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Survey on the D-module f^s

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We discuss various aspects of the singularity invariants with differential origin derived from the D-module generated by f^s.

In this survey we discuss various aspects of the singularity invariants with differential origin derived from the D-module generated by f^s. We should like to point the reader to some other works: [Saito 2007] for V-filtration, Bernstein–Sato polynomials, multiplier ideals; [Budur 2012b] for all these and Milnor fibers; [Torrelli 2007] and [Narváez-Macarro 2008] for homogeneity and free divisors; [Suciu 2014] on details of arrangements, specifically their Milnor fibers, although less focused on D-modules.

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1. Introduction

Notation 1.1. In this article, X will denote a complex manifold. Unless indicated otherwise, X will be $\mathbb{C}^n$.

Throughout, let $R = \mathbb{C}[x_1, \ldots, x_n]$ be the ring of polynomials in n variables over the complex numbers. We denote by $D = R\langle \partial_1, \ldots, \partial_n \rangle$ the Weyl algebra. In particular, ∂_i denotes the partial differentiation operator with respect to x_i. If X is a general manifold, $\mathscr{O}_X$ (the sheaf of regular functions) and $\mathscr{D}_X$ (the sheaf of $\mathbb{C}$-linear differential operators on $\mathscr{O}_X$) take the places of R and D.

If $X = \mathbb{C}^n$ we use Roman letters to denote rings and modules; in the general case we use calligraphic letters to denote corresponding sheaves.

By the ideal J_f we mean the $\mathscr{O}_X$-ideal generated by the partial derivatives $\partial f/\partial x_1, \ldots, \partial f/\partial x_n$; this ideal varies with the choice of coordinate system in which we calculate. In contrast, the Jacobian ideal $\mathrm{Jac}(f) = J_f + (f)$ is independent.

The ring D (resp. the sheaf $\mathscr{D}_X$) is coherent, and both left- and right-Noetherian; it has only trivial two-sided ideals [Björk 1993, Theorem 1.2.5]. Introductions to the theory of D-modules as we use them here can be found in [Kashiwara 2003; Bernstein ca. 1997; Björk 1993; 1979].

The ring D admits the order filtration induced by the weight $x_i \to 0$, $\partial_i \to 1$. The order filtration (and other good filtrations) leads to graded objects $\mathrm{gr}_{(0,1)}(-)$; see [Schapira 1985]. The graded objects obtained from ideals are ideals in the polynomial ring $\mathbb{C}[x, \xi]$, homogeneous in the symbols of the differentiation operators; their radicals are closed under the Poisson bracket, and thus the corresponding varieties are involutive [Kashiwara 1975; Kashiwara and Kawai 1981a]. For a D-module M and a component C of the support of $\mathrm{gr}_{(0,1)}(M)$, attach to the pair (M, C) the multiplicity $\mu(M, C)$ of $\mathrm{gr}_{(0,1)}(M)$ along C. The characteristic cycle of M is $\mathrm{charC}(M) = \sum_C \mu(M, C) \cdot C$, an element of the Chow ring on $T^*\mathbb{C}^n$. The module is *holonomic* if it is zero or if its characteristic variety is of dimension n, the minimal possible value.

Throughout, f will be a regular function on X, with divisor $\mathrm{Var}(f)$. We distinguish several homogeneity conditions on f:

- f is *locally* (*strongly*) *Euler-homogeneous* if for all $p \in \mathrm{Var}(f)$ there is a vector field θ_p defined near p with $\theta_p \bullet (f) = f$ (and θ_p vanishes at p).
- f is *locally* (*weakly*) *quasihomogeneous* if near all $p \in \mathrm{Var}(f)$ there is a local coordinate system $\{x_i\}$ and a positive (resp. nonnegative) weight vector $a = \{a_1, \ldots, a_n\}$ with respect to which $f = \sum_{i=1}^n a_i x_i \partial_i(f)$.
- We reserve *homogeneous* and *quasihomogeneous* for the case when $X = \mathbb{C}^n$ and f is globally homogeneous or quasihomogeneous.

To any nonconstant $f \in R$, one can attach several invariants that measure the singularity structure of the hypersurface $f = 0$. In this article, we are primarily interested in those derived from the (parametric) annihilator $\operatorname{ann}_{D[s]}(f^s)$ of f^s:

Definition 1.2. Let s be a new variable, and denote by $R_f[s]\cdot f^s$ the free module generated by f^s over the localized ring $R_f[s] = R[f^{-1}, s]$. Via the chain rule

$$\partial_i \bullet \left(\frac{g}{f^k} f^s\right) = \partial_i \bullet \left(\frac{g}{f^k}\right) f^s + \frac{sg}{f^{k+1}} \cdot \frac{\partial f}{\partial x_i} f^s \tag{1-1}$$

for each $g(x, s) \in R[s]$, $R_f[s]\cdot f^s$ acquires the structure of a left $D[s]$-module. Denote by

$$\operatorname{ann}_{D[s]}(f^s) = \{P \in D[s] \mid P \bullet f^s = 0\}$$

the *parametric annihilator*, and by

$$\mathscr{M}_f(s) = D[s]/\operatorname{ann}_{D[s]}(f^s)$$

the cyclic $D[s]$-module generated by $1 \cdot f^s \in R_f[s]\cdot f^s$.

Bernstein's functional equation [1972] asserts the existence of a differential operator $P(x, \partial, s)$ and a nonzero polynomial $b_{f,P}(s) \in \mathbb{C}[s]$ such that

$$P(x, \partial, s) \bullet f^{s+1} = b_{f,P}(s) \cdot f^s, \tag{1-2}$$

i.e., the existence of the element $P \cdot f - b_{f,P}(s) \in \operatorname{ann}_{D[s]}(f^s)$. Bernstein's result implies that $D[s]\bullet f^s$ is D-coherent (while $R_f[s]f^s$ is not).

Definition 1.3. The monic generator of the ideal in $\mathbb{C}[s]$ generated by all $b_{f,P}(s)$ appearing in an equation (1-2) is the *Bernstein–Sato polynomial* $b_f(s)$. Denote $\rho_f \subseteq \mathbb{C}$ the set of roots of $b_f(s)$.

Note that the operator P in the functional equation is only determined up to $\operatorname{ann}_{D[s]}(f^s)$. See [Björk 1979] for an elementary proof of the existence of $b_f(s)$. Alternative (and more general) proofs are given in [Kashiwara 2003]; see also [Bernstein ca. 1997; Mebkhout and Narváez-Macarro 1991; Núñez-Betancourt 2013].

The $\mathbb{C}[s]$-module $\mathscr{M}_f(s)/\mathscr{M}_f(s+1)$ is precisely annihilated by $b_f(s)$. It is an interesting problem to determine for any $q(s) \in \mathbb{C}[s]$ the ideals

$$\mathfrak{a}_{f,q(s)} = \left\{g \in R \mid q(s)gf^s \in D[s]\bullet f^{s+1}\right\}$$

from [Walther 2005]. By [Malgrange 1975],

$$\mathfrak{a}_{f,s+1} = R \cap (\operatorname{ann}_{D[s]}(f^s) + D[s]\cdot(f, J_f)).$$

Question 1.4. Is $\mathfrak{a}_{f,s+1} = J_f + (f)$?

A positive answer would throw light on connections between $b_f(s)$ and cohomology of Milnor fibers.

Remark 1.5. At the 1954 International Congress of Mathematics in Amsterdam, I. M. Gelfand asked the following question. Given a real analytic function $f\colon \mathbb{R}^n \to \mathbb{R}$, the assignment ($s \in \mathbb{C}$)

$$f(x)_+^s = \begin{cases} f(x)^s & \text{if } f(x) > 0, \\ 0 & \text{if } f(x) \le 0. \end{cases}$$

is continuous in x and analytic in s where the real part of s is positive. Can one analytically continue $f(x)_+^s$? Sato introduced $b_f(s)$ in order to answer Gelfand's question; Bernstein [1972] established their existence in general.

Remark 1.6. Let $m \in M$ be a nonzero section of a holonomic D-module. Generalizing the case $1 \in R$ there is a functional equation

$$P(x, \partial, s) \bullet (mf^{s+1}) = b_{f,P;m}(s) \cdot mf^s$$

with $b_{f,P;m}(s) \in \mathbb{C}[s]$ nonzero. The monic generator of the ideal $\{b_{f,P;m}(s)\}$ is the b-function $b_{f;m}(s)$ [Kashiwara 1976].

2. Parameters and numbers

For any complex number γ, the expression f^γ represents, locally outside $\mathrm{Var}(f)$, a multivalued analytic function. Via the chain rule as in (1-1), the cyclic R_f-module $R_f \cdot f^\gamma$ becomes a left D-module, and we set

$$\mathscr{M}_f(\gamma) = D \bullet f^\lambda \cong D/\operatorname{ann}_D(f^\gamma).$$

There are natural $D[s]$-linear maps

$$\mathrm{ev}_f(\gamma)\colon \mathscr{M}_f(s) \to \mathscr{M}_f(\gamma), \quad P(x, \partial, s) \bullet f^s \mapsto P(x, \partial, \gamma) \bullet f^\gamma,$$

and D-linear inclusions

$$\mathrm{inc}_f(s)\colon \mathscr{M}_f(s+1) \to \mathscr{M}_f(s), \quad P(x, \partial, s) \bullet f^{s+1} \mapsto P(x, \partial, s) \cdot f \bullet f^s$$

with cokernel $\mathscr{N}_f(s) = \mathscr{M}_f(s)/\mathscr{M}_f(s+1) \cong D[s]/(\operatorname{ann}_{D[s]}(f^s) + D[s]f)$, and

$$\mathrm{inc}_f(\gamma)\colon \mathscr{M}_f(\gamma+1) \to \mathscr{M}_f(\gamma), \quad P(x, \partial) \bullet f^{\lambda+1} \mapsto P(x, \partial) \cdot f \bullet f^\lambda$$

with cokernel $\mathscr{N}_f(\gamma) = \mathscr{M}_f(\gamma)/\mathscr{M}_f(\gamma+1) \cong D/(\operatorname{ann}_D(f^\gamma) + D \cdot f)$.

The kernel of the morphism $\mathrm{ev}_f(\gamma)$ contains the (two-sided) ideal $D[s](s-\gamma)$; the containment can be proper, for example if $\gamma = 0$. If $\{\gamma - 1, \gamma - 2, \ldots\}$ is disjoint from the root set ρ_f then $\ker \mathrm{ev}_f(\gamma) = D[s] \cdot (s - \gamma)$ [Kashiwara 1976]. If $\gamma \notin \rho_f$ then $\mathrm{inc}_f(\gamma)$ is an isomorphism because of the functional equation; if

$\gamma = -1$, or if $b_f(\gamma) = 0$ while ρ_f does not meet $\{\gamma - 1, \gamma - 2, \ldots\}$ then $\mathrm{inc}_f(\gamma)$ is not surjective [Walther 2005].

Question 2.1. Does $\mathrm{inc}_f(\gamma)$ fail to be an isomorphism for all $\gamma \in \rho_f$?

In contrast, the induced maps $\mathscr{M}_f(s)/(s-\gamma-1) \to \mathscr{M}_f(s)/(s-\gamma)$ are isomorphisms exactly when $\gamma \notin \rho_f$ [Björk 1993, 6.3.15]. The morphism $\mathrm{inc}_f(s)$ is never surjective as $s+1$ divides $b_f(s)$. One sets

$$\tilde{b}_f(s) = \frac{b_f(s)}{s+1}.$$

By [Torrelli 2009, 4.2], the following are equivalent for a section $m \neq 0$ of a holonomic module:

- the smallest integral root of $b_{f;m}(s)$ is at least $-\ell$;
- $(D \bullet m) \otimes_R R[f^{-1}]$ is generated by $m/f^\ell = m \otimes 1/f^\ell$;
- $(D \bullet m) \otimes_R R[f^{-1}]/D \bullet (m \otimes 1)$ is generated by m/f^ℓ;
- $D[s] \bullet mf^s \to (D \bullet m) \otimes_R R[f^{-1}]$, $P(s) \bullet (mf^s) \mapsto P(-\ell) \bullet (m/f^\ell)$ is an epimorphism with kernel $D[s] \cdot (s+\ell) mf^s$.

Definition 2.2. We say that f satisfies condition

(A_1) (resp. (A_s)) if $\mathrm{ann}_D(1/f)$ (resp. $\mathrm{ann}_D(f^s)$) is generated by operators of order one;

(B_1) if R_f is generated by $1/f$ over D.

Condition (A_1) implies (B_1) in any case [Torrelli 2004]. Local Euler-homogeneity, (A_s) and (B_1) combined imply (A_1) [Torrelli 2007], and for Koszul free divisors (see Definition 4.7 below) this implication can be reversed [Torrelli 2004].

Condition (A_1) does not imply (A_s): $f = xy(x+y)(x+yz)$ is free (see Definition 4.1), and locally Euler-homogeneous and satisfies (A_1) and (B_1) [Calderón-Moreno 1999; Calderón-Moreno et al. 2002; Calderón-Moreno and Narváez-Macarro 2002b; Castro-Jiménez and Ucha 2001; Torrelli 2004], but $\mathrm{ann}_{D[s]}(f^s)$ and $\mathrm{ann}_D(f^s)$ require a second order generator.

Condition (A_1) implies local Euler-homogeneity if f has isolated singularities [Torrelli 2002], or if it is Koszul-free or of the form $z^n - g(x, y)$ for reduced g [Torrelli 2004]. In [Castro-Jiménez et al. 2007] it is shown that for certain locally weakly quasihomogeneous free divisors $\mathrm{Var}(f)$, (A_1) holds for high powers of f, and even for f itself by [Narváez-Macarro 2008, Remark 1.7.4].

For an isolated singularity, f has (A_1) if and only if it has (B_1) and is quasihomogeneous [Torrelli 2002]. For example, a reduced plane curve (has automatically (B_1) and) has (A_1) if and only if it is quasihomogeneous. See [Schulze 2007] for further results.

Condition (B_1) is equivalent to $\mathrm{inc}_f(-2), \mathrm{inc}_f(-3), \ldots$ all being isomorphisms, and also to -1 being the only integral root of $b_f(s)$ [Kashiwara 1976]. Locally quasihomogeneous free divisors satisfy condition (B_1) at any point [Castro-Jiménez and Ucha 2002].

3. *V*-filtration and Bernstein–Sato polynomials

3A. ***V-filtration.*** The articles [Saito 1994; Maisonobe and Mebkhout 2004; Budur 2005; Budur 2012b] are recommended for material on V-filtrations.

3A1. *Definition and basic properties.* Let Y be a smooth complex manifold (or variety), and let X be a closed submanifold (or -variety) of Y defined by the ideal sheaf $\mathscr{I}$. The V-filtration on $\mathscr{D}_Y$ along X is, for $k \in \mathbb{Z}$, given by

$$V^k(\mathscr{D}_Y) = \{P \in \mathscr{D}_Y \mid P \bullet \mathscr{I}^{k'} \subseteq \mathscr{I}^{k+k'} \text{ for all } k' \in \mathbb{Z}\},$$

with the understanding that $\mathscr{I}^{k'} = \mathscr{O}_Y$ for $k' \leq 0$. The associated graded sheaf of rings $\mathrm{gr}_V(\mathscr{D}_Y)$ is isomorphic to the sheaf of rings of differential operators on the normal bundle $T_X(Y)$, algebraic in the fiber of the bundle.

Suppose that $Y = \mathbb{C}^n \times \mathbb{C}$ with coordinate function t on $\mathbb{C}$, and let X be the hyperplane $t = 0$. Then $V^k(D_Y)$ is spanned by $\{x^u \partial^v t^a \partial_t^b \mid a - b \geq k\}$. Given a coherent holonomic D_Y-module M with regular singularities in the sense of [Kashiwara and Kawai 1981b], Kashiwara [1983] and Malgrange [1983] define an exhaustive decreasing rationally indexed filtration on M that is compatible with the V-filtration on D_Y and has the following properties:

(1) Each $V^\alpha(M)$ is coherent over $V^0(D_Y)$ and the set of α with nonzero $\mathrm{gr}_V^\alpha(M) = V^\alpha(M)/V^{>\alpha}(M)$ has no accumulation point.

(2) For $\alpha \gg 0$, $V^1(D_Y)V^\alpha(M) = V^{\alpha+1}(M)$.

(3) $t\partial_t - \alpha$ acts nilpotently on $\mathrm{gr}_V^\alpha(M)$.

The V-filtration is unique and can be defined in somewhat greater generality [Budur 2005]. Of special interest is the following case considered in [Malgrange 1983; Kashiwara 1983].

Notation 3.1. Denote $R_{x,t}$ the polynomial ring $R[t]$, t a new indeterminate, and let $D_{x,t}$ be the corresponding Weyl algebra. Fix $f \in R$ and consider the regular $D_{x,t}$-module

$$\mathscr{B}_f = H^1_{f-t}(R[t]),$$

the unique local cohomology module of $R[t]$ supported in $f - t$. Then $\mathscr{B}_f$ is naturally isomorphic as $D_{x,t}$-module to the direct image (in the D-category) $i_+(R)$ of R under the graph embedding

$$i\colon X \to X \times \mathbb{C}, \quad x \mapsto (x, f(x)).$$

Moreover, extending (1-1) via

$$t \bullet (g(x,s) f^{s-k}) = g(x,s+1) f^{s+1-k},$$
$$\partial_t \bullet (g(x,s) f^{s-k}) = -s g(x,s-1) f^{s-1-k},$$

the module $R_f[s] \otimes f^s$ becomes a $D_{x,t}$-module extending the $D[s]$-action where $-\partial_t t$ acts as s.

The existence of the V-filtration on $\mathscr{B}_f = i_+(R)$ is equivalent to the existence of generalized b-functions $b_{f;\eta}(s)$ in the sense of [Kashiwara 1976]; see [Kashiwara 1978; Malgrange 1983]. In fact, one can recover one from the other:

$$V^\alpha(\mathscr{B}_f) = \{\eta \in \mathscr{B}_f \mid [b_{f;\eta}(-c) = 0] \Rightarrow [\alpha \le c]\}$$

and the multiplicity of $b_{f;\eta}(s)$ at α is the degree of the minimal polynomial of $s-\alpha$ on $\mathrm{gr}_V^\alpha(D[s]\eta f^s / D[s]\eta f^{s+1})$ [Sabbah 1987a]. For more on this "microlocal approach", see [Saito 1994].

3B. ***The log-canonical threshold.*** By [Kollár 1997] (see also [Lichtin 1989; Yano 1978]), the absolute value of the largest root of $b_f(s)$ is the *log-canonical threshold* $\mathrm{lct}(f)$ given by the supremum of all numbers s such that the local integrals

$$\int_{U \ni p} \frac{|dx|}{|f|^{2s}}$$

converge for all $p \in X$ and all small open U around p. Smaller lct corresponds to worse singularities; the best one can hope for is $\mathrm{lct}(f) = 1$ as one sees by looking at a smooth point. The notion goes back to Arnol'd, who called it (essentially) the complex singular index [Arnold et al. 1985].

The point of *multiplier ideals* is to force the finiteness of the integral by allowing moderating functions in the integral:

$$\mathscr{I}(f,\lambda)_p = \left\{g \in \mathscr{O}_X \mid g/f^\lambda \text{ is } L^2\text{-integrable near } p \in \mathrm{Var}(f)\right\},$$

for $\lambda \in \mathbb{R}$. By [Ein et al. 2004], there is a finite collection of *jumping numbers* for f of rational numbers $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_\ell = 1$ such that $\mathscr{I}(f,\alpha)$ is constant on $[\alpha_i, \alpha_{i+1})$ but $\mathscr{I}(f,\alpha_i) \neq \mathscr{I}(f,\alpha_{i+1})$. The log-canonical threshold appears as α_1. These ideas had appeared before in [Lipman 1982; Loeser and Vaquié 1990].

Generalizing Kollar's approach, each α_i is a root of $b_f(s)$ [Ein et al. 2004]. In [Saito 2007, Theorem 4.4] a partial converse is shown for locally Euler-homogeneous divisors. Extending the idea of jumping numbers to the range $\alpha > 1$ one sees that α is a jumping number if and only if $\alpha + 1$ is a jumping number, but the connection to the Bernstein–Sato polynomial is lost in general. For example, if $f(x,y) = x^2 + y^3$ then jumping numbers are $\{5/6, 1\} + \mathbb{N}$ while $b_f(s) = (s+5/6)(s+1)(s+7/6)$.

3C. ***Bernstein–Sato polynomial.*** The roots of $b_f(s)$ relate to an astounding number of other invariants, see for example [Kollár 1997] for a survey. However, besides the functional equation there is no known way to describe ρ_f.

3C1. *Fundamental results.* Let $p \in \mathbb{C}^n$ be a closed point, cut out by the maximal ideal $\mathfrak{m} \subseteq R$. Extending R to the localization $R_{\mathfrak{m}}$ (or even the ring of holomorphic functions at p) one arrives at potentially larger sets of polynomials $b_{f,P}(s)$ that satisfy a functional equation (1-2) with $P(x, \partial, s)$ now in the correspondingly larger ring of differential operators. The *local* (resp. *local analytic*) Bernstein–Sato polynomial $b_{f,p}(s)$ (resp. $b_{f,p^{an}}(s)$) is the generator of the resulting ideal generated by the $b_{f,P}(s)$ in $\mathbb{C}[s]$. We denote by $\rho_{f,p}$ (resp. $\tilde{\rho}_{f,p}$) the root set of $b_{f,p^{an}}(s)$ (resp. $b_{f,p^{an}}(s)/(s+1)$). From the definitions and [Lyubeznik 1997b; Briançon et al. 2000; Briançon and Maynadier 1999] we have

$$b_{f,p^{an}}(s) | b_{f,p}(s) | b_f(s) = \operatorname{lcm}_{p \in \operatorname{Var}(f)} b_{f,p}(s) = \operatorname{lcm}_{p \in \operatorname{Var}(f)} b_{f,p^{an}}(s), \quad (3\text{-}1)$$

and the function $\mathbb{C}^n \ni p \mapsto \operatorname{Var}(b_f(s))$, counting with multiplicity, is upper semi-continuous in the sense that for p' sufficiently near p one has $b_{f,p'}(s) \,|\, b_{f,p}(s)$. The underlying reason is the coherence of D.

The Bernstein–Sato polynomial $b_f(s)$ factors over $\mathbb{Q}$ into linear factors, $\rho_f \subseteq \mathbb{Q}$, and all roots are negative [Malgrange 1975; Kashiwara 1976]. The proof uses resolution of singularities over $\mathbb{C}$ in order to reduce to simple normal crossing divisors, where rationality and negativity of the roots is evident. For this Kashiwara proves a comparison theorem [Kashiwara 1976, Theorem 5.1] that establishes $b_f(s)$ as a divisor of a shifted product of the least common multiple of the local Bernstein–Sato polynomials of the pullback of f under the resolution map. There is a refinement by Lichtin [1989] for plane curves. The roots of $b_f(s)$, besides being negative, are always greater than $-n$, n being the minimum number of variables required to express f locally analytically [Varchenko 1981; Saito 1994].

3C2. *Constructible sheaves from f^s.* Let $V = V(n, d)$ be the vector space of all complex polynomials in $x_1 \ldots, x_n$ of degree at most d. Consider the function $\beta\colon V \ni f \mapsto b_f(s)$. By [Lyubeznik 1997b; Briançon and Maynadier 1999], there is an algebraic stratification of V such that on each stratum the function β is constant. For varying n, d these stratifications can be made to be compatible.

3C3. *Special cases.* If p is a smooth point of $\operatorname{Var}(f)$ then f can be used as an analytic coordinate near p, hence $b_{f,p^{an}}(s) = s + 1$, and so $b_f(s) = s + 1$ for all smooth hypersurfaces. By Proposition 2.6 in [Briançon and Maisonobe 1996], an extension of [Briançon et al. 1991], the equation $b_f(s) = s + 1$ implies smoothness of $\operatorname{Var}(f)$. Explicit formulas for the Bernstein–Sato polynomial are rare; here are some classes of examples.

- $f = \prod x_i^{a_i}$: $P = \prod \partial_i^{a_i}$ up to a scalar, $b_f(s) = \prod_i \prod_{j=1}^{a_i}(s + j/a_j)$.
- f (quasi)homogeneous with isolated singularity at zero:

$$\tilde{b}_f(s) = \operatorname{lcm}\left(s + \frac{\deg(g\,\mathrm{d}x)}{\deg(f)}\right),$$

  where g runs through a (quasi)homogeneous standard basis for J_f by work of Kashiwara, Sato, Miwa, Malgrange, Kochman [Malgrange 1975; Yano 1978; Torrelli 2005; Kochman 1976]. Note that the Jacobian ring of such a singularity is an Artinian Gorenstein ring, whose duality operator implies symmetry of ρ_f.
- $f = \det(x_{i,j})_1^n$: $P = \det(\partial_{i,j})_1^n$, $b_f(s) = (s+1)\cdots(s+n)$. This is attributed to Cayley, but see the comments in [Caracciolo et al. 2013].
- For some hyperplane arrangements, $b_f(s)$ is known; see [Walther 2005; Budur et al. 2011c].
- A long list of examples is worked out in [Yano 1978].

If V is a complex vector space, G a reductive group acting linearly on V with open orbit U such that $V \smallsetminus U$ is a divisor $\mathrm{Var}(f)$, Sato's theory of prehomogeneous vectors spaces [Sato and Shintani 1974; Muro 1988; Sato 1990; Yano 1977] yields a factorization for $b_f(s)$. For reductive linear free divisors, Granger and Schulze [2010] and Sevenheck [2011] discuss symmetry properties of Bernstein–Sato polynomials. In [Narvaez-Macarro 2013] this theme is taken up again, investigating specifically symmetry properties of ρ_f when $D[s] \bullet f^s$ has a Spencer logarithmic resolution (see [Castro-Jiménez and Ucha 2002] for definitions). This covers locally quasihomogeneous free divisors, and more generally free divisors whose Jacobian is of linear type. The motivation is the fact that roots of $b_f(s)$ seem to come in strands, and whenever roots can be understood the strands appear to be linked to Hodge-theory.

There are several results on ρ_f for other divisors of special shape. Trivially, if $f(x) = g(x_1, \ldots, x_k) \cdot h(x_{k+1}, \ldots, x_n)$ then $b_f(s) \mid b_g(s) \cdot b_h(s)$; the question of equality appears to be open. In contrast, $b_f(s)$ cannot be assembled from the Bernstein–Sato polynomials of the factors of f in general, even if the factors are hyperplanes and one has some control on the intersection behavior; see Section 8 below. If $f(x) = g(x_1, \ldots, x_k) + h(x_{k+1}, \ldots, x_n)$ and at least one is locally Euler-homogeneous, then there are Thom–Sebastiani type formulas [Saito 1994]. In particular, diagonal hypersurfaces are completely understood.

3C4. *Relation to intersection homology module.* Suppose

$$Y = \mathrm{Var}(f_1, \ldots, f_k) \subseteq X$$

is a complete intersection and denote by $\mathscr{H}^k_Y(\mathscr{O}_X)$ the unique (algebraic) local cohomology module of $\mathscr{O}_X$ along Y. Brylinski [1983; 1985], continuing work of Kashiwara, defined $\mathscr{L}(Y, X) \subseteq \mathscr{H}^k_Y(\mathscr{O}_X)$, the *intersection homology $\mathscr{D}_X$-module of Y*, the smallest $\mathscr{D}_X$-module equal to $\mathscr{H}^k_Y(\mathscr{O}_X)$ in the generic point(s). See also [Barlet and Kashiwara 1986]. The module $\mathscr{L}(X, Y)$ contains the fundamental class of Y in X [Barlet 1980].

Question 3.2. When is $\mathscr{L}(X, Y) = \mathscr{H}^k_Y(\mathscr{O}_X)$?

Equality is equivalent to $\mathscr{H}^k_Y(\mathscr{O}_X)$ being generated by the cosets of $\Delta / \prod_{i=1}^k f_i$ over $\mathscr{D}_X$ where Δ is the ideal generated by the k-minors of the Jacobian matrix of $f_1, \ldots, f_k$. A necessary condition is that $1/\prod_{i=1}^k f_i$ generates $\mathscr{H}^k_Y(\mathscr{O}_X)$, but this is not sufficient: consider $xy(x+y)(x+yz)$, where $\rho_f = -\{\frac{1}{2}, \frac{3}{4}, 1, 1, 1, \frac{5}{4}\}$. Indeed by [Torrelli 2009], equality can be characterized in terms of functional equations, as the following are equivalent at $p \in X$:

(1) $\mathscr{L}(X, Y) = \mathscr{H}^k_Y(\mathscr{O}_X)$ in the stalk;

(2) $\tilde{\rho}_{f,p} \cap \mathbb{Z} = \varnothing$;

(3) 1 is not an eigenvalue of the monodromy operator on the reduced cohomology of the Milnor fibers near p.

If $1/\prod_{i=1}^k f_i$ generates $R[1/\prod f_i]$ and $1/\prod_{i=1}^k f_i \in \mathscr{L}(X, Y)$, then $\tilde{b}_f(-1) \neq 0$ [Torrelli 2009]. It seems unknown whether (irrespective of $1/\prod_{i=1}^k f_i$ generating $R[1/\prod f_i]$) the condition $\tilde{b}_f(-1) \neq 0$ is equivalent to $1/\prod_{i=1}^k f_i$ being in $\mathscr{L}(X, Y)$. See also [Massey 2009] for a topological viewpoint. (By the Riemann–Hilbert correspondence of [Kashiwara 1984] and [Mebkhout 1984], $\mathscr{L}(X, Y)$ corresponds to the intersection cohomology complex of Y on X [Brylinski 1983] and $\mathscr{H}^k_Y(\mathscr{O}_X)$ to $\mathbb{C}_Y[n-k]$ [Grothendieck 1966; Kashiwara 1976; Mebkhout 1977]. Equality then says: the link is a rational homology sphere). Barlet [1999] characterizes property (3) above in terms of currents for complexified real f. Equivalence of (1) and (3) for isolated singularities can be derived from [Milnor 1968; Brieskorn 1970]; the general case can be shown using [Saito 1990, 4.5.8] and the formalism of weights. For the case $k = 1$, (1) requires irreducibility; in general, there is a criterion in terms of b-functions [Torrelli 2009, 1.6, 1.10].

4. LCT and logarithmic ideal

4A. ***Logarithmic forms.*** Let $X = \mathbb{C}^n$ be the analytic manifold, f a holomorphic function on X, and $Y = \mathrm{Var}(f)$ a divisor in X with $j\colon U = X \smallsetminus Y \hookrightarrow X$ the embedding. Let $\Omega^\bullet_X(*Y)$ denote the complex of differential forms on X that are (at worst) meromorphic along Y. By [Grothendieck 1966], $\Omega^\bullet_X(*Y) \to \mathbb{R}j_*\mathbb{C}_U$ is a quasiisomorphism.

A form ω is *logarithmic* along Y if $f\omega$ and $f\mathrm{d}\omega$ are holomorphic; these ω form the logarithmic de Rham complex $\Omega^\bullet_X(\log Y)$ on X along Y. The complex $\Omega^\bullet_X(\log Y)$ was first used with great effect on normal crossing divisors by Deligne [1971; 1974] in order to establish mixed Hodge structures, and later by Esnault and Viehweg [1992] in order to prove vanishing theorems. A major reason for the success of normal crossings is that in that case $\Omega^i_X(\log Y)$ is a locally free module over $\mathscr{O}_X$. The logarithmic de Rham complex was introduced in [Saito 1980] for general divisors.

4B. *Free divisors.*

Definition 4.1. A divisor $\mathrm{Var}(f)$ is *free* if (locally) $\Omega^1_X(\log f)$ is a free $\mathscr{O}_X$-module.

For a nonsmooth locally Euler-homogeneous divisor, freeness is equivalent to the Jacobian ring $\mathscr{O}_X/J_f$ being a codimension-2 Cohen–Macaulay $\mathscr{O}_X$-module; in general, freeness is equivalent to the Tjurina algebra $R/(f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ being of projective dimension 2 or less over R. See [Saito 1980; Aleksandrov 1986] for relations to determinantal equations. Free divisors have rather big singular locus, and are in some ways at the opposite end from isolated singularities in the singularity zoo. If $\Omega^1_X(\log f)$ is (locally) free, then $\Omega^i_X(\log f) \cong \bigwedge^i \Omega^i_X(\log f)$ and also (locally) free [Saito 1980]. A weakening is

Definition 4.2. A divisor $\mathrm{Var}(f)$ is *tame* if, for all $i \in \mathbb{N}$, (locally) $\Omega^i_X(\log f)$ has projective dimension at most i as a $\mathscr{O}_X$-module.

Plane curves are trivially free; surfaces in 3-space are trivially tame. Normal crossing divisors are easily shown to be free. Discriminants of (semi)versal deformations of an isolated complete intersection singularity (and some others) are free [Aleksandrov 1986; 1990; Looijenga 1984; Saito 1981; Damon 1998; Buchweitz et al. 2009]. Unitary reflection arrangements are free [Terao 1981].

Definition 4.3. The *logarithmic derivations* $\mathrm{Der}_X(-\log f)$ along $Y = \mathrm{Var}(f)$ are the $\mathbb{C}$-linear derivations $\theta \in \mathrm{Der}(\mathscr{O}_X; \mathbb{C})$ that satisfy $\theta \bullet f \in (f)$.

A derivation θ is logarithmic along Y if and only it is so along each component of the reduced divisor to Y [Saito 1980]. The modules $\mathrm{Der}_X(-\log f)$ and $\Omega^1_X(\log f)$ are reflexive and mutually dual over R. Moreover, $\Omega^i_X(\log f)$ and $\Omega^{n-i}_X(\log f)$ are dual.

4C. *LCT.*

Definition 4.4. If

$$\Omega^\bullet_X(\log Y) \to \Omega^\bullet_X(*Y) \tag{4-1}$$

is a quasiisomorphism, we say that *LCT holds for* Y.

We recommend [Narváez-Macarro 2008].

Remark 4.5. (1) This "logarithmic comparison theorem", a property of a divisor, is very hard to check explicitly. No general algorithms are known, even in $\mathbb{C}^3$ (but see [Castro-Jiménez and Takayama 2009] for $n = 2$).

(2) LCT fails for rather simple divisors such as $f = x_1x_2 + x_3x_4$.

(3) If Y is a reduced normal crossing divisor, Deligne [1970] proved (4-1) to be a filtered (by pole filtration) quasiisomorphism; this provided a crucial step in the development of the theory of mixed Hodge structures [Deligne 1971; 1974].

(4) Limiting the order of poles in forms needed to capture all cohomology of U started with the seminal [Griffiths 1969a; 1969b] and continues; see for example [Deligne and Dimca 1990; Dimca 1991; Karpishpan 1991].

(5) The free case was studied for example in [Castro-Jiménez et al. 1996]. But even in this case, LCT is not understood.

(6) If f is quasihomogeneous with an isolated singularity at the origin, then LCT for f is equivalent to a topological condition (the link of f at the origin being a rational homology sphere), as well as an arithmetic one on the Milnor algebra of f [Holland and Mond 1998]. In [Schulze 2010], using the Gauss–Manin connection, this is extended to a list of conditions on an isolated hypersurface singularity, each one of which forces the implication [D has LCT] $\Rightarrow$ [D is quasihomogeneous].

(7) For a version regarding more general connections, see [Calderón-Moreno and Narváez-Macarro 2009].

A plane curve satisfies LCT if and only it is locally quasihomogeneous [Calderón-Moreno et al. 2002]. By [Castro-Jiménez et al. 1996], free locally quasihomogeneous divisors satisfy LCT in any dimension. By [Granger and Schulze 2006a], in dimension three, free divisors with LCT must be locally Euler-homogeneous. Conjecturally, LCT implies local Euler-homogeneity [Calderón-Moreno et al. 2002]. The converse is false, see for example [Castro-Jiménez and Ucha 2005]. The classical example of rotating lines with varying cross-ratio $f = xy(x + y)(x + yz)$ is free, satisfies LCT and is locally Euler-homogeneous, but only weakly quasihomogeneous [Calderón-Moreno et al. 2002]. In [Castro-Jiménez et al. 2007], the effect of the Spencer property on LCT is discussed in the presence of homogeneity conditions. For locally quasihomogeneous divisors (or if the nonfree locus is zero-dimensional), LCT implies (B_1) [Castro-Jiménez and Ucha 2002; Torrelli 2007]. In particular, LCT implies (B_1) for divisors with isolated singularities. In [Granger and Schulze 2006b] quasihomogeneity of

isolated singularities is characterized in terms of a map of local cohomology modules of logarithmic differentials.

A free divisor is *linear free* if the (free) module $\mathrm{Der}_X(-\log f)$ has a basis of linear vector fields. In [Granger et al. 2009], linear free divisors in dimension at most 4 are classified, and for these divisors LCT holds at least on global sections. In the process, it is shown that LCT is implied if the Lie algebra of linear logarithmic vector fields is reductive. The example of $n \times n$ invertible upper triangular matrices acting on symmetric matrices [Granger et al. 2009, Example 5.1] shows that LCT may hold without the reductivity assumption. Linear free divisors appear naturally, for example in quiver representations and in the theory of prehomogeneous vector spaces and castling transformations [Buchweitz and Mond 2006; Sato and Kimura 1977; Granger et al. 2011]. Linear freeness is related to unfoldings and Frobenius structures [de Gregorio et al. 2009].

Denote by $\mathrm{Der}_{X,0}(-\log f)$ the derivations θ with $\theta \bullet f = 0$. In the presence of a global Euler-homogeneity E on Y there is a splitting

$$\mathrm{Der}_X(-\log f) \cong R \cdot E \oplus \mathrm{Der}_{X,0}(-\log f).$$

Reading derivations as operators of order one,

$$\mathrm{Der}_{X,0}(-\log f) \subseteq \mathrm{ann}_D(f^s).$$

We write S for $\mathrm{gr}_{(0,1)}(D)$; if y_i is the symbol of ∂_i then we have $S = R[y]$.

Definition 4.6. The inclusion $\mathrm{Der}_{X,0}(-\log f) \hookrightarrow \mathrm{ann}_D(f^s)$, via the order filtration, defines a subideal of $\mathrm{gr}_{(0,1)}(\mathrm{ann}_D(f^s)) \subseteq \mathrm{gr}_{(0,1)}(D) = S$ called the *logarithmic ideal* L_f of $\mathrm{Var}(f)$.

Note that the symbols of $\mathrm{Der}_X(-\log f)$ are in the ideal $R \cdot y$ of height n.

Definition 4.7. If $\mathrm{Der}_X(-\log f)$ has a generating set (as an R-module) whose symbols form a regular sequence on S, then Y is called *Koszul free*.

As $\mathrm{Der}_X(-\log f)$ has rank n, a Koszul free divisor is indeed free. Divisors in the plane [Saito 1980] and locally quasihomogeneous free divisors [Calderón-Moreno and Narváez-Macarro 2002b; 2002a] are Koszul free. In the case of normal crossings, this has been used to make resolutions for $D[s] \bullet f^s$ and $D[s]/D[s](\mathrm{ann}_{D[s]} f^s, f)$ [Gros and Narváez-Macarro 2000]. A way to distill invariants from resolutions of $D[s] \bullet f^s$ is given in [Arcadias 2010]. The logarithmic module $\widetilde{M}^{\log f} = D/D \cdot \mathrm{Der}_X(-\log f)$ has in the Spencer case (see [Castro-Jiménez and Ucha 2002; Calderón-Moreno and Narváez-Macarro 2005]) a natural free resolution of Koszul type.

For Koszul-free divisors, the ideal $D \cdot \mathrm{Der}_X(-\log f)$ is holonomic [Calderón-Moreno 1999]. By [Granger et al. 2009, Theorem 7.4], in the presence of freeness, the Koszul property is equivalent to the local finiteness of Saito's logarithmic

stratification. This yields an algorithmic way to certify (some) free divisors as not locally quasihomogeneous, since free locally quasihomogeneous divisors are Koszul free. Based on similar ideas, one may devise a test for strong local Euler-homogeneity [Granger et al. 2009, Lemma 7.5]. See [Calderón-Moreno 1999] and [Torrelli 2007, Section 2] for relations of Koszul freeness to perversity of the logarithmic de Rham complex.

Castro-Jiménez and Ucha established conditions for $Y = \mathrm{Var}(f)$ to have LCT in terms of D-modules [Castro-Jiménez and Ucha 2001; 2002; 2004b] for certain free f. For example, LCT is equivalent to (A_1) for Spencer free divisors. Calderón-Moreno and Narváez-Macarro [2005] proved that free divisors have LCT if and only if the natural morphism $\mathscr{D}_X \otimes^L_{V^0(\mathscr{D}_X)} \mathscr{O}_X(Y) \to \mathscr{O}_X(*Y)$ is a quasiisomorphism, $\mathscr{O}_X(Y)$ being the meromorphic functions with simple pole along f. For Koszul free Y, one has at least

$$\mathscr{D}_X \otimes^L_{V^0(\mathscr{D}_X)} \mathscr{O}_X(Y) \cong \mathscr{D}_X \otimes_{V^0(\mathscr{D}_X)} \mathscr{O}_X(Y).$$

A similar condition ensures that the logarithmic de Rham complex is perverse [Calderón-Moreno 1999; Calderón-Moreno and Narváez-Macarro 2005]. The two results are related by duality between logarithmic connections on $\mathscr{D}_X$ and the V-filtration [Castro-Jiménez and Ucha 2002; 2004a; Calderón-Moreno and Narváez-Macarro 2005].

It is unknown how LCT is related to (A_1) in general, but for quasihomogeneous polynomials with isolated singularities the two conditions are equivalent [Torrelli 2007].

4D. *Logarithmic linearity.*

Definition 4.8. We say that $f \in R$ satisfies (L_s) if the characteristic ideal of $\mathrm{ann}_D(f^s)$ is generated by symbols of derivations.

Condition (L_s) holds for isolated singularities [Yano 1978], locally quasihomogeneous free divisors [Calderón-Moreno and Narváez-Macarro 2002b], and locally strongly Euler-homogeneous holonomic tame divisors [Walther 2015]. Also, (L_s) plus (B_1) yields (A_1) for locally Euler-homogeneous f by [Kashiwara 1976]; see [Torrelli 2007].

The logarithmic ideal supplies an interesting link between $\Omega^\bullet_X(\log f)$ and $\mathrm{ann}_D(f^s)$ via approximation complexes: if f is holonomic, strongly locally Euler-homogeneous and also tame then the complex $(\Omega^\bullet_X(\log f)[y],\, y\,\mathrm{d}x)$ is a resolution of the logarithmic ideal L_f, and S/L_f is a Cohen–Macaulay domain of dimension $n+1$; if f is in fact free, S/L_f is a complete intersection [Narváez-Macarro 2008; Walther 2015].

Question 4.9. For locally Euler-homogeneous divisors, is $\mathrm{ann}_D(f^s)$ related to the cohomology of $(\Omega^\bullet_X(\log f)[y],\, y\,\mathrm{d}x)$?

5. Characteristic variety

We continue to assume that $X = \mathbb{C}^n$. For $f \in R$ let U_f be the open set defined by $\mathrm{d}f \neq 0 \neq f$. Because of the functional equation, $\mathscr{M}_f(s)$ is coherent over D [Bernstein 1972; Kashiwara 1976], and the restriction of $\mathrm{charV}(D[s]\bullet f^s)$ to U_f is the Zariski closure of

$$\left\{ \left(\xi, s\frac{\mathrm{d}f(\xi)}{f(\xi)} \right) \,\middle|\, \xi \in U_f, s \in \mathbb{C} \right\}; \tag{5-1}$$

it is an $(n+1)$-dimensional involutive subvariety of T^*U_f [Kashiwara 2003]. Ginsburg [1986] gives a formula for the characteristic cycle of $D[s]\bullet mf^s$ in terms of an intersection process for holonomic sections m.

In favorable cases, more can be said. By [Calderón-Moreno and Narváez-Macarro 2002b], if the divisor is reduced, free and locally quasihomogeneous then $\mathrm{ann}_{D[s]}(f^s)$ is generated by derivations, both $\mathscr{M}_f(s)$ and $\mathscr{N}_f(s)$ have Koszul–Spencer type resolutions, and so the characteristic varieties are complete intersections. In the more general case where f is locally strongly Euler-homogeneous, holonomic and tame, $\mathrm{ann}_D(f^s)$ is still generated by order one operators and the ideal of symbols of $\mathrm{ann}_D(f^s)$ (and hence the characteristic ideal of $\mathscr{M}_f(s)$ as well) is a Cohen–Macaulay prime ideal [Walther 2015]. Under these hypotheses, the characteristic ideal of $\mathscr{N}_f(s)$ is Cohen–Macaulay but usually not prime.

5A. ***Stratifications.*** By [Kashiwara and Schapira 1979], the resolution theorem of Hironaka can be used to show that there is a stratification of $\mathbb{C}^n$ such that for each holonomic D-module M, $\mathrm{charC}(M) = \bigsqcup_{\sigma\in\Sigma} \mu(M,\sigma)T^*_\sigma$ where T^*_σ is the closure of the conormal bundle of the smooth stratum σ in $\mathbb{C}^n$ and $\mu(M,\sigma) \in \mathbb{N}$.

For $D[s]\bullet f^s/D[s]\bullet f^{s+1}$ Kashiwara proved that if one considers a Whitney stratification S for f (for example the "canonical" stratification in [Damon and Mond 1991]) then the characteristic variety of the D-module $\mathscr{N}_f(s)$ is in the union of the conormal varieties of the strata $\sigma \in S$ [Yano 1978].

If one slices a pair (X, D) of a smooth space and a divisor with a hyperplane, various invariants of the divisor will behave well provided that the hyperplane is not "special". A prime example are Bertini and Lefschetz theorems. For D-modules, Kashiwara defined the notion of *noncharacteristic restriction*: the smooth hypersurface H is noncharacteristic for the D-module M if it meets each component of the characteristic variety of M transversally (see [Pham 1979] for an exposition). The condition assures that the inverse image functor attached to the embedding $H \hookrightarrow X$ has no higher derived functors for M. In [Dimca et al. 2006] these ideas are used to show that the V-filtration, and hence the multiplier ideals as well as nearby and vanishing cycle sheaves, behave nicely under noncharacteristic restriction.

5B. ***Deformations.*** Varchenko proved, via establishing constancy of Hodge numbers, that in a μ-constant family of isolated singularities, the spectrum is constant [Varchenko 1982]. In [Dimca et al. 2006] it is shown that the formation of the spectrum along the divisor $Y \subseteq X$ commutes with the intersection with a hyperplane transversal to any stratum of a Whitney regular stratification of D, and a weak generalization of Varchenko's constancy results for certain deformations of nonisolated singularities is derived.

In contrast, the Bernstein–Sato polynomial may not be constant along μ-constant deformations. Suppose $f(x) + \lambda g(x)$ is a 1-parameter family of plane curves with isolated singularities at the origin. If the Milnor number $\dim_{\mathbb{C}}(R/J_{(f+\lambda g)})$ is constant in the family, the singularity germs in the family are topologically equivalent [Tráng and Ramanujam 1976]; for discussion, see [Dimca 1992, Section 2]. However, in such a family $b_f(s)$ can vary, as it is a differential (but not a topological) invariant. Indeed, $f + \lambda g = x^4 + y^5 + \lambda x y^4$ has constant Milnor number 20, and the general curve (not quasihomogeneous in any coordinate system, as $\rho_{f+\lambda g}$ is not symmetric about -1; see Section 3C) has $-\rho_{f+\lambda g} = \{1\} \cup \frac{1}{20}\{9, 11, 13, 14, 17, 18, 19, 21, 22, 23, 26, 27\}$ while the special curve has $-\rho_f = -\rho_{f+\lambda g} \cup \{-31/20\} \smallsetminus \{-11/20\}$. See [Cassou-Noguès 1986] for details and similar examples based on Newton polytope considerations, and [Stahlke 1997] for deformations of plane diagonal curves.

6. Milnor fiber and monodromy

6A. ***Milnor fibers.*** Let $B(p, \varepsilon)$ denote the ε-ball around $p \in \mathrm{Var}(f) \subseteq \mathbb{C}^n$. Milnor [1968] proved that the diffeomorphism type of the open real manifold

$$M_{p,t_0,\varepsilon} = B(p, \varepsilon) \cap \mathrm{Var}(f - t_0)$$

is independent of ε, t_0 as long as $0 < |t_0| \ll \varepsilon \ll 1$. For $0 < \tau \ll \varepsilon \ll 1$ denote by M_p the fiber of the bundle $B(p, \varepsilon) \cap \{q \in \mathbb{C}^n \mid 0 < |f(q)| < \tau\} \to f(q)$.

The direct image functor for D-modules to the projection $\mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$, $(x, t) \mapsto t$ turns the $D_{x,t}$-module $\mathscr{B}_f$ into the *Gauss–Manin system* $\mathscr{H}_f$. The D-module restriction of $H^k(\mathscr{H}_f)$ to $t = t_0$ is the k-th cohomology of the Milnor fibers along $\mathrm{Var}(f)$ for $0 < |t_0| < \tau$.

Fix a k-cycle $\sigma \in H_p(\mathrm{Var}(f - t_0))$ and choose $\eta \in H^k(\mathscr{H}_f)$. Deforming σ to a k-cycle over t using the Milnor fibration, one can evaluate $\int_{\sigma_t} \eta$. The Gauss–Manin system has Fuchsian singularities and these periods are in the Nilsson class [Malgrange 1974]. For example, the classical Gauss hypergeometric function saw the light of day the first time as solution to a system of differential equations attached to the variation of the Hodge structure on an elliptic curve (expressed as integrals of the first and second kind) [Brieskorn and Knörrer

1981]. In [Pham 1979] this point of view is taken to be the starting point. The techniques explained there form the foundation for many connections between f^s and singularity invariants attached to $\mathrm{Var}(f)$.

In [Budur 2003], a bijection (for $0 < \alpha \leq 1$) is established between a subset of the jumping numbers of f at $p \in \mathrm{Var}(f)$ and the support of the *Hodge spectrum* [Steenbrink 1989]

$$\mathrm{Sp}(f) = \sum_{\alpha \in \mathbb{Q}} n_\alpha(f) t^\alpha,$$

with $n_\alpha(f)$ determined by the size of the α-piece of Hodge component of the cohomology of the Milnor fiber of f at p. See also [Saito 1993; Varchenko 1981], and [Steenbrink 1987] for a survey on Hodge invariants. We refer to [Budur 2012b; Saito 2009] for many more aspects of this part of the story.

6B. ***Monodromy.*** The vector spaces $H^k(M_{p,t_0,\varepsilon}, \mathbb{C})$ form a smooth vector bundle over a punctured disk $\mathbb{C}^*$. The linear transformation $\mu_{f,p,k}$ on $H^k(M_{p,t_0,\varepsilon}, \mathbb{C})$ induced by $p \mapsto p \cdot \exp(2\pi i \lambda)$ is the k-th monodromy of f at p. Let $\chi_{f,p,k}(t)$ denote the characteristic polynomial of $\mu_{f,p,k}$, set

$$e_{f,p,k} = \{\gamma \in \mathbb{C} \mid \gamma \text{ is an eigenvalue of } \mu_{f,p,k}\}$$

and put $e_{f,p} = \bigcup e_{f,p,k}$.

For most (in a quantifiable sense) divisors f with given Newton diagram, a combinatorial recipe can be given that determines the alternating product $\prod(\chi_{f,p,k}(t))^{(-1)^k}$ [Varchenko 1976], similarly to A'Campo's formula in terms of an embedded resolution [A'Campo 1975].

6C. ***Degrees, eigenvalues, and Bernstein–Sato polynomial.*** By [Malgrange 1983; Kashiwara 1983], the exponential function maps the root set of the local analytic Bernstein–Sato polynomial of f at p onto $e_{f,p}$. The set $\exp(-2\pi i \tilde{\rho}_{f,p})$ is the set of eigenvalues of the monodromy on the Grothendieck–Deligne vanishing cycle sheaf $\phi_f(\mathbb{C}_{X,p})$. This was shown in [Saito 1994] by algebraic microlocalization.

If f is an isolated singularity, the Milnor fiber M_f is a bouquet of spheres, and $H^{n-1}(M_f, \mathbb{C})$ can be identified with the Jacobian ring R/J_f. Moreover, if f is quasihomogeneous, then under this identification R/J_f is a $\mathbb{Q}[s]$-module, s acting via the Euler operator, and $\tilde{\rho}_f$ is in bijection with the degree set of the nonzero quasihomogeneous elements in R/J_f. For nonisolated singularities, most of this breaks down, since R/J_f is not Artinian in that case. However, for homogeneous f, consider the *Jacobian module*

$$H^0_{\mathfrak{m}}(R/J_f) = \{g + J_f \mid \exists k \in \mathbb{N}, \forall i, x_i^k g \in J_f\}$$

and the canonical $(n-1)$-form

$$\eta = \sum_i x_i \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n.$$

Every class in $H^{n-1}(M_f; \mathbb{C})$ is of the form $g\eta$ for suitable $g \in R$, and there is a filtration on $H^{n-1}(M_f, \mathbb{C})$ induced by integration of $\mathscr{B}_f$ along $\partial_1, \ldots, \partial_n$, with the following property: if $g \in R$ is the smallest degree homogeneous polynomial such that $g\eta$ represents a chosen element of $H^{n-1}(M_f, \mathbb{C})$ then $b_f(-(\deg(g\eta))/\deg(f)) = 0$ [Walther 2005]. Moreover, let $\bar{g} \neq 0$ by a homogeneous element in the Jacobian module and suppose that its degree $\deg(g\eta) = \deg(g) + \sum_i \deg(x_i)$ is between d and $2d$. Then, by [Walther 2015], $g\eta$ represents a nonzero cohomology class in $H^{n-1}(M_f, \mathbb{C})$ as in the isolated case.

6D. ***Zeta functions.*** The zeta function $Z_f(s)$ attached to a divisor $f \in R$ is the rational function

$$Z_f(s) = \sum_{I \subseteq S} \chi(E_I^*) \prod_{i \in I} \frac{1}{N_i s + \nu_i},$$

where $\pi : \left(Y, \bigcup_I E_i\right) \to (\mathbb{C}^n, \mathrm{Var}(f))$ is an embedded resolution of singularities, and N_i (resp. $\nu_i - 1$) are the multiplicities of E_i in $\pi^*(f)$ (resp. in the Jacobian of π). By results of Denef and Loeser [1992], $Z_f(s)$ is independent of the resolution.

Conjecture 6.1 (Topological Monodromy Conjecture).

(SMC) *Any pole of $Z_f(s)$ is a root of the Bernstein–Sato polynomial $b_f(s)$.*

(MC) *Any pole of $Z_f(s)$ yields under exponentiation an eigenvalue of the monodromy operator at some $p \in \mathrm{Var}(f)$.*

The strong version (SMC) implies (MC) by [Malgrange 1975; Kashiwara 1983]. Each version allows a generalization to ideals.

(SMC), formulated by Igusa [2000] and Denef–Loeser [1992] holds for

- reduced curves by [Loeser 1988] with a discussion on the nature of the poles by Veys [1993; 1990; 1995];
- certain Newton-nondegenerate divisors by [Loeser 1990];
- some hyperplane arrangements (see Section 8);
- monomial ideals in any dimension by [Howald et al. 2007].

Additionally, (MC) holds for

- bivariate ideals by Van Proeyen and Veys [2010];
- all hyperplane arrangements by [Budur et al. 2011b; 2011c];

- some partial cases: [Artal Bartolo et al. 2002; Lemahieu and Veys 2009] some surfaces; [Artal Bartolo et al. 2005] quasiordinary power series; [Lichtin and Meuser 1985; Loeser 1990] in certain Newton nondegenerate cases; [Igusa 1992; Kimura et al. 1990] for invariants of prehomogeneous vector spaces; [Lemahieu and Van Proeyen 2011] for nondegenerate surfaces.

Strong evidence for (MC) for $n = 3$ is procured in [Veys 2006]. The articles [Rodrigues 2004; Némethi and Veys 2012] explore what (MC) could mean on a normal surface as ambient space and gives some results and counterexamples to naive generalizations. See also [Denef 1991] and the introductions of [Bories 2013b; 2013] for more details in survey format.

A root of $b_f(s)$, a monodromy eigenvalue, and a pole of $Z_f(s)$ may have multiplicity; can the monodromy conjecture be strengthened to include multiplicities? This version of (SMC) was proved for reduced bivariate f in [Loeser 1988]; in [Melle-Hernández et al. 2009; 2010] it is proved for certain nonreduced bivariate f, and for some trivariate ones.

A different variation, due to Veys, of the conjecture is the following. Vary the definition of $Z_f(s)$ to $Z_{f;g}(s) = \sum_{I\subseteq S} \chi(E_I^*) \prod_{i\in I} 1/(N_i s + \nu_i')$, where ν_i' is the multiplicity of E_i in the pullback along π of some differential form g. (The standard case is when g is the volume form). Two questions arise: (1) varying over a suitable set G of forms g, can one generate all roots of $b_f(s)$ as poles of the resulting zeta functions? And if so, can one (2) do this such that the pole sets of all zeta functions so constructed are always inside ρ_f, so that

$$\rho_f = \{\alpha \mid \text{ there exists } g \in G, \lim_{s\to\alpha} Z_{f;g}(s) = \infty\}?$$

Némethi and Veys [2010; 2012] prove a weak version: if $n = 2$ then monodromy eigenvalues are exponentials of poles of zeta functions from differential forms.

The following is discussed in [Bories 2013a]. For some ideals with $n = 2$, (1) is false for the topological zeta function (even for divisors: consider $xy^5 + x^3y^2 + x^4y$). For monomial ideals with two generators in $n = 2$, (1) is correct; with more than two generators it can fail. Even in the former case, (2) can be false.

7. Multivariate versions

If $f = (f_1, \ldots, f_r)$ defines a map $f\colon \mathbb{C}^n \to \mathbb{C}^r$, several b-functions can be defined:

(1) the univariate Bernstein–Sato polynomial $b_f(s)$ attached to the ideal $(f) \subseteq R$ from [Budur et al. 2006a];

(2) the multivariate Bernstein–Sato polynomials $b_{f,i}(s)$ of all elements $b(s)$ of $\mathbb{C}[s_1, \ldots, s_r]$ such that there is an equation $P(x, \partial, s) \bullet f_i f^s = b(s) f^s$ in multiindex notation;

(3) the multivariate Bernstein–Sato ideal $B_{f,\mu}(s)$ for $\mu \in \mathbb{N}^r$ of all $b(s) \in \mathbb{C}[s_1, \ldots, s_r]$ such that there is an equation $P(x, \partial, s) \bullet f^{s+\mu} = b(s) f^s$ in multiindex notation. The most interesting case is $\mu = \mathbf{1} = (1, \ldots, 1)$;

(4) the multivariate Bernstein–Sato ideal $B_{f,\Sigma}(s)$ of all $b(s) \in \mathbb{C}[s_1, \ldots, s_r]$ that multiply f^s into $\sum D[s] f_i f^s$ in multiindex notation.

The Bernstein–Sato polynomial in (1) above has been studied in the case of a monomial ideal in [Budur et al. 2006b] and more generally from the point of view of the Newton polygon in [Budur et al. 2006c]. While the roots for monomial ideals do not depend just on the Newton polygon, their residue classes modulo $\mathbb{Z}$ do.

Nontriviality of the quantities in (2)–(4) have been established in [Sabbah 1987c; 1987d; 1987b], but see also [Bahloul 2005]. The ideals $B_{f,\mu}(s)$ and $B_{f,\Sigma}(s)$ do not have to be principal [Ucha and Castro-Jiménez 2004; Bahloul and Oaku 2010]. In [Sabbah 1987c; Gyoja 1993] it is shown that $B_{f,\mu}(s)$ contains a polynomial that factors into linear forms with nonnegative rational coefficients and positive constant term. Bahloul and Oaku [2010] show that these ideals are local in the sense of (3-1).

The following would generalize Kashiwara's result in the univariate case as well as the results of Sabbah and Gyoja above.

Conjecture 7.1 [Budur 2012a]. *The Bernstein–Sato ideal $B_{f,\mu}(s)$ is generated by products of linear forms $\sum \alpha_i s_i + a$ with α_i, a nonnegative rational and $a > 0$.*

For $n = 2$, partial results by Cassou-Noguès and Libgober exist [2011]. In [Budur 2012a] it is further conjectured that the Malgrange–Kashiwara result, exponentiating $\rho_{f,p}$ gives $e_{f,p}$, generalizes: monodromy in this case is defined in [Verdier 1983], and Sabbah's specialization functor ψ_f from [1990] takes on the rôle of the nearby cycle functor, and conjecturally exponentiating the variety of $B_{f,p}(s)$ yields the uniform support (near p) of Sabbah's functor. The latter conjecture would imply Conjecture 7.1.

Similarly to the one-variable case, if $V(n, d, m)$ is the vector space of (ordered) m-tuples of polynomials in $x_1, \ldots, x_n$ of degree at most d, there is an algebraic stratification of $V(n, d, m)$ such that on each stratum the function $V \ni f = (f_1, \ldots, f_m) \mapsto b_f(s)$ is constant. Corresponding results for the Bernstein–Sato ideal $B_{f,\mathbf{1}}(s)$ hold by [Briançon et al. 2000].

8. Hyperplane arrangements

A *hyperplane arrangement* is a divisor of the form

$$\mathscr{A} = \prod_{i \in I} \alpha_i$$

where each α_i is a polynomial of degree one. We denote $H_i = \mathrm{Var}(\alpha_i)$. Essentially all information we are interested in is of local nature, so we assume that each α_i is a form so that $\mathscr{A}$ is *central*. If there is a coordinate change in $\mathbb{C}^n$ such that $\mathscr{A}$ becomes the product of polynomials in disjoint sets of variables, the arrangement is *decomposable*, otherwise it is *indecomposable*.

A *flat* is any (set-theoretic) intersection $\bigcap_{i \in J} H_i$ where $J \subseteq I$. The *intersection lattice* $L(\mathscr{A})$ is the partially ordered set consisting of the collection of all flats, with order given by inclusion.

8A. ***Numbers and parameters.*** Hyperplane arrangements satisfy (B_1) everywhere [Walther 2005]. Arrangements satisfy (A_1) everywhere if they decompose into a union of a generic and a hyperbolic arrangement [Torrelli 2004], and if they are tame [Walther 2015]. Terao conjectured that all hyperplane arrangements satisfy (A_1); some of them fail (A_s) [Walther 2015].

Apart from recasting various of the previously encountered problems in the world of arrangements, a popular study is the following: choose a discrete invariant I of a divisor. Does the function $\mathscr{A} \mapsto I(\mathscr{A})$ factor through the map $\mathscr{A} \mapsto L(\mathscr{A})$? Randell showed that if two arrangements are connected by a one-parameter family of arrangements which have the same intersection lattice, the complements are diffeomorphic [Randell 1989] and the isomorphism type of the Milnor fibration is constant [Randell 1997]. Rybnikov [2011] (see also [Artal Bartolo et al. 2006]) showed on the other hand that there are arrangements (even in the projective plane) with equal lattice but different complement. In particular, not all isotopic arrangements can be linked by a smooth deformation.

8B. ***LCT and logarithmic ideal.*** The most prominent positive result is one by Brieskorn [1973]: the *Orlik–Solomon algebra* $\mathrm{OS}(\mathscr{A}) \subseteq \Omega^\bullet(\log \mathscr{A})$ generated by the forms $\mathrm{d}\alpha_i/\alpha_i$ is quasiisomorphic to $\Omega^\bullet(*\mathscr{A})$, hence to the singular cohomology algebra of $U_{\mathscr{A}}$. The relation with combinatorics was given in [Orlik and Solomon 1980; Orlik and Terao 1992]. For a survey on the Orlik–Solomon algebra, see [Yuzvinsky 2001]. The best known open problem in this area is this:

Conjecture 8.1 [Terao 1978]. $\mathrm{OS}(\mathscr{A}) \to \Omega^\bullet(\log \mathscr{A})$ *is a quasiisomorphism.*

While the general case remains open, Wiens and Yuzvinsky [1997] proved it for tame arrangements, and also if $n \leq 4$. The techniques are based on [Castro-Jiménez et al. 1996].

8C. ***Milnor fibers.*** There is a survey article by Suciu on complements, Milnor fibers, and cohomology jump loci [Suciu 2014], and [Budur 2012b] contains further information on the topic. It is not known whether $L(\mathscr{A})$ determines the Betti numbers (even less the Hodge numbers) of the Milnor fiber of an arrangement. The first Betti number of the Milnor fiber $M_{\mathscr{A}}$ at the origin is stable under intersection with a generic hyperplane (if $n > 2$). But it is unknown whether the first Betti number of an arrangement in 3-space is a function of the lattice alone. By [Dimca et al. 2013], this is so for collections of up to 14 lines with up to 5-fold intersections in the projective plane. See also [Libgober 2012] for the origins of the approach. By [Budur et al. 2011a], a lower combinatorial bound for the λ-eigenspace of $H^1(M_{\mathscr{A}})$ is given under favorable conditions on L. If L satisfies stronger conditions, the bound is shown to be exact. In any case, [Budur et al. 2011a] gives an algebraic, although perhaps noncombinatorial, formula for the Hodge pieces in terms of multiplier ideals.

By [Orlik and Randell 1993], the Betti numbers of $M_{\mathscr{A}}$ are combinatorial if $\mathscr{A}$ is generic. See also [Cohen and Suciu 1995].

8D. ***Multiplier ideals.*** Mustaţă gave a formula for the multiplier ideals of arrangements, and used it to show that the log-canonical threshold is a function of $L(\mathscr{A})$. The formula is somewhat hard to use for showing that each jumping number is a lattice invariant; this problem was solved in [Budur and Saito 2010]. Explicit formulas in low dimensional cases follow from the spectrum formulas given there and in [Yoon 2013]. Teitler [2008] improved Mustaţă's formula [2006] for multiplier ideals to not necessarily reduced hyperplane arrangements.

8E. ***Bernstein–Sato polynomials.*** By [Walther 2005], $\rho_{\mathscr{A}} \cap \mathbb{Z} = \{-1\}$; by [Saito 2006], $\rho_{\mathscr{A}} \subseteq (-2, 0)$. There are few classes of arrangements with explicit formulas for their Bernstein–Sato polynomial:

- Boolean (a normal crossing arrangement, locally given by $x_1 \cdots x_k$);
- hyperbolic (essentially an arrangement in two variables);
- generic (central, and all intersections of n hyperplanes equal the origin).

The first case is trivial, the second is easy, the last is [Walther 2005] with assistance from [Saito 2007]. Some interesting computations are in [Budur et al. 2011c], and [Budur 2012a] has a partial confirmation of the multivariable Kashiwara–Malgrange theorem. The Bernstein–Sato polynomial is not determined by the intersection lattice [Walther 2015].

8F. ***Zeta functions.*** Budur, Mustaţă and Teitler [Budur et al. 2011b] show: (MC) holds for arrangements, and in order to prove (SMC), it suffices to show the following conjecture.

Conjecture 8.2. *For all indecomposable central arrangements with d planes in n-space, $b_{\mathscr{A}}(-n/d) = 0$.*

The idea is to use the resolution of singularities obtained by blowing up the dense edges from [Schechtman et al. 1995]. The corresponding computation of the zeta function is inspired from [Igusa 1974; 1975]. The number $-n/d$ does not have to be the log-canonical threshold. By [Budur et al. 2011b], Conjecture 8.2 holds in a number of cases, including reduced arrangements in dimension 3. By [Walther 2015] it holds for tame arrangements.

Examples of Veys (in 4 variables) show that (SMC) may hold even if Conjecture 8.2 were false in general, since $-n/d$ is not always a pole of the zeta function [Budur et al. 2011c]. However, in these examples, $-n/d$ is in fact a root of $b_f(s)$.

For arrangements, each monodromy eigenvalue can be captured by zeta functions in the sense of Némethi and Veys (see Section 6D), but not necessarily all of $\rho_{\mathscr{A}}$ (Veys and Walther, unpublished).

9. Positive characteristic

Let here $\mathbb{F}$ denote a field of characteristic $p > 0$. The theory of D-modules is rather different in positive characteristic compared to their behavior over the complex numbers. There are several reasons for this:

(1) On the downside, the ring D_p of $\mathbb{F}$-linear differential operators on $R_p = \mathbb{F}[x_1, \ldots, x_n]$ is no longer finitely generated: as an $\mathbb{F}$-algebra it is generated by the elements $\partial^{(\alpha)}$, $\alpha \in \mathbb{N}^n$, which act via $\partial^{(\alpha)} \bullet (x^\beta) = \binom{\beta}{\alpha} x^{\beta-\alpha}$.

(2) As a trade-off, one has access to the Frobenius morphism $x_i \mapsto x_i^p$, as well as the Frobenius functor $F(M) = R' \otimes_R M$ where R' is the $R - R$-bimodule on which R acts via the identity on the left, and via the Frobenius on the right. Lyubeznik [1997a] created the category of F-finite F-modules and proved striking finiteness results. The category includes many interesting D_p-modules, and all F-modules are D_p-modules.

(3) Holonomicity is more complicated; see [Bögvad 2002].

A most surprising consequence of Lyubeznik's ideas is that in positive characteristic the property (B_1) is meaningless: it holds for every $f \in R_p$ [Alvarez-Montaner et al. 2005]. The proof uses in significant ways the difference between D_p and the Weyl algebra. In particular, the theory of Bernstein–Sato polynomials is rather different in positive characteristic. In [Mustaţă 2009] a sequence of Bernstein–Sato polynomials is attached to a polynomial f assuming that the Frobenius morphism is finite on R (e.g., if $\mathbb{F}$ is finite or algebraically closed); these polynomials are then linked to test ideals, the finite characteristic counterparts

to multiplier ideals. In [Blickle et al. 2009] variants of our modules $\mathscr{M}_f(\gamma)$ are introduced and [Núñez-Betancourt and Pérez 2013] shows that simplicity of these modules detects the F-thresholds from [Mustaţă et al. 2005]. These are cousins of the jumping numbers of multiplier ideals and related to the Bernstein–Sato polynomial via base-p-expansions. The Kashiwara–Brylinski intersection homology module was shown to exist in positive characteristic by Blickle in his thesis [Blickle 2004].

Appendix: Computability

by Anton Leykin

Computations around f^s can be carried out by hand in special cases. Generally, the computations are enormous and computers are required (although not often sufficient). One of the earliest such approaches are in [Briançon et al. 1989; Aleksandrov and Kistlerov 1992], but at least implicitly Buchberger's algorithm in a Weyl algebra was discussed as early as [Castro-Jimenez 1984]. An algorithmic approach to the isolated singularities case [Maisonobe 1994] preceded the general algorithms based on Gröbner bases in a noncommutative setting outlined below.

10A. ***Gröbner bases.*** The *monomials* $x^\alpha \partial^\beta$ with $\alpha, \beta \in \mathbb{N}^n$ form a $\mathbb{C}$-basis of D; expressing $p \in D$ as linear combination of monomials leads to its *normal form*. The monomial orders on the commutative monoid $[x, \partial]$ for which for all $i \in [n]$ the leading monomial of $\partial_i x_i = x_i \partial_i + 1$ is $x_i \partial_i$, can be used to run Buchberger's algorithm in D. Modifications are needed in improvements that exploit commutativity, but the naïve Buchberger's algorithm works without any changes. See [Kandri-Rody and Weispfenning 1990] for more general settings in polynomial rings of solvable type. Surprisingly, the worst case complexity of Gröbner bases computations in Weyl algebras is *not* worse than in the commutative polynomial case: it is doubly exponential in the number of indeterminates [Aschenbrenner and Leykin 2009; Grigoriev and Chistov 2008].

10B. ***Characteristic variety.*** A weight vector $(u, v) \in \mathbb{Z}^n \times \mathbb{Z}^n$ with $u + v \geq 0$ induces a filtration of D,

$$F_i = \mathbb{C} \cdot \{x^\alpha \partial^\beta \mid u \cdot \alpha + v \cdot \beta \leq i\}, \quad i \in \mathbb{Z}.$$

The (u, v)-*Gröbner deformation* of a left ideal $I \subseteq D$ is

$$\operatorname{in}_{(u,v)}(I) = \mathbb{C} \cdot \{\operatorname{in}_{(u,v)}(P) \mid P \in I\} \subseteq \operatorname{gr}_{(u,v)} D,$$

the ideal of *initial forms* of elements of I with respect to the given weight in the associated graded algebra. It is possible to compute Gröbner deformations in the

homogenized Weyl algebra

$$D^h = D\langle h\rangle/\langle \partial_i x_i - x_i\partial_i - h^2,\, x_i h - h x_i,\, \partial_i h - h\partial_i \mid 1 \le i \le n\rangle;$$

see [Castro-Jimenez and Narváez-Macarro 1997; Oaku and Takayama 2001b]. Gröbner deformations are the main topic of [Saito et al. 2000].

10C. ***Annihilator.*** Recall the construction appearing in the beginning of Section 6A: $D_{x,t}$ acts on $D[s]f^s$; in particular, the operator $-\partial_t t$ acts as multiplication by s. It is this approach that lead Oaku to an algorithm for $\operatorname{ann}_{D[s]}(f^s)$, $\operatorname{ann}_D(f^s)$ and $b_f(s)$ [Oaku 1997]. We outline the ideas.

Malgrange observed that

$$\operatorname{ann}_{D[s]}(f^s) = \operatorname{ann}_{D_{x,t}}(f^s) \cap D[s], \tag{10-1}$$

$$\text{with} \quad \operatorname{ann}_{D_{x,t}}(f^s) = \left\langle t - f,\, \partial_1 + \frac{\partial f}{\partial x_1}\partial_t,\, \dots,\, \partial_n + \frac{\partial f}{\partial x_n}\partial_t \right\rangle \subseteq D_{x,t}. \tag{10-2}$$

The former can be found from the latter by eliminating t and ∂_t from the ideal

$$\langle s + t\partial_t\rangle + \operatorname{ann}_{D_{x,t}}(f^s) \subseteq D_{x,t}\langle s\rangle; \tag{10-3}$$

of course $s = -\partial_t t$ does not commute with t, ∂_t here.

Oaku's method for $\operatorname{ann}_{D[s]}(f^s)$ accomplished the elimination by augmenting two commuting indeterminates:

$$\begin{aligned} &\operatorname{ann}_{D[s]}(f^s) = I'_f \cap D[s], \\ &I'_f = \left\langle t - uf,\, \partial_1 + u\frac{\partial f}{\partial x_1}\partial_t,\, \dots,\, \partial_n + u\frac{\partial f}{\partial x_n}\partial_t,\, uv - 1 \right\rangle \subseteq D_{x,t}[u, v]. \end{aligned} \tag{10-4}$$

Now outright eliminate u, v. Note that I'_f is quasihomogeneous if the weights are $t, u \rightsquigarrow -1$ and $\partial_t, v \rightsquigarrow 1$, all other variables having weight zero. The homogeneity of the input and the relation $[\partial_t, t] = 1$ assures the termination of the computation. The operators of weight 0 in the output (with $-\partial_t t$ replaced by s) generate $I'_f \cap D[s]$.

A modification given in [Briançon and Maisonobe 2002] and used, e.g., in [Ucha and Castro-Jiménez 2004], reduces the number of algebra generators by one. Consider the subalgebra $D\langle s, \partial_t\rangle \subset D_{x,t}$; the relation $[s, \partial_t] = \partial_t$ shows that it is of solvable type. According to [Briançon and Maisonobe 2002],

$$\begin{aligned} &\operatorname{ann}_{D[s]}(f^s) = I''_f \cap D[s], \\ &I''_f = \left\langle s + f\partial_t,\, \partial_1 + \frac{\partial f}{\partial x_1}\partial_t,\, \dots,\, \partial_n + \frac{\partial f}{\partial x_n}\partial_t \right\rangle \subset D\langle s, \partial_t\rangle. \end{aligned} \tag{10-5}$$

Note that $I_f'' = \operatorname{ann}_{D_{x,t}}(f^s) \cap D\langle s, \partial_t\rangle$. The elimination step is done as in [Oaku 1997]; the decrease of variables usually improves performance. An algorithm to decide (A_1) for arrangements is given in [Àlvarez Montaner et al. 2007].

10D. ***Algorithms for the Bernstein–Sato polynomial.*** As the minimal polynomial of s on $\mathscr{N}_f(s)$, $b_f(s)$ can be obtained by means of linear algebra as a syzygy for the normal forms of powers of s modulo $\operatorname{ann}_{D[s]}(f^s) + D[s]\cdot f$ with respect to any fixed monomial order on $D[s]$. Most methods follow this path, starting with [Oaku 1997]. Variations appear in [Walther 1999; Oaku and Takayama 2001a; Oaku et al. 2000]; see also [Saito et al. 2000].

A different approach is to compute $b_f(s)$ without recourse to $\operatorname{ann}_{D[s]}(f^s)$, via a Gröbner deformation of the ideal $I_f = \operatorname{ann}_{D_{x,t}}(f^s)$ in (10-2) with respect to the weight $(-w, w)$ with $w = (0^n, 1) \in \mathbb{N}^{n+1}$: $\langle b_f(s)\rangle = \operatorname{in}_{(-w,w)}(I_f) \cap \mathbb{Q}[-\partial_t t]$. Here again, computing the minimal polynomial using linear algebra tends to provide some savings in practice.

In [Levandovskyy and Martín-Morales 2012] the authors give a method to check specific numbers for being in ρ_f. A method for $b_f(s)$ in the prehomogeneous vector space setup is in [Muro 2000].

10E. ***Stratification from $b_f(s)$.*** The Gröbner deformation $\operatorname{in}_{(-w,w)}(I_f)$ in the previous subsection can be refined as follows; see [Berkesch and Leykin 2010, Theorem 2.2]. Let $b(x, s)$ be nonzero in the polynomial ring $\mathbb{C}[x, s]$. Then $b(x, s) \in (\operatorname{in}_{(-w,w)} I_f) \cap \mathbb{C}[x, s]$ if and only if there exists $P \in D[s]$ satisfying the functional equation $b(x, s) f^s = P f f^s$. From this one can design an algorithm not only for computing the *local* Bernstein–Sato polynomial $b_{f,p}(s)$ for $p \in \operatorname{Var}(f)$, but also the stratification of $\mathbb{C}^n$ according to local Bernstein–Sato polynomials; see [Nishiyama and Noro 2010; Berkesch and Leykin 2010] for various approaches. Moreover, one can compute the stratifications from Section 3C2; see [Leykin 2001].

For the ideal case, [Andres et al. 2009] gives a method to compute an intersection of a left ideal of an associative algebra over a field with a subalgebra, generated by a single element. An application is a method for the computation of the Bernstein-Sato polynomial of an ideal. Another such was given by Bahloul [2001], and a version on general varieties was given by the same in [2003].

10F. ***Multiplier ideals.*** Consider polynomials $f_1, \ldots, f_r \in \mathbb{C}[x]$, let f stand for $(f_1, \ldots, f_r)$, s for $s_1, \ldots, s_r$, and f^s for $\prod_{i=1}^r f_i^{s_i}$. In this subsection, let $D_{x,t} = \mathbb{C}\langle x, t, \partial_x, \partial_t\rangle$ be the $(n+r)$-th Weyl algebra.

Consider $D_{x,t}(s) \bullet f^s \subseteq R_{x,t}[f^{-1}, s] f^s$ and put

$$t_j \bullet h(x, s_1, \ldots, s_j, \ldots, s_r) f^s = h(x, s_1, \ldots, s_j + 1, \ldots, s_r) f_j f^s,$$

$$\partial_{t_j} \bullet h(x, s_1, \ldots, s_j, \ldots, s_r) f^s = -s_j h(x, s_1, \ldots, s_j - 1, \ldots, s_r) f_j^{-1} f^s,$$

for $h \in \mathbb{C}[x][f^{-1}, s]$, generalizing the univariate constructions.

The *generalized Bernstein–Sato polynomial* $b_{f,g}(\sigma)$ of f at $g \in \mathbb{C}[x]$ is the monic univariate polynomial b of the lowest degree for which there exist $P_k \in D_{x,t}$ such that

$$b(\sigma)gf^s = \sum_{k=1}^{r} P_k g f_k f^s, \quad \sigma = -\left(\sum_{i=1}^{r} \partial_{t_i} t_i\right). \tag{10-6}$$

An algorithm for $b_{f,g}(\sigma)$ is an essential ingredient for the algorithms in [Shibuta 2011; Berkesch and Leykin 2010] that compute the jumping numbers and corresponding multiplier ideals for $I = \langle f_1, \ldots, f_r \rangle$. That $b_{f,g}(\sigma)$ is related to multiplier ideals was worked out in [Budur et al. 2006a].

There are algorithms for special cases: monomial ideals [Howald 2001], hyperplane arrangements [Mustaţă 2006], and determinantal ideals [Johnson 2003]. A Macaulay2 package *MultiplierIdeals* by Teitler collects all implementations available in Macaulay2. See also [Budur 2005].

10G. ***Software.*** Algorithms for computing Bernstein–Sato polynomials have been implemented in *kan/sm1* [Takayama], *Risa/Asir* [Noro et al.], the *dmod_lib* library [Levandovskyy and Morales] for Singular [Decker et al. 2012], and the *D-modules* package [Leykin and Tsai] for Macaulay2 [Grayson and Stillman]. The best source of information of these is documentation in the current versions of the corresponding software. A relatively recent comparison of the performance for several families of examples is given in [Levandovskyy and Martín Morales 2008].

The following are articles by developers discussing their implementations: [Noro 2002; Nishiyama and Noro 2010; Oaku and Takayama 2001a; Andres et al. 2010; Levandovskyy and Morales; Leykin 2002; Berkesch and Leykin 2010].

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