, in $\mathbb{C}[x_{g-1}, x_g]$. We may choose a monomial order such that $\mathbf{L}(f_{ij}) = x_i x_j$. Since $\binom{g-2}{2} = \binom{g+1}{2} - (3g-3)$, these quadrics span $(I_C)_2$. On the other hand, they do not form a Gröbner basis for I_C because the $(g-2)\binom{n+1}{2} + (n+1)$ monomials $x_i^k x_{g-1}^\ell x_g^m$ with $i = 1, \ldots, g-2$ and $k + \ell + m = n$ represent a basis for $(S/\langle x_i x_j \mid 1 \leq i < j \leq g-2\rangle)_n$, which is still larger. We therefore need $g-3$ further cubic Gröbner basis elements. To find these, Petri considers the basepoint-free pencil trick applied to $|\omega_C(-D)|$ and $\mathcal{L} = \omega_C^{\otimes 2}(-D)$. The cokernel of the map

$$H^0(\omega_C(-D)) \otimes H^0(\omega_C^{\otimes 2}(-D)) \to H^0(\omega_C^{\otimes 3}(-2D)) \tag{2-3}$$

has dimension $h^1(\omega_C) = 1$. To find the missing element in $H^0(\omega_C^{\otimes 3}(-2D))$, Petri considers the linear form $\alpha_i = \alpha_i(x_{g-1}, x_g)$ in the pencil spanned by x_{g-1}, x_g that defines a tangent hyperplane to C at p_i. Then $\alpha_i \omega_i^{\otimes 2} \in H^0(\omega_C^{\otimes 3}(-2D))$ because $\omega_i^{\otimes 2}$ vanishes quadratically at all points $p_j \neq p_i$, while α_i vanishes doubly at p_i. Not all of these elements can be contained in the image of (2-3), since otherwise we would find $g-2$ further cubic Gröbner basis elements of type

$$\alpha_i x_i^2 + \text{ lower order terms,}$$

where a lower order term is a term that is at most linear in $x_1, \ldots, x_{g-2}$. As this is too many, at least one of the $\alpha_i \omega_i^{\otimes 2}$ spans the cokernel of the map (2-3).

We now argue by uniform position. Since C is irreducible, the behavior of $\alpha_i \omega_i^{\otimes 2}$ with respect to spanning of the cokernel is the same for any general choice of points $p_1, \ldots, p_g$. So for general choices, each of these elements span the cokernel, and after adjusting scalars, we find that

$$G_{k\ell} := \alpha_k x_k^2 - \alpha_\ell x_\ell^2 + \text{ lower order terms} \tag{2-4}$$

are in I_C. Note that $G_{k\ell} = -G_{\ell k}$ and $G_{k\ell} + G_{\ell m} = G_{km}$. So this gives only $g-3$ further equations with leading terms $x_k^2 x_{g-1}$ for $k = 1, \ldots, g-3$ up to a scalar. The last Gröbner basis element is a quartic H with leading term $\mathbf{L}(H) = x_{g-2}^3 x_{g-1}$, which we can obtain as a remainder of the Buchberger test applied to $x_{g-2} G_{k,g-2}$. There are no further Gröbner basis elements, because the quotient S/J of S by

$$J := \langle x_i x_j, x_k^2 x_{g-1}, x_{g-2}^3 x_{g-1} \mid 1 \le i < j \le g-2, 1 \le k \le g-3 \rangle$$

has the same Hilbert function as S/I_C. Hence $\mathbf{L}(I_C) = J$.

We now apply Buchberger's test to $x_k f_{ij}$ for a triple of distinct indices $1 \le i, j, k \le g-2$. Division with remainder yields a syzygy

$$x_k f_{ij} - x_j f_{ik} + \sum_{r \ne k} a_{ij}^r f_{rk} - \sum_{r \ne j} a_{ik}^r f_{rj} + \rho_{ijk} G_{kj} = 0 \qquad (2\text{-}5)$$

for a suitable coefficient $\rho_{ijk} \in \mathbb{C}$. (Moreover, comparing coefficients, we find that $a_{ij}^k = \rho_{ijk} \alpha_k$ holds. In particular, Petri's coefficients ρ_{ijk} are symmetric in i, j, k, since a_{ij}^k is symmetric in i, j.) Since C is irreducible, we have that for a general choice of $p_1, \dots, p_g$, either all coefficients $\rho_{ijk} \ne 0$ or all $\rho_{ijk} = 0$. In the first case, the cubics lie in the ideal generated by the quadrics.

In the second case, the f_{ij} are a Gröbner basis by themselves. Thus the zero locus $V(f_{ij} \mid 1 \le i < j \le g-2)$ of the quadrics f_{ij} define an ideal of a scheme X of dimension 2 and degree $g-2$. Since C is irreducible and nondegenerate, the surface X is irreducible and nondegenerate as well. Thus $X \subset \mathbb{P}^{g-2}$ is a surface of minimal degree. These were classified by Bertini; see, for instance, [Eisenbud and Harris 1987b]. Either X is a rational normal surface scroll, or X is isomorphic to the Veronese surface $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$. In the case of a scroll, the ruling on X cuts out a g_3^1 on C by Riemann–Roch. In the case of the Veronese surface, the preimage of C in $\mathbb{P}^2$ is a plane quintic. $\qquad \square$

Perhaps the most surprising part of Petri's theorem is this: either I_C is generated by quadrics or there are precisely $g-3$ minimal cubic generators. It is a consequence of the irreducibility of C that no value in between 0 and $g-3$ is possible for the number of cubic generators. If we drop the assumption of irreducibility, then there are canonical curves with $1, \dots, g-5$ or $g-3$ cubic generators. For example, if we take a stable curve $C = C_1 \cup C_2$ with two smooth components of genus $g_i \ge 1$ intersecting in three points, so that C has genus $g = g_1 + g_2 + 2$, then the dualizing sheaf ω_C is very ample and the three intersection points lie on a line by the residue theorem. For general curves C_1 and C_2 of genus $g_i \ge 3$ for $i \in \{1, 2\}$, the ideal I_C has precisely one cubic generator; see [Schreyer 1991]. However, we could not find such an example with precisely $g-4$ generators. For genus $g = 5$, one cubic generator is excluded by the structure theorem of Buchsbaum–Eisenbud, and obstructions for larger g are unclear to us.

Conjecture 2.3. *Let $A = S/I$ be a graded artinian Gorenstein algebra with Hilbert function $\{1, g-2, g-2, 1\}$. Then I has $0, 1, \dots, g-5$ or $g-3$ cubic minimal generators.*

The veracity of this conjecture would imply the corresponding statement for reducible canonical curves because the artinian reduction $A := S/(I_C + \langle \ell_1, \ell_2 \rangle)$ of S/I_C, for general linear forms ℓ_1, ℓ_2, has Hilbert function $\{1, g-2, g-2, 1\}$.

Petri's analysis has been treated by Mumford [1975], and also in [Arbarello et al. 1985; Saint-Donat 1973; Shokurov 1971]. From our point of view, Gröbner bases and the use of uniform position simplify and clarify the treatment quite a bit. Mumford [1975] remarks that we now have seen all curves at least once, following a claim made in [Petri 1923]. We disagree with him on this point. If we introduce indeterminates for all of the coefficients in Petri's equations, then the scheme defined by the condition on the coefficients that f_{ij}, G_{kl}, and H form a Gröbner basis can have many components [Schreyer 1991; Little 1998]. It is not clear to us how to find the component corresponding to smooth curves, much less how to find closed points on this component.

3. Finite length modules and space curves

In the remaining part of these lectures, we report on how to find all curves in a Zariski open subset of the moduli space $\mathfrak{M}_g$ of curves of genus g for small g. In Section 4, we report on the known *unirationality* results for these moduli spaces. But first, we must discuss a method to explicitly construct space curves.

In this section, a *space curve* $C \subset \mathbb{P}^3$ will be a Cohen–Macaulay subscheme of pure dimension 1; in particular, C has no embedded points. We denote by $\mathcal{I}_C$ the ideal sheaf of C and by $I_C = \sum_{n \in \mathbb{Z}} H^0(\mathbb{P}^3, \mathcal{I}_C(n))$ the homogeneous ideal of C. The goal of this section is to construct a curve C of genus g and degree d. To do so, we will use work of Rao, who showed that the construction of C is equivalent to the creation of its Hartshorne–Rao module (see Rao's theorem).

Definition 3.1. The *Hartshorne–Rao module* of C is the finite length module

$$M = M_C := \sum_{n \in \mathbb{Z}} H^1(\mathbb{P}^3, \mathcal{I}_C(n)) \subset \sum_{n \in \mathbb{Z}} H^0(\mathbb{P}^3, \mathcal{O}(n)) \cong S := \mathbb{K}[x_0, .., x_3].$$

The Hartshorne–Rao module measures the deviation of C from being projectively normal. Furthermore, M_C plays an important role in liaison theory of curves in $\mathbb{P}^3$, which we briefly recall now.

Let $S := \mathbb{K}[x_0, \ldots, x_3]$ and $S_C := S/I_C$ denote the homogeneous coordinate ring of $\mathbb{P}^3$ and $C \subset \mathbb{P}^3$, respectively. By the Auslander–Buchsbaum–Serre formula [Eisenbud 1995, Theorem 19.9], S_C has projective dimension $\mathrm{pd}_S S_C \leq 3$. Thus its minimal free resolution has the form

$$0 \leftarrow S_C \leftarrow S \leftarrow F_1 \leftarrow F_2 \leftarrow F_3 \leftarrow 0,$$

with free graded modules $F_i = \oplus S(-j)^{\beta_{ij}}$. By the same formula in the local case, we see that the sheafified $\mathcal{G} := \ker(\widetilde{F}_1 \to \mathcal{O}_{\mathbb{P}^3})$ is always a vector bundle, and

$$0 \leftarrow \mathcal{O}_C \leftarrow \mathcal{O}_{\mathbb{P}^3} \leftarrow \bigoplus_j \mathcal{O}_{\mathbb{P}^3}(-j)^{\beta_{1j}} \leftarrow \mathcal{G} \leftarrow 0 \qquad (3\text{-}1)$$

is a resolution by locally free sheaves. If C is arithmetically Cohen–Macaulay, then $F_3 = 0$ and $\mathcal{G}$ splits into a direct sum of line bundles. In this case, the ideal I_C is generated by the maximal minors of $F_1 \leftarrow F_2$ by the Hilbert–Burch theorem [Hilbert 1890; Burch 1968; Eisenbud 1995]. In general, we have

$$M_C \cong \sum_{n \in \mathbb{Z}} H^2(\mathbb{P}^3, \mathcal{G}(n)) \quad \text{and} \quad \sum_{n \in \mathbb{Z}} H^1(\mathbb{P}^3, \mathcal{G}(n)) = 0. \qquad (3\text{-}2)$$

We explain now why curves linked by an even number of liaison steps have, up to a twist, the same Hartshorne–Rao module, thus illustrating its connection to liaison theory. We will then mention Rao's theorem, which states that the converse also holds.

Suppose that $f, g \in I_C$ are homogeneous forms of degree d and e without common factors. Let $X := V(f, g)$ denote the corresponding complete intersection, and let C' be the residual scheme defined by the homogeneous ideal $I_{C'} := (f, g) : I_C$ [Peskine and Szpiro 1974]. The locally free resolutions of $\mathcal{O}_C$ and $\mathcal{O}_{C'}$ are closely related, as follows. Applying $\mathcal{E}xt^2(-, \omega_{\mathbb{P}^3})$ to the sequence

$$0 \to \mathcal{I}_{C/X} \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

gives

$$0 \leftarrow \mathcal{E}xt^2(\mathcal{I}_{C/X}, \omega_{\mathbb{P}^3}) \leftarrow \omega_X \leftarrow \omega_C \leftarrow 0.$$

From $\omega_X \cong \mathcal{O}_X(d + e - 4)$, we conclude that $\mathcal{E}xt^2(\mathcal{I}_{C/X}, \mathcal{O}_{\mathbb{P}^3}(-d - e)) \cong \mathcal{O}_{C'}$, and hence $\mathcal{I}_{C'/X} \cong \omega_C(-d - e + 4)$. Now the mapping cone of

$$
\begin{array}{ccccccccc}
0 & \longleftarrow & \mathcal{O}_C & \longleftarrow & \mathcal{O}_{\mathbb{P}^3} & \longleftarrow & \bigoplus_j \mathcal{O}_{\mathbb{P}^3}(-j)^{\beta_{1j}} & \longleftarrow & \mathcal{G} & \longleftarrow & 0 \\
& & \uparrow & & \uparrow{\scriptstyle\cong} & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & \mathcal{O}_X & \longleftarrow & \mathcal{O}_{\mathbb{P}^3} & \longleftarrow & \mathcal{O}_{\mathbb{P}^3}(-d) \oplus \mathcal{O}_{\mathbb{P}^3}(-e) & \longleftarrow & \mathcal{O}_{\mathbb{P}^3}(-d - e) & \longleftarrow & 0
\end{array}
$$

dualized with $\mathcal{H}om(-, \mathcal{O}_{\mathbb{P}^3}(-d - e))$ gives

$$
\begin{array}{ccccccccc}
0 \to & \mathcal{O}_{\mathbb{P}^3}(-d - e) & \longrightarrow & \bigoplus_j \mathcal{O}_{\mathbb{P}^3}(j - d - e)^{\beta_{1j}} & \longrightarrow & \mathcal{G}^*(-d - e) & \longrightarrow\!\!\!\!\rightarrow & \mathcal{I}_{C'/X} \\
& {\scriptstyle\cong}\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \to & \mathcal{O}_{\mathbb{P}^3}(-d - e) & \longrightarrow & \mathcal{O}_{\mathbb{P}^3}(-e) \oplus \mathcal{O}_{\mathbb{P}^3}(-d) & \longrightarrow & \mathcal{O}_{\mathbb{P}^3} & \longrightarrow\!\!\!\!\rightarrow & \mathcal{O}_X,
\end{array}
$$

which yields the following locally free resolution of $\mathcal{O}_{C'}$:

$$0 \to \bigoplus_j \mathcal{O}_{\mathbb{P}^3}(j - d - e)^{\beta_{1j}} \to \mathcal{G}^*(-d - e) \oplus \mathcal{O}_{\mathbb{P}^3}(-e) \oplus \mathcal{O}_{\mathbb{P}^3}(-d)$$
$$\to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{C'} \to 0.$$

In particular, after truncating this complex to resolve $I_{C'}$, one sees that

$$M_{C'} := \sum_{n \in \mathbb{Z}} H^1(\mathbb{P}^3, \mathcal{I}_{C'}(n)) \cong \sum_{n \in \mathbb{Z}} H^1(\mathbb{P}^3, \mathcal{G}^*(n - d - e))$$

$$\cong \sum_{n \in \mathbb{Z}} H^2(\mathbb{P}^3, \mathcal{G}(d + e - 4 - n))^* \cong \mathrm{Hom}_{\mathbb{K}}(M_C, \mathbb{K})(4 - d - e).$$

Thus curves that are related via an even number of liaison steps have the same Hartshorne–Rao module up to a twist. Rao's famous result says that the converse is also true.

Rao's theorem (Theorem 3.2) [Rao 1978]. *The even liaison classes of curves in $\mathbb{P}^3$ are in bijection with finite length graded S- modules up to twist.* $\square$

Therefore the difficulty in constructing the desired space curve C (of degree d and genus g) lies completely in the construction of the appropriate Hartshorne–Rao module $M = M_C$. Upon constructing M, we may then obtain the desired ideal sheaf $\mathcal{I}_C$ as follows. Assume that we have a free S-resolution of M_C,

$$0 \leftarrow M_C \leftarrow F_0 \leftarrow F_1 \leftarrow F_2 \leftarrow F_3 \leftarrow F_4 \leftarrow 0,$$

with $F_i = \bigoplus_j S(-j)^{\beta_{ij}}$. Let $\mathcal{F} := \widetilde{N}$ be the sheafification of $N := \ker(F_1 \to F_0)$, the second syzygy module of M. In this case, $\mathcal{F}$ will be a vector bundle without line bundle summands such that $H^1_*(\mathcal{F}) \cong H^1_*(\mathcal{I}_C)$ and $H^2_*(\mathcal{F}) = 0$. Here, we have used the notation $H^i_*(\mathcal{F}) := \bigoplus_n H^i(\mathcal{F}(n))$. If we constructed the correct Hartshorne–Rao module M, then taking $\mathcal{L}_1$ and $\mathcal{L}_2$ to be appropriate choices of direct sums of line bundles on $\mathbb{P}^3$, a general homomorphism $\varphi \in \mathrm{Hom}(\mathcal{L}_1, \mathcal{F} \oplus \mathcal{L}_2)$ will produce the desired curve C, as we will obtain $\mathcal{I}_C$ as the cokernel of a map φ of the bundles

$$0 \longrightarrow \mathcal{L}_1 \overset{\varphi}{\longrightarrow} \mathcal{F} \oplus \mathcal{L}_2 \longrightarrow \mathcal{I}_C \longrightarrow 0.$$

To compute the rank of $\mathcal{F}$ and to choose the direct sums of line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$, we now make plausible assumptions about the Hilbert function of M_C. We illustrate this approach in the example of the construction of a smooth linearly normal curve C of degree $d = 11$ and genus $g = 10$. Since $2d > 2g - 2$, the line bundle $\mathcal{O}_C(2)$ is already nonspecial. Hence by Riemann–Roch, we have that $h^0(\mathcal{O}_C(2)) = 22 + 1 - 10 = 13$.

Remark 3.2. If we assume that C is a curve of maximal rank, i.e., that all maps $H^0(\mathcal{O}_{\mathbb{P}^3}(n)) \to H^0(\mathcal{O}_C(n))$ are either injective or surjective, then we can

compute the Hilbert function of M_C and I_C. Note that being of maximal rank is an open condition, so among the curves in the union $\mathcal{H}_{d,g}$ of the component of the Hilbert scheme $\mathrm{Hilb}_{dt+1-g}(\mathbb{P}^3)$ containing smooth curves, maximal rank curves form an open (and hopefully nonempty) subset. There is a vast literature on the existence of maximal rank curves; see, for example, [Fløystad 1991].

To gain insight into the Betti numbers of $M = M_C$, we use Hilbert's formula for the Hilbert series:

$$h_M(t) = \sum_{n \in \mathbb{Z}} \dim M_n t^n = \frac{\sum_{i=0}^{3}(-1)^i \sum_j \beta_{ij} t^j}{(1-t)^4}.$$

Since $h_{M_C}(t) = 3t^2 + 4t^3$ by our maximal rank assumption (Remark 3.2), we have

$$(1-t)^4 h_M(t) = 3t^2 - 8t^3 + 2t^4 + 12t^5 - 13t^6 + 4t^7,$$

and thus the Betti table of M must be

$$\beta(M) = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 8 & 2 & . & . \\ 3 & . & . & 12 & 13 & 4 \end{array}$$

if we assume that M has a so called *natural resolution*, which means that for each degree j at most one β_{ij} is nonzero. Note that having a natural resolution is an open condition in a family of modules with constant Hilbert function.

Table 1 provides a detailed look at the Hilbert functions relevant to our computation. From these we see that $H_*^0(\mathcal{O}_C)$ and $S_C = S/I_C$ will have the potential Betti tables at the top of the next page, if we assume that they also have natural resolutions.

n	$h^1(\mathcal{I}_C(n))$	$h^0(\mathcal{O}_C(n))$	$h^0(\mathcal{O}_{\mathbb{P}^3}(n))$	$h^0(\mathcal{I}_C(n))$
0	0	1	1	0
1	0	4	4	0
2	3	13	10	0
3	4	24	20	0
4	0	35	35	0
5	0	46	56	10
6	0	55	84	29

Table 1. With our maximal rank assumption of Remark 3.2, this table provides the relevant Hilbert functions in the case $d = 11$ and $g = 10$.

$$\beta(H^0_*(\mathcal{O}_C)) \;=\; \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1 & . & . \\ 1 & . & . & . \\ 2 & 3 & 8 & 2 \\ 3 & . & . & 2 \end{array} \quad \text{and} \quad \beta(S_C) \;=\; \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & . & . & . \\ 1 & . & . & . & . \\ 2 & . & . & . & . \\ 3 & . & . & . & . \\ 4 & . & 10 & 13 & 4 \end{array}.$$

Comparing these Betti tables, we find the following plausible choices of $\mathcal{F}$, $\mathcal{L}_1$, and $\mathcal{L}_2$:

- We choose $\mathcal{F} := \widetilde{N}$, where $N = \ker(\psi : S^8(-3) \to S^3(-2))$ is a sufficiently general 3×8 matrix of linear forms; in particular, rank $\mathcal{F} = 5$.

- Let $\mathcal{L}_1 := \mathcal{O}^2(-4) \oplus \mathcal{O}^2(-5)$ and $\mathcal{L}_2 := 0$.

- The map $\varphi \in \mathrm{Hom}(\mathcal{L}_1, \mathcal{F})$ is a sufficiently general homomorphism. Since the map $F_2 \to H^0_*(\mathcal{F})$ is surjective, the choice of φ amounts to choosing an inclusion $\mathcal{O}^2(-5) \to \mathcal{O}^{12}(-5)$, i.e., a point in the Grassmannian $\mathbb{G}(2, 12)$.

- Finally, $\mathcal{I}_C = \mathrm{coker}\,\varphi$.

It is not clear that general choices as above will necessarily yield a smooth curve. If the sheaf $\mathcal{H}om(\mathcal{L}_1, \mathcal{F} \oplus \mathcal{L}_2)$ happens to be generated by its global sections $\mathrm{Hom}(\mathcal{L}_1, \mathcal{F} \oplus \mathcal{L}_2)$, then a Bertini-type theorem as in [Kleiman 1974] would apply. However, since we have to take all generators of $H^0_*(\mathcal{F})$ in degree 4, this is not the case. On the other hand, there is no obvious reason that $\mathrm{coker}\,\varphi$ should not define a smooth curve, and upon construction, it is easy to check the smoothness of such an example using a computer algebra system, such as Macaulay2 or Singular. Doing this, we find that general choices do lead to a smooth curve.

Exercise 3.3. Construct examples of curves of degree and genus as prescribed in [Hartshorne 1977, Figure 18 on page 354], including those which were open cases at the time of the book's publication.

4. Random curves

In this section, we explain how the ideas of Section 3 lead to a computer-aided proof of the unirationality of the moduli space $\mathfrak{M}_g$ of curves of genus g, when g is small. We will illustrate this approach by example, through the case of genus $g = 12$ and degree $d = 13$ in Theorem 4.5.

Definition 4.1. A variety X is called *unirational* if there exists a dominant rational map $\mathbb{A}^n \dashrightarrow X$. A variety X is called *uniruled* if there exists a dominant rational map $\mathbb{A}^1 \times Y \dashrightarrow X$ for some variety Y that does not factor through Y. A smooth projective variety X has *Kodaira dimension κ* if the section ring

$R_X := \sum_{n \geq 0} H^0(X, \omega_X^{\otimes n})$ of pluri-canonical forms on X has a Hilbert function with growth rate $h^0(\omega_X^{\otimes n}) \in O(n^\kappa)$. We say that X has *general type* if $\kappa = \dim X$, the maximal possible value.

Since the pluri-genera $h^0(\omega_X^{\otimes n})$ are birational invariants, being of general type does not depend on a choice of a smooth compactification. Thus we may also speak of general type for quasiprojective varieties.

Unirationality and general type are on opposite ends of birational geometry. If a variety is of general type, then there exists no rational curve through a general point of X [Kollár 1996, Corollary IV.1.11]. On the other hand, uniruled varieties have the pluri-canonical ring $R_X = (R_X)_0 = \mathbb{C}$ and thus (by convention) have Kodaira dimension $\kappa = -\infty$. In fact, even if X is unirational, then we can connect any two general points of X by a rational curve.

We now recall results concerning the unirationality of the moduli space $\mathfrak{M}_g$. There are positive results for small genus, followed by negative results for large genus.

Theorem 4.2 ([Severi 1921] for $g \leq 10$; [Sernesi 1981; Chang and Ran 1984] for $g = 12, 11, 13$; [Verra 2005] for $g = 14$.). *The moduli space $\mathfrak{M}_g$ of curves of genus g is unirational for $g \leq 14$.*

Theorem 4.3 [Harris and Mumford 1982; Eisenbud and Harris 1987a; Farkas 2006; 2009a; 2009b]. *The moduli space $\mathfrak{M}_g$ of curves of genus g is of general type for $g \geq 24$ or $g = 22$. The moduli space M_{23} has Kodaira dimension ≥ 2.*
                                                                                        $\square$

We call this beautiful theorem a negative result because it says that it will be very difficult to write down explicitly a general curve of large genus. Given a family of curves of genus $g \geq 24$ that pass through a general point of $\mathfrak{M}_g$, say via an explicit system of equations with varying coefficients, none of the essential coefficients is a free parameter. All of the coefficients will satisfy some complicated algebraic relations. On the other hand, in unirational cases, there exists a dominant family of curves whose parameters vary freely.

In principle, we can compute a dominating family explicitly along with a unirationality proof. In practice, this is often out of reach using current computer algebra systems; however, the following approach is feasible today in many cases. By replacing each free parameter in the construction of the family by a randomly chosen value in the ground field, the computation of an explicit example is possible. In particular, over a finite field $\mathbb{F}$, where it is natural to use the constant probability distribution on $\mathbb{F}$, a unirationality proof brings with it the possibility of choosing random points in $\mathfrak{M}_g(\mathbb{F})$, i.e., to compute a random curve. These curves can then be used for further investigations of the moduli space, as well as to considerably simplify the existing unirationality proofs. The

advantage of using such random curves in the unirationality proof is that, with high probability, they will be smooth curves, while in a theoretical treatment, smoothness is always a delicate issue.

To begin this construction, we first need some information on the projective models of a general curve. This is the content of Brill– Noether theory. Let

$$W_d^r(C) := \{L \in \mathrm{Pic}^d(C) \mid h^0(C, L) \geq r + 1\} \subset \mathrm{Pic}^d(C)$$

denote the space of line bundles of degree d on C that give rise to a morphism $C \to \mathbb{P}^r$.

Theorem 4.4. *Let C be a smooth projective curve of genus g.*

(1) [Brill and Noether 1874] *At every point*, $\dim W_d^r(C) \geq \rho := g - (r+1)(g - d + r)$.

(2) [Griffiths and Harris 1980; Fulton and Lazarsfeld 1981] *If $\rho \geq 0$, then $W_d^r(C) \neq 0$, and if $\rho > 0$, then $W_d^r(C)$ is connected. Further, the tangent space of $W_d^r(C)$ at a point $L \in W_d^r(C) \setminus W_d^{r+1}(C)$ is*

$$T_L W_d^r(C) = \mathrm{Im}\, \mu_L^\perp \subset H^1(\mathcal{O}_C) = T_L \mathrm{Pic}^d(C),$$

where $\mu_L \colon H^0(L) \otimes H^0(\omega_C \otimes L^{-1}) \to H^0(\omega_C) = H^1(\mathcal{O}_C)^*$ denotes the Petri map.

(3) [Gieseker 1982] *If $C \in \mathfrak{M}_g$ is a general curve, then $W_d^r(C)$ is smooth of dimension ρ away from $W_d^{r+1}(C)$. More precisely, the Petri map μ_L is injective for all $L \in W_d^r(C) \setminus W_d^{r+1}(C)$.* $\qquad\square$

We now illustrate the computer-aided unirationality proof of $\mathfrak{M}_g$ by example, through the case $g = 12, d = 13$ [Schreyer and Tonoli 2002]. This case is not amongst those covered in [Sernesi 1981] or [Chang and Ran 1984] (which are $g = 11, d = 12$, $g = 12, d = 12$, and $g = 13, d = 13$). We are choosing the case $d = 12, g = 13$ because it illustrates well the difficulty of this construction. For $g = 14$, see [Verra 2005] and, for a computer aided unirationality proof, [Schreyer 2013]. For a related Macaulay2 package, see [von Bothmer et al. 2011].

Theorem 4.5. *Let $g = 12$ and $d = 13$. Then $\mathrm{Hilb}_{dt+1-g}(\mathbb{P}^3)$ has a component $\mathcal{H}_{d,g}$ that is unirational and dominates the moduli space $\mathfrak{M}_g$ of curves of genus g.*

Proof. This proof proceeds as follows. We first compute the Hilbert function and expected syzygies of the Hartshorne–Rao module $M = H_*^1(\mathcal{I}_C)$, the coordinate ring S_C, and the section ring $R := H_*^0(\mathcal{O}_C)$. We then use this information to choose generic matrices which realize the free resolution of M. Finally, we show that this construction leads to a family of curves that dominate $\mathfrak{M}_{12}$ and generically contains smooth curves.

We first choose r so that a general curve has a model of degree $d = 13$ in $\mathbb{P}^r$. In our case, we choose $r = 3$ so that $g - d + r = 2$. To compute the Hilbert function and expected syzygies of the Hartshorne–Rao module $M = H^1_*(\mathcal{I}_C)$, the coordinate ring $S_C = S/I_C$, and the section ring $R = H^0_*(\mathcal{O}_C)$, we assume the open condition that C has maximal rank, i.e.,

$$H^0(\mathbb{P}^3, \mathcal{O}(n)) \to H^0(C, L^n)$$

is of maximal rank for all n, as in Remark 3.2. In this case, $h_M(t) = 5t^2 + 8t^3 + 6t^4$, which has Hilbert numerator

$$h_M(t)(1-t)^4 = 5t^2 - 12t^3 + 4t^4 + 4t^5 + 9t^6 - 16t^{10} + 6t^{11}.$$

If M has a natural resolution, so that for each j at most one $\beta_{ij}(M)$ is nonzero, then M has the Betti table

$$\beta(M) = \quad
\begin{array}{c|ccccc}
 & 0 & 1 & 2 & 3 & 4 \\
\hline
2 & 5 & 12 & 4 & . & . \\
3 & . & . & 4 & . & . \\
4 & . & . & 9 & 16 & 6
\end{array}.$$

If we assume the open condition that S_C and R have natural syzygies as well, then their Betti tables are

$$\beta(S_C) = \quad
\begin{array}{c|ccc}
 & 0 & 1 & 2 \\
\hline
0 & 1 & . & . \\
1 & . & . & . \\
2 & 3 & 12 & 4 \\
3 & . & . & 2
\end{array}
\quad \text{and} \quad
\beta(R) = \quad
\begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\
\hline
0 & 1 & . & . & . \\
1 & . & . & . & . \\
2 & . & . & . & . \\
3 & . & . & . & . \\
4 & . & 2 & . & . \\
5 & . & 9 & 16 & 6
\end{array}$$

We conclude that once we have constructed the Hartshorne–Rao module $M = M_C$, say via its representation

$$0 \leftarrow M \leftarrow S^5(-2) \leftarrow S^{12}(-3),$$

we may choose $\mathcal{F}$ to be the kernel of

$$0 \leftarrow \mathcal{O}^5(-2) \leftarrow \mathcal{O}^{12}(-3) \leftarrow \mathcal{F} \leftarrow 0$$

and set $\mathcal{L}_1 := \mathcal{O}(-4)^4 \oplus \mathcal{O}^2(-5)$ and $\mathcal{L}_2 := 0$. Then C is determined by M and the choice of a point in $\mathbb{G}(2, 4)$. In particular, as mentioned earlier, constructing C is equivalent to constructing the finite length module M with the desired syzygies.

If we choose the presentation matrix ϕ of M to be given by a general (or random) 5×12 matrix of linear forms, then its cokernel will be a module with

Hilbert series $5t^2 + 8t^3 + 2t^4$. In other words, to get the right Hilbert function for M, we must force 4 linear syzygies. To do this, choose a general (or random) 12×4 matrix ψ of linear forms. Then

$$\ker(\psi^t \colon S^{12}(1) \to S^4(2))$$

has at least $12 \cdot 4 - 4 \cdot 10 = 8$ generators in degree 0. In fact, there are precisely 8 and a general point in $\mathbb{G}(5, 8)$ gives rise to a 12×5 matrix φ^t of linear forms. This means that $M := \operatorname{coker}(\varphi \colon S^{1}2(-3) \to S^5(-2))$ to have Hilbert series $5t^2 + 8t^3 + 6t^4$, due to the forced 4 linear syzygies.

Having constructed M, it remains to prove that this construction leads to a family of curves that dominates $\mathfrak{M}_{12}$. To this end, we compute a random example C, say over a finite prime field $\mathbb{F}_p$, and confirm its smoothness. Since we may regard our computation over $\mathbb{F}_p$ as the reduction modulo p of a construction defined over an open part of $\operatorname{Spec} \mathbb{Z}$, semicontinuity allows us to establish the existence of a smooth example defined over $\mathbb{Q}$ with the same syzygies.

We now consider the universal family $\mathfrak{W}_d^r \subset \mathfrak{Pic}^d$ over $\mathfrak{M}_g$ and a neighborhood of our example $(C, L) \in \mathfrak{Pic}^d$. Note that the codimension of $\mathfrak{W}_d^r$ is at most $(r + 1)(g - d + r) = 4 \cdot 2 = 8$. On the other hand, we claim that the Petri map μ_L for (C, L) is injective. (Recall the definition of μ_L from Theorem 4.4.) To see this, note that the Betti numbers of $H_*^0(\omega_C)$ correspond to the dual of the resolution of $H_*^0(\mathcal{O}_C)$, so

$$\beta(H_*^0(\omega_C)) \ = \ \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline -1 & 2 & . & . \\ 0 & 4 & 12 & 3 \\ 1 & . & . & . \\ 2 & . & . & 1 \end{array}$$

Thus there are no linear relations among the two generators in $H^0(\omega_C \otimes L^{-1})$, which means that the $\mu_L \colon H^0(L) \otimes H^0(\omega_C \otimes L^{-1})$ is injective. From this we see that $\dim W_d^r(C)$ has dimension 4 at (C, L), and the constructed family dominates for dimension reasons. □

The unirationality of $\mathfrak{M}_{15}$ and $\mathfrak{M}_{16}$ are open; however, these moduli spaces are uniruled.

Theorem 4.6 (Chang and Ran [1986; 1991]; see also [Bruno and Verra 2005; Farkas 2009a]). *The moduli space $\mathfrak{M}_{15}$ is rationally connected, and $\mathfrak{M}_{16}$ is uniruled.* □

To explain why the unirationality in these cases is more difficult to approach using the method of Theorem 4.5, we conclude with a brief discussion on the space models of curves of genus $g = 16$. By Brill–Noether theory, a general

curve C of genus 16 has finitely many models of degree $d = 15$ in $\mathbb{P}^3$. Again assuming the maximal rank condition of Remark 3.2, the Hartshorne–Rao module $M = H^1_*(\mathcal{I}_C)$ has Hilbert series

$$H_M(t) = 5t^2 + 10t^3 + 10t^4 + 4t^5$$

and expected syzygies

$$\beta(M) \; = \; \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 2 & 5 & 10 & . & & \\ 3 & . & . & 4 & . & . \\ 4 & . & . & 9 & 6 & . \\ 5 & . & . & . & 6 & 4 \end{array} \; \cdot$$

The section ring $H^0_*(\mathcal{O}_C)$ and the coordinate ring S_C have expected syzygies

$$\beta(H^0_*(\mathcal{O}_C)) \; = \; \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1 & . & . \\ 1 & . & . & . \\ 2 & 5 & 10 & . \\ 3 & . & . & 4 \end{array} \quad \text{and} \quad \beta(S_C) \; = \; \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & . & . & . \\ 1 & . & . & . & . \\ 2 & . & . & . & . \\ 3 & . & . & . & . \\ 4 & . & . & . & . \\ 5 & . & 9 & 6 & . \\ 6 & . & . & 6 & 4 \end{array} \; \cdot$$

Proposition 4.7. *A general curve C of genus $g = 16$ and degree $d = 15$ in $\mathbb{P}^3$ has syzygies as above. In particular, the Hartshorne–Rao module M_C uniquely determines C. Furthermore, the rational map from the component $\mathcal{H}_{d,g}$ of the Hilbert scheme $\mathrm{Hilb}_{15t+1-16}(\mathbb{P}^3)$ that dominates $\mathfrak{M}_{16}$ defined by*

$$\mathcal{H}_{d,g} \dashrightarrow \{\, 20 \text{ determinantal points} \,\}$$
$$C \; \mapsto \; \Gamma := \operatorname{supp} \operatorname{coker}(\varphi^t : \mathcal{O}^6(-1) \to \mathcal{O}^4)$$

is dominant. Here $\varphi \colon S^4(-9) \to S^6(-8)$ denotes the linear part of the last syzygy matrix of M.

Proof. For the first statement, it suffices to find an example with the expected syzygies, since Betti numbers behave semicontinuously in a family of modules with constant Hilbert function. We may even take a reducible example, provided that it is smoothable. Consider the union $C := E_1 \cup E_2 \cup E_3$ of three smooth curves of genus 2 and degree 5, such that $E_i \cap E_j$ for $i \neq j$ consists of 4 nodes of C. Then C has degree $d = 3 \cdot 5 = 15$ and genus $g = 3 \cdot 2 + 4 \cdot 3 - 2 = 16$. Clearly, C is smoothable as an abstract curve. For general choices, it is smoothable as an embedded curve because the g^3_{15} on the reducible curve is an isolated smooth

point in W^3_{15} (as we will see), so the smooth curves nearby have an isolated g^3_{15} as well.

It is easy to find such a union over a finite field $\mathbb{F}$. Start with the 12 intersection points $\{p_1, \ldots, p_4\} \cup \{p_5, \ldots, p_8\} \cup \{p_9, \ldots, p_{12}\}$ randomly chosen in $\mathbb{P}^3(\mathbb{F})$. To construct E_1, pick at random a quadric Q_1 in the pencil of quadrics through $\{p_1, \ldots, p_8\}$. Next, we must check if the tangent hyperplane of Q_1 in a point, say p_1, intersects Q_1 in a pair of lines individually defined over $\mathbb{F}$; this will happen about 50% of the time. Once this is true, choose one of the lines, call it L_1. Then $|\mathcal{O}_{Q_1}(3) \otimes \mathcal{O}_{Q_1}(-L_1)|$ is a linear system of class $(3, 2)$ on $Q_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$. We may take E_1 as a general curve in this linear system that passes through $\{p_1, \ldots, p_8\}$. Similarly, we choose E_2 using $\{p_1, \ldots, p_4, p_9, \ldots, p_{12}\}$ and E_3 starting with $\{p_5, \ldots, p_{12}\}$. The union of the E_i yields the desired curve C, and it a straightforward computation to check that C has the expected Hartshorne–Rao module and syzygies.

The second statement can be proved by showing that the appropriate map between tangent spaces is surjective for this example. This involves computing appropriate Ext-groups. Define

$$\begin{aligned}
\overline{M} := \mathrm{coker}(S^6(-2) \oplus S^6(-1) &\to S^4) \\
&= \mathrm{Ext}^4_S(M, S(-9)) \\
&= \mathrm{Hom}_K(M, K)(-5)
\end{aligned}$$

$$\text{and} \quad N := \mathrm{coker}(\varphi^t \colon S^6(-1) \to S^4).$$

Then there is a short exact sequence

$$0 \to P \to N \to \overline{M} \to 0$$

of modules with Hilbert series

$$\begin{aligned}
h_{\overline{M}}(t) &= 4 + 10t + 10t^2 + 5t^3 \\
h_N(t) &= 4 + 10t + 16t^2 + 20t^3 + 20t^4 + 20t^5 + \cdots \\
h_P(t) &= 6t^2 + 15t^3 + 20t^4 + 20t^5 + \cdots.
\end{aligned}$$

The group $\mathrm{Ext}^1_S(\overline{M}, \overline{M})$ governs the deformation theory of $\overline{M}$ (and M). More details can be found in [Hartshorne 2010], for example, Theorem 2.7 applied in the affine case. More precisely, the degree 0 part of this Ext-group is the tangent space of homogeneous deformations of M, which in turn is isomorphic to the tangent space of the Hilbert scheme in C. Similarly, in the given example, $\mathrm{Ext}^1(N, N)_0$ can be identified with the tangent space to the space of twenty

determinantal points. Note that we have the diagram

$$\text{Ext}^1_S(\overline{M}, \overline{M}) \longrightarrow \text{Ext}^1_S(N, \overline{M}) \longrightarrow \text{Ext}^1_S(P, \overline{M}).$$
$$\uparrow$$
$$\text{Ext}^1_S(N, N)$$
$$\uparrow$$
$$\text{Ext}^1_S(N, P)$$

In our example, computation shows that

$$\dim \text{Ext}^1_S(\overline{M}, \overline{M})_0 = 60,$$

$$\dim \text{Ext}^1_S(N, \overline{M})_0 = \dim \text{Ext}^1_S(N, N)_0 = 45, \quad \text{and}$$

$$\dim \text{Ext}^1_S(P, \overline{M})_0 = \dim \text{Ext}^1_S(N, P)_0 = 0.$$

Thus the induced map $\text{Ext}^1_S(\overline{M}, \overline{M})_0 \to \dim \text{Ext}^1_S(N, N)_0$ is surjective with 15-dimensional kernel, as expected.   □

Exercise 4.8. Fill in the computational details in of the proof of Proposition 4.7 and Theorem 4.5 using your favorite computer algebra system.

Remark 4.9. In the proof of Proposition 4.7, the module P has syzygies

$$\beta(P) = \quad \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 2 & 6 & 9 & . & . \\ 3 & . & 4 & 6 & . \\ 4 & . & . & 6 & 5 \end{array}.$$

The cokernel of $\psi^t \colon S^6(-1) \to S^5$ has support on a determinantal curve E of degree 15 and genus 26, which is smooth for general C. The points Γ form a divisor on E with $h^0(E, \mathcal{O}_E(\Gamma)) = 1$. The curves E and C do not intersect; in fact, we have no idea how the curve E is related to C, other than the fact that it can be constructed from the syzygies of M. It is possible that $\mathfrak{M}_{16}$ is not unirational, and, even if $\mathfrak{M}_{16}$ is unirational, it could be that the component of the Hilbert scheme containing C is itself not unirational.

   It is not clear to us whether it is a good idea to start with the determinantal points Γ in Proposition 4.7. Perhaps entirely different purely algebraic methods might lead to a unirational construction of the modules M, and we invite the reader to discover such an approach.

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