

Limits in commutative algebra and algebraic geometry

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In this survey article we explore various limits and asymptotic properties in commutative algebra and algebraic geometry. We show that several important invariants have good asymptotic behavior. We develop this and give some examples of pathological behavior for topics such as multiplicity of graded families of ideals, volumes of line bundles on schemes and regularity of powers of ideals.

1. Introduction

This article is on the general theme of limits arising in commutative algebra and algebraic geometry, in asymptotic multiplicity and regularity. The four sections of this article are based on the four talks that I gave during the Spring semester of the special year on Commutative Algebra, held at MSRI during the 2012–2013 academic year. The first section is based on an Evans lecture I gave at Berkeley.

2. Asymptotic multiplicities

Multiplicity and projection from a point. We begin by discussing a formula involving the multiplicity of a point on a variety, which evolved classically. Proofs of the formula (1) can be found in Theorem 5.11 of [Mumford 1976] (over $k = \mathbb{C}$), in Section 11 of [Abhyankar 1998], Section 12 of [Lipman 1975] and Theorem 12.1 [Cutkosky 2009].

Suppose that $k = \bar{k}$ is an algebraically closed field and $X \subset \mathbb{P}_k^N$ is a d -dimensional projective variety. The degree of X is defined as

$$\#(X \cap L^{N-d})$$

where L^{N-d} is a generic linear subspace of \mathbb{P}^N of dimension $N - d$. Suppose that $z \in \mathbb{P}^N$ is a closed point. Let $\pi_z : \mathbb{P}^N \dashrightarrow \mathbb{P}^{N-1}$ be the projection from z and $Y = \pi_z(X)$. We have the following formula relating $\deg(X)$ and $\deg(Y)$.

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Theorem 2.1. *Suppose that $z \in X$ and X is not a cone over z . Then*

$$\deg(X) = [k(X) : k(Y)]\deg(Y) + \text{mult}_z(X). \quad (1)$$

The index of function fields $[k(X) : k(Y)]$ is equal to the number of points in X above a general point q of Y . In the case when $z \notin X$, $\text{mult}_z(X) = 0$. When $z \in X$, the correction term $\text{mult}_z(X)$ is the multiplicity $e_{m_R}(R)$ of the local ring $R = \mathcal{O}_{X,z}$, which is defined below. Formula (1), together with its role in resolution of singularities, was discussed by Zariski [1935, pp. 21–22]. At that time the correction term $\text{mult}_z(X)$ was not completely understood. In some cases, this number is a local intersection number, but not always. In fact, we have

$$e_{m_R}(R) \leq \dim_k(\mathcal{O}_{X,z}/I(L')\mathcal{O}_{X,z}) \quad (2)$$

where L' is a general linear space through z , and (2) is an equality if and only if $\mathcal{O}_{X,z}$ is Cohen–Macaulay [Zariski and Samuel 1960, Chapter VIII, Section 10, Theorem 23].

Multiplicity has a purely geometric construction (over \mathbb{C}), as explained in [Mumford 1976].

Multiplicity, graded families of ideals, filtrations and the Zariski subspace theorem. From now on in this section we suppose that (R, m_R) is a (Noetherian) local ring of dimension d .

A family of ideals $\{I_n\}_{n \in \mathbb{N}}$ of R is called a graded family of ideals if $I_0 = R$ and $I_m I_n \subseteq I_{m+n}$ for all m, n . $\{I_n\}$ is a filtration of R if, in addition, $I_{n+1} \subseteq I_n$ for all n .

The most basic example is $I_n = J^n$ where J is a fixed ideal of R .

Suppose that N is a finitely generated R -module, and I is an m_R -primary ideal. Let

$$t = \dim R/\text{ann}(N)$$

be the dimension of N . Let ℓ_R be the length of an R -module.

The theorem of Hilbert–Samuel is that the function $\ell_R(N/I^n N)$ is a polynomial in n of degree t for $n \gg 0$. This polynomial is called the Hilbert–Samuel polynomial.

The multiplicity of a finitely generated R -module N with respect to an m_R -primary ideal I is the leading coefficient of this polynomial times $t!$; that is,

$$e_I(N) = \lim_{n \rightarrow \infty} \frac{\ell_R(N/I^n N)}{n^t/t!}. \quad (3)$$

This multiplicity is always a natural number.

Multiplicity is the most basic invariant in resolution of singularities. Suppose X is an algebraic variety. A point $p \in X$ is nonsingular if $\mathcal{O}_{X,p}$ is a regular local

ring [Zariski 1947]. The following theorem shows that $p \in X$ is a nonsingular point if and only if its multiplicity $e_{m_R}(R) = 1$ where $R = \mathcal{O}_{X,p}$.

Theorem 2.2. *If R is a regular local ring then $e_{m_R}(R) = 1$. If $e_{m_R}(R) = 1$ and R is formally equidimensional, then R is a regular local ring.*

The first statement follows since the associated graded ring of a regular local ring is a polynomial ring. The second statement is Theorem 40.6 of [Nagata 1959].

Suppose that $R \subset S$ are local rings with $m_R = m_S \cap R$. $I_n = m_S^n \cap R$ is a filtration of R . Let $\hat{R} = \lim_{\rightarrow} R/m_R^n$ be the m_R -adic completion of R and $\hat{S} = \lim_{\rightarrow} S/m_S^n$ be the m_S -adic completion of S .

The induced map $\hat{R} \rightarrow \hat{S}$ is an inclusion if and only if there exists a function $\sigma(n)$ such that $I_n \subset m_R^{\sigma(n)}$ and $\sigma(n) \rightarrow \infty$ as $n \rightarrow \infty$.

A fundamental theorem in algebraic geometry is the Zariski subspace theorem.

Theorem 2.3 [Zariski 1949; Abhyankar 1998, 10.6]. *Suppose that $R \subset S$ are local domains, essentially of finite type over a field k and R is analytically irreducible (\hat{R} is a domain). Then the induced map $\hat{R} \rightarrow \hat{S}$ is an inclusion.*

While this is a theorem in algebraic geometry, the subspace theorem fails in complex analytic geometry.

Theorem 2.4 [Gabrielov 1971]. *There exist inclusions $R \subset S$ of convergent complex power series such that the induced map $\hat{R} \rightarrow \hat{S}$ of formal power series is not an inclusion.*

Even though the Zariski subspace theorem is true, the induced filtration $m_S^n \cap R$ is not the best behaved.

Theorem 2.5 [Cutkosky and Srinivas 1993]. *There exists an inclusion $R \rightarrow S$ of d -dimensional normal domains, essentially of finite type over the complex numbers, such that*

$$\lim_{n \rightarrow \infty} \frac{\ell_R(R/I_n)}{n^d}$$

exists but is an irrational number, where $I_n = m_S^n \cap R$.

The irrationality of the limit implies that $\bigoplus_{n \geq 0} I_n$ is not a finitely generated R -algebra.

Limits of lengths of graded families of ideals. Suppose that $\{I_n\}_{n \in \mathbb{N}}$ is a graded family of m_R -primary ideals (I_n is m_R -primary for $n \geq 1$) in a d -dimensional (Noetherian) local ring R . We pose the following question:

Question 2.6. When does

$$\lim_{n \rightarrow \infty} \frac{\ell_R(R/I_n)}{n^d} \tag{4}$$

exist?

This problem was considered in [Ein, Lazarsfeld and Smith 2003] and [Mustață 2002].

Lazarsfeld and Mustață [2009] showed that the limit exists for all graded families of m_R -primary ideals in R if R is a domain which is essentially of finite type over an algebraically closed field k with $R/m_R = k$. All of these assumptions are necessary in their proof. Their proof is by reducing the problem to one on graded linear series on a projective variety, and then using a method introduced by Okounkov [2003] to reduce the problem to one of counting points in an integral semigroup.

In [Cutkosky 2013a], it is shown that the limit exists for all graded families of m_R -primary ideals in R if R is analytically unramified (\hat{R} is reduced), equicharacteristic and R/m_R is perfect.

In Example 5.3 of [Cutkosky 2014], an example is given of a nonreduced local ring R with a graded family of m_R -primary ideals $\{I_n\}$ such that the limit (4) does not exist. Dao and Smirnov communicated to me that they found this same example. They further showed that it was universal, proving the following theorem.

The nilradical $N(R)$ of a d -dimensional ring R is

$$N(R) = \{x \in R \mid x^n = 0 \text{ for some positive integer } n\}.$$

Recall that

$$\dim N(R) = \dim R/\text{ann}(N(R)),$$

so that $\dim N(R) = d$ if and only if there exists a minimal prime P of R such that $\dim R/P = d$ and R_P is not reduced.

Theorem 2.7 (Dao and Smirnov). *Suppose that R is a d -dimensional local ring such that $\dim N(R) = d$. Then there exists a graded family of m_R -primary ideals $\{I_n\}$ of R such that*

$$\lim_{n \rightarrow \infty} \frac{\ell_R(R/I_n)}{n^d}$$

does not exist.

We now state our general theorem, which gives necessary and sufficient conditions on a local ring R for all limits of graded families of m_R -primary ideals to exist.

Theorem 2.8 [Cutkosky 2014, Theorem 5.5]. *Suppose that R is a d -dimensional local ring. Then the limit*

$$\lim_{n \rightarrow \infty} \frac{\ell_R(R/I_n)}{n^d}$$

exists for all graded families of m_R -primary ideals $\{I_n\}$ of R if and only if $\dim N(\hat{R}) < d$.

If R is excellent, then $N(\hat{R}) = N(R)\hat{R}$, and the theorem is true with the condition $\dim N(\hat{R}) < d$ replaced with $\dim N(R) < d$. However, there exist Noetherian local domains R (so that $N(R) = 0$) such that $\dim N(\hat{R}) = \dim R$ [Nagata 1959, (E3.2)].

Before giving the proof of Theorem 2.8 in the next section, we turn to some of the theorem’s applications. We start with some general “volume = multiplicity” formulas.

Theorem 2.9 [Cutkosky 2014, Theorem 6.5]. *Suppose that R is a d -dimensional, analytically unramified local ring, and $\{I_n\}$ is a graded family of m_R -primary ideals in R . Then*

$$\lim_{n \rightarrow \infty} \frac{\ell_R(R/I_n)}{n^d/d!} = \lim_{p \rightarrow \infty} \frac{e_{I_p}(R)}{p^d}$$

Related formulas have been proven in [Ein, Lazarsfeld and Smith 2003; Mustař 2002; Lazarsfeld and Mustař 2009]. This last paper proves the formula when R is essentially of finite type over an algebraically closed field k with $R/m_R = k$. All of these assumptions are necessary in their proof.

Suppose that R is a (Noetherian) local ring and I, J are ideals in R . The generalized symbolic power $I_n(J)$ is defined by

$$I_n(J) = I^n : J^\infty = \bigcup_{i=1}^{\infty} I^n : J^i.$$

Theorem 2.10 [Cutkosky 2014, Corollary 6.4]. *Suppose that R is an analytically unramified d -dimensional local ring. Let s be the constant limit dimension $s = \dim I_n(J)/I^n$ for $n \gg 0$. Suppose that $s < d$. Then*

$$\lim_{n \rightarrow \infty} \frac{e_{m_R}(I_n(J)/I^n)}{n^{d-s}}$$

exists.

This was proven in [Herzog, Puthenpurakal and Verma 2008] for ideals I and J in a d -dimensional local ring, with the assumption that $\bigoplus_{n \geq 0} I_n(J)$ is a finitely generated R -algebra.

3. The proof of Theorem 2.8

In this section we outline the proof of Theorem 2.8. See [Cutkosky 2013a; 2014] for details.

Proof that $\dim N(\hat{R}) < d$ implies limits exist. Suppose that R is a d -dimensional local ring with $\dim N(R) < d$ and $\{I_n\}$ is a graded family of m_R -primary ideals in R .

We have $\ell_{\hat{R}}(R/I_n\hat{R}) = \ell_R(R/I_n)$ for all n , so we may assume that $R = \hat{R}$ is complete; in particular, we may assume that R is excellent with $\dim N(R) < d$. There exists a positive integer c such that $m_R^c \subseteq I_1$, which implies that

$$m_R^{nc} \subseteq I_n \text{ for all positive } n. \quad (5)$$

Let $N = N(R)$ and $A = R/N$. We have short exact sequences

$$0 \rightarrow N/N \cap I_i R \rightarrow R/I_i R \rightarrow A/I_i A \rightarrow 0,$$

from which we deduce that there exists a constant $\alpha > 0$ such that

$$\ell_R(N/N \cap I_i R) \leq \ell_R(N/m_R^{ci} N) \leq \alpha i^{\dim N} \leq \alpha i^{d-1}.$$

Replacing R with A and I_n with $I_n A$, we thus reduce to the case that R is reduced. Using the following lemma, we then reduce to the case that R is a complete domain (so that it is analytically irreducible).

Lemma 3.1 [Cutkosky 2013a, Corollary 6.4]. *Suppose that R is a reduced local domain, and $\{I_n\}$ is a graded family of m_R -primary ideals in R . Let $\{P_1, \dots, P_s\}$ be the set of minimal primes of R and let $R_i = R/P_i$. Then there exists $\alpha > 0$ such that*

$$\left| \left(\sum_{i=1}^s \ell_{R_i}(R_i/I_n R_i) \right) - \ell_R(R/I_n) \right| \leq \alpha n^{d-1} \text{ for all } n.$$

We now present a method introduced in [Okounkov 2003] to compute limits of multiplicities. The method has been refined in [Lazarsfeld and Mustařă 2009; Kaveh and Khovanskii 2012].

Suppose that $\Gamma \subset \mathbb{N}^{d+1}$ is a semigroup. Let $\Sigma(\Gamma)$ be the closed convex cone generated by Γ in \mathbb{R}^{d+1} . Define $\Delta(\Gamma) = \Sigma(\Gamma) \cap (\mathbb{R}^d \times \{1\})$. For $i \in \mathbb{N}$, let $\Gamma_i = \Gamma \cap (\mathbb{N}^d \times \{i\})$.

Theorem 3.2 [Okounkov 2003; Lazarsfeld and Mustařă 2009]. *Suppose that Γ satisfies these conditions:*

- 1) *There exist finitely many vectors $(v_i, 1) \in \mathbb{N}^{d+1}$ spanning a semigroup $B \subseteq \mathbb{N}^{d+1}$ such that $\Gamma \subseteq B$ (boundedness).*
- 2) *The subgroup generated by Γ is the full integral lattice \mathbb{Z}^{d+1} .*

Then

$$\lim_{i \rightarrow \infty} \frac{\#\Gamma_i}{i^d} = \text{vol}(\Delta(\Gamma))$$

exists.

We now return to the proof that $\dim N(\hat{R}) < d$ implies limits exist. Recall that we have reduced to the case that R is a complete domain. Let $\pi : X \rightarrow \operatorname{spec}(R)$ be the normalization of the blowup of m_R . Being excellent, X is of finite type over R . X is regular in codimension 1, so there exists a closed point $p \in \pi^{-1}(m_R)$ such that $S = \mathcal{O}_{X,p}$ is regular and dominates R . We have an inclusion $R \rightarrow S$ of d -dimensional local rings such that $m_S \cap R = m_R$ with equality of quotient fields $Q(R) = Q(S)$. Let $k = R/m_R$, $k' = S/m_S$. Since S is essentially of finite type over R , we have that $[k' : k] < \infty$. Let y_1, \dots, y_d be regular parameters in S . Choose $\lambda_1, \dots, \lambda_d \in \mathbb{R}_+$ which are rationally independent with $\lambda_i \geq 1$. Prescribe a rank 1 valuation ν on $Q(R)$ by

$$\nu(y_1^{i_1} \cdots y_d^{i_d}) = i_1 \lambda_1 + \cdots + i_d \lambda_d$$

and $\nu(\gamma) = 0$ if $\gamma \in S$ is a unit. The value group of ν is

$$\Gamma_\nu = \lambda_1 \mathbb{Z} + \cdots + \lambda_d \mathbb{Z} \subseteq \mathbb{R}.$$

Let V_ν be the valuation ring of ν . Then

$$k' = S/m_S \cong V_\nu/m_\nu.$$

For $\lambda \in \mathbb{R}_+$, define valuation ideals in V_ν by

$$K_\lambda = \{f \in Q(R) \mid \nu(f) \geq \lambda\}$$

and

$$K_\lambda^+ = \{f \in Q(R) \mid \nu(f) > \lambda\}.$$

Now suppose that $I \subset R$ is an ideal and $\lambda \in \Gamma_\nu$ is nonnegative. We have an inclusion

$$I \cap K_\lambda / I \cap K_\lambda^+ \subseteq K_\lambda / K_\lambda^+ \cong k'.$$

Thus

$$\dim_k I \cap K_\lambda / I \cap K_\lambda^+ \leq [k' : k].$$

Lemma 3.3 [Cutkosky 2013a, Lemma 4.3]. *There exists $\alpha \in \mathbb{Z}_+$ such that $K_{\alpha n} \cap R \subset m_R^n$ for all $n \in \mathbb{Z}_+$.*

The proof uses Huebl's linear Zariski subspace theorem [Hübl 2001] or Rees' Izumi theorem [Rees 1989]. The assumption that R is analytically irreducible is necessary for the lemma. Recalling the constant c of (5), let $\beta = \alpha c$. Then

$$K_{\beta n} \cap R \subset m_R^{nc} \subseteq I_n \tag{6}$$

for all n . For $1 \leq t \leq [k' : k]$, define

$$\begin{aligned}\Gamma^{(t)} &= \{(n_1, \dots, n_d, i) \in \mathbb{N}^{d+1} \mid \dim_k I_i \cap K_{n_1\lambda_1 + \dots + n_d\lambda_d} / I_i \cap K_{n_1\lambda_1 + \dots + n_d\lambda_d}^+ \geq t \\ &\quad \text{and } n_1 + \dots + n_d \leq \beta i\}, \\ \hat{\Gamma}^{(t)} &= \{(n_1, \dots, n_d, i) \in \mathbb{N}^{d+1} \mid \dim_k R \cap K_{n_1\lambda_1 + \dots + n_d\lambda_d} / R \cap K_{n_1\lambda_1 + \dots + n_d\lambda_d}^+ \geq t \\ &\quad \text{and } n_1 + \dots + n_d \leq \beta i\}.\end{aligned}$$

Lemma 3.4 [Cutkosky 2014, Lemma 4.4]. *Suppose that $t \geq 1$, $0 \neq f \in I_i$, $0 \neq g \in I_j$ and*

$$\dim_k I_i \cap K_{v(f)} / I_i \cap K_{v(f)}^+ \geq t.$$

Then

$$\dim_k I_{i+j} \cap K_{v(fg)} / I_{i+j} \cap K_{v(fg)}^+ \geq t.$$

Since $v(fg) = v(f) + v(g)$, we conclude that when they are nonempty, $\Gamma^{(t)}$ and $\hat{\Gamma}^{(t)}$ are subsemigroups of \mathbb{N}^{d+1} .

Given $\lambda = n_1\lambda_1 + \dots + n_d\lambda_d$ such that $n_1 + \dots + n_d \leq \beta i$, we have

$$\dim_k K_\lambda \cap I_i / K_\lambda^+ \cap I_i = \#\{t \mid (n_1, \dots, n_d, i) \in \Gamma^{(t)}\}.$$

Recalling (6), we obtain

$$\begin{aligned}\ell_R(R/I_i) &= \ell_R(R/K_{\beta i} \cap R) - \ell_R(I_i/K_{\beta i} \cap I_i) \\ &= \left(\sum_{0 \leq \lambda < \beta i} \dim_k K_\lambda \cap R / K_\lambda^+ \cap R \right) - \left(\sum_{0 \leq \lambda < \beta i} \dim_k K_\lambda \cap I_i / K_\lambda^+ \cap I_i \right) \\ &= \left(\sum_{t=1}^{[k':k]} \#\hat{\Gamma}_i^{(t)} \right) - \left(\sum_{t=1}^{[k':k]} \#\Gamma_i^{(t)} \right),\end{aligned}$$

where $\Gamma_i^{(t)} = \Gamma^{(t)} \cap (\mathbb{N}^d \times \{i\})$ and $\hat{\Gamma}_i^{(t)} = \hat{\Gamma}^{(t)} \cap (\mathbb{N}^d \times \{i\})$. The semigroups $\Gamma^{(t)}$ and $\hat{\Gamma}^{(t)}$ satisfy the hypotheses of Theorem 3.2. Thus

$$\lim_{i \rightarrow \infty} \frac{\#\Gamma_i^{(t)}}{i^d} = \text{vol}(\Delta(\Gamma^{(t)}))$$

and

$$\lim_{i \rightarrow \infty} \frac{\#\hat{\Gamma}_i^{(t)}}{i^d} = \text{vol}(\Delta(\hat{\Gamma}^{(t)}))$$

so that

$$\lim_{i \rightarrow \infty} \frac{\ell_R(R/I_i)}{i^d}$$

exists.

An example where limits do not exist. Let $i_1 = 2$ and $r_1 = i_1/2$. For $j \geq 1$, inductively define i_{j+1} so that i_{j+1} is even and $i_{j+1} > 2^j i_j$. Let $r_{j+1} = i_{j+1}/2$. For $n \in \mathbb{Z}_+$, define

$$\sigma(n) = \begin{cases} 1 & \text{if } n = 1, \\ i_j/2 & \text{if } i_j \leq n < i_{j+1}. \end{cases} \quad (7)$$

The limit

$$\lim_{n \rightarrow \infty} \frac{\sigma(n)}{n} \quad (8)$$

does not exist, even when n is constrained to lie in an arithmetic sequence [Cutkosky 2014, Lemmas 6.1 and 6.2]. The following example shows that limits do not always exist on nonreduced local rings.

Example 3.5 [Cutkosky 2014, Example 5.3]. Let k be a field, $d > 0$ and R be the nonreduced d -dimensional local ring $R = k[[x_1, \dots, x_d, y]]/(y^2)$. There exists a graded family of m_R -primary ideals $\{I_n\}$ in R such that the limit

$$\lim_{n \rightarrow \infty} \frac{\ell_R(R/I_n)}{n^d} \quad (9)$$

does not exist, even when n is constrained to lie in an arithmetic sequence.

This example was also found by Dao and Smirnov. They further showed that it is universal (Theorem 2.7).

Proof. Let $\bar{x}_1, \dots, \bar{x}_d, \bar{y}$ be the classes of x_1, \dots, x_d, y in R . Let N_i be the set of monomials of degree i in the variables $\bar{x}_1, \dots, \bar{x}_d$. Let $\sigma(n)$ be the function defined in (7). Define m_R -primary ideals I_n in R by $I_n = (N_n, \bar{y}N_{n-\sigma(n)})$ for $n \geq 1$ (and $I_0 = R$).

We first verify that $\{I_n\}$ is a graded family of ideals, by showing that $I_m I_n \subset I_{m+n}$ for all $m, n > 0$. This follows since

$$I_m I_n = (N_{m+n}, \bar{y}N_{(m+n)-\sigma(m)}, \bar{y}N_{(m+n)-\sigma(n)})$$

and $\sigma(j) \leq \sigma(k)$ for $k \geq j$.

R/I_n has a k -basis consisting of

$$\{N_i \mid i < n\} \quad \text{and} \quad \{\bar{y}N_j \mid j < n - \sigma(n)\}.$$

Thus

$$\ell_R(R/I_n) = \binom{n}{d} + \binom{n-\sigma(n)}{d},$$

and the limit (9) does not exist, even when n is constrained to lie in an arithmetic sequence, by (8). \square

4. Volumes on schemes

Unless stated otherwise, we will assume that X is a d -dimensional proper scheme over an arbitrary field k . The vector space $N(X)$ of \mathbb{R} -divisors modulo numerical equivalence is defined in [Kleiman 1966], Chapter 1 of [Lazarsfeld 2004a], and extended to this level of generality (proper schemes over a field) in [Cutkosky 2013b]. In the case of a nonsingular variety, $N(X)$ is defined in the last section of this article.

If \mathcal{L} is a line bundle on X , then the volume of \mathcal{L} is

$$\text{vol}(\mathcal{L}) = \limsup_{n \rightarrow \infty} \frac{\dim_k \Gamma(X, \mathcal{L}^n)}{n^d/d!}.$$

If \mathcal{L} is ample, then $\text{vol}(\mathcal{L}) = (\mathcal{L})^d$ (the self intersection number). $\text{vol}(\mathcal{L})$ is well defined on $N(X)$. The volume can be irrational [Cutkosky and Srinivas 1993] even on a nonsingular projective variety over \mathbb{C} . The following is a fundamental result.

Theorem 4.1. *Suppose that X is a projective variety over an algebraically closed field k , and \mathcal{L} is a line bundle on X . Then*

$$\text{vol}(\mathcal{L}) = \lim_{n \rightarrow \infty} \frac{\dim_k \Gamma(X, \mathcal{L}^n)}{n^d/d!}$$

exists as a limit.

The function vol extends to a continuous, even continuously differentiable d -homogeneous function on $N(X)$ [Lazarsfeld 2004b; Boucksom, Favre and Jonsson 2009; Lazarsfeld and Mustață 2009]. This last result is true for $\text{vol}(\mathcal{L})$ on $N(X)$ when X is a proper variety over an arbitrary field [Cutkosky 2013b].

There are several proofs of Theorem 4.1. Lazarsfeld [2004b] gave one when k is algebraically closed of characteristic zero, using Fujita approximation ([Fujita 1994], which requires resolution of singularities). Satoshi Takagi [2007] gave a proof when k is algebraically closed of characteristic $p > 0$ using de Jong's resolution [1996] after alterations and Fujita approximation.

There are also proofs by Okounkov [2003] for ample line bundles and by Lazarsfeld and Mustață [2009] using the cone method (using Theorem 3.2).

Definition 4.2. A graded linear series (for a line bundle \mathcal{L}) on a d -dimensional proper scheme X over a field k is a graded k -subalgebra

$$L = \bigoplus_{n \geq 0} L_n \subseteq \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^n).$$

Of course L need not be a finitely generated k -algebra. For the definition of Iitaka–Kodaira dimension, we need the following definition:

$$\sigma(L) = \max\{s \mid \text{there exist } y_1, \dots, y_s \in L \text{ homogenous of positive degree and algebraically independent over } k\}.$$

We define the Iitaka–Kodaira dimension of L to be

$$\kappa(L) = \begin{cases} \sigma(L) - 1 & \text{if } \sigma(L) > 0, \\ -\infty & \text{if } \sigma(L) = 0. \end{cases}$$

The index of \mathcal{L} is $m(L) = [\mathbb{Z} : G]$ where G is the subgroup generated by $\{n \mid L_n \neq 0\}$.

If X is reduced and $\kappa(L) = -\infty$, then $L_m = 0$ for all $m > 0$ and if X is reduced and $\kappa(L) \geq 0$, then there exist constants $0 < \alpha < \beta$ such that

$$\alpha n^{\kappa(L)} < \dim_k L_{mn} < \beta n^{\kappa(L)}$$

for $n \gg 0$. However, if X is not reduced, then we can have $\kappa(L) = -\infty$ and $\dim_k L_n > n^d$ for all $n \gg 0$ [Cutkosky 2014, Section 12].

Theorem 4.3 ([Lazarsfeld and Mustařa 2009] when $\dim X = \kappa(L)$ and $m(L) = 1$; [Kaveh and Khovanskii 2012]). *Suppose that X is a projective variety over an algebraically closed field k , and L is a graded linear series on X with $\kappa(L) \geq 0$, index $m = m(L)$. Then*

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_{nm}}{n^{\kappa(L)}}$$

exists.

In analogy with our Theorem 2.8 for existence of limits of graded families of ideals, we have the following necessary and sufficient conditions for the existence of limits on projective schemes.

Theorem 4.4 [Cutkosky 2014, Theorem 10.6]. *Suppose that X is a d -dimensional projective scheme over a field k with $d > 0$. Let*

$$\mathcal{N}_X = \{f \in \mathcal{O}_X \mid f^s = 0 \text{ for some positive integer } s\},$$

the nilradical of X . Let $\alpha \in \mathbb{N}$. Then the following are equivalent:

- 1) *For every graded linear series L on X with $\alpha \leq \kappa(L)$, there exists a positive integer r such that*

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_{a+nr}}{n^{\kappa(L)}}$$

exists for every positive integer a .

2) For every graded linear series L on X with $\alpha \leq \varkappa(L)$, there exists an arithmetic sequence $a + nr$ (for fixed r and a depending on L) such that

$$\lim_{n \rightarrow \infty} \frac{\dim_k L_{a+nr}}{n^{\varkappa(L)}}$$

exists.

3) $\dim \mathcal{N}_X < \alpha$.

The implication 3) \implies 1) holds if X is a proper scheme over a field k , or if X is a compact analytic space.

Corollary 4.5 [Cutkosky 2014, Theorem 10.7]. *Suppose that X is a proper d -dimensional scheme over a field k with $\dim \mathcal{N}_X < d$ and \mathcal{L} is a line bundle on X . Then*

$$\text{vol}(\mathcal{L}) = \lim_{n \rightarrow \infty} \frac{\dim_k \Gamma(X, \mathcal{L}^n)}{n^d/d!}.$$

5. Asymptotic regularity

We first give a comparison of local cohomology and sheaf cohomology. Let $R = k[x_0, \dots, x_n]$, a polynomial ring over a field with the standard grading, and $\mathfrak{m} = (x_0, \dots, x_n)$. Suppose that M is a graded module over R . Let \tilde{M} be the sheafification of M on \mathbb{P}^n . We have an exact sequence of graded R -modules

$$0 \rightarrow H_{\mathfrak{m}}^0(M) \rightarrow M \rightarrow \bigoplus_{j \in \mathbb{Z}} H^0(\mathbb{P}^n, \tilde{M}(j)) \rightarrow H_{\mathfrak{m}}^1(M) \rightarrow 0$$

and isomorphisms

$$\bigoplus_{j \in \mathbb{Z}} H^i(\mathbb{P}^n, \tilde{M}(j)) \cong H_{\mathfrak{m}}^{i+1}(M) \text{ for } i \geq 1.$$

We have the interpretation

$$\bigoplus_{j \in \mathbb{Z}} H^0(\mathbb{P}^n, \tilde{M}(j)) \cong \varinjlim \text{Hom}_R(\mathfrak{m}^n, M)$$

as an ideal transform.

We now define the regularity of a finitely generated graded R -module M :

$$a^i(M) = \begin{cases} \sup\{j \mid H_{\mathfrak{m}}^i(M)_j \neq 0\} & \text{if } H_{\mathfrak{m}}^i(M) \neq 0, \\ -\infty & \text{otherwise.} \end{cases}$$

The regularity of M is defined to be

$$\text{reg}(M) = \max_i \{a^i(M) + i\}.$$

Interpreting R as the coordinate ring of $\mathbb{P}^n = \text{proj}(R)$, and considering the sheaf \tilde{M} on \mathbb{P}^n associate to M , we can define the regularity of \tilde{M} to be

$$\begin{aligned} \text{reg}(\tilde{M}) &= \max \{m \mid H^i(\mathbb{P}^n, \tilde{M}(m-i-1)) \neq 0 \text{ for some } i \geq 1\} \\ &= \max_{i \geq 2} \{a^i(M) + i\}. \end{aligned}$$

Thus

$$\text{reg}(\tilde{M}) \leq \text{reg}(M).$$

We now give some interpretations of regularity of modules. Let

$$F_* : 0 \rightarrow \cdots \rightarrow F_j \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

be a minimal free resolution of M as a graded R -module. Let b_j be the maximum degree of the generators of F_j . Then

$$\text{reg}(M) = \max \{b_j - j \mid j \geq 0\}.$$

In fact, we have (see [Eisenbud and Goto 1984])

$$\begin{aligned} \text{reg}(M) &= \max \{b_j - j \mid j \geq 0\} \\ &= \max \{n \mid \exists j \text{ such that } \text{Tor}_j^R(k, M)_{n+j} \neq 0\} \\ &= \max \{n \mid \exists j \text{ such that } H_m^j(M)_{n-j} \neq 0\}. \end{aligned}$$

We define

$$\text{reg}_i(M) = \max \{n \mid \text{Tor}_i^R(k, M)_n \neq 0\} - i.$$

Then

$$\text{reg}(M) = \max \{\text{reg}_i(M) \mid i \geq 0\}.$$

Further, $\text{reg}_0(M)$ is the maximum degree of a homogeneous generator of M .

We now discuss the regularity of powers of ideals. We outline the proof of [Cutkosky, Herzog and Trung 1999] showing that $\text{reg}(I^n)$ is a linear polynomial for large n .

Let F_1, \dots, F_s be homogeneous generators of $I \subset R = k[x_0, \dots, x_n]$, with $\deg(F_i) = d_i$. The map $y_i \mapsto F_i$ induces a surjection of bigraded R -algebras

$$S = R[y_1, \dots, y_s] \rightarrow R(I) = \bigoplus_{m \geq 0} I^m$$

where we have $\text{bideg}(x_i) = (1, 0)$ for $0 \leq i \leq n$, and $\text{bideg}(y_j) = (d_j, 1)$ for $1 \leq j \leq s$. We have

$$\text{Tor}_i^R(k, I^m)_a \cong \text{Tor}_i^S(S/\mathfrak{m}S, R(I))_{(a,m)}.$$

Theorem 5.1. *Let E be a finitely generated bigraded module over $k[y_1, \dots, y_s]$. Then the function*

$$\rho_E(m) = \max\{a \mid E_{(a,m)} \neq 0\}$$

is a linear polynomial for $m \gg 0$.

Since $\text{Tor}_i^S(S/mS, R(I))$ is a finitely generated bigraded S/mS module, we have:

Theorem 5.2 [Cutkosky, Herzog and Trung 1999]. *For all $i \geq 0$, the function $\text{reg}_i(I^n)$ is a linear polynomial for $n \gg 0$.*

Theorem 5.3 [Cutkosky, Herzog and Trung 1999; Kodiyalam 2000]. *$\text{reg}(I^n)$ is a linear polynomial for $n \gg 0$.*

In the expression

$$\text{reg}(I^n) = an + b$$

for $n \gg 0$, we have

$$a = \lim_{n \rightarrow \infty} \frac{\text{reg}(I^n)}{n} = \lim_{n \rightarrow \infty} \frac{d(I^n)}{n} = \rho(I)$$

where $d(I^n) = \text{reg}_0(I^n)$ is the maximal degree of a homogeneous generator of I^n , and (see [Kodiyalam 2000])

$$\rho(I) = \min\{\max\{d(J) \mid J \text{ is a graded reduction of } I\}\}.$$

Theorem 5.4 [Tài Hà 2011]. *Suppose that R is standard graded over a commutative Noetherian ring with unity, I is a graded ideal of R and M is a finitely generated graded R -module. Then there exists a constant e such that for $n \gg 0$,*

$$\text{reg}(I^n M) = \rho_M(I)n + e$$

where

$$\rho_M(I) = \min\{d(J) \mid J \text{ is a } M\text{-reduction of } I\}.$$

J is an M -reduction of I if $I^{n+1}M = JI^nM$ for some $n \geq 0$.

Theorem 5.5 [Eisenbud and Harris 2010]. *Let $R = k[x_1, \dots, x_m]$. If I is R_+ -primary and generated in a single degree, then the constant term of $\text{reg}(I^n)$ (for $n \gg 0$) is the maximum of the regularity of the fibers of the morphism defined by a minimal set of generators.*

Theorem 5.6 [Tài Hà 2011; Chardin 2013]. *The constant term of the regularity $\text{reg}(I^n)$, for I homogeneous in $R = k[x_0, \dots, x_m]$ with generators all of the same degree, can be computed as the maximum of regularities of the localization of the structure sheaf of the graph of a rational map of \mathbb{P}^m determined by I above points in the projection of the graph onto its second factor (the image of the rational map).*

We now give a comparison of $\text{reg}_i(I^m)$, $a^i(I^m)$ and $\text{reg}(I^m)$. We continue to study the graded polynomial ring $R = k[x_0, \dots, x_n]$, and assume that I is a homogeneous ideal. Recall that

$$a^i(I^m) = \begin{cases} \sup\{j \mid H_m^i(I^m)_j \neq 0\} & \text{if } H_m^i(I^m) \neq 0, \\ -\infty & \text{otherwise,} \end{cases}$$

$$\text{reg}_i(I^m) = \max\{n \mid \text{Tor}_i^R(k, I^m)_n \neq 0\} - i,$$

and the regularity is

$$\text{reg}(I^m) = \max_i \{a^i(I^m) + i\} = \max\{\text{reg}_i(I^m) \mid i \geq 0\}.$$

We have shown that all of the functions $\text{reg}_i(I^m)$ are eventually linear polynomials, so $\text{reg}(I^m)$ is eventually a linear polynomial.

We first discuss the behavior of $a^i(I^m)$.

Theorem 5.7 [Cutkosky 2000]. *There is a homogeneous height two prime ideal I in $k[x_0, x_1, x_2, x_3]$ of a nonsingular space curve, such that*

$$a^2(I^m) = \lfloor m(9 + \sqrt{2}) \rfloor + 1 + \sigma(m)$$

for $m > 0$, where $\lfloor x \rfloor$ is the greatest integer in a real number x and

$$\sigma(m) = \begin{cases} 0 & \text{if } m = q_{2n} \text{ for some } n \in \mathbb{N}, \\ 1 & \text{otherwise,} \end{cases}$$

where q_n is defined recursively by

$$q_0 = 1, q_1 = 2, q_n = 2q_{n-1} + q_{n-2},$$

computed from the convergents $\frac{p_n}{q_n}$ of the continued fraction expansion of $\sqrt{2}$.

The m such that $\sigma(m) = 0$ are very sparse, as $q_{2n} \geq 3^n$.

We also compute that

$$a^3(I^m) = \lfloor m(9 - \sqrt{2}) \rfloor - \tau(m)$$

where $0 \leq \tau(m) \leq \text{constant}$ is a bounded function, and

$$a^4(I^m) = -4.$$

Since $9 + \sqrt{2} \leq \lim_{m \rightarrow \infty} \frac{\text{reg}(I^m)}{m} \in \mathbb{Z}_+$, we have

$$a^1(I^m) = \text{reg}(I^m) = \text{linear function for } m \gg 0.$$

We now discuss the proof of this theorem. We first need to review numerical equivalence, from [Kleiman 1966] and [Lazarsfeld 2004a, Chapter 1]. Let k be

an algebraically closed field, and X be a nonsingular projective variety over k . Define

$$\begin{aligned} \text{Div}(X) &= \text{divisors on } X \\ &:= \text{formal sums of codimension-1 subvarieties of } X. \end{aligned}$$

Numerical equivalence is defined by

$$D_1 \equiv D_2 \iff (D_1 \cdot C) = (D_2 \cdot C) \text{ for all curves } C \text{ on } X.$$

The \mathbb{R} -vector space

$$N(X) = (\text{Div}(X)/\equiv) \otimes_{\mathbb{Z}} \mathbb{R}$$

is finite-dimensional (this is proved in [Lazarsfeld 2004a, Proposition 1.1.16], for instance).

A divisor D on X is ample if $H^0(\mathcal{O}_X(mD))$ gives a projective embedding of X for some $m \gg 0$.

Theorem 5.8. *A divisor D is ample if and only if $(D^d \cdot V) > 0$ for all d -dimensional irreducible subvarieties V of X .*

Taking $V = X$ this condition is $(D^{\dim X}) > 0$.

Set

$A(X)$ = ample cone = convex cone in $N(X)$ generated by ample divisors

$\text{Nef}(X)$ = nef cone

= convex cone generated by numerically effective divisors

$$((D \cdot C) \geq 0 \text{ for all curves } C \text{ on } X,)$$

$\text{NE}(X)$ = convex cone generated by effective divisors

$$(h^0(\mathcal{O}_X(nD)) > 0 \text{ for some } n > 0.).$$

Theorem 5.9.

$$A(X) \subseteq \overline{A(X)} = \text{Nef}(X) \subseteq \overline{\text{NE}(X)}$$

Here \overline{T} denotes closure of T in the euclidean topology.

Suppose that S is a nonsingular projective surface. Then $N(S)$ has an intersection form $q(D) = (D^2)$ for D a divisor on S .

A K3 surface is a nonsingular projective surface with $H^1(\mathcal{O}_S) = 0$ and such that K_S is trivial. From the theory of K3 surfaces (as reviewed in Section 2 of [Cutkosky 2000] and using a theorem of Morrison [Morrison 1984]) it follows that there exists a K3 surface S such that $N(S) \cong \mathbb{R}^3$ and

$$q(D) = 4x^2 - 4y^2 - 4z^2$$

for $D = (x, y, z) \in \mathbb{R}^3$.

Lemma 5.10. *Suppose that C is an integral curve on S . Then $(C^2) \geq 0$.*

Proof. Suppose otherwise. Then $(C^2) = -2$, since S is a K3 surface. But 4 divides $q(C) = (C^2)$, a contradiction. \square

Corollary 5.11. $\overline{NE}(S) = \overline{A}(S)$, and

$$\overline{NE}(S) = \{(x, y, z) \mid 4x^2 - 4y^2 - 4z^2 \geq 0, x \geq 0\}.$$

Let $H = (1, 0, 0)$ (that is, let H be a divisor whose class is $(1, 0, 0)$). Then $H^0(\mathcal{O}_S(H))$ gives an embedding of S as a quartic surface in \mathbb{P}^3 . Choose $(a, b, c) \in \mathbb{Z}^3$ such that $a > 0$, $a^2 - b^2 - c^2 > 0$ and $\sqrt{b^2 + c^2} \notin \mathbb{Q}$. (a, b, c) is in the interior of $\overline{NE}(S)$, which is equal to $A(S)$. There exists a nonsingular curve C on S such that $C = (a, b, c)$ in $N(X)$. Let

$$\lambda_2 = a + \sqrt{b^2 + c^2} \text{ and } \lambda_1 = a - \sqrt{b^2 + c^2}.$$

Suppose that $m, r \in \mathbb{N}$. Then

$$\begin{aligned} mH - rC &\in \overline{NE}(S) && \text{and is ample if } r\lambda_2 < m, \\ mH - rC, rC - mH &\notin \overline{NE}(S) && \text{if } r\lambda_1 < m < r\lambda_2, \\ rC - mH &\in \overline{NE}(S) && \text{and is ample if } m < r\lambda_1. \end{aligned}$$

Choose $C = (a, b, c)$ so that $7 < \lambda_1 < \lambda_2$ and $\lambda_2 - \lambda_1 > 2$. Then by Riemann–Roch,

$$\begin{aligned} \chi(mH - rC) &= h^0(mH - rC) - h^1(mH - rC) + h^2(mH - rC) \\ &= \frac{1}{2}(mH - rC)^2 + 2. \end{aligned}$$

Theorem 5.12.

$$\begin{aligned} h^1(mH - rC) &= \begin{cases} 0 & \text{if } r\lambda_2 < m, \\ -\frac{1}{2}(mH - rC)^2 - 2 & \text{if } r\lambda_1 < m < r\lambda_2, \end{cases} \\ h^2(mH - rC) &= 0 \quad \text{if } r\lambda_1 < m. \end{aligned}$$

Let \bar{H} be a linear hyperplane on \mathbb{P}^3 such that $\bar{H} \cdot S = H$. Let $\mathcal{I}_C = \tilde{\mathcal{I}}_C$, where \mathcal{I}_C is the homogeneous ideal of C in the coordinate ring R of \mathbb{P}^3 .

Let $\pi : X \rightarrow \mathbb{P}^3$ be the blowup of C . Let $E = \pi^*(C)$, the exceptional surface, \bar{S} be the strict transform of S on X . $\bar{S} \cong S$ and $E \cdot \bar{S} = C$. For $m, r \in \mathbb{N}$ and $i \geq 0$,

$$H^i(\mathbb{P}^3, \mathcal{I}_C^r(m)) \cong H^i(X, \mathcal{O}_X(m\bar{H} - rE)).$$

In particular,

$$I_C^{(r)} = (I_C^r)^{\text{sat}} \cong \bigoplus_{m \geq 0} H^0(\mathbb{P}^3, \mathcal{I}_C^r(m)) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(m\bar{H} - rE)).$$

In the exact sequence

$$0 \rightarrow \mathcal{O}_X(-\bar{S}) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{\bar{S}} \rightarrow 0$$

we have $\bar{S} \sim 4\bar{H} - E$. Tensor with $\mathcal{O}_X((m+4)\bar{H} - (r+1)E)$ to get

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_X(m\bar{H} - rE) \longrightarrow \mathcal{O}_X((m+4)\bar{H} - (r+1)E) \\ \longrightarrow \mathcal{O}_S((m+4)H - (r+1)C) \longrightarrow 0. \end{aligned}$$

Since $h^i(\mathcal{O}_X(m\bar{H})) = 0$ for $i > 0$ and $m \geq 0$, our calculation of cohomology on S and induction gives

$$\begin{aligned} h^1(m\bar{H} - rE) &= \begin{cases} 0 & \text{if } m > r\lambda_2, \\ h^1(mH - rE) & \text{if } m = \lfloor r\lambda_2 \rfloor \text{ or } m = \lfloor r\lambda_2 \rfloor - 1, \end{cases} \\ h^2(mH - rE) &= 0 & \text{if } m > \lambda_1 r, \\ h^3(mH - rE) &= 0 & \text{if } m > 4r. \end{aligned}$$

By our calculation of cohomology on S , we have for $r, t \in \mathbb{N}$,

$$h^1(\mathcal{I}_C^r(t-1)) = h^1((t-1)\bar{H} - rE) \begin{cases} = 0 & \text{if } t \geq \lfloor r\lambda_2 \rfloor + 1 + \sigma(r), \\ \neq 0 & \text{if } t = \lfloor r\lambda_2 \rfloor + \sigma(r), \end{cases}$$

where

$$\sigma(r) = \begin{cases} 0 & \text{if } h^1(\lfloor r\lambda_2 \rfloor H - rC) = 0, \\ 1 & \text{if } h^1(\lfloor r\lambda_2 \rfloor H - rC) \neq 0. \end{cases}$$

We obtain, for $r \in \mathbb{N}$,

$$a^2(I_C^r) = \text{reg}(I_C^{(r)}) = \text{reg}((I_C^r)^{\text{sat}}) = \lfloor r\lambda_2 \rfloor + 1 + \sigma(r)$$

with

$$\lim_{r \rightarrow \infty} \frac{a^2(I_C^r)}{r} = \lim_{r \rightarrow \infty} \frac{\text{reg}((I_C^r)^{\text{sat}})}{r} = \lambda_2 \notin \mathbb{Q}.$$

In this example, we have shown that the function

$$\text{reg}((I_C^n)^{\text{sat}}) = \text{reg}(I_C^{(n)})$$

has irrational behavior asymptotically. This is perhaps not so surprising, as its symbolic algebra

$$\bigoplus_{n \geq 0} I_C^{(n)}$$

is not a finitely generated R -algebra. An example of an ideal of a union of generic points in the plane whose symbolic algebras is not finitely generated was found and used by Nagata [1959] to give his counterexample to Hilbert's 14th problem. Roberts [1985] interpreted this example to give an example of a prime

ideal of a space curve. Even for rational monomial curves this algebra may not be finitely generated, by an example in [Goto, Nishida and Watanabe 1994].

Holger Brenner [2013] has recently given a remarkable example showing that the Hilbert–Kunz multiplicity can be irrational.

We now discuss the regularity of coherent sheaves. Suppose that X is a projective variety, over a field k , and H is a very ample divisor on X . Suppose that $\mathcal{J} \subset \mathcal{O}_X$ is an ideal sheaf. Let $\pi : B(\mathcal{J}) \rightarrow X$ be the blowup of \mathcal{J} , with exceptional divisor F . The Seshadri constant of \mathcal{J} is defined to be

$$s_H(\mathcal{J}) = \inf\{s \in \mathbb{R} \mid \pi^*(sH) - F \text{ is a very ample } \mathbb{R}\text{-divisor on } B(\mathcal{J}).\}$$

The regularity of \mathcal{J} is defined to be

$$\text{reg}_H(\mathcal{J}) = \max\{m \mid H^i(X, \mathcal{H} \otimes \mathcal{O}_X((m - i - 1)H)) \neq 0.\}$$

Theorem 5.13 [Cutkosky, Ein and Lazarsfeld 2001]. *Suppose that $\mathcal{I} \subset \mathcal{O}_X$ is an ideal sheaf. Then*

$$\lim_{m \rightarrow \infty} \frac{\text{reg}_H(\mathcal{I}^m)}{m} = \lim_{m \rightarrow \infty} \frac{d_H(\mathcal{I}^m)}{m} = s_H(\mathcal{I}).$$

For an ideal sheaf \mathcal{J} ,

$$d_H(\mathcal{J}) = \text{least integer } d \text{ such that } \mathcal{J}(dH) \text{ is globally generated.}$$

If H is a linear hyperplane on \mathbb{P}^n , and $\mathcal{I} = \tilde{I}$, we get the statement that the limit

$$\lim_{m \rightarrow \infty} \frac{\text{reg}((I^m)^{\text{sat}})}{m} = \lim_{m \rightarrow \infty} \frac{d((I^m)^{\text{sat}})}{m}.$$

exists, where $d((I^m)^{\text{sat}})$ is the maximal degree of a generator of $(I^m)^{\text{sat}}$.

We now give an example of an irrational Seshadri constant. The ideal I of a nonsingular curve in \mathbb{P}^3 contained in a quartic which we considered earlier gives an example (see [Cutkosky 2000]):

$$s_H(\tilde{I}) = \lim_{m \rightarrow \infty} \frac{\text{reg}_H(\tilde{I}^m)}{m} = \lim_{m \rightarrow \infty} \frac{\text{reg}((I^m)^{\text{sat}})}{m} = 9 + \sqrt{2}.$$

We do have something like linear growth of regularity reg_H in the example. Recall that the example is of the homogeneous height two prime ideal I in $k[x_0, x_1, x_2, x_3]$ of a nonsingular projective space curve, such that

$$\begin{aligned} \text{reg}_H(\tilde{I}^n) &= \text{reg}((I^n)^{\text{sat}}) = \max\{a^i(I^n) + i \mid 2 \leq i \leq 4\} \\ &= \lfloor m(9 + \sqrt{2}) \rfloor + 1 + \sigma(m) \end{aligned}$$

for $m > 0$, where $\lfloor x \rfloor$ is the greatest integer in a real number x and

$$\sigma(m) = \begin{cases} 0 & \text{if } m = q_{2n} \text{ for some } n \in \mathbb{N}, \\ 1 & \text{otherwise,} \end{cases}$$

where q_n is defined recursively by

$$q_0 = 1, q_1 = 2, q_n = 2q_{n-1} + q_{2n-2}.$$

Theorem 5.14 (Wenbo Niu [2011]). *Suppose that $\mathcal{I} = \tilde{I}$ is an ideal sheaf on \mathbb{P}^n . Then there is a bounded function $\tau(m)$, with $0 \leq \tau(m) \leq \text{constant}$, such that*

$$\text{reg}_H(\mathcal{I}^m) = \text{reg}((I^n)^{\text{sat}}) = \lfloor s_H m \rfloor + \tau(m).$$

for all $m > 0$.

We conclude this section with some questions. Suppose that I is a homogeneous ideal in a polynomial ring S .

- Does

$$\lim_{n \rightarrow \infty} \frac{a^i(I^n)}{n}$$

exist for all i ?

- Does

$$\lim_{n \rightarrow \infty} \frac{\text{reg}(I^{(n)})}{n}$$

exist? (The answer is yes if the singular locus of S/I has dimension ≤ 1 ; see [Herzog, Hoa and Trung 2002].)

- Does

$$\lim_{n \rightarrow \infty} \frac{\text{reg}(\text{in}(I^n))}{n}$$

exist?

- David Eisenbud has posed the following problem. Suppose that I is generated in a single degree. Explain (geometrically) the constant term b in the linear polynomial

$$\text{reg}(I^n) = an + b \quad \text{for } n \gg 0.$$

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