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Vector bundles and ideal closure operations

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This is an introduction to the use of vector bundle techniques to ideal closure operations, in particular to tight closure and related closures like solid closure and plus closure. We also briefly introduce the theory of vector bundles in general, with an emphasis on smooth projective curves, and discuss the relationship between forcing algebras and closure operations.

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Introduction

An ideal operation is an assignment which provides for every ideal I in a commutative ring R a further ideal I' fulfilling certain structural conditions such as $I \subseteq I'$, $I'' = I'$, and an inclusion $I \subseteq J$ should induce an inclusion $I' \subseteq J'$. The most important examples are the radical of an ideal, whose importance stems from Hilbert's Nullstellensatz, the integral closure $\bar{I}$, which plays a crucial role in the normalization of blow-up algebras, and tight closure I^*, which is a closure operation in positive characteristic invented by Hochster and Huneke. In the context of tight closure, many other closure operations were introduced such as plus closure I^+, Frobenius closure I^F, solid closure $I^\star$, dagger closure $I^\dagger$, parasolid closure.

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In this survey article we want to describe how the concepts of forcing algebras, vector bundles and their torsors can help to understand closure operations. This approach can be best understood by looking at the fundamental question whether $f \in (f_1, \dots, f_n)' = I'$. In his work on solid closure, Hochster considered the forcing algebra

$$A = R[T_1, \dots, T_n]/(f_1T_1 + \dots + f_nT_n + f)$$

and put properties of this R-algebra in relation to the closure operation. Suppose that the ideal I is primary to the maximal ideal $\mathfrak{m}$ in a local normal domain R of dimension d. Then the containment $f \in I^\star$ is equivalent to the property that the local cohomology $H^d_{\mathfrak{m}}(A)$ does not vanish. This is still a difficult property, however the situation becomes somehow geometrically richer. Because of $H^d_{\mathfrak{m}}(A) \cong H^{d-1}(T, \mathcal{O}_A)$, where $T = D(\mathfrak{m}A)$ is the open subset of $\operatorname{Spec} A$ above the punctured spectrum $U = D(\mathfrak{m})$, we can study global cohomological properties of T. The first syzygy module $\operatorname{Syz}(f_1, \dots, f_n)$ is a locally free sheaf on U and acts on T in a locally trivial way. The scheme T is therefore a torsor for the syzygy bundle, which are classified by $H^1(U, \operatorname{Syz}(f_1, \dots, f_n))$, and the element f determines this class. So the ideal operation is reflected by global properties of T, which is locally just an affine space over the base.

If the ring and the ideal are graded, then these objects have their counterpart on the corresponding projective varieties. This allows to apply results and machinery from algebraic geometry to closure operations, like intersection theory, semistability conditions, ampleness, moduli spaces of vector bundles, deformations. As this translation works best in dimension two, where the corresponding projective varieties are curves, we will focus here on this case. This approach has led in dimension two over a finite field to a positive solution to the tantalizing question whether tight closure is plus closure, and to negative solutions to arithmetic variation and to the localization problem in tight closure theory. The aim of this article is to provide an introduction to this techniques and to show how they help to solve problems from tight closure theory.

Throughout we assume a basic knowledge of commutative algebra and algebraic geometry including local cohomology and sheaf cohomology; once in a while we will use some notions and results from tight closure theory.

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1. Linear equations, forcing algebras and closure operations

Systems of linear equations. We start with some linear algebra. Let K be a field. We consider a system of linear homogeneous equations over K,

$$\begin{aligned} f_{11}t_1 + \cdots + f_{1n}t_n &= 0, \\ f_{21}t_1 + \cdots + f_{2n}t_n &= 0, \\ &\vdots \\ f_{m1}t_1 + \cdots + f_{mn}t_n &= 0, \end{aligned}$$

where the f_{ij} are elements in K. The solution set to this system of homogeneous equations is a vector space V over K (a subvector space of K^n), its dimension is $n - \operatorname{rk}(A)$, where $A = (f_{ij})_{ij}$ is the matrix given by these elements. Additional elements $f_1, \ldots, f_m \in K$ give rise to the system of inhomogeneous linear equations,

$$\begin{aligned} f_{11}t_1 + \cdots + f_{1n}t_n &= f_1, \\ f_{21}t_1 + \cdots + f_{2n}t_n &= f_2, \\ &\vdots \\ f_{m1}t_1 + \cdots + f_{mn}t_n &= f_m. \end{aligned}$$

The solution set T of this inhomogeneous system may be empty, but nevertheless it is tightly related to the solution space of the homogeneous system. First of all, there exists an action

$$V \times T \to T, \quad (v, t) \mapsto v + t,$$

because the sum of a solution of the homogeneous system and a solution of the inhomogeneous system is again a solution of the inhomogeneous system. Theis action is a group action of the group $(V, +, 0)$ on the set T. Moreover, if we fix one solution $t_0 \in T$ (supposing that at least one solution exists), then there exists a bijection

$$V \to T, \quad v \mapsto v + t_0.$$

This means that the group V acts simply transitive on T, and so T can be identified with the vector space V, however not in a canonical way.

Suppose now that X is a geometric object (a topological space, a manifold, a variety, a scheme, the spectrum of a ring) and that instead of elements in the field K we have functions

$$f_{ij} : X \to K$$

on X (which are continuous, or differentiable, or algebraic). We form the matrix of functions $A = (f_{ij})_{ij}$, which yields for every point $P \in X$ a matrix $A(P)$ over K. Then we get from these data the space

$$V = \left\{ (P; t_1, \dots, t_n) \mid A(P) \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} = 0 \right\} \subseteq X \times K^n$$

together with the projection to X. For a fixed point $P \in X$, the fiber V_P of V over P is the solution space to the corresponding system of homogeneous linear equations given by inserting P into f_{ij}. In particular, all fibers of the map

$$V \to X$$

are vector spaces (maybe of nonconstant dimension). These vector space structures yield an addition[1]

$$V \times_X V \to V, \quad (P; s_1, \dots, s_n; t_1, \dots, t_n) \mapsto (P; s_1 + t_1, \dots, s_n + t_n)$$

(only points in the same fiber can be added). The mapping

$$X \to V, \quad P \mapsto (P; 0, \dots, 0)$$

is called the *zero-section*.

Suppose now that additional functions

$$f_1, \dots, f_m : X \to K$$

are given. Then we can form the set

$$T = \left\{ (P; t_1, \dots, t_n) \mid A(P) \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} = \begin{pmatrix} f_1(P) \\ \vdots \\ f_n(P) \end{pmatrix} \right\} \subseteq X \times K^n$$

with the projection to X. Again, every fiber T_P of T over a point $P \in X$ is the solution set to the system of inhomogeneous linear equations which arises by inserting P into f_{ij} and f_i. The actions of the fibers V_P on T_P (coming from linear algebra) extend to an action

$$V \times_X T \to T, \quad (P; t_1, \dots, t_n; s_1, \dots, s_n) \mapsto (P; t_1 + s_1, \dots, t_n + s_n).$$

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[1] $V \times_X V$ is the fiber product of $V \to X$ with itself.

Also, if a (continuous, differentiable, algebraic) map

$$s : X \to T$$

with $s(P) \in T_P$ exists, then we can construct a (continuous, differentiable, algebraic) isomorphism between V and T. However, different from the situation in linear algebra (which corresponds to the situation where X is just one point), such a section does rarely exist.

These objects T have new and sometimes difficult global properties which we try to understand in these lectures. We will work mainly in an algebraic setting and restrict to the situation where just one equation

$$f_1 T_1 + \cdots + f_n T_n = f$$

is given. Then in the homogeneous case ($f = 0$) the fibers are vector spaces of dimension $n - 1$ or n, and the later holds exactly for the points $P \in X$ where $f_1(P) = \cdots = f_n(P) = 0$. In the inhomogeneous case the fibers are either empty or of dimension (as a scheme) $n - 1$ or n. We give some typical examples.

Example 1.1. We consider the line $X = \mathbb{A}^1_K$ (or $X = K, \mathbb{R}, \mathbb{C}$ etc.) with the (identical) function x. For $f_1 = x$ and $f = 0$, i.e., for the homogeneous equation $xt = 0$, the geometric object V consists of a horizontal line (corresponding to the zero-solution) and a vertical line over $x = 0$. So all fibers except one are zero-dimensional vector spaces. For the inhomogeneous equation $xt = 1$, T is a hyperbola, and all fibers are zero-dimensional with the exception that the fiber over $x = 0$ is empty.

For the homogeneous equation $0t = 0$, V is just the affine cylinder over the base line. For the inhomogeneous equation $0t = x$, T consists of one vertical line, almost all fibers are empty.

Example 1.2. Let X denote a plane ($K^2, \mathbb{R}^2, \mathbb{A}^2_K$) with coordinate functions x and y. We consider an inhomogeneous linear equation of type

$$x^a t_1 + y^b t_2 = x^c y^d.$$

The fiber of the solution set T over a point $\neq (0, 0)$ is one-dimensional, whereas the fiber over $(0, 0)$ has dimension two (for $a, b, c, d \geq 1$). Many properties of T depend on these four exponents.

In (most of) these examples we can observe the following behavior. On an open subset, the dimension of the fibers is constant and equals $n - 1$, whereas the fiber over some special points degenerates to an n-dimensional solution set (or becomes empty).

Forcing algebras. We describe now the algebraic setting of systems of linear equations depending on a base space. For a commutative ring R, its spectrum $X = \operatorname{Spec}(R)$ is a topological space on which the ring elements can be considered as functions. The value of $f \in R$ at a prime ideal $P \in \operatorname{Spec}(R)$ is just the image of f under the ring homomorphism $R \to R/P \to \kappa(P) = Q(R/P)$. In this interpretation, a ring element is a function with values in different fields. Suppose that R contains a field K. Then an element $f \in R$ gives rise to the ring homomorphism

$$K[Y] \to R, \quad Y \mapsto f,$$

which gives rise to a scheme morphism

$$\operatorname{Spec}(R) \to \operatorname{Spec}(K[Y]) \cong \mathbb{A}^1_K.$$

This is another way to consider f as a function on $\operatorname{Spec}(R)$ with values in the affine line.

The following construction appeared first in [Hochster 1994] in the context of solid closure.

Definition 1.3. Let R be a commutative ring and let $f_1, \dots, f_n$ and f be elements in R. Then the R-algebra

$$R[T_1, \dots, T_n]/(f_1T_1 + \cdots + f_nT_n - f)$$

is called the *forcing algebra* of these elements (or these data).

The forcing algebra B forces f to lie inside the extended ideal $(f_1, \dots, f_n)B$ (hence the name). For every R-algebra S such that $f \in (f_1, \dots, f_n)S$ there exists a (non unique) ring homomorphism $B \to S$ by sending T_i to the coefficient $s_i \in S$ in an expression $f = s_1 f_1 + \cdots + s_n f_n$.

The forcing algebra induces the spectrum morphism

$$\varphi : \operatorname{Spec}(B) \to \operatorname{Spec}(R).$$

Over a point $P \in X = \operatorname{Spec}(R)$, the fiber of this morphism is given by

$$\operatorname{Spec}\bigl(B \otimes_R \kappa(P)\bigr),$$

and we can write

$$B \otimes_R \kappa(P) = \kappa(P)[T_1, \dots, T_n]/(f_1(P)T_1 + \cdots + f_n(P)T_n - f(P)),$$

where $f_i(P)$ means the evaluation of the f_i in the residue class field. Hence the $\kappa(P)$-points in the fiber are exactly the solutions to the inhomogeneous linear equation $f_1(P)T_1 + \cdots + f_n(P)T_n = f(P)$. In particular, all the fibers are (empty or) affine spaces.

Forcing algebras and closure operations. Let R denote a commutative ring and let $I = (f_1, \dots, f_n)$ be an ideal. Let $f \in R$ and let

$$B = R[T_1, \dots, T_n]/(f_1T_1 + \cdots + f_nT_n - f)$$

be the corresponding forcing algebra and

$$\varphi : \operatorname{Spec}(B) \to \operatorname{Spec}(R)$$

the corresponding spectrum morphism. How are properties of φ (or of the R-algebra B) related to certain ideal closure operations?

We start with some examples. The element f belongs to the ideal I if and only if we can write $f = r_1 f_1 + \cdots + r_n f_n$ with $r_i \in R$. By the universal property of the forcing algebra this means that there exists an R-algebra homomorphism

$$B \to R,$$

hence $f \in I$ holds if and only if φ admits a scheme section. This is also equivalent to

$$R \to B$$

admitting an R-module section (R being a direct module summand of B) or B being a pure R-algebra (so for forcing algebras properties might be equivalent which are not equivalent for arbitrary algebras).

The radical of an ideal. Now we look at the *radical* of the ideal I,

$$\operatorname{rad}(I) = \left\{ f \in R \mid f^k \in I \text{ for some } k \right\}.$$

The importance of the radical comes mainly from Hilbert's Nullstellensatz, saying that for algebras of finite type over an algebraically closed field there is a natural bijection between radical ideals and closed algebraic zero-sets. So geometrically one can see from an ideal only its radical. As this is quite a coarse closure operation we should expect that this corresponds to a quite coarse property of the morphism φ as well. Indeed, it is true that $f \in \operatorname{rad}(I)$ if and only if φ is surjective. This is true since the radical of an ideal is the intersection of all prime ideals in which it is contained. Hence an element f belongs to the radical if and only if for all residue class homomorphisms

$$\theta : R \to \kappa(\mathfrak{p})$$

where I is sent to 0, also f is sent to 0. But this means for the forcing equation that whenever the equation degenerates to 0, then also the inhomogeneous part becomes zero, and so there will always be a solution to the inhomogeneous equation.

Exercise. Define the radical of a submodule inside a module.

Integral closure of an ideal. Another closure operation is *integral closure* (see [Huneke and Swanson 2006]). It is defined by

$$\overline{I} = \left\{ f \in R \mid f^k + a_1 f^{k-1} + \cdots + a_{k-1} f + a_k = 0 \; for some \; k \text{ and } a_i \in I^i \right\}.$$

This notion is important for describing the normalization of the blow up of the ideal I. Another characterization (assume that R is noetherian) is that there exists a $z \in R$, not contained in any minimal prime ideal of R, such that $zf^n \in I^n$ holds for all n. Another equivalent property — the valuative criterion — is that for all ring homomorphisms

$$\theta : R \to D$$

to a discrete valuation domain D the containment $\theta(f) \in \theta(I)D$ holds.

The characterization of the integral closure in terms of forcing algebras requires some notions from topology. A continuous map

$$\varphi : X \to Y$$

between topological spaces X and Y is called a *submersion*, if it is surjective and if Y carries the image topology (quotient topology) under this map. This means that a subset $W \subseteq Y$ is open if and only if its preimage $\varphi^{-1}(W)$ is open. Since the spectrum of a ring endowed with the Zarisiki topology is a topological space, this notion can be applied to the spectrum morphism of a ring homomorphism. With this notion we can state that $f \in \bar{I}$ if and only if the forcing morphism

$$\varphi : \operatorname{Spec}(B) \to \operatorname{Spec}(R)$$

is a universal submersion (universal means here that for any ring change $R \to R'$ to a noetherian ring R', the resulting homomorphism $R' \to B'$ still has this property). The relation between these two notions stems from the fact that also for universal submersions there exists a criterion in terms of discrete valuation domains: A morphism of finite type between two affine noetherian schemes is a universal submersion if and only if the base change to any discrete valuation domain yields a submersion (see [SGA 1 1971, Remarque 2.6]). For a morphism

$$Z \to \operatorname{Spec}(D)$$

(D a discrete valuation domain) to be a submersion means that above the only chain of prime ideals in $\operatorname{Spec}(D)$, namely $(0) \subset \mathfrak{m}_D$, there exists a chain of prime ideals $\mathfrak{p}' \subseteq \mathfrak{q}'$ in Z lying over this chain. This pair-lifting property holds for a universal submersion

$$\operatorname{Spec}(S) \longrightarrow \operatorname{Spec}(R)$$

for any pair of prime ideals $\mathfrak{p} \subseteq \mathfrak{q}$ in $\operatorname{Spec}(R)$. This property is stronger than lying over (which means surjective) but weaker than the going-down or the going-up property (in the presence of surjectivity).

If we are dealing only with algebras of finite type over the complex numbers $\mathbb{C}$, then we may also consider the corresponding complex spaces with their natural topology induced from the euclidean topology of $\mathbb{C}^n$. Then universal submersive with respect to the Zariski topology is the same as submersive in the complex topology (the target space needs to be normal).

Example 1.4. Let K be a field and consider $R = K[X]$. Since this is a principal ideal domain, the only interesting forcing algebras (if we are only interested in the local behavior around (X)) are of the form $K[X, T]/(X^n T - X^m)$. For $m \geq n$ this $K[X]$-algebra admits a section (corresponding to the fact that $X^m \in (X^n)$), and if $n \geq 1$ there exists an affine line over the maximal ideal (X). So now assume $m < n$. If $m = 0$, then we have a hyperbola mapping to an affine line, with the fiber over (X) being empty, corresponding to the fact that 1 does not belong to the radical of (X^n) for $n \geq 1$. So assume finally $1 \leq m < n$. Then X^m belongs to the radical of (X^n), but not to its integral closure (which is the identical closure on a one-dimensional regular ring). We can write the forcing equation as $X^n T - X^m = X^m(X^{n-m}T - 1)$. So the spectrum of the forcing algebra consists of a (thickened) line over (X) and of a hyperbola. The forcing morphism is surjective, but it is not a submersion. For example, the preimage of $V(X) = \{(X)\}$ is a connected component hence open, but this single point is not open.

Example 1.5. Let K be a field and let $R = K[X, Y]$ be the polynomial ring in two variables. We consider the ideal $I = (X^2, Y)$ and the element X. This element belongs to the radical of this ideal; hence the forcing morphism

$$\operatorname{Spec}\bigl(K[X, Y, T_1, T_2]/(X^2 T_1 + Y T_2 - X)\bigr) \to \operatorname{Spec}\bigl(K[X, Y]\bigr)$$

is surjective. We claim that it is not a submersion. For this we look at the reduction modulo Y. In $K[X, Y]/(Y) \cong K[X]$ the ideal I becomes (X^2) which does not contain X. Hence by the valuative criterion for integral closure, X does not belong to the integral closure of the ideal. One can also say that the chain $V(X, Y) \subset V(Y)$ in the affine plane does not have a lift (as a chain) to the spectrum of the forcing algebra.

For the ideal $I = (X^2, Y^2)$ and the element XY the situation looks different. Let

$$\theta : K[X, Y] \to D$$

be a ring homomorphism to a discrete valuation domain D. If X or Y is mapped to 0, then also XY is mapped to 0 and hence belongs to the extended ideal. So assume that $\theta(X) = u\pi^r$ and $\theta(Y) = v\pi^s$, where π is a local parameter of D and u and v are units. Then $\theta(XY) = uv\pi^{r+s}$ and the exponent is at least the minimum of $2r$ and $2s$, hence

$$\theta(XY) \in (\pi^{2r}, \pi^{2s}) = (\theta(X^2), \theta(Y^2))D.$$

So XY belongs to the integral closure of (X^2, Y^2) and the forcing morphism

$$\operatorname{Spec}\bigl(K[X, Y, T_1, T_2]/(X^2T_1 + Y^2T_2 - XY)\bigr) \to \operatorname{Spec}\bigl(K[X, Y]\bigr)$$

is a universal submersion.

Continuous closure. Suppose now that $R = \mathbb{C}[X_1, \dots, X_k]$. Then every polynomial $f \in R$ can be considered as a continuous function

$$f : \mathbb{C}^k \to \mathbb{C}, \quad (x_1, \dots, x_k) \mapsto f(x_1, \dots, x_k),$$

in the complex topology. If $I = (f_1, \dots, f_n)$ is an ideal and $f \in R$ is an element, we say that f belongs to the *continuous closure* of I, if there exist continuous functions

$$g_1, \dots, g_n : \mathbb{C}^k \to \mathbb{C}$$

such that

$$f = \sum_{i=1}^{n} g_i f_i$$

(as an identity of functions). The same definition works for $\mathbb{C}$-algebras of finite type; see [Brenner 2006a; Epstein and Hochster 2011; Kollár 2012].

It is not at all clear at once that there may exist polynomials $f \notin I$ but inside the continuous closure of I. For $\mathbb{C}[X]$ it is easy to show that the continuous closure is (like the integral closure) just the ideal itself. We also remark that when we would only allow holomorphic functions $g_1, \dots, g_n$ then we could not get something larger. However, with continuous functions $g_1, g_2 : \mathbb{C}^2 \to \mathbb{C}$ we can for example write

$$X^2Y^2 = g_1X^3 + g_2Y^3.$$

Continuous closure is always inside the integral closure and hence also inside the radical. The element XY does not belong to the continuous closure of $I = (X^2, Y^2)$, though it belongs to the integral closure of I. In terms of forcing algebras, an element f belongs to the continuous closure if and only if the complex forcing mapping

$$\varphi_{\mathbb{C}} : \operatorname{Spec}(B)_{\mathbb{C}} \to \operatorname{Spec}(R)_{\mathbb{C}}$$

(between the corresponding complex spaces) admits a continuous section.

2. Vector bundles and torsors

Geometric vector bundles. We have seen that the fibers of the spectrum of a forcing algebra are (empty or) affine spaces. However, this is not only fiberwise true, but more general: If we localize the forcing algebra at f_i we get

$$(R[T_1, \dots, T_n]/(f_1T_1 + \cdots + f_nT_n - f))_{f_i} \cong R_{f_i}[T_1, \dots, T_{i-1}, T_{i+1}, \dots, T_n],$$

since we can write

$$T_i = -\sum_{j \neq i} \frac{f_j}{f_i} T_j + \frac{f}{f_i}.$$

So over every $D(f_i)$ the spectrum of the forcing algebra is an $(n-1)$-dimensional affine space over the base. So locally, restricted to $D(f_i)$, we have isomorphisms

$$T|_{D(f_i)} \cong D(f_i) \times \mathbb{A}^{n-1}.$$

On the intersections $D(f_i) \cap D(f_j)$ we get two identifications with affine space, and the transition morphisms are linear if $f = 0$, but only affine-linear in general (because of the translation with $\frac{f}{f_i}$).

So the forcing algebra has locally the form $R_{f_i}[T_1, \dots, T_{i-1}, T_{i+1}, \dots, T_n]$ and its spectrum $\operatorname{Spec}(B)$ has locally the form $D(f_i) \times \mathbb{A}_K^{n-1}$. This description holds on the union $U = \bigcup_{i=1}^n D(f_i)$. Moreover, in the homogeneous case ($f = 0$) the transition mappings are linear. Hence $V|_U$, where V is the spectrum of a homogeneous forcing algebra, is a geometric vector bundle according to the following definition.

Definition 2.1. Let X denote a scheme. A scheme V equipped with a morphism

$$p : V \to X$$

is called a *geometric vector bundle* of rank r over X if there exists an open covering $X = \bigcup_{i \in I} U_i$ and U_i-isomorphisms

$$\psi_i : U_i \times \mathbb{A}^r = \mathbb{A}^r_{U_i} \to V|_{U_i} = p^{-1}(U_i)$$

such that for every open affine subset $U \subseteq U_i \cap U_j$ the transition mappings

$$\psi_j^{-1} \circ \psi_i : \mathbb{A}^r_{U_i}|_U \to \mathbb{A}^r_{U_j}|_U$$

are linear automorphisms; that is, they are induced by an automorphism of the polynomial ring $\Gamma(U, \mathcal{O}_X)[T_1, \dots, T_r]$ given by $T_i \mapsto \sum_{j=1}^r a_{ij} T_j$.

Here we can restrict always to affine open coverings. If X is separated then the intersection of two affine open subschemes is again affine and then it is enough to check the condition on the intersections. The trivial bundle of rank r is the r-dimensional affine space $\mathbb{A}^r_X$ over X, and locally every vector bundle looks like this. Many properties of an affine space are enjoyed by general vector bundles. For example, in the affine space we have the natural addition

$$+ : \mathbb{A}^r_U \times_U \mathbb{A}^r_U \to \mathbb{A}^r_U, \quad (v_1, \ldots, v_r, w_1, \ldots, w_r) \mapsto (v_1 + w_1, \ldots, v_r + w_r),$$

and this carries over to a vector bundle, that is, we have an addition

$$\alpha : V \times_X V \to V.$$

The reason for this is that the isomorphisms occurring in the definition of a geometric vector bundle are linear, hence the addition on $V|_U$ coming from an isomorphism with some affine space over U is independent of the chosen isomorphism. For the same reason there is a unique closed subscheme of V called the *zero-section* which is locally defined to be $0 \times U \subseteq \mathbb{A}^r_U$. Also, multiplication by a scalar, i.e., the mapping

$$\cdot : \mathbb{A}_U \times_U \mathbb{A}^r_U \to \mathbb{A}^r_U, \quad (s, v_1, \ldots, v_r) \mapsto (sv_1, \ldots, sv_r),$$

carries over to a scalar multiplication

$$\cdot : \mathbb{A}_X \times_X V \to V.$$

In particular, for every point $P \in X$ the fiber $V_P = V \times_X P$ is an affine space over $\kappa(P)$.

For a geometric vector bundle $p : V \to X$ and an open subset $U \subseteq X$ one sets

$$\Gamma(U, V) = \{s : U \to V|_U \mid p \circ s = \mathrm{Id}_U\},$$

so this is the set of sections in V over U. This gives in fact for every scheme over X a set-valued sheaf. Because of the observations just mentioned, these sections can also be added and multiplied by elements in the structure sheaf, and so we get for every vector bundle a locally free sheaf, which is free on the open subsets where the vector bundle is trivial.

Definition 2.2. A coherent $\mathcal{O}_X$-module $\mathcal{F}$ on a scheme X is called *locally free* of rank r, if there exists an open covering $X = \bigcup_{i \in I} U_i$ and $\mathcal{O}_{U_i}$-module-isomorphisms $\mathcal{F}|_{U_i} \cong (\mathcal{O}_{U_i})^r$ for every $i \in I$.

Vector bundles and locally free sheaves are essentially the same objects.

Theorem 2.3. *Let X denote a scheme. Then the category of locally free sheaves on X and the category of geometric vector bundles on X are equivalent. A geometric vector bundle $V \to X$ corresponds to the sheaf of its sections, and a locally free sheaf $\mathcal{F}$ corresponds to the (relative) spectrum of the symmetric algebra of the dual module $\mathcal{F}^\vee$.*

The free sheaf of rank r corresponds to the affine space $\mathbb{A}^r_X$ over X.

Torsors of vector bundles. We have seen that

$$V = \operatorname{Spec}\bigl(R[T_1, \dots, T_n]/(f_1T_1 + \cdots + f_nT_n)\bigr)$$

acts on the spectrum of a forcing algebra

$$T = \operatorname{Spec}\bigl(R[T_1, \dots, T_n]/(f_1T_1 + \cdots + f_nT_n - f)\bigr)$$

by addition. The restriction of V to $U = D(f_1, \dots, f_n)$ is a vector bundle, and T restricted to U becomes a V-torsor.

Definition 2.4. Let V denote a geometric vector bundle over a scheme X. A scheme $T \to X$ together with an action

$$\beta : V \times_X T \to T$$

is called a geometric (Zariski) *torsor* for V (or a V-principal fiber bundle or a *principal homogeneous space*) if there exists an open covering $X = \bigcup_{i \in I} U_i$ and isomorphisms

$$\varphi_i : T|_{U_i} \to V|_{U_i}$$

such that the diagrams (we set $U = U_i$ and $\varphi = \varphi_i$)

$$\begin{tikzcd} V|_U \times_U T|_U \arrow[r, "\beta"] \arrow[d, "\operatorname{Id} \times \varphi"'] & T|_U \arrow[d, "\varphi"] \\ V|_U \times_U V|_U \arrow[r, "\alpha"] & V|_U \end{tikzcd}$$

commute, where α is the addition on the vector bundle.

The torsors of vector bundles can be classified in the following way.

Proposition 2.5. *Let X denote a noetherian separated scheme and let*

$$p : V \to X$$

denote a geometric vector bundle on X with sheaf of sections $\mathcal{S}$. Then there exists a correspondence between first cohomology classes $c \in H^1(X, \mathcal{S})$ and geometric V-torsors.

Proof. We describe only the correspondence. Let T denote a V-torsor. There exists by definition an open covering $X = \bigcup_{i \in I} U_i$ such that there exist isomorphisms

$$\varphi_i : T|_{U_i} \to V|_{U_i}$$

which are compatible with the action of $V|_{U_i}$ on itself. The isomorphisms φ_i induce automorphisms

$$\psi_{ij} = \varphi_j \circ \varphi_i^{-1} : V|_{U_i \cap U_j} \to V|_{U_i \cap U_j}.$$

These automorphisms are compatible with the action of V on itself, and this means that they are of the form

$$\psi_{ij} = \mathrm{Id}_V |_{U_i \cap U_j} + s_{ij}$$

with suitable sections $s_{ij} \in \Gamma(U_i \cap U_j, \mathcal{S})$. This family defines a Čech cocycle for the covering and gives therefore a cohomology class in $H^1(X, \mathcal{S})$.

For the reverse direction, suppose that the cohomology class $c \in H^1(X, \mathcal{S})$ is represented by a Čech cocycle $s_{ij} \in \Gamma(U_i \cap U_j, \mathcal{S})$ for an open covering $X = \bigcup_{i \in I} U_i$. Set $T_i := V|_{U_i}$. We take the morphisms

$$\psi_{ij} : T_i|_{U_i \cap U_j} = V|_{U_i \cap U_j} \to V|_{U_i \cap U_j} = T_j|_{U_i \cap U_j}$$

given by $\psi_{ij} := \mathrm{Id}_V |_{U_i \cap U_j} + s_{ij}$ to glue the T_i together to a scheme T over X. This is possible since the cocycle condition guarantees the gluing condition for schemes (see [EGA I 1960, Chapter 0, §4.1.7]). The action of $T_i = V|_{U_i}$ on itself glues also together to give an action on T. ☐

It follows immediately that for an affine scheme (a scheme of type $\mathrm{Spec}(R)$) there is no nontrivial torsor for any vector bundle. There will however be in general many nontrivial torsors on the punctured spectrum (and on a projective variety).

Forcing algebras and induced torsors. As T_U is a V_U-torsor, and as every V-torsor is represented by a unique cohomology class, there should be a natural cohomology class coming from the forcing data. To see this, let R be a noetherian ring and $I = (f_1, \dots, f_n)$ be an ideal. Then on $U = D(I)$ we have the short exact sequence

$$0 \to \mathrm{Syz}(f_1, \dots, f_n) \to \mathcal{O}_U^n \to \mathcal{O}_U \to 0.$$

On the left we have a locally free sheaf of rank $n-1$ which we call the *syzygy sheaf* or *syzygy bundle*. It is the sheaf of sections in the geometric vector bundle

$$\mathrm{Spec}\big(R[T_1, \dots, T_n]/(f_1T_1 + \cdots + f_nT_n)\big)|_U.$$

An element $f \in R$ defines an element $f \in \Gamma(U, \mathcal{O}_U)$ and hence a cohomology class $\delta(f) \in H^1(U, \mathrm{Syz}(f_1, \dots, f_n))$. Hence f defines in fact a $\mathrm{Syz}(f_1, \dots, f_n)$-torsor over U. We will see that this torsor is induced by the forcing algebra given by $f_1, \dots, f_n$ and f.

Theorem 2.6. *Let R denote a noetherian ring, let $I = (f_1, \dots, f_n)$ denote an ideal and let $f \in R$ be another element. Let $c \in H^1(D(I), \mathrm{Syz}(f_1, \dots, f_n))$ be the corresponding cohomology class and let*

$$B = R[T_1, \dots, T_n]/(f_1T_1 + \cdots + f_nT_n - f)$$

denote the forcing algebra for these data. Then the scheme $\operatorname{Spec}(B)|_{D(I)}$ *together with the natural action of the syzygy bundle on it is isomorphic to the torsor given by* c*.*

Proof. We compute the cohomology class $\delta(f) \in H^1(U, \operatorname{Syz}(f_1, \dots, f_n))$ and the cohomology class given by the forcing algebra. For the first computation we look at the short exact sequence

$$0 \to \operatorname{Syz}(f_1, \dots, f_n) \to \mathcal{O}_U^n \overset{f_1,\dots,f_n}{\to} \mathcal{O}_U \to 0.$$

On $D(f_i)$, the element f is the image of $(0, \dots, 0, f/f_i, 0, \dots, 0)$ (the nonzero entry is at the i-th place). The cohomology class is therefore represented by the family of differences

$$\left(0, \dots, 0, \frac{f}{f_i}, 0, \dots, 0, -\frac{f}{f_j}, 0, \dots, 0\right) \in \Gamma(D(f_i) \cap D(f_j), \operatorname{Syz}(f_1, \dots, f_n)).$$

On the other hand, there are isomorphisms

$$V|_{D(f_i)} \to T|_{D(f_i)}, \quad (s_1, \dots, s_n) \mapsto \left(s_1, \dots, s_{i-1}, s_i + \frac{f}{f_i}, s_{i+1}, \dots, s_n\right).$$

The composition of two such isomorphisms on $D(f_i f_j)$ is the identity plus the same section as before. ☐

Example 2.7. Let $(R, \mathfrak{m})$ denote a two-dimensional normal local noetherian domain and let f and g be two parameters in R. On $U = D(\mathfrak{m})$ we have the short exact sequence

$$0 \to \mathcal{O}_U \cong \operatorname{Syz}(f, g) \to \mathcal{O}_U^2 \overset{f,g}{\to} \mathcal{O}_U \to 0$$

and its corresponding long exact sequence of cohomology,

$$0 \to R \to R^2 \overset{f,g}{\to} R \overset{\delta}{\to} H^1(U, \mathcal{O}_X) \to \cdots .$$

The connecting homomorphism δ sends an element $h \in R$ to $\frac{h}{fg}$. The torsor given by such a cohomology class $c = \frac{h}{fg} \in H^1(U, \mathcal{O}_X)$ can be realized by the forcing algebra

$$R[T_1, T_2]/(fT_1 + gT_2 - h).$$

Note that different forcing algebras may give the same torsor, because the torsor depends only on the spectrum of the forcing algebra restricted to the punctured spectrum of R. For example, the cohomology class $\frac{1}{fg} = \frac{fg}{f^2g^2}$ defines one torsor, but the two fractions yield the two forcing algebras $R[T_1, T_2]/(fT_1 + gT_2 - 1)$ and $R[T_1, T_2]/(f^2T_1 + g^2T_2 - fg)$, which are quite different. The fiber over the maximal ideal of the first one is empty, whereas the fiber over the maximal ideal of the second one is a plane.

If R is regular, say $R = K[X, Y]$ (or the localization of this at (X, Y) or the corresponding power series ring) then the first cohomology classes are K-linear combinations of terms $1/(x^i y^j)$, for $i, j \geq 1$. They are realized by the forcing algebras

$$K[X, Y, T_1, T_2]/(X^i T_1 + Y^j T_2 - 1).$$

Since the fiber over the maximal ideal is empty, the spectrum of the forcing algebra equals the torsor. Or, the other way round, the torsor is itself an affine scheme.

3. Tight closure and cohomological properties of torsors

The closure operations we have considered so far can be characterized by some property of the forcing algebra. However, they can not be characterized by a property of the corresponding torsor alone. For example, for $R = K[X, Y]$, we may write

$$\frac{1}{XY} = \frac{X}{X^2Y} = \frac{XY}{X^2Y^2} = \frac{X^2Y^2}{X^3Y^3},$$

so the torsors given by the forcing algebras

$$R[T_1, T_2]/(XT_1 + YT_2 - 1), \quad R[T_1, T_2]/(X^2T_1 + YT_2 - X),$$
$$R[T_1, T_2]/(X^2T_1 + Y^2T_2 - XY) \quad \text{and} \quad R[T_1, T_2]/(X^3T_1 + Y^3T_2 - X^2Y^2)$$

are all the same (the restriction over $D(X, Y)$), but their global properties are quite different. We have a nonsurjection, a surjective nonsubmersion, a submersion which does not admit (for $K = \mathbb{C}$) a continuous section and a map which admits a continuous section.

We deal now with closure operations which depend only on the torsor which the forcing algebra defines, so they only depend on the cohomology class of the forcing data inside the syzygy bundle. Our main example is tight closure, a theory developed by Hochster and Huneke, and related closure operations like solid closure and plus closure. For background on tight closure see [Hochster and Huneke 1990; Huneke 1996; 1998].

Tight closure and solid closure. Let R be a noetherian domain of positive characteristic, let

$$F : R \to R, \quad f \mapsto f^p,$$

be the *Frobenius homomorphism*, and

$$F^e : R \to R, \quad f \mapsto f^q, \quad q = p^e,$$

its e-th iteration. Let I be an ideal and set

$$I^{[q]} = \text{extended ideal of } I \text{ under } F^e.$$

Then define the *tight closure* of I to be the ideal

$$I^* := \{f \in R \mid \text{there exists } z \neq 0 \text{ such that } zf^q \in I^{[q]} \text{ for all } q = p^e\}.$$

The element f defines the cohomology class $c \in H^1(D(I), \operatorname{Syz}(f_1, \dots, f_n))$. Suppose that R is normal and that I has height at least 2 (think of a local normal domain of dimension at least 2 and an $\mathfrak{m}$-primary ideal I). Then the e-th Frobenius pull-back of the cohomology class is

$$F^{e*}(c) \in H^1\big(D(I), F^{e*}(\operatorname{Syz}(f_1, \dots, f_n))\big) \cong H^1(D(I), \operatorname{Syz}(f_1^q, \dots, f_n^q))$$

$(q = p^e)$ and this is the cohomology class corresponding to f^q. By the height assumption, $zF^{e*}(c) = 0$ if and only if $zf^q \in (f_1^q, \dots, f_n^q)$, and if this holds for all e then $f \in I^*$ by definition. This shows already that tight closure under the given conditions does only depend on the cohomology class.

This is also a consequence of the following theorem, which gives a characterization of tight closure in terms of forcing algebra and local cohomology.

Theorem 3.1 [Hochster 1994, Theorem 8.6]. *Let R be a normal excellent local domain with maximal ideal $\mathfrak{m}$ over a field of positive characteristic. Let $f_1, \dots, f_n$ generate an $\mathfrak{m}$-primary ideal I and let f be another element in R. Then $f \in I^*$ if and only if*

$$H^{\dim(R)}_{\mathfrak{m}}(B) \neq 0,$$

where $B = R[T_1, \dots, T_n]/(f_1T_1 + \cdots + f_nT_n - f)$ denotes the forcing algebra of these elements.

If the dimension d is at least two, then

$$H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(B) \cong H^d_{\mathfrak{m}B}(B) \cong H^{d-1}(D(\mathfrak{m}B), \mathcal{O}_B).$$

This means that we have to look at the cohomological properties of the complement of the exceptional fiber over the closed point, i.e., the torsor given by these data. If $H^{d-1}(D(\mathfrak{m}B), \mathcal{O}_B) = 0$ then this is true for all quasicoherent sheaves instead of the structure sheaf. This property can be expressed by saying that the *cohomological dimension* of $D(\mathfrak{m}B)$ is $\leq d-2$ and thus smaller than the cohomological dimension of the punctured spectrum $D(\mathfrak{m})$, which is exactly $d-1$. So belonging to tight closure can be rephrased by saying that the formation of the corresponding torsor does not change the cohomological dimension.

If the dimension is two, then we have to look at whether the first cohomology of the structure sheaf vanishes. This is true (by Serre's cohomological criterion for affineness, see below) if and only if the open subset $D(\mathfrak{m}B)$ is an *affine scheme* (the spectrum of a ring).

The right-hand side of the equivalence in Theorem 3.1 — the nonvanishing of the top-dimensional local cohomology — is independent of any characteristic assumption, and can be taken as the basis for the definition of another closure operation, called *solid closure*. So the theorem above says that in positive characteristic tight closure and solid closure coincide. There is also a definition of tight closure for algebras over a field of characteristic 0 by reduction to positive characteristic; see [Hochster 1996].

An important property of tight closure is that it is trivial for regular rings: $I^* = I$ for every ideal I. This rests upon Kunz's theorem [1969, 3.3] saying that the Frobenius homomorphism for regular rings is flat. This property implies the following cohomological property of torsors.

Corollary 3.2. *Let* $(R, \mathfrak{m})$ *denote a regular local ring of dimension* d *and of positive characteristic*, *let* $I = (f_1, \dots, f_n)$ *be an* $\mathfrak{m}$*-primary ideal and* $f \in R$ *an element with* $f \notin I$. *Let* $B = R[T_1, \dots, T_n]/(f_1T_1 + \cdots + f_nT_n - f)$ *be the corresponding forcing algebra. Then for the extended ideal* $\mathfrak{m}B$ *we have*

$$H^d_{\mathfrak{m}B}(B) = H^{d-1}(D(\mathfrak{m}B), \mathcal{O}_B) = 0.$$

Proof. This follows from Theorem 3.1 and $f \notin I^*$. □

In dimension two this is true in every (even mixed) characteristic.

Theorem 3.3. *Let* $(R, \mathfrak{m})$ *denote a two-dimensional regular local ring*, *let* $I = (f_1, \dots, f_n)$ *be an* $\mathfrak{m}$*-primary ideal and* $f \in R$ *an element with* $f \notin I$. *Let* $B = R[T_1, \dots, T_n]/(f_1T_1 + \cdots + f_nT_n - f)$ *be the corresponding forcing algebra. Then the extended ideal* $\mathfrak{m}B$ *satisfies*

$$H^2_{\mathfrak{m}B}(B) = H^1(D(\mathfrak{m}B), \mathcal{O}_B) = 0.$$

In particular, *the open subset* $T = D(\mathfrak{m}B)$ *is an affine scheme if and only if* $f \notin I$.

The main point for the proof of this result is that for $f \notin I$, the natural mapping

$$H^1(U, \mathcal{O}_X) \to H^1(T, \mathcal{O}_T)$$

is not injective by a Matlis duality argument. Since the local cohomology of a regular ring is explicitly known, this map annihilates some cohomology class of the form $1/(fg)$ where f, g are parameters. But then it annihilates the complete local cohomology module and then T is an affine scheme.

For nonregular two-dimensional rings it is a difficult question in general to decide whether a torsor is affine or not. A satisfactory answer is only known in the normal two-dimensional graded case over a field, which we will deal with in the final lectures.

In higher dimension in characteristic zero it is not true that a regular ring is *solidly closed* (meaning that every ideal equals its solid closure), as was shown by the following example of Paul Roberts.

Example 3.4 [Roberts 1994]. Let K be a field of characteristic 0 and let

$$B = K[X, Y, Z][U, V, W]/(X^3U + Y^3V + Z^3W - X^2Y^2Z^2).$$

Then the ideal $\mathfrak{a} = (X, Y, Z)B$ has the property that $H^3_{\mathfrak{a}}(B) \neq 0$. This means that in $R = K[X, Y, Z]$ the element $X^2Y^2Z^2$ belongs to the solid closure of the ideal (X^3, Y^3, Z^3); hence the three-dimensional polynomial ring is not solidly closed.

This example was the motivation for the introduction of parasolid closure [Brenner 2003a], which has all the good properties of solid closure but which is also trivial for regular rings.

If R is a normal local domain of dimension 2 and $I = (f_1, \dots, f_n)$ an $\mathfrak{m}$-primary ideal, then $f \in I^*$ (or inside the solid closure) if and only if $D(\mathfrak{m}) \subseteq \operatorname{Spec}(B)$ is not an affine scheme, where B denotes the forcing algebra. Here we will discuss in more detail, with this application in mind, when a scheme is affine.

Affine schemes. A scheme U is called *affine* if it is isomorphic to the spectrum of some commutative ring R. If the scheme is of finite type over a field (or a ring) K, then this is equivalent to saying that there exist global functions

$$g_1, \dots, g_m \in \Gamma(U, \mathcal{O}_U)$$

such that the mapping

$$U \to \mathbb{A}^m_K, \quad x \mapsto (g_1(x), \dots, g_m(x)),$$

is a closed embedding. The relation to cohomology is given by the following well-known theorem of Serre [Hartshorne 1977, Theorem III.3.7].

Theorem 3.5. *For U a noetherian scheme, the following properties are equivalent.*

(1) *U is an affine scheme.*

(2) *For every quasicoherent sheaf $\mathcal{F}$ on U and all $i \geq 1$ we have $H^i(U, \mathcal{F}) = 0$.*

(3) *For every coherent ideal sheaf $\mathcal{I}$ on U we have $H^1(U, \mathcal{I}) = 0$.*

It is in general a difficult question whether a given scheme U is affine. For example, suppose that $X = \operatorname{Spec}(R)$ is an affine scheme and

$$U = D(\mathfrak{a}) \subseteq X$$

is an open subset (such schemes are called *quasiaffine*) defined by an ideal $\mathfrak{a} \subseteq R$. When is U itself affine? The cohomological criterion above simplifies to the condition that $H^i(U, \mathcal{O}_X) = 0$ for $i \geq 1$.

Of course, if $\mathfrak{a} = (f)$ is a principal ideal (or up to radical a principal ideal), then $U = D(f) \cong \operatorname{Spec}(R_f)$ is affine. On the other hand, if $(R, \mathfrak{m})$ is a local ring of dimension ≥ 2, then

$$D(\mathfrak{m}) \subset \operatorname{Spec}(R)$$

is not affine, since

$$H^{d-1}(U, \mathcal{O}_X) = H^d_{\mathfrak{m}}(R) \neq 0$$

by the relation between sheaf cohomology and local cohomology and a theorem of Grothendieck [Bruns and Herzog 1993, Theorem 3.5.7].

Codimension condition. One can show that for an open affine subset $U \subseteq X$ the closed complement $Y = X \setminus U$ must be of pure codimension one (U must be the complement of the support of an effective divisor). In a regular or (locally $\mathbb{Q}$-) factorial domain the complement of every effective divisor is affine, since the divisor can be described (at least locally geometrically) by one equation. But it is easy to give examples to show that this is not true for normal three-dimensional domains. The following example is a standard example for this phenomenon and it is in fact given by a forcing algebra (we write here and in the following often small letters for the classes of the variables in the residue class ring).

Example 3.6. Let K be a field and consider the ring

$$R = K[X, Y, U, V]/(XU - YV).$$

The ideal $\mathfrak{p} = (x, y)$ is a prime ideal in R of height one. Hence the open subset $U = D(x, y)$ is the complement of an irreducible hypersurface. However, U is not affine. For this we consider the closed subscheme

$$\mathbb{A}^2_K \cong Z = V(u, v) \subseteq \operatorname{Spec}(R)$$

and

$$Z \cap U \subseteq U.$$

If U were affine, then also the closed subscheme $Z \cap U \cong \mathbb{A}^2_K \setminus \{(0, 0)\}$ would be affine, but this is not true, since the complement of the punctured plane has codimension 2.

Ring of global sections of affine schemes. For an open subset $U = D(\mathfrak{a}) \subseteq \operatorname{Spec}(R)$ its ring of global sections $\Gamma(U, \mathcal{O}_X)$ is difficult to compute in general. If R is a domain and $\mathfrak{a} = (f_1, \dots, f_n)$, then

$$\Gamma(U, \mathcal{O}_X) = R_{f_1} \cap R_{f_2} \cap \cdots \cap R_{f_n}.$$

This ring is not always of finite type over R, but it is if U is affine.

Lemma 3.7. *Let R be a noetherian ring and $U = D(\mathfrak{a}) \subseteq \operatorname{Spec}(R)$ an open subset.*

(1) *U is an affine scheme if and only if $\mathfrak{a}\Gamma(U, \mathcal{O}_X) = (1)$.*

(2) *If this holds, and $q_1 f_1 + \cdots + q_n f_n = 1$ with $f_1, \dots, f_n \in \mathfrak{a}$ and $q_i \in \Gamma(U, \mathcal{O}_X)$, then $\Gamma(U, \mathcal{O}_X) = R[q_1, \dots, q_n]$. In particular, the ring of global sections over U is finitely generated over R.*

Sketch of proof. (1) There always exists a natural scheme morphism

$$U \to \operatorname{Spec}\bigl(\Gamma(U, \mathcal{O}_X)\bigr),$$

and U is affine if and only if this morphism is an isomorphism. It is always an open embedding (because it is an isomorphism on the $D(f)$, $f \in \mathfrak{a}$), and the image is $D(\mathfrak{a}\Gamma(U, \mathcal{O}_X))$. This is everything if and only if the extended ideal is the unit ideal.

(2) We write $1 = q_1 f_1 + \cdots + q_n f_n$ and consider the natural morphism

$$U \to \operatorname{Spec}\bigl(R[q_1, \dots, q_n]\bigr)$$

corresponding to the ring inclusion $R[q_1, \dots, q_n] \subseteq \Gamma(U, \mathcal{O}_X)$. This morphism is again an open embedding and its image is everything. $\square$

We give some examples of tight closure computations on the Fermat cubic $x^3 + y^3 + z^3 = 0$, a standard example in tight closure theory, with the methods we have developed so far.

Example 3.8. We consider the Fermat cubic $R = K[X, Y, Z]/(X^3 + Y^3 + Z^3)$, the ideal $I = (X, Y)$ and the element Z. We claim that for characteristic $\neq 3$ the element Z does not belong to the solid closure of I. Equivalently, the open subset

$$D(X, Y) \subseteq \operatorname{Spec}\bigl(R[S, T]/(XS + YT - Z)\bigr)$$

is affine. For this we show that the extended ideal inside the ring of global sections is the unit ideal. First of all we get the equation

$$X^3 + Y^3 = (XS + YT)^3 = X^3S^3 + 3X^2S^2YT + 3XSY^2T^2 + Y^3T^3$$

or, equivalently,

$$X^3(S^3-1)+3X^2YS^2T+3XY^2ST^2+Y^3(T^3-1)=0.$$

We write this as

$$\begin{aligned}X^3(S^3-1)&=-3X^2YS^2T-3XY^2ST^2-Y^3(T^3-1)\\&=Y\bigl(-3X^2S^2T-3XYST^2-Y^2(T^3-1)\bigr),\end{aligned}$$

which yields on $D(X,Y)$ the rational function

$$Q=\frac{S^3-1}{Y}=\frac{-3X^2S^2T-3XYST^2-Y^2\left(T^3-1\right)}{X^3}.$$

This shows that $S^3-1=QY$ belongs to the extended ideal. Similarly, one can show that also the other coefficients $3S^2T$, $3ST^2$, T^3-1 belong to the extended ideal. Therefore in characteristic different from 3, the extended ideal is the unit ideal.

Example 3.9. We consider the Fermat cubic $R=K[X,Y,Z]/(X^3+Y^3+Z^3)$, the ideal $I=(X,Y)$ and the element Z^2. We claim that in positive characteristic $\neq 3$ the element Z^2 does belong to the tight closure of I. Equivalently, the open subset

$$D(X,Y)\subseteq \operatorname{Spec}\bigl(R[S,T]/(XS+YT-Z^2)\bigr)$$

is not affine. The element Z^2 defines the cohomology class

$$c=\frac{Z^2}{XY}\in H^1(D(X,Y),\mathcal{O}_X)$$

and its Frobenius pull-backs are $F^{e*}(c)=\frac{Z^{2q}}{X^qY^q}\in H^1(D(X,Y),\mathcal{O}_X)$. This cohomology module has a $\mathbb{Z}$-graded structure (the degree is given by the difference of the degree of the numerator and the degree of the denominator) and, moreover, it is 0 in positive degree (this is related to the fact that the corresponding projective curve is elliptic). Therefore for any homogeneous element $t\in R$ of positive degree we have $tF^{e*}(c)=0$ and so Z^2 belongs to the tight closure.

From this it follows also that in characteristic 0 the element Z^2 belongs to the solid closure, because affineness is an open property in an arithmetic (or any) family, which follows from Lemma 3.7 (1).

We give now a cohomological proof of a tight closure containment on the Fermat cubic for a nonparameter ideal. M. McDermott has raised the question whether

$$xyz\in(x^2,y^2,z^2)^*\text{ in }K[X,Y,Z]/(X^3+Y^3+Z^3).$$

This was answered positively by A. Singh [1998] by a long “equational” argument.

Example 3.10. Let $R = K[X, Y, Z]/(X^3 + Y^3 + Z^3)$, where K is a field of positive characteristic $p \neq 3$, $I = (x^2, y^2, z^2)$ and $f = xyz$. We consider the short exact sequence

$$0 \to \operatorname{Syz}(x^2, y^2, z^2) \to \mathcal{O}_U^3 \overset{x^2,y^2,z^2}{\to} \mathcal{O}_U \to 0$$

and the cohomology class

$$c = \delta(xyz) \in H^1\left(U, \operatorname{Syz}(x^2, y^2, z^2)\right).$$

We want to show that $zF^{e*}(c) = 0$ for all $e \geq 0$ (here the test element z equals the element z in the ring). It is helpful to work with the graded structure on this syzygy sheaf (or to work on the corresponding elliptic curve Proj R directly). Now the equation $x^3 + y^3 + z^3 = 0$ can be considered as a syzygy (of total degree 3) for x^2, y^2, z^2, yielding an inclusion

$$0 \to \mathcal{O}_U \to \operatorname{Syz}(x^2, y^2, z^2).$$

Since this syzygy does not vanish anywhere on U the quotient sheaf is invertible and in fact isomorphic to the structure sheaf. Hence we have

$$0 \to \mathcal{O}_U \to \operatorname{Syz}(x^2, y^2, z^2) \to \mathcal{O}_U \to 0$$

and the cohomology sequence

$$\to H^1(U, \mathcal{O}_U)_s \to H^1\big(U, \operatorname{Syz}(x^2, y^2, z^2)\big)_{s+3} \to H^1(U, \mathcal{O}_U)_s \to 0,$$

where s denotes the degree-s piece. The class c lives in $H^1\left(U, \operatorname{Syz}(x^2, y^2, z^2)\right)_3$, so its Frobenius pull-backs live in $H^1\left(U, F^{e*}\operatorname{Syz}(x^2, y^2, z^2)\right)_{3q}$, and we can have a look at the cohomology of the pull-backs of the sequence, i.e.,

$$\to H^1(U, \mathcal{O}_U)_0 \to H^1\big(U, F^{e*}\operatorname{Syz}(x^2, y^2, z^2)\big)_{3q} \to H^1(U, \mathcal{O}_U)_0 \to 0.$$

The class $zF^{e*}(c)$ lives in $H^1\big(U, F^{e*}\operatorname{Syz}(x^2, y^2, z^2)\big)_{3q+1}$. It is mapped on the right to $H^1(U, \mathcal{O}_U)_1$, which is 0 (because we are working over an elliptic curve), hence it comes from the left, which is $H^1(U, \mathcal{O}_U)_1 = 0$. So $zF^{e*}(c) = 0$ and $f \in \left(x^2, y^2, z^2\right)^*$.

Affineness and superheight. We have mentioned above that the complement of an affine open subset must have pure codimension 1. We have also seen in Example 3.6 that the nonaffineness can be established by looking at the behavior of the codimension when the situation is restricted to closed subschemes. The following definition and theorem is an algebraic version of this observation [Brenner 2002].

Definition 3.11. Let R be a noetherian commutative ring and let $I \subseteq R$ be an ideal. The (noetherian) *superheight* is the supremum

$$\sup\big(\operatorname{ht}(IS) : S \text{ is a notherian } R\text{-algebra}\big).$$

Theorem 3.12. *Let R be a noetherian commutative ring and let $I \subseteq R$ be an ideal and $U = D(I) \subseteq X = \operatorname{Spec}(R)$. Then the following are equivalent.*

(1) *U is an affine scheme.*

(2) *I has superheight ≤ 1 and $\Gamma(U, \mathcal{O}_X)$ is a finitely generated R-algebra.*

It is not true at all that the ring of global sections of an open subset U of the spectrum X of a noetherian ring is of finite type over this ring. This is not even true if X is an affine variety. This problem is directly related to Hilbert's fourteenth problem, which has a negative answer. We will later present examples (see Example 3.13) where U has superheight one, yet is not affine, hence its ring of global sections is not finitely generated.

Plus closure. For an ideal $I \subseteq R$ in a domain R define its *plus closure* by

$$I^+ = \{f \in R \mid \text{there exists a finite domain extension } R \subseteq T \text{ such that } f \in IT\}.$$

Equivalent: let R^+ be the *absolute integral closure* of R. This is the integral closure of R in an algebraic closure of the quotient field $Q(R)$ (first considered in [Artin 1971]). Then

$$f \in I^+ \text{ if and only if } f \in IR^+.$$

The plus closure commutes with localization; see [Huneke 1996, Exercise 12.2].

We also have the inclusion $I^+ \subseteq I^*$; see [Huneke 1996, Theorem 1.7].

Question. Is $I^+ = I^*$?

This is known as the *tantalizing question* in tight closure theory.

In terms of forcing algebras and their torsors, the containment inside the plus closure has the following geometric meaning (see [Brenner 2003c] for details): If R is a d-dimensional domain of finite type over a field, and $I = (f_1, \ldots, f_n)$ is an $\mathfrak{m}$-primary ideal for some maximal ideal $\mathfrak{m}$ and $f \in R$, then $f \in I^+$ if and only if the spectrum of the forcing algebra contains a d-dimensional closed subscheme which meets the exceptional fiber (the fiber over the maximal ideal) in isolated points. This means that the superheight of the extended ideal to the forcing algebra is d or that the torsor contains a punctured d-dimensional closed subscheme. In this case the local cohomological dimension of the torsor must be d as well, since it contains a closed subscheme with this cohomological dimension. So also the plus closure depends only on the torsor.

In characteristic zero, the plus closure behaves very differently compared with positive characteristic. If R is a normal domain of characteristic 0, then the trace map shows that the plus closure is trivial, $I^+ = I$ for every ideal I.

Examples. In the following two examples we use results from tight closure theory to establish (non) -affineness properties of certain torsors.

Example 3.13. Let K be a field and consider the Fermat ring

$$R = K[X, Y, Z]/(X^d + Y^d + Z^d)$$

together with the ideal $I = (X, Y)$ and $f = Z^2$. For $d \geq 3$ we have $Z^2 \notin (X, Y)$. This element is however in the tight closure $(X, Y)^*$ of the ideal in positive characteristic (assume that the characteristic p does not divide d) and is therefore also in characteristic 0 inside the tight closure (in the sense of [Hochster 1996, Definition 3.1]) and inside the solid closure. Hence the open subset

$$D(X, Y) \subseteq \operatorname{Spec}\bigl(K[X, Y, Z, S, T]/(X^d + Y^d + Z^d, SX + TY - Z^2)\bigr)$$

is not an affine scheme. In positive characteristic, Z^2 is also contained in the plus closure $(X, Y)^+$ and therefore this open subset contains punctured surfaces (the spectrum of the forcing algebra contains two-dimensional closed subschemes which meet the exceptional fiber $V(X, Y)$ in only one point; the ideal (X, Y) has superheight two in the forcing algebra). In characteristic zero however, the superheight is one because plus closure is trivial for normal domains in characteristic 0, and therefore by Theorem 3.12 the algebra $\Gamma(D(X, Y), \mathcal{O}_B)$ is not finitely generated. For $K = \mathbb{C}$ and $d = 3$ one can also show that $D(X, Y)_{\mathbb{C}}$ is, considered as a complex space, a Stein space.

Example 3.14. Let K be a field of positive characteristic $p \geq 7$ and consider the ring

$$R = K[X, Y, Z]/(X^5 + Y^3 + Z^2)$$

together with the ideal $I = (X, Y)$ and $f = Z$. Since R has a rational singularity, it is F-regular, so all ideals are tightly closed. Therefore $Z \notin (X, Y)^*$ and so the torsor

$$D(X, Y) \subseteq \operatorname{Spec}\bigl(K[X, Y, Z, S, T]/(X^5 + Y^3 + Z^2, SX + TY - Z)\bigr)$$

is an affine scheme. In characteristic zero this can be proved by either using that R is a quotient singularity or by using the natural grading ($\deg X = 6$, $\deg Y = 10$, $\deg Z = 15$) where the corresponding cohomology class $Z/(XY)$ gets degree -1 and then applying the geometric criteria (see below) on the corresponding projective curve (rather the corresponding curve of the standard homogenization $U^{30} + V^{30} + W^{30} = 0$).

4. Cones over projective curves

We continue with the question when the torsors given by a forcing algebra over a two-dimensional ring are affine? We will look at the graded situation to be able to work on the corresponding projective curve.

In particular we want to address the following questions:

1. Is there a procedure to decide whether the torsor is affine?
2. Is it nonaffine if and only if there exists a geometric reason for it not to be affine (because the superheight is too large) ?
3. How does the affineness vary in an arithmetic family, when we vary the prime characteristic?
4. How does the affineness vary in a geometric family, when we vary the base ring?

In terms of tight closure, these questions are directly related to the tantalizing question of tight closure (is it the same as plus closure), the dependence of tight closure on the characteristic and the localization problem of tight closure.

Geometric interpretation in dimension two. We will restrict now to the two-dimensional homogeneous case in order to work on the corresponding projective curve. We want to find an object over the curve which corresponds to the forcing algebra or its induced torsor. The results of this part were developed in [Brenner 2003b; 2004; 2006c]; see also [Brenner 2008].

Let R be a two-dimensional standard-graded normal domain over an algebraically closed field K. Let $C = \operatorname{Proj}(R)$ be the corresponding smooth projective curve and let

$$I = (f_1, \dots, f_n)$$

be an R_+-primary homogeneous ideal with generators of degrees $d_1, \dots, d_n$. Then we get on C the short exact sequence

$$0 \to \operatorname{Syz}(f_1, \dots, f_n)(m) \to \bigoplus_{i=1}^{n} \mathcal{O}_C(m - d_i) \overset{f_1, \dots, f_n}{\to} \mathcal{O}_C(m) \to 0.$$

Here $\operatorname{Syz}(f_1, \dots, f_n)(m)$ is a vector bundle, called the *syzygy bundle*, of rank $n-1$ and of degree

$$((n-1)m - \sum_{i=1}^{n} d_i) \deg(C).$$

Recall that the degree of a vector bundle $\mathcal{S}$ on a projective curve is defined as the degree of the invertible sheaf $\bigwedge^r \mathcal{S}$, where r is the rank of $\mathcal{S}$. The degree is additive on short exact sequences.

A homogeneous element f of degree m defines an element in $\Gamma(C, \mathcal{O}_C(m))$ and thus a cohomology class $\delta(f) \in H^1(C, \mathrm{Syz}(f_1, \dots, f_n)(m))$, so this defines a torsor over the projective curve. We mention an alternative description of the torsor corresponding to a first cohomology class in a locally free sheaf which is better suited for the projective situation.

Remark 4.1. Let $\mathcal{S}$ denote a locally free sheaf on a scheme X. For a cohomology class $c \in H^1(X, \mathcal{S})$ one can construct a geometric object: Because of $H^1(X, \mathcal{S}) \cong \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{S})$, the class defines an extension

$$0 \to \mathcal{S} \to \mathcal{S}' \to \mathcal{O}_X \to 0.$$

This extension is such that under the connecting homomorphism of cohomology, $1 \in \Gamma(X, \mathcal{O}_X)$ is sent to $c \in H^1(X, \mathcal{S})$. The extension yields a projective subbundle

$$\mathbb{P}(\mathcal{S}^\vee) \subset \mathbb{P}(\mathcal{S}'^\vee).$$

If V is the corresponding geometric vector bundle of $\mathcal{S}$, one may think of $\mathbb{P}(\mathcal{S}^\vee)$ as $\mathbb{P}(V)$ which consists for every base point $x \in X$ of all the lines in the fiber V_x passing through the origin. The projective subbundle $\mathbb{P}(V)$ has codimension one inside $\mathbb{P}(V')$, for every point it is a projective space lying (linearly) inside a projective space of one dimension higher. The complement is then over every point an affine space. One can show that the global complement

$$T = \mathbb{P}(\mathcal{S}'^\vee) \setminus \mathbb{P}(\mathcal{S}^\vee)$$

is another model for the torsor given by the cohomology class. The advantage of this viewpoint is that we may work, in particular when X is projective, in an entirely projective setting.

Semistability of vector bundles. In the situation of a forcing algebra of homogeneous elements, this torsor T can also be obtained as Proj B, where B is the (not necessarily positively) graded forcing algebra. In particular, it follows that the containment $f \in I^*$ is equivalent to the property that T is not an affine variety. For this properties, positivity (ampleness) properties of the syzygy bundle are crucial. We need the concept of (Mumford) - semistability.

Definition 4.2. Let $\mathcal{S}$ be a vector bundle on a smooth projective curve C. It is called *semistable* if

$$\mu(\mathcal{T}) = \frac{\deg(\mathcal{T})}{\mathrm{rk}(\mathcal{T})} \leq \frac{\deg(\mathcal{S})}{\mathrm{rk}(\mathcal{S})} = \mu(\mathcal{S})$$

for all subbundles $\mathcal{T}$.

Suppose that the base field has positive characteristic $p > 0$. Then $\mathcal{S}$ is called *strongly semistable*, if all (absolute) Frobenius pull-backs $F^{e*}(\mathcal{S})$ are semistable.

An important property of a semistable bundle of negative degree is that it can not have any global section $\neq 0$. Note that a semistable vector bundle need not be strongly semistable, the following is probably the simplest example.

Example 4.3. Let C be the smooth Fermat quartic given by $x^4+y^4+z^4$ and consider on it the syzygy bundle $\operatorname{Syz}(x, y, z)$ (which is also the restricted cotangent bundle from the projective plane). This bundle is semistable. Suppose that the characteristic is 3. Then its Frobenius pull-back is $\operatorname{Syz}(x^3, y^3, z^3)$. The curve equation gives a global nontrivial section of this bundle of total degree 4. But the degree of $\operatorname{Syz}(x^3, y^3, z^3)(4)$ is negative, hence it can not be semistable anymore.

The following example is related to Example 3.10.

Example 4.4. Let $R = K[X, Y, Z]/\left(X^3+Y^3+Z^3\right)$, where K is a field of positive characteristic $p \neq 3$, $I = \left(x^2, y^2, z^2\right)$, and $C = \operatorname{Proj}(R)$. The equation $x^3+y^3+z^3=0$ yields the short exact sequence

$$0 \to \mathcal{O}_C \to \operatorname{Syz}(x^2, y^2, z^2)(3) \to \mathcal{O}_C \to 0.$$

This shows that $\operatorname{Syz}(x^2, y^2, z^2)$ is strongly semistable, since the Frobenius pull-backs of this sequence show that all $F^{e*}\left(\operatorname{Syz}(x^2, y^2, z^2)\right)$ are semistable.

For a strongly semistable vector bundle $\mathcal{S}$ on C and a cohomology class $c \in H^1(C, \mathcal{S})$ with corresponding torsor we obtain the following affineness criterion (in characteristic zero we mean by strongly semistable just semistable).

Theorem 4.5. *Let C denote a smooth projective curve over an algebraically closed field K and let $\mathcal{S}$ be a strongly semistable vector bundle over C together with a cohomology class $c \in H^1(C, \mathcal{S})$. Then the torsor $T(c)$ is an affine scheme if and only if* $\deg(\mathcal{S}) < 0$ *and* $c \neq 0$ ($F^e(c) \neq 0$ *for all e in positive characteristic*).

This result rests on the ampleness of $\mathcal{S}'^{\vee}$ occurring in the dual exact sequence $0 \to \mathcal{O}_C \to \mathcal{S}'^{\vee} \to \mathcal{S}^{\vee} \to 0$ given by c (this rests on [Gieseker 1971] and [1971]). It implies for a strongly semistable syzygy bundle the following *degree formula* for tight closure.

Theorem 4.6. *Suppose that* $\operatorname{Syz}(f_1, \dots, f_n)$ *is strongly semistable. Then*

$$R_m \subseteq I^* \quad \textit{for } m \geq \frac{\sum d_i}{n-1}$$

and

$$R_m \cap I^* \subseteq I^F \quad \textit{for } m < \frac{\sum d_i}{n-1},$$

where I^F is the Frobenius closure. In a relative setting, if R' is a finitely generated $\mathbb{Z}$-algebra and $R = R'/(p)$, then

$$R_m \cap I^* \subseteq I \text{ for } m < \frac{\sum d_i}{n-1}$$

for almost all prime numbers.

We indicate the proof of the inclusion result. The degree condition implies that $c = \delta(f) \in H^1(C, \mathcal{S})$ is such that $\mathcal{S} = \operatorname{Syz}(f_1, \dots, f_n)(m)$ has nonnegative degree. Then also all Frobenius pull-backs $F^*(\mathcal{S})$ have nonnegative degree. Let $\mathcal{L} = \mathcal{O}(k)$ be a twist of the tautological line bundle on C such that its degree is larger than the degree of ω_C^{-1}, the dual of the canonical sheaf. Let $z \in H^0(Y, \mathcal{L})$ be a nonzero element. Then $zF^{e*}(c) \in H^1(C, F^{e*}(\mathcal{S}) \otimes \mathcal{L})$, and by Serre duality we have

$$H^1(C, F^{e*}(\mathcal{S}) \otimes \mathcal{L}) \cong H^0\big(F^{e*}(\mathcal{S}^\vee) \otimes \mathcal{L}^{-1} \otimes \omega_C\big)^\vee.$$

On the right we have a semistable sheaf of negative degree, which can not have a nontrivial section. Hence $zF^{e*}(c) = 0$ and therefore f belongs to the tight closure.

Harder–Narasimhan filtration. In general, there exists an exact criterion for the affineness of the torsor $T(c)$ depending on c and the *strong Harder–Narasimhan filtration* of $\mathcal{S}$. For this we give the definition of the Harder–Narasimhan filtration.

Definition 4.7. Let $\mathcal{S}$ be a vector bundle on a smooth projective curve C over an algebraically closed field K. Then the filtration

$$0 = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_{t-1} \subset \mathcal{S}_t = \mathcal{S}$$

of subbundles such that all quotient bundles $\mathcal{S}_k/\mathcal{S}_{k-1}$ are semistable with decreasing slopes $\mu_k = \mu(\mathcal{S}_k/\mathcal{S}_{k-1})$, is called the *Harder–Narasimhan filtration* of $\mathcal{S}$. This object exists uniquely by a theorem of Harder and Narasimhan [1975].

A Harder–Narasimhan filtration is called *strong* if all the quotients $\mathcal{S}_i/\mathcal{S}_{i-1}$ are strongly semistable. A Harder–Narasimhan filtration is not strong in general; however, by a theorem of A. Langer [2004, Theorem 2.7], there exists some Frobenius pull-back $F^{e*}(\mathcal{S})$ such that its Harder–Narasimhan filtration is strong.

Theorem 4.8. *Let C denote a smooth projective curve over an algebraically closed field K and let $\mathcal{S}$ be a vector bundle over C together with a cohomology class $c \in H^1(C, \mathcal{S})$. Let*

$$\mathcal{S}_1 \subset \mathcal{S}_2 \subset \cdots \subset \mathcal{S}_{t-1} \subset \mathcal{S}_t = F^{e*}(\mathcal{S})$$

be a strong Harder–Narasimhan filtration. We choose i *such that* $\mathcal{S}_i/\mathcal{S}_{i-1}$ *has degree* ≥ 0 *and that* $\mathcal{S}_{i+1}/\mathcal{S}_i$ *has degree* < 0. *We set* $\mathcal{Q} = F^{e*}(\mathcal{S})/\mathcal{S}_i$. *Then the following are equivalent.*

(1) *The torsor* $T(c)$ *is not an affine scheme.*

(2) *Some Frobenius power of the image of* $F^{e*}(c)$ *inside* $H^1(X, \mathcal{Q})$ *is* 0.

Plus closure in dimension two. Let K be a field and let R be a normal two-dimensional standard-graded domain over K with corresponding smooth projective curve C. A homogeneous $\mathfrak{m}$-primary ideal with homogeneous ideal generators $f_1, \dots, f_n$ and another homogeneous element f of degree m yield a cohomology class

$$c = \delta(f) \in H^1(C, \operatorname{Syz}(f_1, \dots, f_n)(m)).$$

Let $T(c)$ be the corresponding torsor. We have seen that the affineness of this torsor over C is equivalent to the affineness of the corresponding torsor over $D(\mathfrak{m}) \subseteq \operatorname{Spec}(R)$. Now we want to understand what the property $f \in I^+$ means for c and for $T(c)$. Instead of the plus closure we will work with the *graded plus closure* $I^{+\mathrm{gr}}$, where $f \in I^{+\mathrm{gr}}$ holds if and only if there exists a finite graded extension $R \subseteq S$ such that $f \in IS$. The existence of such an S translates into the existence of a finite morphism

$$\varphi : C' = \operatorname{Proj}(S) \to \operatorname{Proj}(R) = C$$

such that $\varphi^*(c) = 0$. Here we may assume that C' is also smooth. Therefore we discuss the more general question when a cohomology class $c \in H^1(C, \mathcal{S})$, where $\mathcal{S}$ is a locally free sheaf on C, can be annihilated by a finite morphism

$$C' \to C$$

of smooth projective curves. The advantage of this more general approach is that we may work with short exact sequences (in particular, the sequences coming from the Harder–Narasimhan filtration) in order to reduce the problem to semistable bundles which do not necessarily come from an ideal situation.

Lemma 4.9. *Let* C *denote a smooth projective curve over an algebraically closed field* K, *let* $\mathcal{S}$ *be a locally free sheaf on* C *and let* $c \in H^1(C, \mathcal{S})$ *be a cohomology class with corresponding torsor* $T \to C$. *Then the following conditions are equivalent.*

(1) *There exists a finite morphism*

$$\varphi : C' \to C$$

from a smooth projective curve C' *such that* $\varphi^*(c) = 0$.

(2) *There exists a projective curve* $Z \subseteq T$.

Proof. If (1) holds, then the pull-back $\varphi^*(T) = T \times_C C'$ is trivial (as a torsor), as it equals the torsor given by $\varphi^*(c) = 0$. Hence $\varphi^*(T)$ is isomorphic to a vector bundle and contains in particular a copy of C'. The image Z of this copy is a projective curve inside T. If (2) holds, then let C' be the normalization of Z. Since Z dominates C, the resulting morphism

$$\varphi : C' \to C$$

is finite. Since this morphism factors through T and since T annihilates the cohomology class by which it is defined, it follows that $\varphi^*(c) = 0$. □

We want to show that the cohomological criterion for (non) -affineness of a torsor along the Harder–Narasimhan filtration of the vector bundle also holds for the existence of projective curves inside the torsor, under the condition that the projective curve is defined over a finite field. This implies that tight closure is (graded) plus closure for graded $\mathfrak{m}$-primary ideals in a two-dimensional graded domain over a finite field.

Annihilation of cohomology classes of strongly semistable sheaves. We deal first with the situation of a strongly semistable sheaf $\mathcal{S}$ of degree 0. The following two results are from [Lange and Stuhler 1977]. We say that a locally free sheaf is *étale trivializable* if there exists a finite étale morphism $\varphi : C' \to C$ such that $\varphi^*(\mathcal{S}) \cong \mathcal{O}_{C'}^r$. Such bundles are directly related to linear representations of the étale fundamental group.

Lemma 4.10. *Let K denote a finite field* (*or the algebraic closure of a finite field*) *and let C be a smooth projective curve over K. Let $\mathcal{S}$ be a locally free sheaf over C. Then $\mathcal{S}$ is étale trivializable if and only if there exists some n such that $F^{n*}\mathcal{S} \cong \mathcal{S}$.*

Theorem 4.11. *Let K denote a finite field* (*or the algebraic closure of a finite field*) *and let C be a smooth projective curve over K. Let $\mathcal{S}$ be a strongly semistable locally free sheaf over C of degree* 0. *Then there exists a finite morphism*

$$\varphi : C' \to C$$

such that $\varphi^*(\mathcal{S})$ is trivial.

Proof. We consider the family of locally free sheaves $F^{e*}(\mathcal{S})$, $e \in \mathbb{N}$. Because these are all semistable of degree 0, and defined over the same finite field, we must have (by the existence of the moduli space for vector bundles) a repetition: $F^{e*}(\mathcal{S}) \cong F^{e'*}(\mathcal{S})$ for some $e' > e$. By Lemma 4.10 the bundle $F^{e*}(\mathcal{S})$ admits an étale trivialization $\varphi : C' \to C$. Hence the finite map $F^e \circ \varphi$ trivializes the bundle. □

Theorem 4.12. *Let K denote a finite field (or the algebraic closure of a finite field) and let C be a smooth projective curve over K. Let $\mathcal{S}$ be a strongly semistable locally free sheaf over C of nonnegative degree and let $c \in H^1(C, \mathcal{S})$ denote a cohomology class. Then there exists a finite morphism*

$$\varphi : C' \to C$$

such that $\varphi^*(c)$ is trivial.

Proof. If the degree of $\mathcal{S}$ is positive, then a Frobenius pull-back $F^{e*}(\mathcal{S})$ has arbitrary large degree and is still semistable. By Serre duality we get that $H^1(C, F^{e*}(\mathcal{S})) = 0$. So in this case we can annihilate the class by an iteration of the Frobenius alone. So suppose that the degree is 0. Then there exists by Theorem 4.11 a finite morphism which trivializes the bundle. So we may assume that $\mathcal{S} \cong \mathcal{O}_C^r$. Then the cohomology class has several components $c_i \in H^1(C, \mathcal{O}_C)$ and it is enough to annihilate them separately by finite morphisms. But this is possible by the parameter theorem of K. Smith [1994] (or directly using Frobenius and Artin–Schreier extensions). ☐

The general case. We look now at an arbitrary locally free sheaf $\mathcal{S}$ on C, a smooth projective curve over a finite field. We want to show that the same numerical criterion (formulated in terms of the Harder–Narasimhan filtration) for nonaffineness of a torsor holds also for the finite annihilation of the corresponding cohomology class (or the existence of a projective curve inside the torsor).

Theorem 4.13. *Let K denote a finite field (or the algebraic closure of a finite field) and let C be a smooth projective curve over K. Let $\mathcal{S}$ be a locally free sheaf over C and let $c \in H^1(C, \mathcal{S})$ denote a cohomology class. Let $\mathcal{S}_1 \subset \cdots \subset \mathcal{S}_t$ be a strong Harder–Narasimhan filtration of $F^{e*}(\mathcal{S})$. We choose i such that $\mathcal{S}_i/\mathcal{S}_{i-1}$ has degree ≥ 0 and that $\mathcal{S}_{i+1}/\mathcal{S}_i$ has degree < 0. We set $\mathcal{Q} = F^{e*}(\mathcal{S})/\mathcal{S}_i$. Then the following are equivalent.*

(1) *The class c can be annihilated by a finite morphism.*

(2) *Some Frobenius power of the image of $F^{e*}(c)$ inside $H^1(C, \mathcal{Q})$ is 0.*

Proof. Suppose that (1) holds. Then the torsor is not affine and hence by Theorem 4.8 also (2) holds. So suppose that (2) is true. By applying a certain power of the Frobenius we may assume that the image of the cohomology class in $\mathcal{Q}$ is 0. Hence the class stems from a cohomology class $c_i \in H^1(C, \mathcal{S}_i)$. We look at the short exact sequence

$$0 \to \mathcal{S}_{i-1} \to \mathcal{S}_i \to \mathcal{S}_i/\mathcal{S}_{i-1} \to 0,$$

where the sheaf of the right-hand side has a nonnegative degree. Therefore the image of c_i in $H^1(C, \mathcal{S}_i/\mathcal{S}_{i-1})$ can be annihilated by a finite morphism due to

Theorem 4.12. Hence after applying a finite morphism we may assume that c_i stems from a cohomology class $c_{i-1} \in H^1(C, \mathcal{S}_{i-1})$. Going on inductively we see that c can be annihilated by a finite morphism. ☐

Theorem 4.14. *Let C denote a smooth projective curve over the algebraic closure of a finite field K, let $\mathcal{S}$ be a locally free sheaf on C and let $c \in H^1(C, \mathcal{S})$ be a cohomology class with corresponding torsor $T \to C$. Then T is affine if and only if it does not contain any projective curve.*

Proof. Due to Theorem 4.8 and Theorem 4.13, for both properties the same numerical criterion does hold. ☐

These results imply the following theorem in the setting of a two-dimensional graded ring.

Theorem 4.15. *Let R be a standard-graded, two-dimensional normal domain over (the algebraic closure of) a finite field. Let I be an R_+-primary graded ideal. Then*

$$I^* = I^+.$$

This is also true for nonprimary graded ideals and also for submodules in finitely generated graded submodules. Moreover, G. Dietz [2006] has shown that one can get rid also of the graded assumption (of the ideal or module, but not of the ring).

5. Tight closure in families

After having understood tight closure and plus closure in the two-dimensional situation we proceed to a special three-dimensional situation, namely families of two-dimensional rings parametrized by a one-dimensional base scheme.

Affineness under deformations. We consider a base scheme B and a morphism

$$Z \to B$$

together with an open subscheme $W \subseteq Z$. For every base point $b \in B$ we get the open subset

$$W_b \subseteq Z_b$$

inside the fiber Z_b. It is a natural question to ask how properties of W_b vary with b. In particular we may ask how the cohomological dimension of W_b varies and how the affineness may vary.

In the algebraic setting we have a D-algebra S and an ideal $\mathfrak{a} \subseteq S$ (so $B = \operatorname{Spec}(D)$, $Z = \operatorname{Spec}(S)$ and $W = D(\mathfrak{a})$) which defines for every prime ideal $\mathfrak{p} \in \operatorname{Spec}(D)$ the extended ideal $\mathfrak{a}_{\mathfrak{p}}$ in $S \otimes_D \kappa(\mathfrak{p})$.

This question is already interesting when $B = \operatorname{Spec}(D)$ is an affine one-dimensional integral scheme, in particular in the following two situations.

(1) $B = \operatorname{Spec}(\mathbb{Z})$. Then we speak of an *arithmetic deformation* and want to know how affineness varies with the characteristic and what the relation is to characteristic zero.

(2) $B = \mathbb{A}^1_K = \operatorname{Spec}(K[t])$, where K is a field. Then we speak of a *geometric deformation* and want to know how affineness varies with the parameter t, in particular how the behavior over the special points where the residue class field is algebraic over K is related to the behavior over the generic point.

It is fairly easy using Lemma 3.7 (1) to show that if the open subset in the generic fiber is affine, then also the open subsets are affine for almost all special points.

We deal with this question where W is a torsor over a family of smooth projective curves (or a torsor over a punctured two-dimensional spectrum). The arithmetic as well as the geometric variant of this question are directly related to questions in tight closure theory. Because of the above mentioned degree criteria in the strongly semistable case, a weird behavior of the affineness property of torsors is only possible if we have a weird behavior of strong semistability.

Arithmetic deformations. We start with the arithmetic situation.

Example 5.1 [Brenner and Katzman 2006]. Consider $\mathbb{Z}[X, Y, Z]/(X^7+Y^7+Z^7)$ and take the ideal $I = (x^4, y^4, z^4)$ and the element $f = x^3y^3$. Consider reductions $\mathbb{Z} \to \mathbb{Z}/(p)$. Then

$$f \in I^* \text{ holds in } \mathbb{Z}/(p)[x, y, z]/(x^7 + y^7 + z^7) \text{ for } p \equiv 3 \mod 7$$

and

$$f \notin I^* \text{ holds in } \mathbb{Z}/(p)[x, y, z]/(x^7 + y^7 + z^7) \text{ for } p \equiv 2 \mod 7.$$

In particular, the bundle $\operatorname{Syz}(x^4, y^4, z^4)$ is semistable in the generic fiber, but not strongly semistable for any reduction $p \equiv 2 \mod 7$. The corresponding torsor is an affine scheme for infinitely many prime reductions and not an affine scheme for infinitely many prime reductions.

In terms of affineness (or local cohomology) this example has the following properties: the ideal

$$(x, y, z) \subseteq \mathbb{Z}/(p)[x, y, z, s_1, s_2, s_3]/(x^7 + y^7 + z^7, s_1x^4 + s_2y^4 + s_3z^4 + x^3y^3)$$

has cohomological dimension 1 if $p = 3 \mod 7$ and has cohomological dimension 0 (equivalently, $D(x, y, z)$ is an affine scheme) if $p = 2 \mod 7$.

Geometric deformations: a counterexample to the localization problem. Let $S \subseteq R$ be a multiplicative system and I an ideal in R. Then the *localization problem* of tight closure is the question whether the identity

$$(I^*)_S = (IR_S)^*$$

holds.

Here the inclusion $\subseteq$ is always true and $\supseteq$ is the problem. This means explicitly:

Question. If $f \in (IR_S)^*$, can we find an $h \in S$ such that $hf \in I^*$ holds in R?

Proposition 5.2. *Let $\mathbb{Z}/(p) \subset D$ be a one-dimensional domain and $D \subseteq R$ of finite type, and I an ideal in R. Suppose that localization holds and that*

$$f \in I^* \text{ holds in } R \otimes_D Q(D) = R_{D^*} = R_{Q(D)}$$

($S = D^* = D \setminus \{0\}$ is the multiplicative system). Then $f \in I^*$ holds in $R \otimes_D \kappa(\mathfrak{p})$ for almost all $\mathfrak{p}$ in Spec D.

Proof. By localization, there exists $h \in D$, $h \neq 0$, such that

$$hf \in I^* \text{ in } R.$$

By persistence of tight closure (under a ring homomorphism) we get

$$hf \in I^* \text{ in } R_{\kappa(\mathfrak{p})}.$$

The element h does not belong to $\mathfrak{p}$ for almost all $\mathfrak{p}$, so h is a unit in $R_{\kappa(\mathfrak{p})}$ and hence

$$f \in I^* \text{ in } R_{\kappa(\mathfrak{p})}$$

for almost all $\mathfrak{p}$. ☐

In order to get a counterexample to the localization property we will look now at geometric deformations:

$$D = \mathbb{F}_p[t] \subset \mathbb{F}_p[t][X, Y, Z]/(g) = S,$$

where t has degree 0 and X, Y, Z have degree 1 and g is homogeneous. Then (for every field $\mathbb{F}_p[t] \subseteq K$)

$$S \otimes_{\mathbb{F}_p[t]} K$$

is a two-dimensional standard-graded ring over K. For residue class fields of points of $\mathbb{A}^1_{\mathbb{F}_p} = \text{Spec}(\mathbb{F}_p[t])$ we have basically two possibilities.

- $K = \mathbb{F}_p(t)$, the function field. This is the *generic* or *transcendental* case.
- $K = \mathbb{F}_q$, the *special* or *algebraic* or *finite* case.

How does $f \in I^*$ vary with K? To analyze the behavior of tight closure in such a family we can use what we know in the two-dimensional standard-graded situation.

In order to establish an example where tight closure does not behave uniformly under a geometric deformation we first need a situation where strong semistability does not behave uniformly. Such an example was given by Paul Monsky in terms of Hilbert–Kunz theory:

Example 5.3 [Monsky 1998]. Let

$$g = Z^4 + Z^2XY + Z(X^3+Y^3) + (t+t^2)X^2Y^2.$$

Consider

$$S = \mathbb{F}_2[t, X, Y, Z]/(g).$$

Monsky proved the following results on the *Hilbert–Kunz multiplicity* of the maximal ideal (x, y, z) in $S \otimes_{\mathbb{F}_2[t]} L$, L a field:

$$e_{HK}(S \otimes_{\mathbb{F}_2[t]} L) = \begin{cases} 3 & \text{for } L = \mathbb{F}_2(t), \\ 3 + 4^{-d} & \text{for } L = \mathbb{F}_q = \mathbb{F}_2(\alpha) \ (t \mapsto \alpha,\ d = \deg(\alpha)). \end{cases}$$

By the geometric interpretation of Hilbert–Kunz theory (see [Brenner 2006b; 2007; Trivedi 2005]) this means that the restricted cotangent bundle

$$\operatorname{Syz}(x, y, z) = (\Omega_{\mathbb{P}^2})|_C$$

is strongly semistable in the transcendental case, but not strongly semistable in the algebraic case. In fact, for $d = \deg(\alpha)$, $t \mapsto \alpha$, where $L = \mathbb{F}_2(\alpha)$, the d-th Frobenius pull-back destabilizes (meaning that it is not semistable anymore).

The maximal ideal (x, y, z) can not be used directly. However, we look at the second Frobenius pull-back which is (characteristic two) just

$$I = (x^4, y^4, z^4).$$

By the degree formula we have to look for an element of degree 6. Let's take

$$f = y^3z^3.$$

This is our example (x^3y^3 does not work). First, by strong semistability in the transcendental case we have

$$f \in I^* \text{ in } S \otimes \mathbb{F}_2(t)$$

by the degree formula. If localization would hold, then f would also belong to the tight closure of I for almost all algebraic instances $\mathbb{F}_q = \mathbb{F}_2(\alpha)$, $t \mapsto \alpha$. Contrary to that we show that for all algebraic instances the element f belongs never to the tight closure of I.

Lemma 5.4. *Let* $\mathbb{F}_q = \mathbb{F}_p(\alpha)$, $t \mapsto \alpha$, $\deg(\alpha) = d$. *Set* $Q = 2^{d-1}$. *Then*

$$xyf^Q \notin I^{[Q]}.$$

Proof. This is an elementary but tedious computation [Brenner and Monsky 2010]. ☐

Theorem 5.5. *Tight closure does not commute with localization.*

Proof. One knows in our situation that xy is a so-called test element. Hence the previous lemma shows that $f \notin I^*$. ☐

In terms of affineness (or local cohomology) this example has the following properties: the ideal

$$(x, y, z) \subseteq \mathbb{F}_2(t)[x, y, z, s_1, s_2, s_3]/(g, s_1x^4 + s_2y^4 + s_3z^4 + y^3z^3)$$

has cohomological dimension 1 if t is transcendental and has cohomological dimension 0 (equivalently, $D(x, y, z)$ is an affine scheme) if t is algebraic.

Corollary 5.6. *Tight closure is not plus closure in graded dimension two for fields with transcendental elements.*

Proof. Consider

$$R = \mathbb{F}_2(t)[X, Y, Z]/(g).$$

In this ring $y^3z^3 \in I^*$, but it can not belong to the plus closure. Else there would be a curve morphism $Y \to C_{\mathbb{F}_2(t)}$ which annihilates the cohomology class c and this would extend to a morphism of relative curves almost everywhere. ☐

Corollary 5.7. *There is an example of a smooth projective* (*relatively over the affine line*) *variety* Z *and an effective divisor* $D \subset Z$ *and a morphism*

$$Z \to \mathbb{A}^1_{\mathbb{F}_2}$$

such that $(Z \setminus D)_\eta$ *is not an affine variety over the generic point* η, *but for every algebraic point* x *the fiber* $(Z \setminus D)_x$ *is an affine variety.*

Proof. Take $C \to \mathbb{A}^1_{\mathbb{F}_2}$ to be the Monsky quartic and consider the syzygy bundle

$$\mathcal{S} = \mathrm{Syz}(x^4, y^4, z^4)(6)$$

together with the cohomology class c determined by $f = y^3z^3$. This class defines an extension

$$0 \to \mathcal{S} \to \mathcal{S}' \to \mathcal{O}_C \to 0$$

and hence $\mathbb{P}(\mathcal{S}^*) \subset \mathbb{P}(\mathcal{S}'^*)$. Then $\mathbb{P}(\mathcal{S}'^*) \setminus \mathbb{P}(\mathcal{S}^*)$ is an example with the stated properties by the previous results. ☐

It is an open question whether such an example can exist in characteristic zero.

Generic results. Is it more difficult to decide whether an element belongs to the tight closure of an ideal or to the ideal itself? We discuss one situation where this is easier for tight closure.

Suppose that we are in a graded situation of a given ring (or a given ring dimension) and have fixed a number (at least the ring dimension) of homogeneous generators and their degrees. Suppose that we want to know the degree bound for (tight closure or ideal) inclusion for generic choice of the ideal generators. Generic means that we write the coefficients of the generators as indeterminates and consider the situation over the (large) affine space corresponding to these indeterminates or over its function field. This problem is already interesting and difficult for the polynomial ring: Suppose we are in $P = K[X, Y, Z]$ and want to study the generic inclusion bound for, say, $n \geq 4$ generic polynomials $F_1, \dots, F_n$ all of degree a. What is the minimal degree number m such that $P_{\geq m} \subseteq (F_1, \dots, F_n)$. The answer is

$$\left\lceil \frac{1}{2(n-1)}\bigl(3 - 3n + 2an + \sqrt{1 - 2n + n^2 + 4a^2 n}\,\bigr) \right\rceil .$$

This rests on the fact that the Fröberg conjecture has been solved in dimension 3, by D. Anick [1986]. (The Fröberg conjecture gives a precise description of the Hilbert function for an ideal in a polynomial ring which is generically generated. Here we only need to know in which degree the Hilbert function of the residue class ring becomes 0.)

The corresponding generic ideal inclusion bound for arbitrary graded rings depends heavily (already in the parameter case) on the ring itself. Surprisingly, the generic ideal inclusion bound for tight closure does not depend on the ring and is only slightly worse than the bound for the polynomial ring. The following theorem is due to Brenner and Fischbacher–Weitz [Brenner and Fischbacher-Weitz 2011].

Theorem 5.8. *Let* $d \geq 1$ *and* $a_1, \dots, a_n$ *be natural numbers* (*the degree type*), $n \geq d+1$. *Let* $K[x_0, x_1, \dots, x_d] \subseteq R$ *be a finite extension of standard-graded domains* (*a graded Noether normalization*). *Suppose that there exist n homogeneous polynomials* $g_1, \dots, g_n$ *in* $P = K[x_0, x_1, \dots, x_d]$ *with* $\deg(g_i) = a_i$ *such that* $P_{\geq m} \subseteq (g_1, \dots, g_n)$.

(1) $R_{m+d} \subseteq (f_1, \dots, f_n)^*$ *holds over the generic point of the parameter space* (*after the base change to the function field of this space*) *of homogeneous elements* $f_1, \dots, f_n$ *in R of this degree type* (*the coefficients of the* f_i *are taken as indeterminates*).

(2) *If R is normal, then $R_{m+d+1} \subseteq (f_1, \dots, f_n)^F \subseteq (f_1, \dots, f_n)^*$ holds for (open) generic choice of homogeneous elements $f_1, \dots, f_n$ in R of this degree type.*

Example 5.9. Suppose that we are in $K[x, y, z]$ and that $n = 4$ and $a = 10$. Then the generic degree bound for ideal inclusion in the polynomial ring is 19. Therefore by Theorem 5.8 the generic degree bound for tight closure inclusion in a three-dimensional graded ring is 21.

Example 5.10. Suppose that $n = d + 1$ in the situation of Theorem 5.8. Then the generic elements $f_1, \dots, f_{d+1}$ are parameters. In the polynomial ring $P = K[x_0, x_1, \dots, x_d]$ we have for parameters of degree $a_1, \dots, a_{d+1}$ the inclusion

$$P_{\geq \sum_{i=0}^{d} a_i - d} \subseteq (f_1, \dots, f_{d+1}),$$

because the graded Koszul resolution ends in $R(-\sum_{i=0}^{d} a_i)$ and

$$(H_{\mathfrak{m}}^{d+1}(P))_k = 0 \quad \text{for } k \geq -d.$$

So the theorem implies for a graded ring R finite over P that

$$P_{\geq \sum_{i=0}^{d} a_i} \subseteq (f_1, \dots, f_{d+1})^*$$

holds for generic elements. But by the graded Briançon–Skoda theorem [Huneke 1998] this holds for parameters even without the generic assumption.

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